OCR
Oxford Cambridge and RSA

Friday 14 June 2019 – Afternoon

A Level Mathematics B (MEI)

H640/03 Pure Mathematics and Comprehension

Time allowed: 2 hours

7694608764

You must have:
- Printed Answer Booklet
- Insert

You may use:
- a scientific or graphical calculator

INSTRUCTIONS
- Use black ink. HB pencil may be used for graphs and diagrams only.
- Answer **all** the questions.
- **Write your answer to each question in the space provided in the Printed Answer Booklet.** If additional space is required, you should use the lined page(s) at the end of the Printed Answer Booklet. The question number(s) must be clearly shown.
- You are permitted to use a scientific or graphical calculator in this paper.
- Final answers should be given to a degree of accuracy appropriate to the context.

INFORMATION
- The total mark for this paper is **75**.
- The marks for each question are shown in brackets **[ ]**.
- You are advised that an answer may receive **no marks** unless you show sufficient detail of the working to indicate that a correct method is used. You should communicate your method with correct reasoning.
- The Printed Answer Booklet consists of **16** pages. The Question Paper consists of **8** pages.

© OCR 2019 [603/1002/9]
DC (KS/SW) 174424/4

OCR is an exempt Charity

stemup.co.uk

Formulae A Level Mathematics B (MEI) (H640)

Arithmetic series

$S_n = \frac{1}{2}n(a+l) = \frac{1}{2}n\{2a+(n-1)d\}$

Geometric series

$S_n = \dfrac{a(1-r^n)}{1-r}$

$S_\infty = \dfrac{a}{1-r}$ for $|r| < 1$

Binomial series

$(a+b)^n = a^n + {}^n\mathrm{C}_1 a^{n-1}b + {}^n\mathrm{C}_2 a^{n-2}b^2 + \ldots + {}^n\mathrm{C}_r a^{n-r}b^r + \ldots + b^n \qquad (n \in \mathbb{N}),$

where ${}^n\mathrm{C}_r = {}_n\mathrm{C}_r = \binom{n}{r} = \dfrac{n!}{r!(n-r)!}$

$(1+x)^n = 1 + nx + \dfrac{n(n-1)}{2!}x^2 + \ldots + \dfrac{n(n-1)\ldots(n-r+1)}{r!}x^r + \ldots \qquad (|x| < 1,\ n \in \mathbb{R})$

Differentiation

$\mathrm{f}(x)$	$\mathrm{f}'(x)$
$\tan kx$	$k\sec^2 kx$
$\sec x$	$\sec x \tan x$
$\cot x$	$-\mathrm{cosec}^2 x$
$\mathrm{cosec}\, x$	$-\mathrm{cosec}\, x \cot x$

Quotient Rule $y = \dfrac{u}{v},\ \dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{v\frac{\mathrm{d}u}{\mathrm{d}x} - u\frac{\mathrm{d}v}{\mathrm{d}x}}{v^2}$

Differentiation from first principles

$\mathrm{f}'(x) = \lim_{h\to 0}\dfrac{\mathrm{f}(x+h)-\mathrm{f}(x)}{h}$

Integration

$\displaystyle\int \frac{\mathrm{f}'(x)}{\mathrm{f}(x)}\,\mathrm{d}x = \ln|\mathrm{f}(x)| + c$

$\displaystyle\int \mathrm{f}'(x)\big(\mathrm{f}(x)\big)^n\,\mathrm{d}x = \frac{1}{n+1}\big(\mathrm{f}(x)\big)^{n+1} + c$

Integration by parts $\displaystyle\int u\frac{\mathrm{d}v}{\mathrm{d}x}\,\mathrm{d}x = uv - \int v\frac{\mathrm{d}u}{\mathrm{d}x}\,\mathrm{d}x$

Small angle approximations

$\sin\theta \approx \theta,\ \cos\theta \approx 1 - \frac{1}{2}\theta^2,\ \tan\theta \approx \theta$ where θ is measured in radians

© OCR 2019

stemup.co.uk

Trigonometric identities

$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$

$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$

$\tan(A \pm B) = \dfrac{\tan A \pm \tan B}{1 \mp \tan A \tan B} \qquad \left(A \pm B \neq (k + \frac{1}{2})\pi\right)$

Numerical methods

Trapezium rule: $\displaystyle\int_a^b y\,\mathrm{d}x \approx \tfrac{1}{2}h\{(y_0 + y_n) + 2(y_1 + y_2 + \ldots + y_{n-1})\}$, where $h = \dfrac{b-a}{n}$

The Newton-Raphson iteration for solving $\mathrm{f}(x) = 0$: $x_{n+1} = x_n - \dfrac{\mathrm{f}(x_n)}{\mathrm{f}'(x_n)}$

Probability

$\mathrm{P}(A \cup B) = \mathrm{P}(A) + \mathrm{P}(B) - \mathrm{P}(A \cap B)$

$\mathrm{P}(A \cap B) = \mathrm{P}(A)\mathrm{P}(B \mid A) = \mathrm{P}(B)\mathrm{P}(A \mid B)$ **or** $\mathrm{P}(A \mid B) = \dfrac{\mathrm{P}(A \cap B)}{\mathrm{P}(B)}$

Sample variance

$s^2 = \dfrac{1}{n-1}S_{xx}$ where $S_{xx} = \Sigma(x_i - \bar{x})^2 = \Sigma x_i^2 - \dfrac{(\Sigma x_i)^2}{n} = \Sigma x_i^2 - n\bar{x}^2$

Standard deviation, $s = \sqrt{\text{variance}}$

The binomial distribution

If $X \sim \mathrm{B}(n, p)$ then $\mathrm{P}(X = r) = {}^n\mathrm{C}_r p^r q^{n-r}$ where $q = 1 - p$

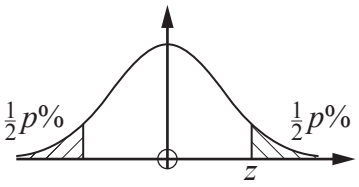
Mean of X is np

Hypothesis testing for the mean of a Normal distribution

If $X \sim \mathrm{N}(\mu, \sigma^2)$ then $\overline{X} \sim \mathrm{N}\left(\mu, \dfrac{\sigma^2}{n}\right)$ and $\dfrac{\overline{X} - \mu}{\sigma/\sqrt{n}} \sim \mathrm{N}(0, 1)$

Percentage points of the Normal distribution

p	10	5	2	1
z	1.645	1.960	2.326	2.576



Kinematics

Motion in a straight line	Motion in two dimensions
$v = u + at$	$\mathbf{v} = \mathbf{u} + \mathbf{a}t$
$s = ut + \frac{1}{2}at^2$	$\mathbf{s} = \mathbf{u}t + \frac{1}{2}\mathbf{a}t^2$
$s = \frac{1}{2}(u + v)t$	$\mathbf{s} = \frac{1}{2}(\mathbf{u} + \mathbf{v})t$
$v^2 = u^2 + 2as$	
$s = vt - \frac{1}{2}at^2$	$\mathbf{s} = \mathbf{v}t - \frac{1}{2}\mathbf{a}t^2$

Answer **all** the questions.

Section A (60 marks)

1 The function $f(x)$ is defined for all real x by

$f(x) = 3x - 2$.

(a) Find an expression for $f^{-1}(x)$. **[2]**

(b) Sketch the graphs of $y = f(x)$ and $y = f^{-1}(x)$ on the same diagram. **[2]**

(c) Find the set of values of x for which $f(x) > f^{-1}(x)$. **[2]**

2 **(a)** Find the transformation which maps the curve $y = x^2$ to the curve $y = x^2 + 8x - 7$. **[4]**

(b) Write down the coordinates of the turning point of $y = x^2 + 8x - 7$. **[1]**

3 **(a)** Express $\frac{1}{(x+2)(x+3)}$ in partial fractions. **[3]**

(b) Find $\int \frac{1}{(x+2)(x+3)}\,dx$ in the form $\ln(f(x)) + c$, where c is the constant of integration and $f(x)$ is a function to be determined. **[3]**

4 **In this question you must show detailed reasoning.**

Show that $\frac{1}{\sqrt{10}+\sqrt{11}} + \frac{1}{\sqrt{11}+\sqrt{12}} + \frac{1}{\sqrt{12}+\sqrt{13}} = \frac{3}{\sqrt{10}+\sqrt{13}}$. **[3]**

© OCR 2019

stemup.co.uk

5 A student's attempt to prove by contradiction that there is no largest prime number is shown below.

> If there is a largest prime, list all the primes.
>
> Multiply all the primes and add 1.
>
> The new number is not divisible by any of the primes in the list and so it must be a new prime.

The proof is incorrect and incomplete.
Write a correct version of the proof. **[3]**

6 A circle has centre C (10, 4). The x-axis is a tangent to the circle, as shown in Fig. 6.

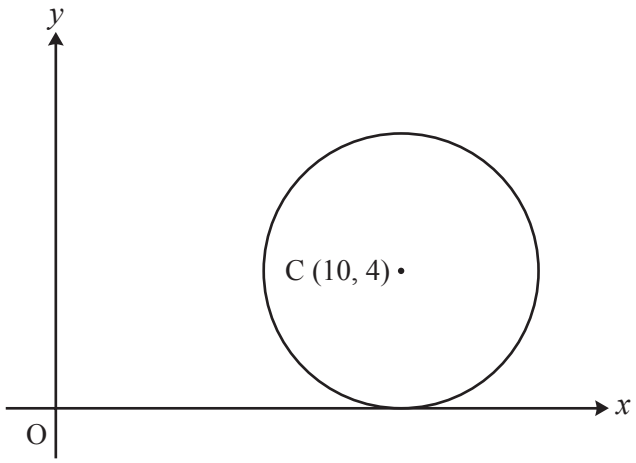


Fig. 6

(a) Find the equation of the circle. **[2]**

(b) Show that the line $y = x$ is not a tangent to the circle. **[4]**

(c) Write down the position vector of the midpoint of OC. **[1]**

7 **In this question you must show detailed reasoning.**

(a) Express $\ln 3 \times \ln 9 \times \ln 27$ in terms of $\ln 3$. **[2]**

(b) Hence show that $\ln 3 \times \ln 9 \times \ln 27 > 6$. **[2]**

© OCR 2019

stemup.co.uk

8 In this question you must show detailed reasoning.

A is the point (1, 0), B is the point (1, 1) and D is the point where the tangent to the curve $y = x^3$ at B crosses the x-axis, as shown in Fig. 8. The tangent meets the y-axis at E.

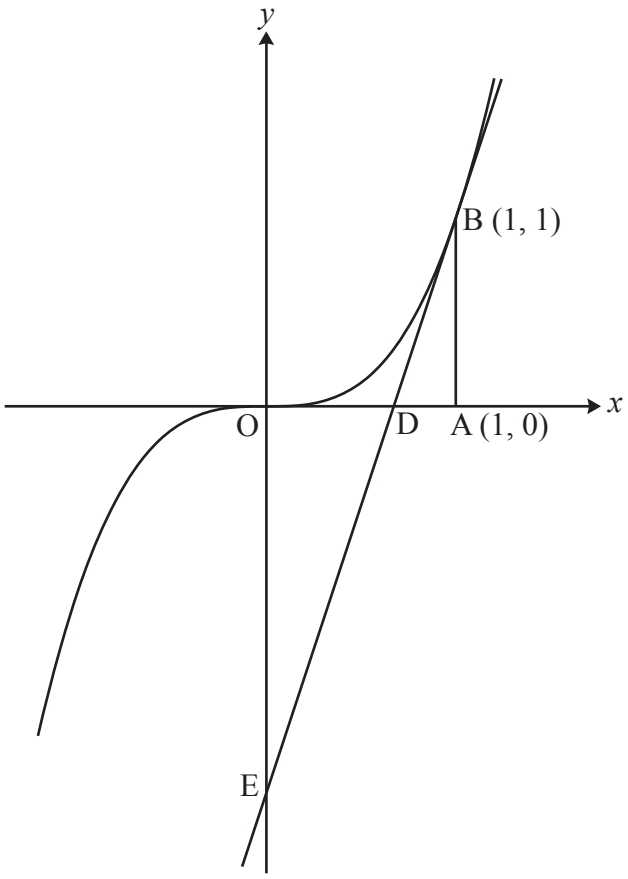


Fig. 8

(a) Find the area of triangle ODE. **[6]**

(b) Find the area of the region bounded by the curve $y = x^3$, the tangent at B and the y-axis. **[4]**

© OCR 2019

stemup.co.uk

9 **In this question you must show detailed reasoning.**

The curve $xy+y^2 = 8$ is shown in Fig. 9.

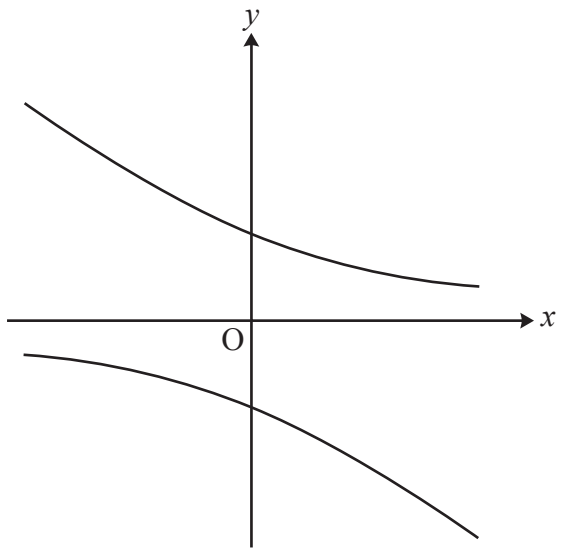


Fig. 9

Find the coordinates of the points on the curve at which the normal has gradient 2. **[6]**

10 Show that $f(x) = \dfrac{e^x}{1+e^x}$ is an increasing function for all values of x. **[4]**

11 By using the substitution $u = 1+\sqrt{x}$, find $\displaystyle\int \frac{x}{1+\sqrt{x}}\,dx$. **[6]**

stemup.co.uk

Answer **all** the questions.

Section B (15 marks)

The questions in this section refer to the article on the Insert. You should read the article before attempting the questions.

12 Show that the equation of the line in Fig. C2 is $ry + hx = hr$, as given in line 24. **[2]**

13 **(a)** **(i)** Show that the cross-sectional area in Fig. C3.2 is $\pi x(2r - x)$. **[2]**

(ii) Hence show that the cross-sectional area is $\frac{\pi r^2}{h^2}(h^2 - y^2)$, as given in line 37. **[2]**

(b) Verify that the formula $\frac{\pi r^2}{h^2}(h^2 - y^2)$ for the cross-sectional area is also valid for

(i) Fig. C3.1, **[1]**

(ii) Fig. C3.3. **[1]**

14 **(a)** Express $\lim_{\delta y \to 0} \sum_{0}^{h} (h^2 - y^2)\delta y$ as an integral. **[1]**

(b) Hence show that $V = \frac{2}{3}\pi r^2 h$, as given in line 41. **[3]**

15 A typical tube of toothpaste measures 5.4 cm across the straight edge at the top and is 12 cm high. It contains 75 ml of toothpaste so it needs to have an internal volume of 75 cm^3.

Comment on the accuracy of the formula $V = \frac{2}{3}\pi r^2 h$, as given in line 41, for the volume in this case. **[3]**

END OF QUESTION PAPER

OCR
Oxford Cambridge and RSA

Copyright Information

OCR is committed to seeking permission to reproduce all third-party content that it uses in its assessment materials. OCR has attempted to identify and contact all copyright holders whose work is used in this paper. To avoid the issue of disclosure of answer-related information to candidates, all copyright acknowledgements are reproduced in the OCR Copyright Acknowledgements Booklet. This is produced for each series of examinations and is freely available to download from our public website (www.ocr.org.uk) after the live examination series.

If OCR has unwittingly failed to correctly acknowledge or clear any third-party content in this assessment material, OCR will be happy to correct its mistake at the earliest possible opportunity.

For queries or further information please contact The OCR Copyright Team, The Triangle Building, Shaftesbury Road, Cambridge CB2 8EA.

OCR is part of the Cambridge Assessment Group; Cambridge Assessment is the brand name of University of Cambridge Local Examinations Syndicate (UCLES), which is itself a department of the University of Cambridge.

stemup.co.uk

Surname	StemUp Model Solution
Answer booklet	Write the question number in the left-hand margin at the start of each answer.

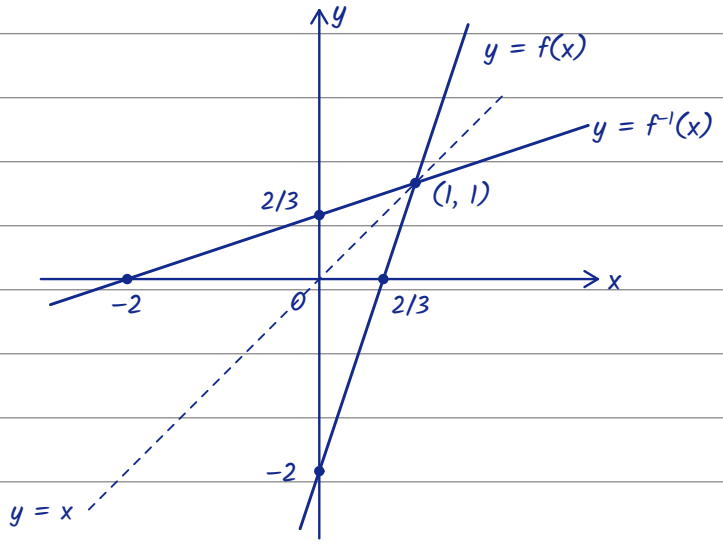
Question number

1 (a)

$y = 3x - 2 \Rightarrow x = \frac{y + 2}{3}$

$f^{-1}(x) = \frac{x + 2}{3}$

1 (b)



1 (c)

$f(x) = f^{-1}(x)$ where they meet on $y = x$:

$3x - 2 = x \Rightarrow 2x = 2 \Rightarrow x = 1$

f has gradient $3 >$ gradient of f^{-1} (which is $\frac{1}{3}$),

so $f(x)$ is above $f^{-1}(x)$ to the right of $x = 1$

$\therefore\ f(x) > f^{-1}(x)$ for $x > 1$

2 (a)

$x^2 + 8x - 7 = (x + 4)^2 - 16 - 7$

$= (x + 4)^2 - 23$

$y = x^2 \rightarrow y = (x + 4)^2 - 23$ is a translation by vector $\begin{pmatrix} -4 \\ -23 \end{pmatrix}$

2 (b)

Turning point $(-4, -23)$

Question number

3 (a)

$$\frac{1}{(x+2)(x+3)} \equiv \frac{A}{x+2} + \frac{B}{x+3}$$

$1 \equiv A(x+3) + B(x+2)$

$x = -3$: $1 = -B \Rightarrow B = -1$

$x = -2$: $1 = A \Rightarrow A = 1$

$$\therefore \frac{1}{(x+2)(x+3)} = \frac{1}{x+2} - \frac{1}{x+3}$$

3 (b)

$$\int \frac{1}{(x+2)(x+3)}\,dx = \int \left(\frac{1}{x+2} - \frac{1}{x+3}\right) dx$$

$$= \ln|x+2| - \ln|x+3| + c$$

$$= \ln\left|\frac{x+2}{x+3}\right| + c$$

4

$$\frac{1}{\sqrt{10}+\sqrt{11}} = \frac{\sqrt{11}-\sqrt{10}}{(\sqrt{11}+\sqrt{10})(\sqrt{11}-\sqrt{10})} = \frac{\sqrt{11}-\sqrt{10}}{11-10} = \sqrt{11}-\sqrt{10}$$

Similarly $\frac{1}{\sqrt{11}+\sqrt{12}} = \frac{\sqrt{12}-\sqrt{11}}{12-11} = \sqrt{12}-\sqrt{11}$

and $\frac{1}{\sqrt{12}+\sqrt{13}} = \frac{\sqrt{13}-\sqrt{12}}{13-12} = \sqrt{13}-\sqrt{12}$

LHS $= \sqrt{11}-\sqrt{10}+\sqrt{12}-\sqrt{11}+\sqrt{13}-\sqrt{12} = \sqrt{13}-\sqrt{10}$

RHS $= \frac{3}{\sqrt{10}+\sqrt{13}} \times \frac{\sqrt{13}-\sqrt{10}}{\sqrt{13}-\sqrt{10}} = \frac{3(\sqrt{13}-\sqrt{10})}{13-10} = \sqrt{13}-\sqrt{10}$

LHS = RHS, so the two expressions are equal

5

Assume there is a largest prime, p.

List all the primes: 2, 3, 5, 7, ..., p.

Let $N = (2 \times 3 \times 5 \times 7 \times \ldots \times p) + 1$. Then $N > p$.

Dividing N by any prime in the list leaves remainder 1,

so N is not divisible by any of the primes in the list.

So N is prime, or it has a prime factor not in the list.

Either way there is a prime that is not in the list,

i.e. a prime larger than p. This is a contradiction,

so there is no largest prime number.

6 (a)

x-axis is a tangent so radius = 4

$(x-10)^2 + (y-4)^2 = 16$

Question number	
6 (b)	Substitute $y = x$: $(x - 10)^2 + (x - 4)^2 = 16$
	$x^2 - 20x + 100 + x^2 - 8x + 16 = 16$
	$2x^2 - 28x + 100 = 0$
	$x^2 - 14x + 50 = 0$
	$b^2 - 4ac = (-14)^2 - 4 \times 1 \times 50 = 196 - 200 = -4$
	$-4 < 0$ so there are no real roots: the line $y = x$ does not
	meet the circle, so $y = x$ is not a tangent to the circle.
6 (c)	Midpoint of OC = (5, 2): position vector $5i + 2j$
7 (a)	$\ln 9 = \ln 3^2 = 2 \ln 3$, $\ln 27 = \ln 3^3 = 3 \ln 3$
	$\ln 3 \times \ln 9 \times \ln 27 = \ln 3 \times 2 \ln 3 \times 3 \ln 3$
	$= 6(\ln 3)^3$
7 (b)	$3 > e$, so $\ln 3 > \ln e = 1$
	$\therefore (\ln 3)^3 > 1^3 = 1$
	$\therefore 6(\ln 3)^3 > 6$, i.e. $\ln 3 \times \ln 9 \times \ln 27 > 6$
8 (a)	$y = x^3 \Rightarrow \frac{dy}{dx} = 3x^2$
	At B (1, 1): gradient $= 3 \times 1^2 = 3$
	Tangent: $y - 1 = 3(x - 1) \Rightarrow y = 3x - 2$
	E: $x = 0 \Rightarrow y = -2$, so $OE = 2$
	D: $y = 0 \Rightarrow 3x - 2 = 0 \Rightarrow x = \frac{2}{3}$, so $OD = \frac{2}{3}$
	Area $ODE = \frac{1}{2} \times \frac{2}{3} \times 2 = \frac{2}{3}$
8 (b)	Area $= \int_0^1 (x^3 - (3x - 2))\,dx = \int_0^1 (x^3 - 3x + 2)\,dx$
	$= \left[\frac{x^4}{4} - \frac{3x^2}{2} + 2x\right]_0^1$
	$= \left(\frac{1}{4} - \frac{3}{2} + 2\right) - 0$
	$= \frac{3}{4}$

Question number	
9	$xy + y^2 = 8$
	Differentiate implicitly: $y + x\frac{dy}{dx} + 2y\frac{dy}{dx} = 0$
	$\frac{dy}{dx} = \frac{-y}{x + 2y}$
	Normal has gradient 2 $\Rightarrow$ tangent has gradient $-\frac{1}{2}$
	$\frac{-y}{x + 2y} = -\frac{1}{2} \Rightarrow 2y = x + 2y \Rightarrow x = 0$
	$x = 0$: $0 + y^2 = 8 \Rightarrow y = \pm\sqrt{8} = \pm 2\sqrt{2}$
	Points $(0, 2\sqrt{2})$ and $(0, -2\sqrt{2})$
10	$f(x) = \frac{e^x}{1 + e^x}$ Quotient rule:
	$f'(x) = \frac{(1 + e^x)e^x - e^x \times e^x}{(1 + e^x)^2} = \frac{e^x + e^{2x} - e^{2x}}{(1 + e^x)^2} = \frac{e^x}{(1 + e^x)^2}$
	$e^x > 0$ for all x and $(1 + e^x)^2 > 0$ for all x
	so $f'(x) > 0$ for all $x \Rightarrow f(x)$ is an increasing function
	for all values of x.
11	$u = 1 + \sqrt{x} \Rightarrow \sqrt{x} = u - 1, \quad x = (u - 1)^2$
	$\frac{dx}{du} = 2(u - 1) \Rightarrow dx = 2(u - 1)\,du$
	$\int \frac{x}{1 + \sqrt{x}}\,dx = \int \frac{(u - 1)^2}{u} \times 2(u - 1)\,du = \int \frac{2(u - 1)^3}{u}\,du$
	$= \int \frac{2(u^3 - 3u^2 + 3u - 1)}{u}\,du$
	$= \int 2\left(u^2 - 3u + 3 - \frac{1}{u}\right)du$
	$= \frac{2u^3}{3} - 3u^2 + 6u - 2\ln\lvert u\rvert + c$
	$= \frac{2}{3}(1 + \sqrt{x})^3 - 3(1 + \sqrt{x})^2 + 6(1 + \sqrt{x}) - 2\ln(1 + \sqrt{x}) + c$
12	Line passes through $(r, 0)$ and $(0, h)$
	gradient $= \frac{h - 0}{0 - r} = -\frac{h}{r}$
	$y = -\frac{h}{r}x + h \Rightarrow ry = -hx + hr \Rightarrow ry + hx = hr$
13(a)(i)	Area = rectangle + two semicircles $= 2x \times 1 + \pi x^2$
	$= 2x \times \pi(r - x) + \pi x^2$
	$= 2\pi rx - 2\pi x^2 + \pi x^2 = 2\pi rx - \pi x^2$

Question number	
13(a)(i)	$= \pi x(2r - x)$
13(a)(ii)	$ry + hx = hr \Rightarrow x = \frac{hr - ry}{h} = \frac{r(h - y)}{h}$ $2r - x = 2r - \frac{r(h - y)}{h} = \frac{2rh - rh + ry}{h} = \frac{r(h + y)}{h}$ $\text{Area} = \pi x(2r - x) = \pi \times \frac{r(h - y)}{h} \times \frac{r(h + y)}{h}$ $= \frac{\pi r^2}{h^2}(h - y)(h + y) = \frac{\pi r^2}{h^2}(h^2 - y^2)$
13(b)(i)	Fig. C3.1 is at the bottom, $y = 0$: $\text{Area} = \frac{\pi r^2}{h^2}(h^2 - 0^2) = \frac{\pi r^2}{h^2} \times h^2 = \pi r^2$, the area of the circle
13(b)(ii)	Fig. C3.3 is at the top, $y = h$: $\text{Area} = \frac{\pi r^2}{h^2}(h^2 - h^2) = 0$, correct as a line has zero area
14 (a)	$\int_0^h (h^2 - y^2)\,dy$
14 (b)	$V = \frac{\pi r^2}{h^2}\int_0^h (h^2 - y^2)\,dy = \frac{\pi r^2}{h^2}\left[h^2 y - \frac{y^3}{3}\right]_0^h$ $= \frac{\pi r^2}{h^2}\left(h^3 - \frac{h^3}{3}\right) - 0 = \frac{\pi r^2}{h^2} \times \frac{2h^3}{3}$ $= \frac{2}{3}\pi r^2 h$
15	The straight edge at the top has length πr (Fig. C3.3) $\pi r = 5.4 \Rightarrow r = \frac{5.4}{\pi} = 1.71887\ldots$ cm $V = \frac{2}{3} \times \pi \times 1.71887\ldots^2 \times 12 = 74.2546\ldots \approx 74.3$ cm^3 This is very close to 75 cm^3 (only about 1% too small), so the formula is accurate here; the small difference could be because the measurements are not exact or the shape is only approximately as modelled.