Day 307 **IGCSE 10-a-day: Model Solutions**

1 Rectangle under a parabola

The diagram shows the curve $y = 75 - x^2$. Rectangle ABCD has A, B on the x-axis and C, D on the curve. B is (x, 0).
(a) Show that the area of ABCD is $-2x^3 + 150x$.
(b) Find the maximum area of ABCD, showing that it is a maximum.

AB = 2x and BC = $75 - x^2$, so area = $2x(75 - x^2) = -2x^3 + 150x$

$\frac{dA}{dx} = -6x^2 + 150 = 0$, so $x^2 = 25$ and $x = 5$ ($x > 0$)

$\frac{d^2A}{dx^2} = -12x < 0$ when $x = 5$, so a maximum

Area = $150 \times 5 - 2 \times 5^3 = 500$

Answer: 500

2 Equations with fractions

Solve $\frac{x - 3}{8} + \frac{x + 4}{4} = 1$

Multiply every term by 8: $(x - 3) + 2(x + 4) = 8$
Expand: $3x + 5 = 8$
$3x = 3$
$x = 1$
Answer: x = 1

3 Angle in a semicircle

AB is a diameter of a circle, centre O. C is a point on the circle. Angle BAC = 62°.
(a) Work out the size of angle ACB.
(b) Work out the size of angle COB.
The diagram is not drawn accurately.

(a) The angle in a semicircle is 90°, so angle ACB = 90°
(b) OA = OC (radii), so triangle OAC is isosceles and angle OCA = 62°
Angle COB is an exterior angle of triangle OAC: 62 + 62 = 124°
(or: the angle at the centre is twice the angle at the circumference)
Answer: (a) 90° (b) 124°

4 Composite functions in x

$f(x) = 2x + 2$ and $g(x) = x^2 - 2x + 1$.
(a) Find fg(x).
(b) Find gf(x).

(a) $fg(x) = f(x^2 - 2x + 1) = 2(x^2 - 2x + 1) + 2$
$= 2x^2 - 4x + 4$
(b) $gf(x) = g(2x + 2) = (2x + 2)^2 - 2(2x + 2) + 1$
$= 4x^2 + 4x + 1$
Answer: (a) fg(x) = $2x^2 - 4x + 4$ (b) gf(x) = $4x^2 + 4x + 1$

5 Three-part ratios

Tariq, Leila and Rhys share some sweets.
Tariq : Leila = 8 : 9
Leila : Rhys = 6 : 4
(a) Work out the ratio Tariq : Leila : Rhys.
(b) They share 276 sweets altogether. How many sweets does Leila get?

(a) Make the Leila part the same: 8 : 9 = 16 : 18 and 6 : 4 = 18 : 12
Tariq : Leila : Rhys = 16 : 18 : 12 = 8 : 9 : 6
(b) Total parts = 23, so 1 part = 276 ÷ 23 = 12
Leila gets 9 × 12 = 108 sweets
Answer: (a) 8 : 9 : 6 (b) 108 sweets

6 Upper and lower bounds

a = 66 and b = 32,
both correct to the nearest whole number.
Work out the lower bound of a × b.

Each value is rounded, so it can be up to 0.5 above or below
$65.5 \leq a < 66.5$; $31.5 \leq b < 32.5$
Lower bound = 65.5 × 31.5 = 2063.25
Answer: 2063.25

7 Rates of work

A tank is filled by a tap in 24 minutes.
A drain can empty the full tank in 60 minutes.
The tap and the drain are both open, and the tank is empty to begin with.
How long does it take for the tank to fill?

In 1 minute: the tap fills $\frac{1}{24}$ and the drain empties $\frac{1}{60}$

Net: $\frac{1}{24} - \frac{1}{60} = \frac{36}{1440}$

Time = $\frac{1440}{36}$ = 40 minutes

Answer: 40 minutes

8 Fencing with a partition

A rectangular field is x metres wide and y metres long. It is fenced all round and divided into two equal parts by a fence each of length x metres, parallel to the width.
The total length of fencing used is 48 m.

(a) Show that the area of the field is $A = -\frac{3}{2}x^2 + 24x$.

(b) Find the maximum possible area, justifying that it is a maximum.

Fencing: $3x + 2y = 48$, so $y = \frac{48 - 3x}{2}$

$A = xy = \frac{x(48 - 3x)}{2} = -\frac{3}{2}x^2 + 24x$

$\frac{dA}{dx} = -3x + 24 = 0$ gives $x = 8$

$\frac{d^2A}{dx^2} = -3 < 0$, so a maximum

A = 96
Answer: 96 m²

9 Three sets from information

72 pupils were asked which activities they do.
A: play football B: play netball C: play rugby
n(A) = 30, n(B) = 36, n(C) = 39, $n(A \cap B) = 8$, $n(A \cap C) = 13$, $n(B \cap C) = 18$, $n(A \cap B \cap C) = 4$.
(a) Work out how many are in none of A, B and C.
(b) Work out how many are in exactly one of A, B and C.

$n(A \cup B \cup C) = 30 + 36 + 39 - 8 - 13 - 18 + 4 = 70$
(a) Number in none of the sets = 72 − 70 = 2
(b) Only A = 30 − 8 − 13 + 4 = 13
Only B = 14, only C = 12, so exactly one = 13 + 14 + 12 = 39
Answer: (a) 2 (b) 39

10 Open box of fixed surface area

An open container has a square base of side x cm and height h cm. The total outside surface area is 108 cm^2.

(a) Show that the volume of the box is $V = \frac{1}{4}(108x - x^3)$ cm^3.

(b) Find the maximum volume of the box, proving that it is a maximum.

$x^2 + 4xh = 108$, so $h = \frac{108 - x^2}{4x}$

$V = x^2h = \frac{x(108 - x^2)}{4} = \frac{1}{4}(108x - x^3)$

$\frac{dV}{dx} = -\frac{3}{4}x^2 + 27 = 0$, so $x^2 = 36$ and $x = 6$

$\frac{d^2V}{dx^2} = -\frac{3}{2}x = -9 < 0$, so a maximum

$V = 108$

Answer: 108 cm^3