Day 133 **IGCSE 10-a-day: Model Solutions**

1 Composite with an inverse function

$f(x) = 5x - 4$ and $g(x) = 3x^2 + 3x + 5$.
(a) Find $f^{-1}g(x)$.
(b) Work out $f^{-1}g(1)$.

(a) $f^{-1}(x) = \frac{x + 4}{5}$

$f^{-1}g(x) = \frac{3x^2 + 3x + 5 + 4}{5} = \frac{3x^2 + 3x + 9}{5}$

(b) $g(1) = 11$, so $f^{-1}g(1) = \frac{11 + 4}{5}$

Answer: (a) $\frac{3x^2 + 3x + 9}{5}$ (b) 3

2 Exact trigonometry in triangles

ABC is a right-angled triangle.
Work out the exact value of x. Give your answer in its simplest form.
The diagram is not drawn accurately.

x is the hypotenuse AC and the known side is the opposite side BC, so use sin 60°

$\sin 60° = \frac{\sqrt{3}}{2}$

$x = 6 \div \sin 60° = 6 \div \frac{\sqrt{3}}{2}$

$x = 4\sqrt{3}$ (after rationalising the denominator)

Answer: $x = 4\sqrt{3}$

3 Weighted percentages

The ratio of Year 10 students to Year 11 students in a school is 4 : 2.
85% of the Year 10 students and 10% of the Year 11 students have a part-time job.
What percentage of all of these students have a part-time job?

Imagine 120 students in total: 80 Year 10 students and 40 Year 11 students
85% of 80 = 68 and 10% of 40 = 4
Together: 72 out of 120
72 ÷ 120 × 100 = 60%

Answer: 60%

4 Finding a matrix from MA = B

$A = \begin{pmatrix} 3 & -1 \\ 4 & -1 \end{pmatrix}$ and $B = \begin{pmatrix} 11 & 9 \\ 15 & 13 \end{pmatrix}$.
AM = B, where M is a 2 × 2 matrix.
Find M.

AM = B so $M = A^{-1}B$

$A^{-1} = \begin{pmatrix} -1 & 1 \\ -4 & 3 \end{pmatrix}$

$M = \begin{pmatrix} -1 & 1 \\ -4 & 3 \end{pmatrix}\begin{pmatrix} 11 & 9 \\ 15 & 13 \end{pmatrix} = \begin{pmatrix} 4 & 4 \\ 1 & 3 \end{pmatrix}$

Answer: $\mathbf{M} = \begin{pmatrix} \mathbf{4} & \mathbf{4} \\ \mathbf{1} & \mathbf{3} \end{pmatrix}$

5 Fencing with a partition

A rectangular pen is x metres wide and y metres long. It is fenced all round and divided into two equal parts by a fence each of length x metres, parallel to the width.
The total length of fencing used is 48 m.

(a) Show that the area of the pen is $A = -\frac{3}{2}x^2 + 24x$.

(b) Find the maximum possible area, justifying that it is a maximum.

Fencing: $3x + 2y = 48$, so $y = \frac{48 - 3x}{2}$

$A = xy = \frac{x(48 - 3x)}{2} = -\frac{3}{2}x^2 + 24x$

$\frac{dA}{dx} = -3x + 24 = 0$ gives $x = 8$

$\frac{d^2A}{dx^2} = -3 < 0$, so a maximum

$A = 96$

Answer: 96 m²

6 Rearranging when the subject appears twice

Make r the subject of $P = \frac{4r + 4}{r - 6}$

Multiply by $r - 6$: $P(r - 6) = 4r + 4$
Expand: $rP - 6P = 4r + 4$
Collect the r terms: $rP - 4r = 6P + 4$
Factorise: $r(P - 4) = 6P + 4$

$r = \frac{6P + 4}{P - 4}$

Answer: $\mathbf{r = \frac{6P + 4}{P - 4}}$

7 Same gradient on two curves

Two curves have equations $y = 3x^2 + 2x - 7$ and $y = 2x^2 + 8x + 8$.
Find the value of x at which the curves have the same gradient.

First curve: $\frac{dy}{dx} = 6x + 2$

Second curve: $\frac{dy}{dx} = 4x + 8$

Equal gradients: $6x + 2 = 4x + 8$
$2x = 6$, so $x = 3$

Answer: x = 3

8 Tangency conditions

The line $y = kx - 5$ is a tangent to the curve $y = x^2 - 1$.
Find the possible values of k.
Give any surds in their simplest form.

Equate: $x^2 - 1 = kx - 5$
Rearrange: $x^2 - kx + 4 = 0$
One point of contact, so $b^2 - 4ac = 0$: $k^2 - 4 \times 4 = 0$
$k^2 = 16$
$k = \pm\sqrt{16} = \pm 4$

Answer: k = ±4

9 Tangent meets the curve again

A curve has equation $y = x^3 + 7x + 4$.
The tangent to the curve at the point A, where $x = 1$, meets the curve again at the point B.
Find the coordinates of B.

$\frac{dy}{dx} = 3x^2 + 7$, so the gradient at A is 10, and A = (1, 12)
Tangent: $y = 10x + 2$
Equate: $x^3 - 3x + 2 = 0$, which has a repeated root $x = 1$
Sum of the roots = 0, so $1 + 1 + x = 0$ and $x = -2$
$y = -18$
Answer: B (−2, −18)

10 Bounds of area and perimeter

A rectangle has length 21 cm and width 20 cm.
Both measurements are correct to the nearest centimetre.
Chen says that the perimeter of the rectangle could be exactly 84 cm.
Is Chen correct? You must show your working.

$20.5 \leq$ length < 21.5 and $19.5 \leq$ width < 20.5
Lower bound of perimeter = $2 \times 20.5 + 2 \times 19.5 = 80$
Upper bound of perimeter = $2 \times 21.5 + 2 \times 20.5 = 84$
So $80 \leq$ perimeter < 84
84 is equal to the upper bound, which cannot be reached, so Chen is not correct
Answer: No, 84 is the upper bound, which cannot be reached