OCR
Oxford Cambridge and RSA

Friday 22 October 2021 – Afternoon

A Level Further Mathematics B (MEI)

Y436/01 Further Pure with Technology

Time allowed: 1 hour 45 minutes

8314759111

You must have:
- the Printed Answer Booklet
- the Formulae Booklet for Further Mathematics B (MEI)
- a computer with appropriate software
- a scientific or graphical calculator

INSTRUCTIONS
- Use black ink. You can use an HB pencil, but only for graphs and diagrams.
- Write your answer to each question in the space provided in the **Printed Answer Booklet**. You can use extra paper if you need to, but you must clearly show your candidate number, the centre number and the question numbers.
- Fill in the boxes on the front of the Printed Answer Booklet.
- Answer **all** the questions.
- Where appropriate, your answer should be supported with working. Marks might be given for using a correct method, even if your answer is wrong.
- Give your final answers to a degree of accuracy that is appropriate to the context.
- Do **not** send this Question Paper for marking. Keep it in the centre or recycle it.

INFORMATION
- The total mark for this paper is **60**.
- The marks for each question are shown in brackets **[ ]**.
- This document has **8** pages.

ADVICE
- Read each question carefully before you start your answer.

© OCR 2021 [K/508/5600]
DC (NF) 300541/5

OCR is an exempt Charity

stemup.co.uk

Answer **all** the questions.

1 A family of circles is given by the equation

$$(x-2\cos a)^2+(y-2\sin a)^2=1 \ (*)$$

where the parameter a satisfies $0 \leqslant a < 2\pi$.

(a) Use a slider (for a) to investigate this family of circles. Write down the cartesian equation of the curve which contains the centre of each circle in the family. **[1]**

(b) Let b and c be real numbers with $0 \leqslant b < c < \pi$. Find and simplify an expression, in terms of b and c, for the distance between the centre of the circle corresponding to $a = b$ and the centre of the circle corresponding to $a = c$. **[2]**

(c) Hence, or otherwise, find a condition on b and c for the two circles in part **(b)** to touch. **[2]**

A curve which every member of a family of curves or lines touches tangentially is called an *envelope* of the family.

(d) By tracing the family of curves using a slider (for a), or otherwise, sketch the envelope of the family (*) in the Printed Answer Booklet. **[2]**

(e) Write down the equations of the curves which make up the envelope for this family (*). **[2]**

2 This question is about the family of straight lines which pass through the points $(0, a)$ and $(1, a^2)$ where the parameter a is any real number.

(a) In terms of a, find the equation of the straight line which passes through the points $(0, a)$ and $(1, a^2)$. **[2]**

(b) Let b and c be distinct real numbers. Given that the straight line corresponding to $a = b$ and the straight line corresponding to $a = c$ are parallel, find b in terms of c. **[3]**

(c) By tracing the family using a slider (for a), or otherwise, sketch the envelope of this family in the Printed Answer Booklet. **[2]**

(d) Determine, in the form $y = \mathrm{h}(x)$, the cartesian equation of the envelope for this family. **[5]**

© OCR 2021

stemup.co.uk

3 **(a)** **(i)** Create a program which returns the highest common factor of positive integers m and n. Write out your program in full in the Printed Answer Booklet. **[3]**

In the rest of this question the highest common factor of positive integers m and n is denoted by (m, n).

(ii) Use your program to find (74333, 89817). **[1]**

(b) Euler's totient function $\varphi(n)$, where n is a positive integer, is defined to be the number of integers m with $1 \leqslant m \leqslant n$ such that $(m, n) = 1$.
For example $\varphi(6) = 2$ because $(1, 6) = 1$, $(2, 6) = 2$, $(3, 6) = 3$, $(4, 6) = 2$, $(5, 6) = 1$ and $(6, 6) = 6$.

(i) Extend your program in **(a)(i)** to create a program which returns $\varphi(n)$ for a given positive integer n. **[3]**

(ii) Use your program to find $\varphi(128)$ and $\varphi(1000)$. **[2]**

(iii) For a positive integer n, determine $\varphi(2^n)$ in terms of n. **[2]**

(iv) For a positive integer n, determine $\varphi(10^n)$ in terms of n. **[3]**

(c) For any positive integer k, let $\mathrm{F}(k)$ be the number of distinct fractions $\frac{m}{n}$ where $0 < m < n \leqslant k$.
For example F(4) = 5, since there are five fractions which satisfy the required condition, namely $\frac{1}{4}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}$.

(i) Find $\mathrm{F}(5)$ and $\mathrm{F}(6)$. **[2]**

(ii) Explain why, for any positive integer l, $\mathrm{F}(l+1) = \mathrm{F}(l) + \varphi(l+1)$. **[2]**

(iii) Determine $\mathrm{F}(100)$. **[2]**

© OCR 2021

stemup.co.uk

4 This question concerns the family of differential equations

$$\frac{dy}{dx} = \frac{1-x}{2(x+1)} + a\arctan(y) \quad (x \geqslant 0) \ (*)$$

where a is a constant.

(a) (i) Find the solution to (*) in the case $a = 0$ in which $y = 0$ when $x = 0$. **[1]**

(ii) Sketch this solution for $0 \leqslant x \leqslant 5$ in the Printed Answer Booklet. **[1]**

(iii) For this solution, determine the maximum value of y for $0 \leqslant x \leqslant 5$. **[2]**

(b) Fig 4.1 and Fig 4.2 show tangent fields for two distinct but unspecified values of a. In each case a sketch of the solution curve $y = g(x)$ which passes through the origin is shown for $0 \leqslant x \leqslant 1$.

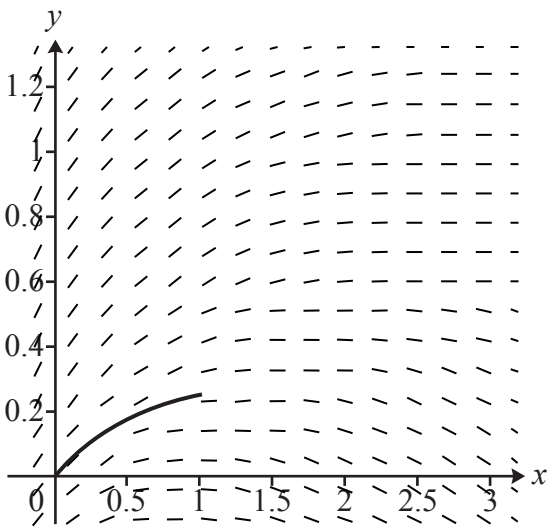


Fig 4.1

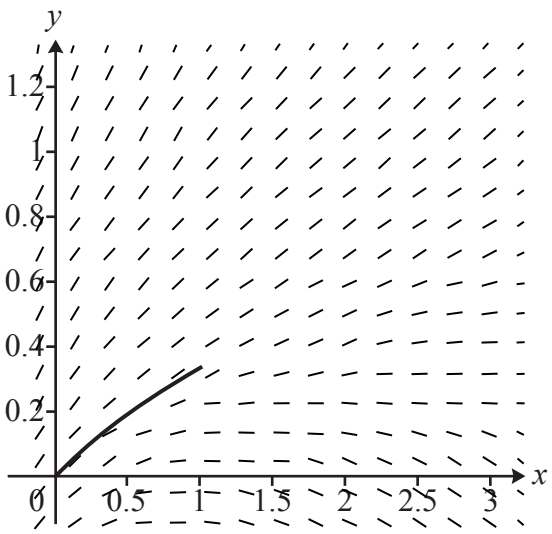


Fig 4.2

(i) For the case in Fig 4.1 suggest a possible value of a. **[1]**

(ii) For the case in Fig 4.2 suggest a possible value of a. **[1]**

(iii) In each case, continue the sketch of the solution curves for $1 \leqslant x \leqslant 3$ in the Printed Answer Booklet. **[2]**

(iv) State a feature which is present in one of the curves in part **(iii)** but not in the other. **[1]**

© OCR 2021

stemup.co.uk

(c) (i) A modified Euler method for the solution of the differential equation $\mathrm{f}(x, y) = \dfrac{\mathrm{d}y}{\mathrm{d}x}$ is as follows.

$$k_1 = h\mathrm{f}(x_n, y_n)$$

$$k_2 = h\mathrm{f}(x_n + h, y_n + k_1)$$

$$x_{n+1} = x_n + h$$

$$y_{n+1} = y_n + \frac{1}{2}(k_1 + k_2).$$

Construct a spreadsheet to solve (*), so that the value of a and the value of h can be varied, in the case $x_0 = 0$ and $y_0 = 0$. State the formulae you have used in your spreadsheet. **[4]**

(ii) In this part of the question $a = 0.5$. Use your spreadsheet with $h = 0.1$ to approximate the value of y when $x = 5$ for the solution to (*) in which $y = 0$ when $x = 0$. **[1]**

(iii) In this part of the question $a = 1$. Use your spreadsheet with $h = 0.1$ to approximate the value of y when $x = 5$ for the solution to (*) in which $y = 0$ when $x = 0$. **[1]**

There is a value c such that

- if $a > c$ then the solution in which $y = 0$ when $x = 0$ increases without bound as x increases from 0

and

- if $a < c$ then the solution in which $y = 0$ when $x = 0$ increases initially but then peaks and decreases as x increases from 0.

(iv) Use your spreadsheet to find c correct to 2 decimal places. **[4]**

END OF QUESTION PAPER

© OCR 2021

stemup.co.uk

BLANK PAGE

© OCR 2021

stemup.co.uk

BLANK PAGE

© OCR 2021

stemup.co.uk

OCR
Oxford Cambridge and RSA

Copyright Information

OCR is committed to seeking permission to reproduce all third-party content that it uses in its assessment materials. OCR has attempted to identify and contact all copyright holders whose work is used in this paper. To avoid the issue of disclosure of answer-related information to candidates, all copyright acknowledgements are reproduced in the OCR Copyright Acknowledgements Booklet. This is produced for each series of examinations and is freely available to download from our public website (www.ocr.org.uk) after the live examination series.

If OCR has unwittingly failed to correctly acknowledge or clear any third-party content in this assessment material, OCR will be happy to correct its mistake at the earliest possible opportunity.

For queries or further information please contact The OCR Copyright Team, The Triangle Building, Shaftesbury Road, Cambridge CB2 8EA.

OCR is part of the Cambridge Assessment Group; Cambridge Assessment is the brand name of University of Cambridge Local Examinations Syndicate (UCLES), which is itself a department of the University of Cambridge.

© OCR 2021

stemup.co.uk

Surname	StemUp Model Solution
Answer booklet	Write the question number in the left-hand margin at the start of each answer.

Question number

1 (a) Centre of each circle is $(2\cos a, 2\sin a)$

$x^2 + y^2 = 4\cos^2 a + 4\sin^2 a = 4$

Curve containing the centres: $x^2 + y^2 = 4$

1 (b) Centres $(2\cos b, 2\sin b)$ and $(2\cos c, 2\sin c)$

$d^2 = (2\cos c - 2\cos b)^2 + (2\sin c - 2\sin b)^2$

$d^2 = 4(\cos^2 c + \sin^2 c) + 4(\cos^2 b + \sin^2 b) - 8(\cos c \cos b + \sin c \sin b)$

$d^2 = 8 - 8\cos(c - b)$

$d = 2\sqrt{2}\sqrt{1 - \cos(c - b)} \quad \left(= 4\sin\left(\frac{c - b}{2}\right) \right)$

1 (c) Both circles have radius 1, so they touch when $d = 1 + 1 = 2$

$2\sqrt{2}\sqrt{1 - \cos(c - b)} = 2$

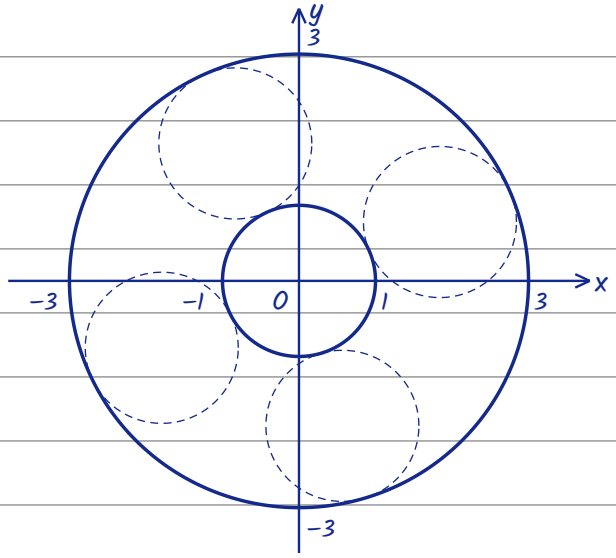
$1 - \cos(c - b) = \frac{1}{2}$

$\cos(c - b) = \frac{1}{2}$

Since $0 < c - b < 2\pi$: $c - b = \frac{\pi}{3}$ (or $c - b = \frac{5\pi}{3}$)

Question number

1 (d)



Envelope (solid): two circles centre 0, radii 1 and 3

(dashed circles are some members of the family)

1 (e)

$x^2 + y^2 = 1$

$x^2 + y^2 = 9$

2 (a)

Gradient $= \frac{a^2 - a}{1 - 0} = a(a - 1)$

Line passes through $(0, a)$, so the y-intercept is a

$y = a(a - 1)x + a$

2 (b)

Lines $y = b(b - 1)x + b$ and $y = c(c - 1)x + c$

Parallel $\Rightarrow$ equal gradients: $b(b - 1) = c(c - 1)$

$b^2 - c^2 - (b - c) = 0$

$(b - c)(b + c) - (b - c) = 0$

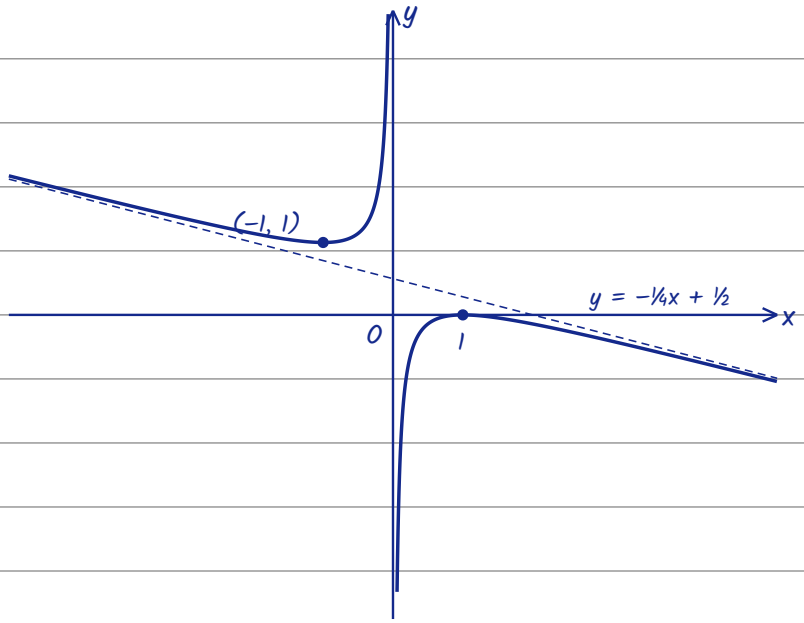
$(b - c)(b + c - 1) = 0$

$b \neq c$, so $b + c - 1 = 0$

$b = 1 - c$

Question number

2 (c)



Two branches, asymptotes $x = 0$ and $y = -\frac{1}{4}x + \frac{1}{2}$,

touches the x-axis at $(1, 0)$, turning point $(-1, 1)$

2 (d) Let $f(x, y, a) = a(a-1)x + a - y = a^2x - ax + a - y$

The envelope satisfies $f = 0$ and $\frac{\partial f}{\partial a} = 0$ simultaneously

$$\frac{\partial f}{\partial a} = 2ax - x + 1 = 0$$

$$\Rightarrow a = \frac{x-1}{2x} \qquad (x \neq 0)$$

Substitute into $f = 0$: $y = x\left(\frac{x-1}{2x}\right)^2 - (x-1)\left(\frac{x-1}{2x}\right)$

$$y = \frac{(x-1)^2}{4x} - \frac{(x-1)^2}{2x}$$

$$y = -\frac{(x-1)^2}{4x}$$

3 (a)(i) def hcf(m, n): (Python)

Question number

```
    k = min(m, n)
    h = 1
    for i in range(1, k + 1):
        if m % i == 0 and n % i == 0:
            h = i
    return h
print(hcf(m, n))
```

3 (a)(ii) hcf(74333, 89817) returns 49

(74333, 89817) = 49

3 (b)(i) (uses the function hcf from (a)(i))

```
def phi(n):
    count = 0
    for m in range(1, n + 1):
        if hcf(m, n) == 1:
            count = count + 1
    return count
print(phi(n))
```

3 (b)(ii) $\varphi(128) = 64$

$\varphi(1000) = 400$

3 (b)(iii) The only prime factor of 2^n is 2, so m is coprime to 2^n exactly when m is odd.

Of the numbers 1, 2, ..., 2^n, half are odd.

$\varphi(2^n) = 2^{n-1}$

Question number	
3 (b)(iv)	$10^n = 2^n \times 5^n$, so m is coprime to 10^n exactly when m is not divisible by 2 or 5, i.e. its last digit is 1, 3, 7 or 9. Each block 1 to 10, 11 to 20, 21 to 30, ... contains 4 such numbers. There are $10^n \div 10 = 10^{n-1}$ blocks up to 10^n. $\varphi(10^n) = 4 \times 10^{n-1}$
3 (c)(i)	$F(5)$: $\frac{1}{5}, \frac{1}{4}, \frac{1}{3}, \frac{2}{5}, \frac{1}{2}, \frac{3}{5}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}$ so $F(5) = 9$ $F(6)$: the above plus $\frac{1}{6}$ and $\frac{5}{6}$ so $F(6) = 11$
3 (c)(ii)	The fractions counted by $F(1 + 1)$ are those counted by $F(1)$ plus the new ones with denominator $1 + 1$, i.e. $\frac{m}{1+1}$ with $1 \leq m \leq 1$. If $(m, 1 + 1) > 1$, then $\frac{m}{1+1}$ cancels to a fraction with a smaller denominator, which is already counted in $F(1)$. If $(m, 1 + 1) = 1$, then $\frac{m}{1+1}$ is in lowest terms, so it is new. ($m = 1 + 1$ is not coprime to $1 + 1$ since $1 \geq 1$.) Number of new fractions $= \varphi(1 + 1)$ $\therefore F(1 + 1) = F(1) + \varphi(1 + 1)$

3 (c)(iii)	$F(1) = 0$, so $F(100) = \sum_{k=2}^{100} \varphi(k)$
	def F(k):
	    total = 0
	    for i in range(2, k + 1):

Question number

total = total + phi(i)

return total

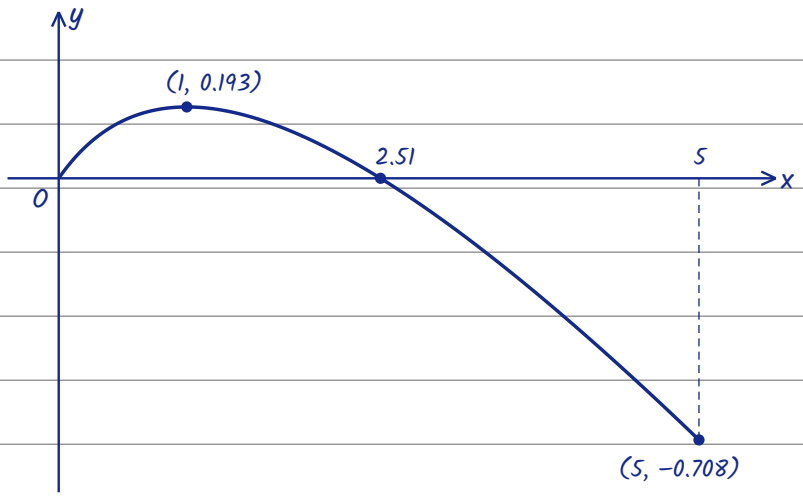
F(100) = 3043

4 (a)(i) $a = 0$: $\frac{dy}{dx} = \frac{1 - x}{2(x + 1)} = \frac{1}{x + 1} - \frac{1}{2}$

$y = \ln(x + 1) - \frac{x}{2} + C, \quad y(0) = 0 \Rightarrow C = 0$

$y = \ln(x + 1) - \frac{x}{2}$

4 (a)(ii)



4 (a)(iii) $\frac{dy}{dx} = \frac{1}{x + 1} - \frac{1}{2} = 0 \Rightarrow x + 1 = 2 \Rightarrow x = 1$

$\frac{dy}{dx} > 0$ for $x < 1$ and < 0 for $x > 1$, so this is a maximum

Maximum $y = \ln 2 - \frac{1}{2} = 0.193$ (3 s.f.)

4 (b)(i) Fig 4.1: $a = 0.5$

Question number

4 (b)(ii) Fig 4.2: $a = 1$

4 (b)(iii)

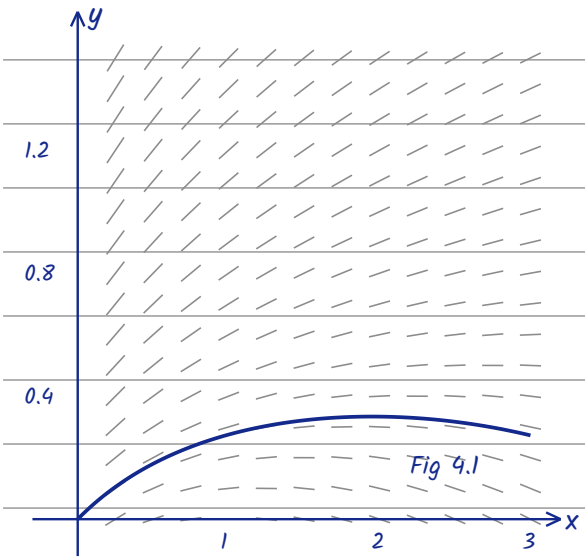


Fig 4.1

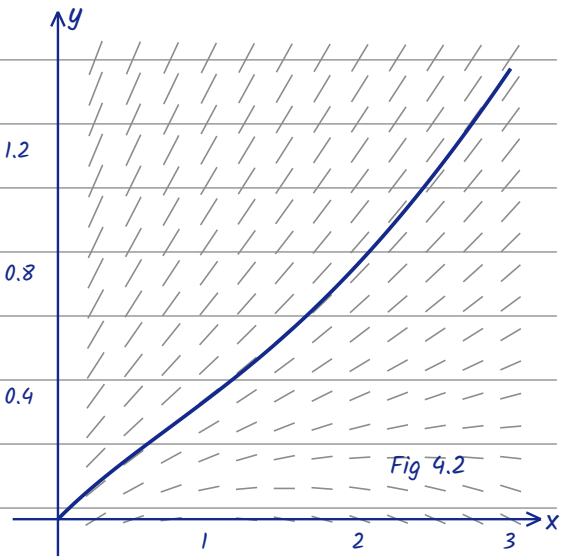


Fig 4.2

4 (b)(iv) The Fig 4.1 curve has a stationary point (a local maximum, near $x = 2$), whereas the Fig 4.2 curve keeps increasing for $1 \leqslant x \leqslant 3$.

4 (c)(i) Parameters: I1 = h (0.1), K1 = a

A1: 0 (x_0) B1: 0 (y_0)

C1 (k_1): =$I$1*((1−A1)/(2*(A1+1))+$K$1*ATAN(B1))

D1 (k_2): =$I$1*((1−(A1+$I$1))/(2*(A1+$I$1+1))
+$K$1*ATAN(B1+C1))

A2 (x): =A1+$I$1

B2 (y): =B1+0.5*(C1+D1)

Copy C1:D1 down to row 2, then copy A2:D2 down until x = 5.

4 (c)(ii) $a = 0.5$, $h = 0.1$: when $x = 5$, $y = -0.249889$

$y \approx -0.250$ (3 s.f.)

Question number	
4 (c)(iii)	$a = 1$, $h = 0.1$: when $x = 5$, $y = 3.160809$
	$y \approx 3.16$ (3 s.f.)
4 (c)(iv)	Vary a with $h = 0.1$ and extend the x column (eg to $x = 80$):
	$a = 0.645$: y increases, peaks ($y \approx 0.64$) then decreases
	$a = 0.655$: y increases without bound
	So $0.645 < c < 0.655$ (narrowing: $a = 0.648$ peaks, $a = 0.649$ grows)
	Repeating with $h = 0.05$ and $h = 0.01$ also gives $c \approx 0.650$
	$c = 0.65$ (2 d.p.)