Day 274 **IGCSE 10-a-day: Model Solutions**

1 Average speed

A car travels the first half of a journey at 30 km/h.
It travels the second half of the same journey at 70 km/h.
Work out the average speed for the whole journey.

Let each half of the journey be 210 km (a common multiple of 30 and 70)
Time for first half = 210 ÷ 30 = 7 hours
Time for second half = 210 ÷ 70 = 3 hours
Average speed = total distance ÷ total time = 420 ÷ 10 = 42 km/h
Answer: 42 km/h

2 Multiplying 2 by 2 matrices

$A = \begin{pmatrix} 4 & -4 \\ 5 & -2 \end{pmatrix}$ and $B = \begin{pmatrix} -1 & 1 \\ -4 & 2 \end{pmatrix}$.
(a) Work out AB.
(b) Work out BA.

(a) AB: row 1 is $4 \times (-1) + (-4) \times (-4)$ and $4 \times 1 + (-4) \times 2$; row 2 is $5 \times (-1) + (-2) \times (-4)$ and $5 \times 1 + (-2) \times 2$
$AB = \begin{pmatrix} 12 & -4 \\ 3 & 1 \end{pmatrix}$
(b) $BA = \begin{pmatrix} 1 & 2 \\ -6 & 12 \end{pmatrix}$
Answer: (a) AB $= \begin{pmatrix} \mathbf{12} & \mathbf{-4} \\ \mathbf{3} & \mathbf{1} \end{pmatrix}$ **(b) BA** $= \begin{pmatrix} \mathbf{1} & \mathbf{2} \\ \mathbf{-6} & \mathbf{12} \end{pmatrix}$

3 LCM problems

A red light flashes every 4 minutes.
A blue light flashes every 6 minutes.
A green light flashes every 21 minutes.
They all flash together at 6:00 am.
(a) At what time do they next flash together?

Find the LCM of 4, 6 and 21
$4 = 2^2$, $6 = 2 \times 3$, $21 = 3 \times 7$
LCM $= 2^2 \times 3 \times 7 = 84$
84 minutes after 6:00 am is 7:24 am
Answer: 7:24 am

4 Increasing for x greater than k

The function $f(x) = -2x^2 - 12x + 6$ is decreasing when $x > k$.
Find the value of k.

$f'(x) = -4x - 12$
The turning point is where $f'(x) = 0$: $-4x - 12 = 0$
$x = -3$
$f'(x)$ is negative for $x > -3$, so $k = -3$
Answer: k = −3

5 Cosine rule and area of a triangle

The diagram shows triangle ABC. (Calculator)
(a) Work out angle ACB, marked θ, to 1 decimal place.
(b) Work out the area of triangle ABC to 3 significant figures.
The diagram is not drawn accurately.

(a) $\cos\theta = \frac{12^2 + 10^2 - 15^2}{2 \times 12 \times 10} = \frac{19}{240}$

$\theta = 85.5°$ (1 d.p.)

(b) Area $= \frac{1}{2} \times 12 \times 10 \times \sin 85.5°$

$= 59.8\ \text{cm}^2$ (3 s.f.)

Answer: (a) 85.5° (b) 59.8 cm^2

6 Three stationary points of a quartic

$y = 2x^4 - 36x^2 - 5$
Find the coordinates of the stationary points of the curve and determine the nature of each.

$\frac{dy}{dx} = 8x^3 - 72x = 8x(x - 3)(x + 3)$

$\frac{dy}{dx} = 0$ gives $x = -3$, $x = 0$ or $x = 3$

Points: $(-3, -167)$, $(0, -5)$, $(3, -167)$

$\frac{d^2y}{dx^2} = 24x^2 - 72$

$x = -3$: $\frac{d^2y}{dx^2} = 144$; $x = 0$: $\frac{d^2y}{dx^2} = -72$; $x = 3$: $\frac{d^2y}{dx^2} = 144$

So $(-3, -167)$ is a minimum, $(0, -5)$ is a maximum, $(3, -167)$ is a minimum

Answer: (−3, −167) minimum; (0, −5) maximum; (3, −167) minimum

7 Exponential graphs

The diagram shows the graph of $y = ka^x$, where k and a are constants and $a > 0$.
The curve passes through the points (1, 12) and (3, 108).
Find the value of k and the value of a.

Substitute the points: $k\,a^1 = 12$ and $k\,a^3 = 108$

Divide: $a^2 = \frac{108}{12} = 9$

$a = 3$ (positive, since $a > 0$)

From the first point: $k \times 3 = 12$, so $k = 4$

Answer: k = 4, a = 3

8 Normal to y = kx^2: show that

P is the point $(a, 4a^2)$ on the curve $y = 4x^2$, where $a \neq 0$.
The normal to the curve at P meets the y-axis at the point Q.
Show that the y coordinate of Q is $4a^2 + \frac{1}{8}$.

$\frac{dy}{dx} = 8x$, so the gradient of the tangent at P is $8a$

Normal gradient $= -\frac{1}{8a}$

Normal: $y - 4a^2 = -\frac{1}{8a}(x - a)$

Put $x = 0$: $y = 4a^2 + \frac{a}{8a} = 4a^2 + \frac{1}{8}$

Answer: shown: $y = 4a^2 + \frac{1}{8}$

9 Probability and quadratic equations

A bag contains x black discs and 2 white discs.
Two discs are taken at random without replacement.
The probability that both are black is $\frac{10}{21}$.

(a) Show that $11x^2 - 51x - 20 = 0$.
(b) Hence work out the number of discs in the bag.

(a) P(both black) = $\frac{x}{x+2} \times \frac{x-1}{x+1} = \frac{10}{21}$

$21\,x(x-1) = 10(x+2)(x+1)$

$21x^2 - 21x = 10x^2 + 30x + 20$ so $11x^2 - 51x - 20 = 0$

(b) Solve: $x = 5$ or $x = -\frac{4}{11}$

x cannot be negative, so $x = 5$

Number of discs = 5 + 2 = 7

Answer: x = 5, so 7 discs in the bag

10 Functions that are their own inverse

$f(x) = \frac{-x+3}{x+1}$ where $x \neq -1$

(a) Show that ff(x) = x.
(b) Write down $f^{-1}(x)$.

(a) Replace x in f(x) by f(x), then multiply the top and the bottom by (x + 1)

$ff(x) = \frac{(-1)(-x+3) + 3(x+1)}{(-x+3) - (-1)(x+1)}$

Top = $x - 3 + 3x + 3 = 4x$

Bottom = $-x + 3 + x + 1 = 4$

$ff(x) = \frac{4x}{4} = x$

(b) ff(x) = x, so f is its own inverse: $f^{-1}(x) = \frac{-x+3}{x+1}$

Answer: (b) $f^{-1}(x) = \frac{-x+3}{x+1}$