OCR
Oxford Cambridge and RSA

Monday 19 October 2020 – Afternoon

A Level Mathematics B (MEI)

H640/03 Pure Mathematics and Comprehension

Time allowed: 2 hours

8160771901

You must have:
- the Printed Answer Booklet
- the Insert
- a scientific or graphical calculator

INSTRUCTIONS
- Use black ink. You can use an HB pencil, but only for graphs and diagrams.
- Write your answer to each question in the space provided in the **Printed Answer Booklet**. If you need extra space use the lined pages at the end of the Printed Answer Booklet. The question numbers must be clearly shown.
- Fill in the boxes on the front of the Printed Answer Booklet.
- Answer **all** the questions.
- Where appropriate, your answer should be supported with working. Marks might be given for using a correct method, even if your answer is wrong.
- Give your final answers to a degree of accuracy that is appropriate to the context.
- Do **not** send this Question Paper for marking. Keep it in the centre or recycle it.

INFORMATION
- The total mark for this paper is **75**.
- The marks for each question are shown in brackets **[ ]**.
- This document has **12** pages.

ADVICE
- Read each question carefully before you start your answer.

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Formulae A Level Mathematics B (MEI) (H640)

Arithmetic series

$S_n = \frac{1}{2}n(a+l) = \frac{1}{2}n\{2a+(n-1)d\}$

Geometric series

$S_n = \dfrac{a(1-r^n)}{1-r}$

$S_\infty = \dfrac{a}{1-r}$ for $|r| < 1$

Binomial series

$(a+b)^n = a^n + {}^n\mathrm{C}_1 a^{n-1}b + {}^n\mathrm{C}_2 a^{n-2}b^2 + \ldots + {}^n\mathrm{C}_r a^{n-r}b^r + \ldots + b^n \qquad (n \in \mathbb{N}),$

where ${}^n\mathrm{C}_r = {}_n\mathrm{C}_r = \binom{n}{r} = \dfrac{n!}{r!(n-r)!}$

$(1+x)^n = 1 + nx + \dfrac{n(n-1)}{2!}x^2 + \ldots + \dfrac{n(n-1)\ldots(n-r+1)}{r!}x^r + \ldots \qquad (|x| < 1,\ n \in \mathbb{R})$

Differentiation

$\mathrm{f}(x)$	$\mathrm{f}'(x)$
$\tan kx$	$k\sec^2 kx$
$\sec x$	$\sec x \tan x$
$\cot x$	$-\mathrm{cosec}^2 x$
$\mathrm{cosec}\, x$	$-\mathrm{cosec}\, x \cot x$

Quotient Rule $y = \dfrac{u}{v},\ \dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{v\frac{\mathrm{d}u}{\mathrm{d}x} - u\frac{\mathrm{d}v}{\mathrm{d}x}}{v^2}$

Differentiation from first principles

$\mathrm{f}'(x) = \lim_{h\to 0}\dfrac{\mathrm{f}(x+h)-\mathrm{f}(x)}{h}$

Integration

$\displaystyle\int \frac{\mathrm{f}'(x)}{\mathrm{f}(x)}\,\mathrm{d}x = \ln|\mathrm{f}(x)| + c$

$\displaystyle\int \mathrm{f}'(x)\big(\mathrm{f}(x)\big)^n\,\mathrm{d}x = \frac{1}{n+1}\big(\mathrm{f}(x)\big)^{n+1} + c$

Integration by parts $\displaystyle\int u\frac{\mathrm{d}v}{\mathrm{d}x}\,\mathrm{d}x = uv - \int v\frac{\mathrm{d}u}{\mathrm{d}x}\,\mathrm{d}x$

Small angle approximations

$\sin\theta \approx \theta,\ \cos\theta \approx 1 - \frac{1}{2}\theta^2,\ \tan\theta \approx \theta$ where θ is measured in radians

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Trigonometric identities

$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$

$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$

$\tan(A \pm B) = \dfrac{\tan A \pm \tan B}{1 \mp \tan A \tan B} \qquad \left(A \pm B \neq (k + \frac{1}{2})\pi\right)$

Numerical methods

Trapezium rule: $\displaystyle\int_a^b y\,\mathrm{d}x \approx \tfrac{1}{2}h\{(y_0 + y_n) + 2(y_1 + y_2 + \ldots + y_{n-1})\}$, where $h = \dfrac{b-a}{n}$

The Newton-Raphson iteration for solving $\mathrm{f}(x) = 0$: $x_{n+1} = x_n - \dfrac{\mathrm{f}(x_n)}{\mathrm{f}'(x_n)}$

Probability

$\mathrm{P}(A \cup B) = \mathrm{P}(A) + \mathrm{P}(B) - \mathrm{P}(A \cap B)$

$\mathrm{P}(A \cap B) = \mathrm{P}(A)\mathrm{P}(B \mid A) = \mathrm{P}(B)\mathrm{P}(A \mid B)$ **or** $\mathrm{P}(A \mid B) = \dfrac{\mathrm{P}(A \cap B)}{\mathrm{P}(B)}$

Sample variance

$s^2 = \dfrac{1}{n-1}S_{xx}$ where $S_{xx} = \Sigma(x_i - \bar{x})^2 = \Sigma x_i^2 - \dfrac{(\Sigma x_i)^2}{n} = \Sigma x_i^2 - n\bar{x}^2$

Standard deviation, $s = \sqrt{\text{variance}}$

The binomial distribution

If $X \sim \mathrm{B}(n, p)$ then $\mathrm{P}(X = r) = {}^n\mathrm{C}_r p^r q^{n-r}$ where $q = 1 - p$

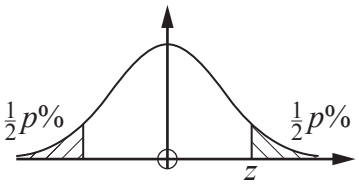
Mean of X is np

Hypothesis testing for the mean of a Normal distribution

If $X \sim \mathrm{N}(\mu, \sigma^2)$ then $\overline{X} \sim \mathrm{N}\left(\mu, \dfrac{\sigma^2}{n}\right)$ and $\dfrac{\overline{X} - \mu}{\sigma/\sqrt{n}} \sim \mathrm{N}(0, 1)$

Percentage points of the Normal distribution

p	10	5	2	1
z	1.645	1.960	2.326	2.576



Kinematics

Motion in a straight line	Motion in two dimensions
$v = u + at$	$\mathbf{v} = \mathbf{u} + \mathbf{a}t$
$s = ut + \frac{1}{2}at^2$	$\mathbf{s} = \mathbf{u}t + \frac{1}{2}\mathbf{a}t^2$
$s = \frac{1}{2}(u + v)t$	$\mathbf{s} = \frac{1}{2}(\mathbf{u} + \mathbf{v})t$
$v^2 = u^2 + 2as$	
$s = vt - \frac{1}{2}at^2$	$\mathbf{s} = \mathbf{v}t - \frac{1}{2}\mathbf{a}t^2$

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Answer **all** the questions.

Section A (60 marks)

1 Find the value of $\sum_{r=1}^{5} 2^r(r-1)$. **[2]**

2 The graph of $y = |1-x| - 2$ is shown in Fig. 2.

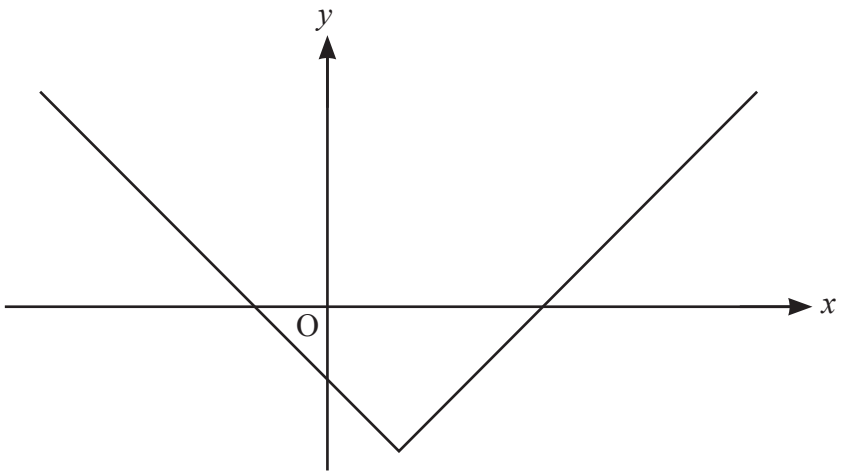


Fig. 2

Determine the set of values of x for which $|1-x| > 2$. **[4]**

3 A particular phone battery will last 10 hours when it is first used. Every time it is recharged, it will only last 98% of its previous time.

Find the maximum total length of use for the battery. **[3]**

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4 Fig. 4 shows the regular octagon ABCDEFGH.

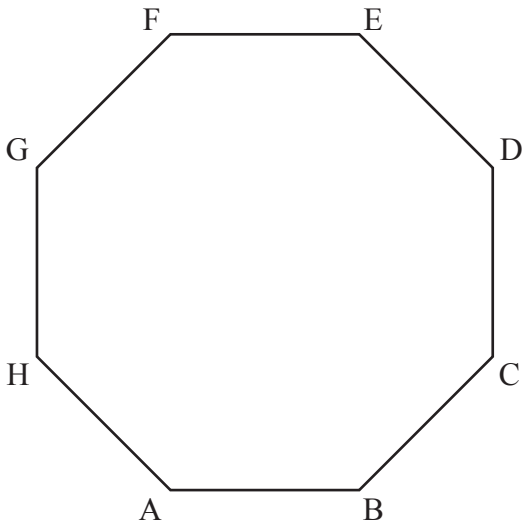


Fig. 4

$\overrightarrow{AB} = \mathbf{i}$, $\overrightarrow{CD} = \mathbf{j}$, where **i** is a unit vector parallel to the x-axis and **j** is a unit vector parallel to the y-axis.

Find an exact expression for $\overrightarrow{BC}$ in terms of **i** and **j**. **[3]**

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5 Fig. 5 shows part of the curve $y = \text{cosec}\, x$ together with the x- and y-axes.

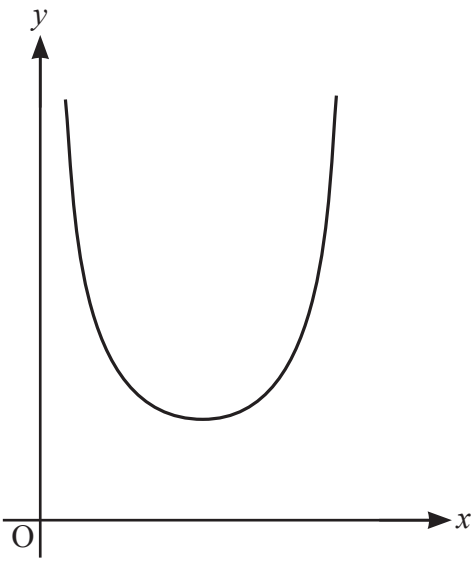


Fig. 5

(a) For the section of the curve which is shown in Fig. 5, write down

(i) the equations of the two vertical asymptotes, **[2]**

(ii) the coordinates of the minimum point. **[1]**

(b) Show that the equation $x = \text{cosec}\, x$ has a root which lies between $x = 1$ and $x = 2$. **[2]**

(c) Use the iteration $x_{n+1} = \text{cosec}\,(x_n)$, with $x_0 = 1$, to find

(i) the values of x_1 and x_2, correct to 5 decimal places, **[1]**

(ii) this root of the equation, correct to 3 decimal places. **[1]**

(d) There is another root of $x = \text{cosec}\, x$ which lies between $x = 2$ and $x = 3$.

Determine whether the iteration $x_{n+1} = \text{cosec}\,(x_n)$ with $x_0 = 2.5$ converges to this root. **[1]**

(e) Sketch the staircase or cobweb diagram for the iteration, starting with $x_0 = 2.5$, on the diagram in the Printed Answer Booklet. **[3]**

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6 **(a)** **(i)** Write down the derivative of e^{kx}, where k is a constant. **[1]**

(ii) A business has been running since 2009. They sell maths revision resources online.

Give a reason why an exponential growth model might be suitable for the annual profits for the business. **[1]**

Fig. 6 shows the relationship between the annual profits of the business in thousands of pounds (y) and the time in years after 2009 (x). The graph of $\ln y$ plotted against x is approximately a straight line.

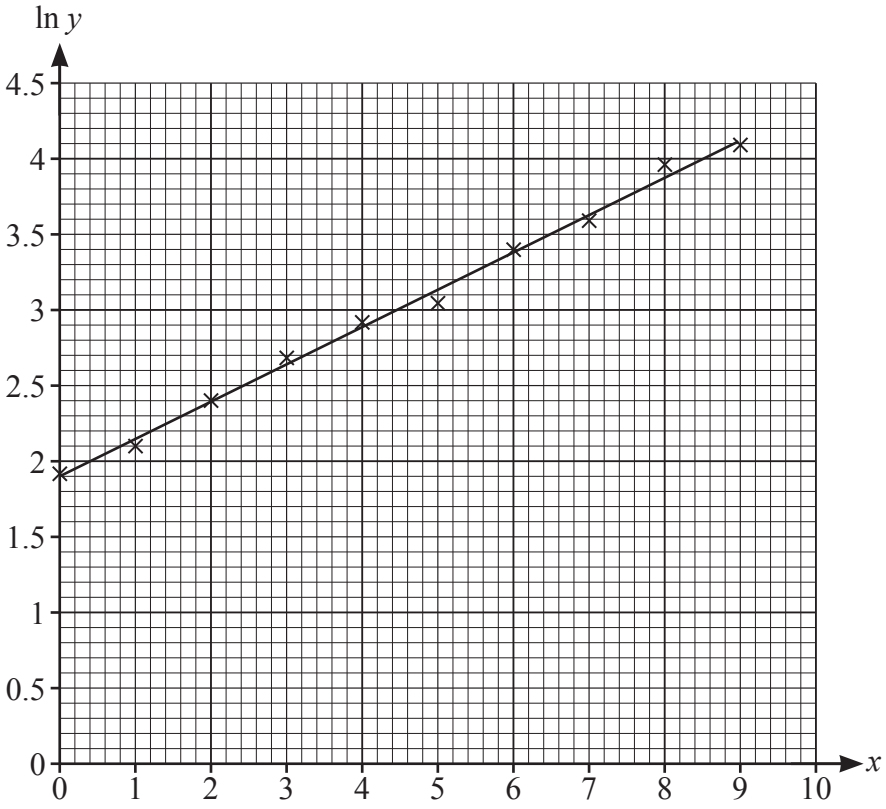


Fig. 6

(b) Show that the straight line is consistent with a model of the form $y = Ae^{kx}$, where A and k are constants. **[2]**

(c) Estimate the values of A and k. **[4]**

(d) Use the model to predict the profit in the year 2020. **[3]**

(e) How reliable do you expect the prediction in part **(d)** to be? Justify your answer. **[1]**

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7 **(a)** Express $\frac{1}{x}+\frac{1}{A-x}$ as a single fraction. **[1]**

The population of fish in a lake is modelled by the differential equation

$$\frac{\mathrm{d}x}{\mathrm{d}t}=\frac{x(400-x)}{400}$$

where x is the number of fish and t is the time in years.

When $t = 0$, $x = 100$.

(b) **In this question you must show detailed reasoning.**

Find the number of fish in the lake when $t = 10$, as predicted by the model. **[8]**

8 **(a)** The curve $y=\frac{1}{(1+x^2)^2}$ is shown in Fig. 8.

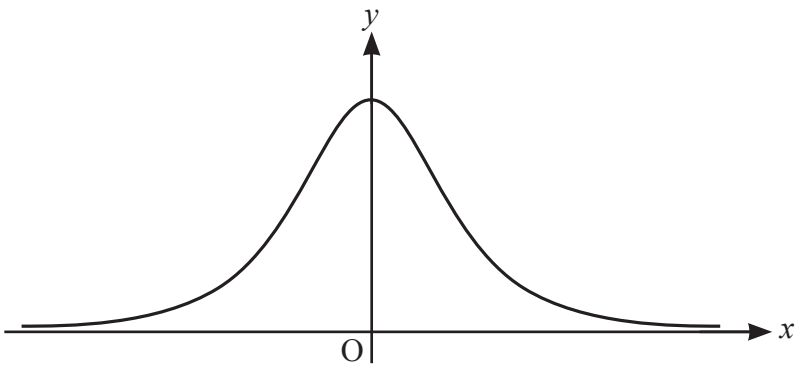


Fig. 8

(i) Show that $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=\frac{20x^2-4}{(1+x^2)^4}$. **[5]**

(ii) **In this question you must show detailed reasoning.**

Find the set of values of x for which the curve is concave downwards. **[3]**

(b) Use the substitution $x = \tan\theta$ to find the exact value of $\int_{-1}^{1}\frac{1}{(1+x^2)^2}\,\mathrm{d}x$. **[8]**

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Answer **all** the questions.

Section B (15 marks)

The questions in this section refer to the article on the Insert. You should read the article before attempting the questions.

9 **(a)** Show that if $a = 1$ and $b > 1$ then $a^b < b^a$. **[2]**

(b) Find integer values of a and b with $b > a > 1$ and a^b not greater than b^a (a counter example to the conjecture given in lines 7–8). **[1]**

10 **In this question you must show detailed reasoning.**

Show that $\int_{\mathrm{e}}^{\pi} \frac{1}{x}\,\mathrm{d}x = \ln \pi - 1$ as given in line 37. **[2]**

11 Show that e^x is an increasing function for all values of x, as stated in line 39. **[2]**

12 **(a)** Show that the only stationary point on the curve $y = \frac{\ln x}{x}$ occurs where $x = \mathrm{e}$, as given in line 45. **[3]**

(b) Show that the stationary point is a maximum. **[3]**

(c) It follows from part **(b)** that, for any positive number a with $a \neq \mathrm{e}$,

$\frac{\ln \mathrm{e}}{\mathrm{e}} > \frac{\ln a}{a}$.

Use this fact to show that $\mathrm{e}^a > a^{\mathrm{e}}$. **[2]**

END OF QUESTION PAPER

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Surname	StemUp Model Solution
Answer booklet	Write the question number in the left-hand margin at the start of each answer.

Question number

1

$\sum_{r=1}^{5} 2^r(r-1) = 2\times0 + 4\times1 + 8\times2 + 16\times3 + 32\times4$

$= 0 + 4 + 16 + 48 + 128$

$= 196$

2

Graph crosses x-axis where $|1 - x| = 2$:

$1 - x = 2 \Rightarrow x = -1$ or $1 - x = -2 \Rightarrow x = 3$

$|1 - x| > 2$ where the graph is above the x-axis:

$x < -1$ or $x > 3$

3

Total $= 10 + 10\times0.98 + 10\times0.98^2 + \ldots$ (GP, $a = 10$, $r = 0.98$)

$|r| < 1$ so use sum to infinity:

$S_\infty = \frac{a}{1-r} = \frac{10}{1-0.98} = \frac{10}{0.02} = 500$

Maximum total length of use = 500 hours

4

Regular octagon: all sides length 1, exterior angle $= 360° \div 8 = 45°$

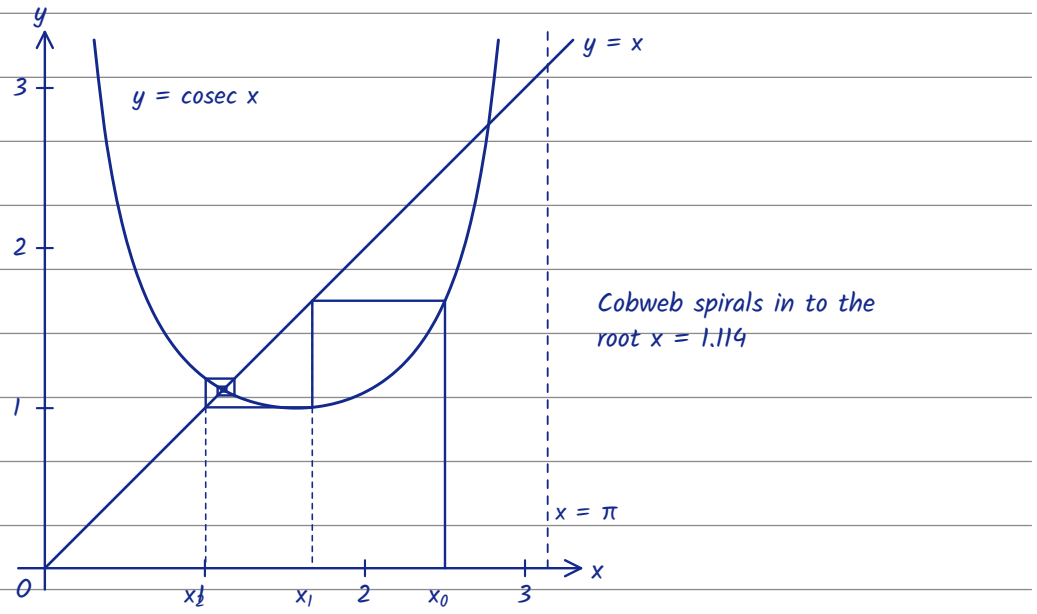
so BC is at 45° to i, i.e. BC is parallel to $i + j$

$|i + j| = \sqrt{2}$, so unit vector in this direction is $\frac{1}{\sqrt{2}}(i + j)$

$BC = \cos45°\, i + \sin45°\, j = \frac{1}{\sqrt{2}}(i + j) = \frac{\sqrt{2}}{2} i + \frac{\sqrt{2}}{2} j$

5 (a)(i)

$x = 0$ and $x = \pi$

Question number	
5 (a)(ii)	Minimum point $\left(\frac{\pi}{2}, 1\right)$
5 (b)	Let $f(x) = x - \operatorname{cosec} x$ (continuous for $1 \leq x \leq 2$)
	$f(1) = 1 - \operatorname{cosec} 1 = -0.188... < 0$
	$f(2) = 2 - \operatorname{cosec} 2 = 0.900... > 0$
	Change of sign, so $x = \operatorname{cosec} x$ has a root between $x = 1$ and $x = 2$
5 (c)(i)	$x_1 = \operatorname{cosec}(1) = 1.188395... = 1.18840$
	$x_2 = \operatorname{cosec}(1.188395...) = 1.077852... = 1.07785$
5 (c)(ii)	$x_3 = 1.13515$, $x_4 = 1.10303$, $x_5 = 1.12035$, $x_6 = 1.11080$, ...
	$x_{10} = 1.11386$, $x_{11} = 1.11432$, $x_{12} = 1.11407$
	Root = 1.114 (3 d.p.)
5 (d)	$x_1 = \operatorname{cosec} 2.5 = 1.67092$, $x_2 = 1.00503$, $x_3 = 1.18458$, $x_4 = 1.07952$ ...
	No: the iteration does not converge to the root between 2 and 3,
	it converges to the root 1.114 (the root between 1 and 2).
5 (e)	

Question number

6 (a)(i) $\frac{d}{dx}(e^{kx}) = ke^{kx}$

6 (a)(ii) The more customers the business has, the more people hear about it, so the rate of growth of profit is likely to be proportional to the profit.

6 (b) $y = Ae^{kx} \Rightarrow \ln y = \ln(Ae^{kx}) = \ln A + kx$

This is of the form $Y = mx + c$ with $Y = \ln y$, gradient k and intercept $\ln A$, so $\ln y$ against x is a straight line: the graph is consistent with the model.

6 (c) Intercept: $\ln A = 1.9 \Rightarrow A = e^{1.9} = 6.6858\ldots \approx 6.69$

Gradient: $k = \frac{4.1 - 1.9}{9 - 0} = \frac{2.2}{9} = 0.2444\ldots \approx 0.244$

$A \approx 6.69$, $k \approx 0.244$

6 (d) $2020 \Rightarrow x = 11$

$y = 6.686 \times e^{0.2444 \times 11} = e^{1.9 + 2.6889} = e^{4.5889}$

$y = 98.385\ldots$ (thousands of pounds)

Predicted profit ≈ £98 400

6 (e) Not very reliable: $x = 11$ is outside the range of the data (extrapolation), and the growth may not stay exponential, e.g. there is a limit to the market for revision resources.

7 (a) $\frac{1}{x} + \frac{1}{A - x} = \frac{(A - x) + x}{x(A - x)} = \frac{A}{x(A - x)}$

7 (b) Separate variables: $\int \frac{400}{x(400 - x)}\, dx = \int 1\, dt$

Question number

7 cont.

Using (a) with A = 400: $\int \left(\frac{1}{x} + \frac{1}{400 - x} \right) dx = \int 1 \, dt$

$\ln x - \ln(400 - x) = t + c$

$\ln \frac{x}{400 - x} = t + c$

When $t = 0$, $x = 100$: $\ln \frac{100}{300} = c \Rightarrow c = \ln \frac{1}{3} = -\ln 3$

$t = \ln \frac{x}{400 - x} + \ln 3 \Rightarrow t = \ln \frac{3x}{400 - x}$

When $t = 10$: $10 = \ln \frac{3x}{400 - x} \Rightarrow \frac{3x}{400 - x} = e^{10}$

$3x = e^{10}(400 - x) \Rightarrow 3x + e^{10}x = 400e^{10}$

$x = \frac{400e^{10}}{3 + e^{10}} = 399.945...$

Number of fish when $t = 10$ is 400 (to the nearest whole fish)

8 (a)(i)

$y = (1 + x^2)^{-2} \Rightarrow \frac{dy}{dx} = -2(1 + x^2)^{-3} \times 2x = \frac{-4x}{(1 + x^2)^3}$

Quotient rule with $u = -4x$, $v = (1 + x^2)^3$:

$u' = -4$, $v' = 3(1 + x^2)^2 \times 2x = 6x(1 + x^2)^2$

$\frac{d^2y}{dx^2} = \frac{(1 + x^2)^3(-4) - (-4x) \times 6x(1 + x^2)^2}{(1 + x^2)^6}$

$= \frac{-4(1 + x^2) + 24x^2}{(1 + x^2)^4}$ (dividing top and bottom by $(1 + x^2)^2$)

$= \frac{-4 - 4x^2 + 24x^2}{(1 + x^2)^4} = \frac{20x^2 - 4}{(1 + x^2)^4}$ as required

8 (a)(ii)

Concave downwards when $\frac{d^2y}{dx^2} < 0$: $\frac{20x^2 - 4}{(1 + x^2)^4} < 0$

Question number

8 cont.

$(1 + x^2)^4 > 0$ for all x, so $20x^2 - 4 < 0$

$5x^2 < 1 \Rightarrow x^2 < \frac{1}{5}$

Set of values: $-\frac{1}{\sqrt{5}} < x < \frac{1}{\sqrt{5}}$

8 (b)

$x = \tan\theta \Rightarrow \frac{dx}{d\theta} = \sec^2\theta$, so $dx = \sec^2\theta\, d\theta$

$x = -1 \Rightarrow \theta = -\frac{\pi}{4}$; $x = 1 \Rightarrow \theta = \frac{\pi}{4}$

$$\int_{-1}^{1} \frac{1}{(1+x^2)^2}\, dx = \int_{-\pi/4}^{\pi/4} \frac{\sec^2\theta}{(1+\tan^2\theta)^2}\, d\theta = \int_{-\pi/4}^{\pi/4} \frac{\sec^2\theta}{\sec^4\theta}\, d\theta$$

using $1 + \tan^2\theta = \sec^2\theta$, $1/\sec\theta = \cos\theta$ and $\cos 2\theta = 2\cos^2\theta - 1$:

$$= \int_{-\pi/4}^{\pi/4} \frac{1}{\sec^2\theta}\, d\theta = \int_{-\pi/4}^{\pi/4} \cos^2\theta\, d\theta = \int_{-\pi/4}^{\pi/4} \frac{1}{2}(\cos 2\theta + 1)\, d\theta$$

$$= \frac{1}{2}\left[\frac{1}{2}\sin 2\theta + \theta\right]_{-\pi/4}^{\pi/4} = \frac{1}{2}\left[\left(\frac{1}{2} + \frac{\pi}{4}\right) - \left(-\frac{1}{2} - \frac{\pi}{4}\right)\right]$$

$$= \frac{1}{2}\left(1 + \frac{\pi}{2}\right) = \frac{1}{2} + \frac{\pi}{4}$$

9 (a)

$a = 1$ and $b > 1$: $a^b = 1^b = 1$ and $b^a = b^1 = b$

Since $b > 1$, $b^a = b > 1 = a^b$, hence $a^b < b^a$

9 (b)

$a = 2$, $b = 3$: $a^b = 2^3 = 8$, $b^a = 3^2 = 9$

$8 < 9$ so a^b is not greater than b^a $(a = 2, b = 3)$

10

$$\int_e^{\pi} \frac{1}{x}\, dx = [\ln x]_e^{\pi} = \ln \pi - \ln e = \ln \pi - 1 \quad (\text{since } \ln e = 1)$$

Question number

11 $y = e^x \Rightarrow dy/dx = e^x$

$e^x > 0$ for all x, so $dy/dx > 0$ for all x,

hence e^x is an increasing function for all values of x.

12 (a) Quotient rule with $u = \ln x$, $v = x$: $u' = 1/x$, $v' = 1$

$$\frac{dy}{dx} = \frac{x \times (1/x) - \ln x \times 1}{x^2} = \frac{1 - \ln x}{x^2}$$

Stationary point: $\frac{1 - \ln x}{x^2} = 0 \Rightarrow 1 - \ln x = 0 \Rightarrow \ln x = 1 \Rightarrow x = e$

This is the only solution, so the only stationary point is where $x = e$.

12 (b)

$$\frac{d^2y}{dx^2} = \frac{x^2(-1/x) - (1 - \ln x)(2x)}{x^4} = \frac{-x - 2x + 2x \ln x}{x^4} = \frac{-3 + 2\ln x}{x^3}$$

At $x = e$: $\frac{d^2y}{dx^2} = \frac{-3 + 2}{e^3} = -\frac{1}{e^3} = -0.0497... < 0$

so the stationary point is a maximum.

12 (c)

$\frac{\ln e}{e} > \frac{\ln a}{a}$ multiply by ae (> 0): $a \ln e > e \ln a$

$\Rightarrow \ln(e^a) > \ln(a^e)$

ln is an increasing function, so $e^a > a^e$