Day 204 **IGCSE 10-a-day: Model Solutions**

1 Radius from chord and height of arc

A circular arc has a chord AB of length 10 cm.
The greatest distance between the chord and the arc is 1 cm.
Work out the radius of the circle.

Let the radius be r. The centre O is on the line through the midpoint M of AB and the highest point of the arc, so OM = $r - 1$

AM = $10 \div 2 = 5$. Pythagoras in triangle OMA: $r^2 = (r - 1)^2 + 5^2$

$r^2 = r^2 - 2r + 1 + 25$

$2r = 26$, so $r = 13$ cm

Answer: 13 cm

2 Exchange rates

Evan changes £1000 into euros at £1 = €1.15.
Evan spends €750 and then changes the rest back into pounds at £1 = €1.25.
How many pounds does Evan get back?

$£1000 \times 1.15 = €1150$

$1150 - 750 = €400$

$400 \div 1.25 = £320$

Answer: £320

3 Two numbers: maximise or minimise

Two numbers x and y have a sum of 24.
Find the least possible value of $x^2 + y^2$, showing that it is the least.

$y = 24 - x$, so $x^2 + y^2 = x^2 + (24 - x)^2 = 2x^2 - 48x + 576$

$\frac{d}{dx}(x^2 + y^2) = 4x - 48 = 0$ gives $x = 12$

Second derivative = $4 > 0$, so a minimum

$x = y = 12$, so the least value is 288

Answer: 288

4 Speed-time graphs

The graph shows the speed of a cyclist during 10 seconds.
(a) Work out the acceleration during the first 4 seconds.
(b) Work out the average speed over the whole 10 seconds.

(a) Acceleration = change in speed ÷ time = $(20 - 4) \div 4 = 4$ m/s^2

(b) Distance = area under the graph

Trapezium: $\frac{4 + 20}{2} \times 4 = 48$; rectangle: $20 \times 4 = 80$; triangle: $\frac{1}{2} \times 2 \times 20 = 20$

Total = 148 m, so average speed = $148 \div 10 = 14.8$ m/s

Answer: (a) 4 m/s^2 (b) 14.8 m/s

5 Rate of change of area or volume

The area, A cm^2, of a circle of radius r cm is $A = \pi r^2$.

(a) Find $\frac{dA}{dr}$ when $r = 5$.

(b) Find the value of r when $\frac{dA}{dr} = 12\pi$.

$\frac{dA}{dr} = 2\pi r$

When $r = 5$: $\frac{dA}{dr} = 10\pi$

$2\pi r = 12\pi$, so $2r = 12$ and $r = 6$ (lengths are positive)

Answer: (a) 10π (b) r = 6

6 Area scale factor and determinant

Shape S has area 5 cm^2.

S is transformed by the matrix $\begin{pmatrix} k & 4 \\ 3 & 3 \end{pmatrix}$ and the image has area 15 cm^2.

Find the two possible values of k.

Area scale factor = 15 ÷ 5 = 3

Determinant = 3k − 12 = ±3

3k − 12 = 3 gives k = 5; 3k − 12 = −3 gives k = 3

Answer: k = 3 or k = 5

7 Inverse functions

$f(x) = \frac{4x - 6}{3x + 4}$

(a) Find $f^{-1}(x)$.

(b) Find the value of $f^{-1}(-10)$.

(a) Let $y = \frac{4x - 6}{3x + 4}$. Then $y(3x + 4) = 4x - 6$

$3xy + 4y = 4x - 6$

Collect x terms: $3xy - 4x = -6 - 4y$

Factorise: $x(3y - 4) = -4y - 6$, so $x = \frac{-4y - 6}{3y - 4}$

$f^{-1}(x) = \frac{-4x - 6}{3x - 4}$

(b) $f^{-1}(-10) = \frac{(-4) \times (-10) + (-6)}{3 \times (-10) - 4} = \frac{34}{-34} = -1$

Answer: (a) $f^{-1}(x) = \frac{-4x - 6}{3x - 4}$ (b) −1

8 Self-inverse function

$f(x) = \frac{4x + 5}{x - k}$, where k is a constant and $x \neq k$.

The function f is its own inverse, so $f^{-1}(x) = f(x)$.

Find the value of k.

Let $y = \frac{4x + 5}{x - k}$, so $y(x - k) = 4x + 5$

$xy - ky = 4x + 5$, so $x(y - 4) = ky + 5$

$f^{-1}(x) = \frac{kx + 5}{x - 4}$

For $f^{-1}(x) = f(x)$ the denominators must match: $x - 4 = x - k$

Answer: k = 4

9 Fractional indices

Given that $x > 0$, solve $x^{-\frac{1}{4}} = \frac{1}{4}$

Raise both sides to the power −4

$x = (\frac{1}{4})^{-4}$

$x = 256$

Answer: x = 256

10 Minimum distance with 1/t

A particle P moves on a straight line. At time t seconds (t > 0), its distance from a point O is s metres, where

$s = 2t^2 + \frac{108}{t}$

Show that P is at its minimum distance from O when t = 3, and find this minimum distance.

$s = 2t^2 + 108t^{-1}$, so $v = \frac{ds}{dt} = 4t - 108t^{-2}$

$v = 0$: $4t = \frac{108}{t^2}$, so $t^3 = 27$ and $t = 3$

$\frac{d^2s}{dt^2} = 4 + \frac{216}{t^3}$, which is 12 when t = 3, so a minimum

$s = 54$

Answer: 54 m