OCR
Oxford Cambridge and RSA

A Level Mathematics B (MEI)

H640/03 Pure Mathematics and Comprehension
Question Paper

Friday 15 June 2018 – Afternoon

Time allowed: 2 hours

7021023250

You must have:
- Printed Answer Booklet
- Insert

You may use:
- a scientific or graphical calculator

INSTRUCTIONS
- Use black ink. HB pencil may be used for graphs and diagrams only.
- Complete the boxes provided on the Printed Answer Booklet with your name, centre number and candidate number.
- Answer **all** the questions.
- **Write your answer to each question in the space provided in the Printed Answer Booklet.** If additional space is required, you should use the lined page(s) at the end of the Printed Answer Booklet. The question number(s) must be clearly shown.
- Do **not** write in the barcodes.
- You are permitted to use a scientific or graphical calculator in this paper.
- Final answers should be given to a degree of accuracy appropriate to the context.

INFORMATION
- The total mark for this paper is **75**.
- The marks for each question are shown in brackets **[ ]**.
- You are advised that an answer may receive **no marks** unless you show sufficient detail of the working to indicate that a correct method is used. You should communicate your method with correct reasoning.
- The Printed Answer Booklet consists of **20** pages. The Question Paper consists of **8** pages.

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Formulae A Level Mathematics B (MEI) (H640)

Arithmetic series

$$S_n = \frac{1}{2}n(a+l) = \frac{1}{2}n\{2a+(n-1)d\}$$

Geometric series

$$S_n = \frac{a(1-r^n)}{1-r}$$

$$S_\infty = \frac{a}{1-r} \text{ for } |r| < 1$$

Binomial series

$$(a+b)^n = a^n + {}^n\mathrm{C}_1 a^{n-1}b + {}^n\mathrm{C}_2 a^{n-2}b^2 + \ldots + {}^n\mathrm{C}_r a^{n-r}b^r + \ldots + b^n \qquad (n \in \mathbb{N}),$$

where ${}^n\mathrm{C}_r = {}_n\mathrm{C}_r = \binom{n}{r} = \frac{n!}{r!(n-r)!}$

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \ldots + \frac{n(n-1)\ldots(n-r+1)}{r!}x^r + \ldots \qquad (|x| < 1,\ n \in \mathbb{R})$$

Differentiation

$\mathrm{f}(x)$	$\mathrm{f}'(x)$
$\tan kx$	$k\sec^2 kx$
$\sec x$	$\sec x \tan x$
$\cot x$	$-\mathrm{cosec}^2 x$
$\mathrm{cosec}\, x$	$-\mathrm{cosec}\, x \cot x$

Quotient Rule $y = \frac{u}{v}, \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{v\frac{\mathrm{d}u}{\mathrm{d}x} - u\frac{\mathrm{d}v}{\mathrm{d}x}}{v^2}$

Differentiation from first principles

$$\mathrm{f}'(x) = \lim_{h \to 0} \frac{\mathrm{f}(x+h) - \mathrm{f}(x)}{h}$$

Integration

$$\int \frac{\mathrm{f}'(x)}{\mathrm{f}(x)}\,\mathrm{d}x = \ln|\mathrm{f}(x)| + c$$

$$\int \mathrm{f}'(x)(\mathrm{f}(x))^n\,\mathrm{d}x = \frac{1}{n+1}(\mathrm{f}(x))^{n+1} + c$$

Integration by parts $\int u\frac{\mathrm{d}v}{\mathrm{d}x}\,\mathrm{d}x = uv - \int v\frac{\mathrm{d}u}{\mathrm{d}x}\,\mathrm{d}x$

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Small angle approximations

$\sin\theta \approx \theta$, $\cos\theta \approx 1 - \frac{1}{2}\theta^2$, $\tan\theta \approx \theta$ where θ is measured in radians

Trigonometric identities

$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$

$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$

$\tan(A \pm B) = \dfrac{\tan A \pm \tan B}{1 \mp \tan A \tan B} \qquad \left(A \pm B \neq (k + \frac{1}{2})\pi\right)$

Numerical methods

Trapezium rule: $\displaystyle\int_a^b y\,\mathrm{d}x \approx \tfrac{1}{2}h\{(y_0 + y_n) + 2(y_1 + y_2 + \ldots + y_{n-1})\}$, where $h = \dfrac{b-a}{n}$

The Newton-Raphson iteration for solving $\mathrm{f}(x) = 0$: $x_{n+1} = x_n - \dfrac{\mathrm{f}(x_n)}{\mathrm{f}'(x_n)}$

Probability

$\mathrm{P}(A \cup B) = \mathrm{P}(A) + \mathrm{P}(B) - \mathrm{P}(A \cap B)$

$\mathrm{P}(A \cap B) = \mathrm{P}(A)\mathrm{P}(B \,|\, A) = \mathrm{P}(B)\mathrm{P}(A \,|\, B)$ **or** $\mathrm{P}(A \,|\, B) = \dfrac{\mathrm{P}(A \cap B)}{\mathrm{P}(B)}$

Sample variance

$s^2 = \dfrac{1}{n-1}S_{xx}$ where $S_{xx} = \Sigma(x_i - \bar{x})^2 = \Sigma x_i^2 - \dfrac{(\Sigma x_i)^2}{n} = \Sigma x_i^2 - n\bar{x}^2$

Standard deviation, $s = \sqrt{\text{variance}}$

The binomial distribution

If $X \sim \mathrm{B}(n, p)$ then $\mathrm{P}(X = r) = {}^n\mathrm{C}_r p^r q^{n-r}$ where $q = 1 - p$

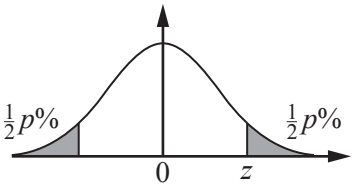
Mean of X is np

Hypothesis testing for the mean of a Normal distribution

If $X \sim \mathrm{N}(\mu, \sigma^2)$ then $\bar{X} \sim \mathrm{N}\left(\mu, \dfrac{\sigma^2}{n}\right)$ and $\dfrac{\bar{X} - \mu}{\sigma/\sqrt{n}} \sim \mathrm{N}(0, 1)$

Percentage points of the Normal distribution

p	10	5	2	1
z	1.645	1.960	2.326	2.576



Kinematics

Motion in a straight line	Motion in two dimensions
$v = u + at$	$\mathbf{v} = \mathbf{u} + \mathbf{a}t$
$s = ut + \frac{1}{2}at^2$	$\mathbf{s} = \mathbf{u}t + \frac{1}{2}\mathbf{a}t^2$
$s = \frac{1}{2}(u + v)t$	$\mathbf{s} = \frac{1}{2}(\mathbf{u} + \mathbf{v})t$
$v^2 = u^2 + 2as$	
$s = vt - \frac{1}{2}at^2$	$\mathbf{s} = \mathbf{v}t - \frac{1}{2}\mathbf{a}t^2$

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Answer **all** the questions.

Section A (60 marks)

1 Triangle ABC is shown in Fig. 1.

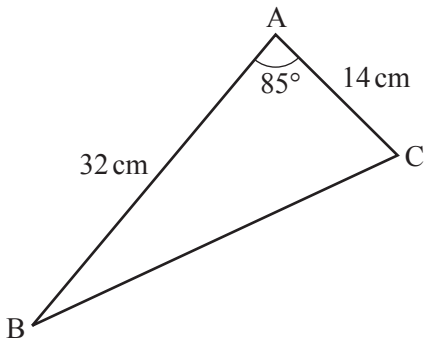


Fig. 1

Find the perimeter of triangle ABC. **[3]**

2 The curve $y = x^3 - 2x$ is translated by the vector $\begin{pmatrix} 1 \\ -4 \end{pmatrix}$. Write down the equation of the translated curve. **[2]**

3 Fig. 3 shows a circle with centre O and radius 1 unit. Points A and B lie on the circle with angle AOB = θ radians. C lies on AO, and BC is perpendicular to AO.

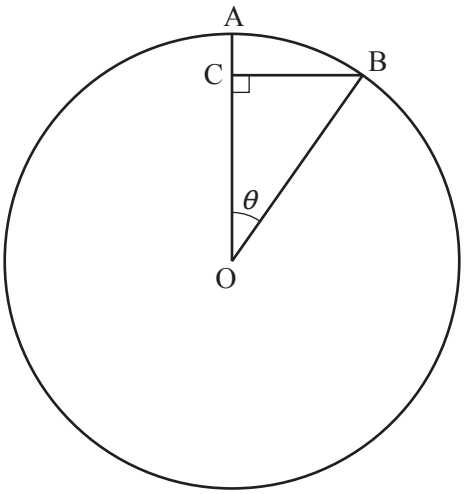


Fig. 3

Show that, when θ is small, AC $\approx \frac{1}{2}\theta^2$. **[2]**

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4 In this question you must show detailed reasoning.

A curve has equation $y = x - 5 + \frac{1}{x-2}$. The curve is shown in Fig. 4.

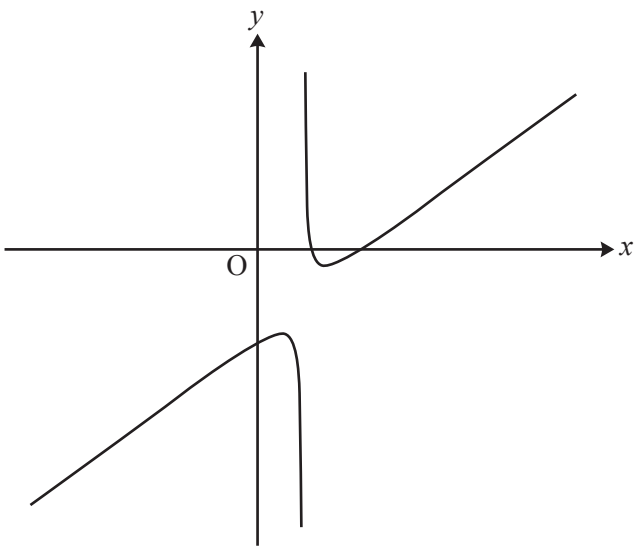


Fig. 4

(i) Determine the coordinates of the stationary points on the curve. **[5]**

(ii) Determine the nature of each stationary point. **[3]**

(iii) Write down the equation of the vertical asymptote. **[1]**

(iv) Deduce the set of values of x for which the curve is concave upwards. **[1]**

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5 A social media website launched on 1 January 2017. The owners of the website report the number of users the site has at the start of each month. They believe that the relationship between the number of users, n, and the number of months after launch, t, can be modelled by $n = a \times 2^{kt}$ where a and k are constants.

(i) Show that, according to the model, the graph of $\log_{10} n$ against t is a straight line. **[2]**

(ii) Fig. 5 shows a plot of the values of t and $\log_{10} n$ for the first seven months. The point at $t = 1$ is for 1 February 2017, and so on.

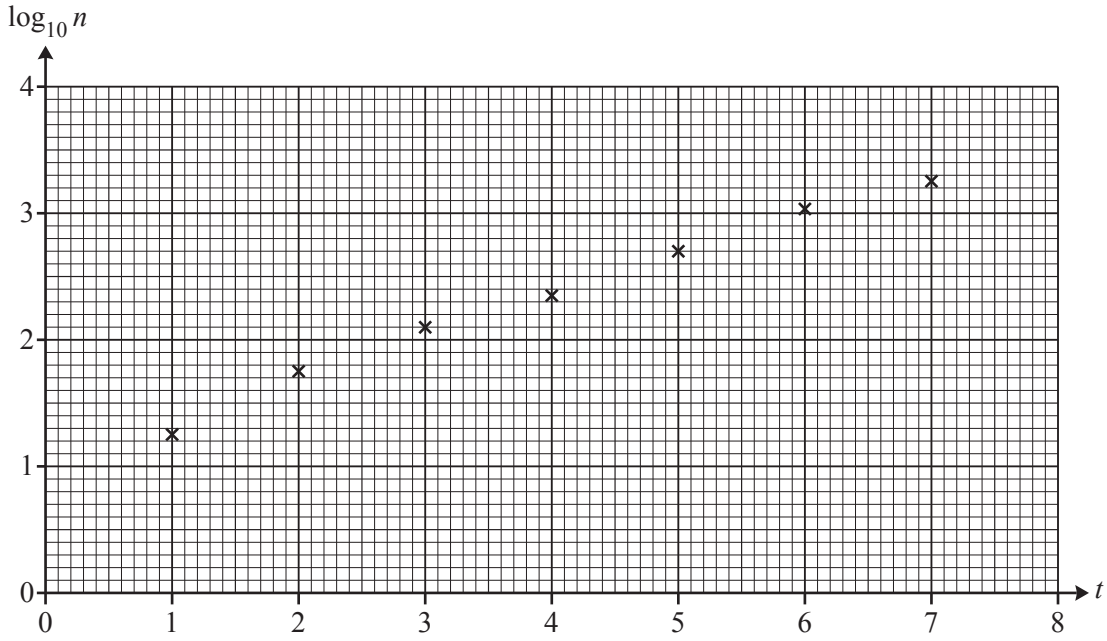


Fig. 5

Find estimates of the values of a and k. **[4]**

(iii) The owners of the website wanted to know the date on which they would report that the website had half a million users. Use the model to estimate this date. **[4]**

(iv) Give a reason why the model may not be appropriate for large values of t. **[1]**

6 Find the constant term in the expansion of $\left(x^2 + \frac{1}{x}\right)^{15}$. **[2]**

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7 **In this question you must show detailed reasoning.**

Fig. 7 shows the curve $y = 5x - x^2$.

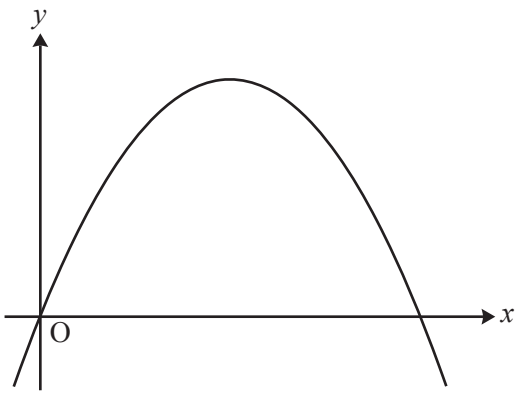


Fig. 7

The line $y = 4 - kx$ crosses the curve $y = 5x - x^2$ on the x-axis and at one other point.

Determine the coordinates of this other point. **[8]**

8 A curve has parametric equations $x = \dfrac{t}{1+t^3}$, $y = \dfrac{t^2}{1+t^3}$, where $t \neq -1$.

(i) **In this question you must show detailed reasoning.**

Determine the gradient of the curve at the point where $t = 1$. **[5]**

(ii) Verify that the cartesian equation of the curve is $x^3 + y^3 = xy$. **[3]**

9 The function $\mathrm{f}(x) = \dfrac{\mathrm{e}^x}{1-\mathrm{e}^x}$ is defined on the domain $x \in \mathbb{R}$, $x \neq 0$.

(i) Find $\mathrm{f}^{-1}(x)$. **[3]**

(ii) Write down the range of $\mathrm{f}^{-1}(x)$. **[1]**

10 Point A has position vector $\begin{pmatrix} a \\ b \\ 0 \end{pmatrix}$ where a and b can vary, point B has position vector $\begin{pmatrix} 4 \\ 2 \\ 0 \end{pmatrix}$ and point C has position vector $\begin{pmatrix} 2 \\ 4 \\ 2 \end{pmatrix}$. ABC is an isosceles triangle with AC = AB.

(i) Show that $a - b + 1 = 0$. **[4]**

(ii) Determine the position vector of A such that triangle ABC has minimum area. **[6]**

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Answer **all** the questions.

Section B (15 marks)

The questions in this section refer to the article on the Insert. You should read the article before attempting the questions.

11 Line 8 states that $\frac{a+b}{2} \geqslant \sqrt{ab}$ for a, $b \geqslant 0$. Explain why the result cannot be extended to apply in each of the following cases.

(i) One of the numbers a and b is positive and the other is negative. **[1]**

(ii) Both numbers a and b are negative. **[1]**

12 Lines 5 and 6 outline the stages in a proof that $\frac{a+b}{2} \geqslant \sqrt{ab}$. Starting from $(a-b)^2 \geqslant 0$, give a detailed proof of the inequality of arithmetic and geometric means. **[3]**

13 Consider a geometric sequence in which all the terms are positive real numbers. Show that, for any three consecutive terms of this sequence, the middle one is the geometric mean of the other two. **[3]**

14 **(i)** In Fig. C1.3, angle CBD = θ. Show that angle CDA is also θ, as given in line 23. **[2]**

(ii) Prove that $h = \sqrt{ab}$, as given in line 24. **[2]**

15 It is given in lines 31–32 that the square has the smallest perimeter of all rectangles with the same area. Using this fact, prove by contradiction that among rectangles of a given perimeter, $4L$, the square with side L has the largest area. **[3]**

END OF QUESTION PAPER

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Surname	StemUp Model Solution
Answer booklet	Write the question number in the left-hand margin at the start of each answer.

Question number

1

$BC^2 = 32^2 + 14^2 - 2 \times 32 \times 14 \times \cos 85°$

$BC^2 = 1141.908...$

$BC = 33.792... = 33.8$ cm

Perimeter $= 32 + 14 + 33.792... = 79.792...$

Perimeter $= 79.8$ cm (3 s.f.)

2

Replace x by $(x - 1)$ and subtract 4:

$y = (x - 1)^3 - 2(x - 1) - 4$ (i.e. $y = x^3 - 3x^2 + x - 3$)

3

Triangle OCB is right-angled at C with $OB = 1$, so $OC = \cos\theta$

$AC = AO - OC = 1 - \cos\theta$

For small θ, $\cos\theta \approx 1 - \frac{1}{2}\theta^2$

$AC \approx 1 - (1 - \frac{1}{2}\theta^2) = \frac{1}{2}\theta^2$ as required

4 (i)

$y = x - 5 + (x - 2)^{-1}$

$dy/dx = 1 - (x - 2)^{-2} = 1 - \dfrac{1}{(x - 2)^2}$

At stationary points $dy/dx = 0$: $1 - \dfrac{1}{(x - 2)^2} = 0$

$(x - 2)^2 = 1 \Rightarrow x - 2 = \pm 1 \Rightarrow x = 1$ or $x = 3$

$x = 1$: $y = 1 - 5 + 1/(-1) = -5$

$x = 3$: $y = 3 - 5 + 1/1 = -1$

Stationary points are $(1, -5)$ and $(3, -1)$

Question number

4 (ii) $d^2y/dx^2 = 2(x-2)^{-3} = \dfrac{2}{(x-2)^3}$

At $x = 1$: $d^2y/dx^2 = 2/(-1)^3 = -2 < 0$, so $(1, -5)$ is a maximum

At $x = 3$: $d^2y/dx^2 = 2/1^3 = 2 > 0$, so $(3, -1)$ is a minimum

4 (iii) $x = 2$

4 (iv) Concave upwards when $d^2y/dx^2 > 0$: $2/(x-2)^3 > 0 \Rightarrow (x-2)^3 > 0$

Set of values: $x > 2$

5 (i) $n = a \times 2^{kt}$

$\log_{10} n = \log_{10} a + \log_{10} 2^{kt} = \log_{10} a + kt \log_{10} 2$

$\log_{10} n = (k \log_{10} 2)\, t + \log_{10} a$

This is of the form $y = mx + c$ with $y = \log_{10} n$ and $x = t$,

where the gradient $k \log_{10} 2$ and the intercept $\log_{10} a$ are constants,

so the graph of $\log_{10} n$ against t is a straight line.

5 (ii)

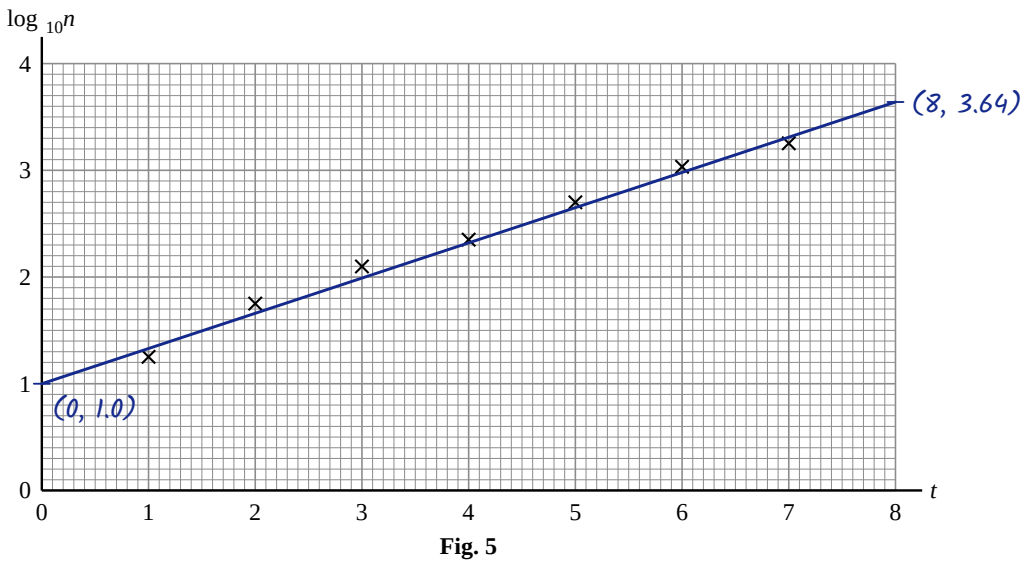


Fig. 5

Line of best fit passes through (0, 1.0) and (8, 3.64)

Intercept: $\log_{10} a = 1.0 \Rightarrow a = 10^{1.0} = 10$

Question number

Gradient $= \frac{3.64 - 1.0}{8 - 0} = 0.33$ so $k \log_{10} 2 = 0.33$

$k = \frac{0.33}{\log_{10} 2} = 1.096...$

$a \approx 10$, $k \approx 1.1$

5 (iii) $n = 10 \times 2^{1.1t}$: $500\,000 = 10 \times 2^{1.1t} \Rightarrow 2^{1.1t} = 50\,000$

$1.1t \log_{10} 2 = \log_{10} 50\,000$

$t = \frac{\log_{10} 50\,000}{1.1 \log_{10} 2} = 14.190... \approx 14.2$

$t = 12$ is 1 January 2018, so $t = 14$ is 1 March 2018

n reaches 500 000 during March 2018, so the owners first report half a million users at the start of the next month: 1 April 2018

5 (iv) Exponential growth cannot continue for ever: the number of users is limited (e.g. by the number of people online), and the model is based on only seven months of data, so it cannot be extrapolated far.

6 General term: ${}^{15}C_r (x^2)^{15-r} \frac{1}{x^r} = {}^{15}C_r x^{30-3r}$

Constant term when $30 - 3r = 0 \Rightarrow r = 10$

${}^{15}C_{10} (x^2)^5 \left(\frac{1}{x}\right)^{10} = 3003 \times x^{10} \times x^{-10}$

Constant term = 3003

7 Curve meets the x-axis when $5x - x^2 = 0 \Rightarrow x(5 - x) = 0$

$x = 0$ or $x = 5$

The line $y = 4 - kx$ passes through $(0, 4)$, not the origin, so it crosses the curve on the x-axis at $(5, 0)$.

Question number

$0 = 4 - 5k \Rightarrow k = \frac{4}{5}$

Intersection: $4 - \frac{4}{5}x = 5x - x^2$

$20 - 4x = 25x - 5x^2 \Rightarrow 5x^2 - 29x + 20 = 0$

$(5x - 4)(x - 5) = 0 \Rightarrow x = 0.8 \text{ or } x = 5$

$x = \frac{4}{5}: \quad y = 5 \times \frac{4}{5} - \left(\frac{4}{5}\right)^2 = 4 - \frac{16}{25} = \frac{84}{25}$

Other point is $\left(\frac{4}{5}, \frac{84}{25}\right)$ i.e. (0.8, 3.36)

8 (i)

$\frac{dx}{dt} = \frac{(1 + t^3) \times 1 - t \times 3t^2}{(1 + t^3)^2} = \frac{1 - 2t^3}{(1 + t^3)^2}$

$\frac{dy}{dt} = \frac{(1 + t^3) \times 2t - t^2 \times 3t^2}{(1 + t^3)^2} = \frac{2t - t^4}{(1 + t^3)^2}$

$\frac{dy}{dx} = \frac{dy}{dt} \div \frac{dx}{dt} = \frac{2t - t^4}{1 - 2t^3}$

At $t = 1$: $\frac{dy}{dx} = \frac{2 - 1}{1 - 2} = \frac{1}{-1} = -1$

Gradient at $t = 1$ is -1

8 (ii)

$x^3 + y^3 = \frac{t^3}{(1 + t^3)^3} + \frac{t^6}{(1 + t^3)^3} = \frac{t^3(1 + t^3)}{(1 + t^3)^3} = \frac{t^3}{(1 + t^3)^2}$

$xy = \frac{t}{1 + t^3} \times \frac{t^2}{1 + t^3} = \frac{t^3}{(1 + t^3)^2}$

So $x^3 + y^3 = xy$, which is the cartesian equation.

9 (i)

$y = \frac{e^x}{1 - e^x} \Rightarrow y(1 - e^x) = e^x \Rightarrow y - ye^x = e^x$

Question number

$y = e^x(1 + y) \Rightarrow e^x = \frac{y}{1+y} \Rightarrow x = \ln\left(\frac{y}{1+y}\right)$

$f^{-1}(x) = \ln\left(\frac{x}{1+x}\right)$

9 (ii) Range: $f^{-1}(x) \in \mathbb{R}$, $f^{-1}(x) \neq 0$

10 (i) $\overrightarrow{AB} = \begin{pmatrix} 4-a \\ 2-b \\ 0 \end{pmatrix}$, $\overrightarrow{AC} = \begin{pmatrix} 2-a \\ 4-b \\ 2 \end{pmatrix}$

$AB = AC \Rightarrow AB^2 = AC^2$

$(4-a)^2 + (2-b)^2 = (2-a)^2 + (4-b)^2 + 2^2$

$16 - 8a + a^2 + 4 - 4b + b^2 = 4 - 4a + a^2 + 16 - 8b + b^2 + 4$

$20 - 8a - 4b = 24 - 4a - 8b$

$4b - 4a - 4 = 0 \Rightarrow a - b + 1 = 0$ as required

10 (ii) Let D be the midpoint of BC: $\overrightarrow{OD} = \frac{1}{2}\left(\begin{pmatrix} 4 \\ 2 \\ 0 \end{pmatrix} + \begin{pmatrix} 2 \\ 4 \\ 2 \end{pmatrix}\right) = \begin{pmatrix} 3 \\ 3 \\ 1 \end{pmatrix}$

Triangle ABC is isosceles (AB = AC), so AD is perpendicular to BC.

From (i) $b = a + 1$, so $\overrightarrow{AD} = \begin{pmatrix} 3-a \\ 3-b \\ 1 \end{pmatrix} = \begin{pmatrix} 3-a \\ 2-a \\ 1 \end{pmatrix}$

$\overrightarrow{BC} = \begin{pmatrix} -2 \\ 2 \\ 2 \end{pmatrix}$, $|BC| = \sqrt{(4+4+4)} = 2\sqrt{3}$

Area $= \frac{1}{2} \times |BC| \times |AD| = \sqrt{3} \times \sqrt{((3-a)^2 + (2-a)^2 + 1)}$

$(3-a)^2 + (2-a)^2 + 1 = 2a^2 - 10a + 14 = 2[(a-2.5)^2 + 0.75]$

This is least when $a = 2.5$, so $b = 3.5$ (minimum area $= \sqrt{3} \times \sqrt{1.5} = 2.12$)

Question number

Position vector of A is $\begin{pmatrix} 2.5 \\ 3.5 \\ 0 \end{pmatrix}$

11 (i) If one of a, b is negative then $ab < 0$, so $\sqrt{(ab)}$ is not a real number: the geometric mean cannot be calculated.

11 (ii) Then $ab > 0$, so $\sqrt{(ab)}$ exists and is positive, but $(a + b)/2 < 0$, so the arithmetic mean is less than the geometric mean and the inequality is false.

12 $(a - b)^2 \geq 0 \Rightarrow a^2 - 2ab + b^2 \geq 0 \Rightarrow a^2 + b^2 \geq 2ab$

Adding $2ab$ to each side: $a^2 + 2ab + b^2 \geq 4ab \Rightarrow (a + b)^2 \geq 4ab$

Since $a, b \geq 0$, $a + b \geq 0$ and $ab \geq 0$, so take positive square roots:

$a + b \geq \sqrt{(4ab)} = 2\sqrt{(ab)} \Rightarrow \frac{a + b}{2} \geq \sqrt{(ab)}$

13 Let three consecutive terms be $\frac{c}{r}$, c, cr where $c > 0$ and $r > 0$

Geometric mean of the first and last $= \sqrt{\frac{c}{r} \times cr} = \sqrt{(c^2)} = c \quad (c > 0)$

which is the middle term, as required.

14 (i) In triangle BCD, angle BCD = 90° (CD is perpendicular to AB), so angle BDC = 180° − 90° − θ = 90° − θ (angles in a triangle sum to 180°)

Angle ADB = 90° (angle in a semicircle)

Angle CDA = angle ADB − angle BDC = 90° − (90° − θ) = θ

14 (ii) Triangle ACD: $\tan\theta = \frac{AC}{CD} = \frac{a}{h}$ Triangle BCD: $\tan\theta = \frac{CD}{CB} = \frac{h}{b}$

Question number

$\frac{a}{h} = \frac{h}{b} \Rightarrow h^2 = ab \Rightarrow h = \sqrt{(ab)}$ (since $h > 0$)

15 Suppose there is a rectangle with perimeter $4L$ whose area A is larger than the area of the square with side L, i.e. $A > L^2$.

By lines 31–32, the square with area A, side $\sqrt{A}$, has a perimeter no larger than this rectangle: $4\sqrt{A} \leq 4L$, so $\sqrt{A} \leq L$ and $A \leq L^2$.

This contradicts $A > L^2$, so no such rectangle exists. Hence among rectangles with perimeter $4L$, the square with side L has the largest area.