# GCE

# Further Mathematics B (MEI)

## Y436/01: Further pure with technology

Advanced GCE

# Mark Scheme for Autumn 2021

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This mark scheme is published as an aid to teachers and students, to indicate the requirements of the examination. It shows the basis on which marks were awarded by examiners. It does not indicate the details of the discussions which took place at an examiners' meeting before marking commenced.

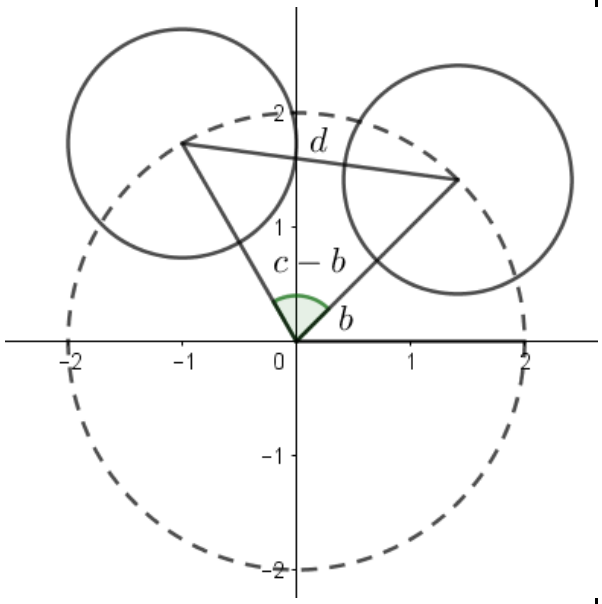
All examiners are instructed that alternative correct answers and unexpected approaches in candidates' scripts must be given marks that fairly reflect the relevant knowledge and skills demonstrated.

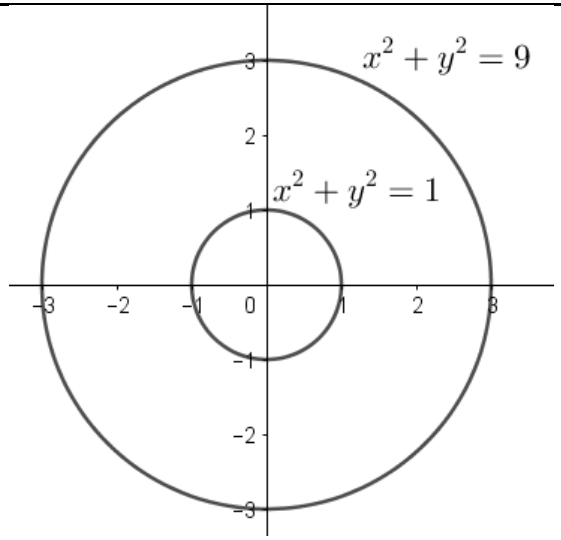
Mark schemes should be read in conjunction with the published question papers and the report on the examination.

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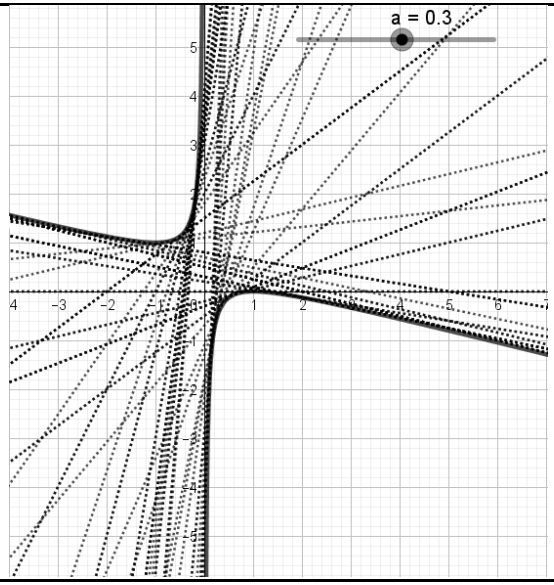
## Annotations and abbreviations

| Annotation in scoris | Meaning |
|---|---|
| ✓and ✗ | |
| BOD | Benefit of doubt |
| FT | Follow through |
| ISW | Ignore subsequent working |
| M0, M1 | Method mark awarded 0, 1 |
| A0, A1 | Accuracy mark awarded 0, 1 |
| B0, B1 | Independent mark awarded 0, 1 |
| E | Explanation mark 1 |
| SC | Special case |
| ^ | Omission sign |
| MR | Misread |
| BP | Blank page |
| Highlighting | |
| | |
| **Other abbreviations in mark scheme** | **Meaning** |
| E1 | Mark for explaining a result or establishing a given result |
| dep* | Mark dependent on a previous mark, indicated by *. The * may be omitted if only previous M mark. |
| cao | Correct answer only |
| oe | Or equivalent |
| rot | Rounded or truncated |
| soi | Seen or implied |
| www | Without wrong working |
| AG | Answer given |
| awrt | Anything which rounds to |
| BC | By Calculator |
| DR | This indicates that the instruction **In this question you must show detailed reasoning** appears in the question. |

| Question | | | Answer | Marks | AOs | | Guidance |
|---|---|---|---|---|---|---|---|
| 1 | (a) | | $x^2 + y^2 = 4$ | B1<br>[1] | 1.2 | | |
| | (b) | | Let $d$ be the distance concerned. Then $d$ is as in the diagram below.<br><br> | | | Diagram not necessary. | |
| | | | By reference to a diagram using the cosine rule<br>$d^2 = 2^2 + 2^2 - 2\times 2\times 2\cos(c-b)$<br>$= 8(1-\cos(c-b))$<br>Therefore $d = 2\sqrt{2}\sqrt{1-\cos(c-b)}$. | M1<br>A1<br>[2] | 2.1<br>1.1 | Alternatives here include $d^2 = (2\cos(c) - 2\cos(b))^2$<br>$+(2\sin(c) - 2\sin(b))^2$ | |

| Question | Answer | Marks | AOs | | Guidance |
|---|---|---|---|---|---|
| (c) | For the circles to touch, need $d = 2$ in the above.<br><br>$2 = 2\sqrt{2}\sqrt{1-\cos(c-b)}$<br>$\Rightarrow 1 = \sqrt{2}\sqrt{1-\cos(c-b)}$<br>$\Rightarrow \frac{1}{\sqrt{2}} = \sqrt{1-\cos(c-b)}$<br>Then $\Rightarrow \frac{1}{2} = 1-\cos(c-b)$<br>$\Rightarrow \cos(c-b) = \frac{1}{2}$<br>$\Rightarrow c-b = \frac{\pi}{3}$, since $0 \le b < c < \pi$ | **M1**<br><br>**A1**<br>**[2]** | **2.1**<br><br>**1.1** | Or just state that it's an equilateral triangle in this case. | |
| (d) |  | **B1**<br><br>**B1**<br><br>**[2]** | **1.1**<br><br>**1.1** | Equations are not required in this part. | |

| Question | | | Answer | Marks | AOs | | Guidance |
|---|---|---|---|---|---|---|---|
| | (e) | | $x^2 + y^2 = 1$<br>$x^2 + y^2 = 9$ | **B1**<br>**B1**<br><br>**[2]** | **1.2**<br>**1.2** | | |
| **2** | (a) | | Gradient of the line through $(0, a)$ and $(1, a^2)$ is<br>$\frac{a^2 - a}{1 - 0} = a(1 - a)$ .<br><br>The line crosses the $y$ – axis at $(0, a)$ so the equation of the line is<br>$y = a(a-1)x + a$ | <br><br><br>**M1**<br><br>**A1**<br>**[2]** | <br><br><br>**1.1a**<br><br>**1.1b** | | |
| | (b) | | The two straight lines are $y = b(b-1)x + b$<br>and $y = c(c-1)x + c$.<br>These are parallel if<br>$b(b-1) = c(c-1)$<br>$\Rightarrow 0 = c^2 - b^2 + b - c$<br>$\Rightarrow 0 = (c-b)(c+b) - (c-b)$<br>$\Rightarrow 0 = (c-b)(c+b-1)$<br>$\Rightarrow 0 = c + b - 1$ (since $c \neq b$)<br>$\Rightarrow c + b = 1$ | <br><br><br>**M1**<br><br><br><br>**M1**<br>**A1**<br>**[3]** | <br><br><br>**3.1a**<br><br><br><br>**2.4**<br>**2.1** | Note that equation can be solved using CAS which is an acceptable method. | |

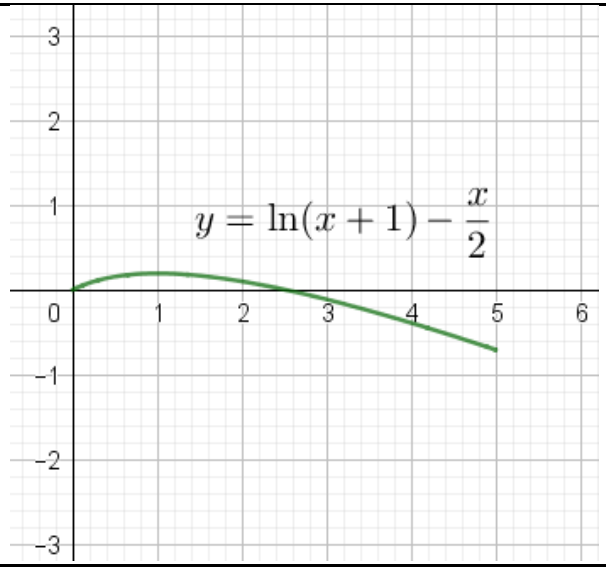
| Question | | | Answer | Marks | AOs | | Guidance |
|---|---|---|---|---|---|---|---|
| | (c) | |  | **B1**<br><br>**B1**<br><br>**[2]** | **1.1b**<br><br>**1.1b** | One mark for each branch of the curve, check for shape, asymptotes, max/min and quadrants and and points on the axes. Note that question requires 'sketch'. | |

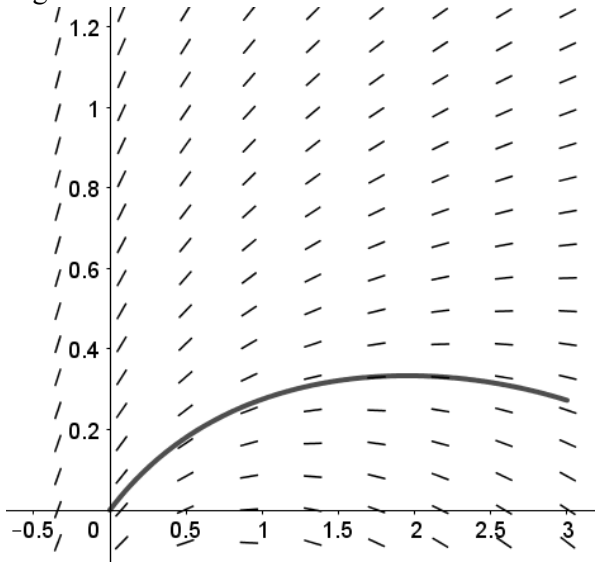
| Question | | | Answer | Marks | AOs | | Guidance |
|---|---|---|---|---|---|---|---|
| | (d) | | Let $f(x,y,a)=a(a-1)x-y+a$ .<br>The envelope is the set of points for which, simultaneously $f(x,y,a)=0$ and<br>$\frac{\partial f}{\partial a}(x,y,a)=0$ , for some value of $a$.<br>$f(x,y,a)=a^2x-ax-y+a$<br>$\Rightarrow \frac{\partial f}{\partial a}=2ax-x+1$ . | **M1**<br><br>**M1** | **3.1a**<br><br>**1.1** | | |
| | | | $\frac{\partial f}{\partial a}(x,y,a)=0$ implies that $a=\frac{x-1}{2x}$ (note x not zero) ,<br>If, in addition, f $(x, y, a)=0$ then<br>$0=\left(\frac{x-1}{2x}\right)\left(\frac{x-1}{2x}-1\right)x-y+\frac{x-1}{2x}$<br>$\Rightarrow y=\frac{x-1}{2x}\left(\frac{-1-x}{2}+1\right)$ oe isw | **M1**<br><br>**M1**<br><br>**A1**<br><br>**[5]** | **1.1**<br><br>**2.2a**<br><br>**2.1** | This is awarded for any correct expression with $y$ in terms of $x$ explicitly.<br><br>Might also see $y=-\frac{(x-1)^2}{4x}$ or $y=\left(\frac{x-1}{2x}\right)\left(\frac{1-x}{2}\right)$ for example. | |

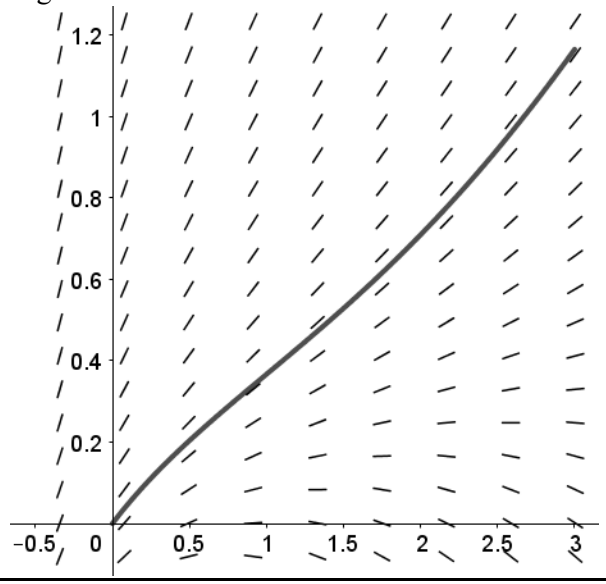
| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 3 | (a) | (i) | | | | Pseudo code accepted, condone lack of syntax, give reasonable BOD on possible transcription errors | |
| | | | Appropriate structure program | **M1** | **3.3** | Example code for Python<br>def hcf(m,n):<br>  k = min(m,n)<br>  hcf=1<br>  for i in range(1,k+1):<br>  if m%i==0 and n%i==0:<br>  hcf=i<br>  return hcf<br>print(hcf(m,n) | |
| | | | Loop with correct range dependent on m, n. Check for common divisors with if statement and tracking greatest one found. | **M1** | **2.1** | | |
| | | | Fully correct program. | **A1**<br>**[3]** | **2.5** | | |
| | | (ii) | (74333, 89817) = 49 | **B1**<br>**[1]** | **1.1** | Set *m* = 74333 and *n* =89817 in the above. | |

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| | (b) | (i) | Appropriate structure program<br><br>Loop with correct range and counts number of values coprime to k.<br><br>Fully correct programme | **M1**<br><br>**M1**<br><br>**A1**<br><br>**[3]** | **3.3**<br><br>**2.1**<br><br>**2.5** | Pseudo code accepted, condone lack of syntax, give reasonable BOD on possible transcription errors<br><br>Example code for Python with hcf function as in 2(i) above.<br>def phi(k):<br>count = 0<br>for i in range(1,k):<br>if hcf(i,k)==1:<br>count = count + 1<br>return count<br>print(phi(k)) | |
| | | (ii) | $\varphi(128) = 64$ and $\varphi(1000) = 400$ | **B1**<br>**B1**<br>**[2]** | **1.1**<br>**1.1** | | |
| | | (iii) | $\varphi(2^n) = 2^{n-1}$. This is because all the odd numbers less than $2^n$ are coprime to $2^n$ and all the even numbers less than $2^n$ are not. The are $2^{n-1}$ such odd numbers. | **M1**<br><br>**A1**<br>**[2]** | **2.1**<br><br>**3.2a** | Spotting odd/even property.<br><br>Correct value in terms of *n*. | |
| | | (iv) | $\varphi(10^n) = 4\times10^{n-1}$. All numbers less than $10^n$ with final digit 1, 3, 7 and 9 are coprime to $10^n$, any other number is not. There are four such numbers in 1, 2,…,10, four in 11, 12,…,20, four in 21,22,…,30, and so on. There are $10^{n-1}$ such groups before reaching $10^n$. So there are $4\times10^{n-1}$ number less than $10^n$ which are coprime to $10^n$. | **M1**<br><br>**M1**<br><br>**A1**<br>**[3]** | **2.1**<br><br>**2.2a**<br><br>**3.2a** | Spotting end digit property.<br><br>Applying it across all numbers less than $10^n$.<br><br>Correct value in terms of *n*. | |

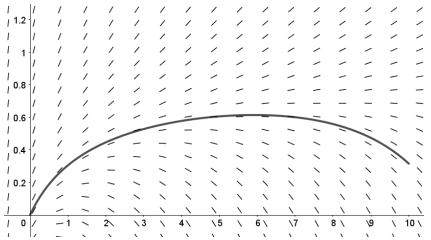
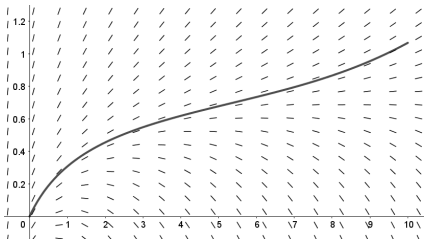
| | | | | | | | |
|---|---|---|---|---|---|---|---|
| | (c) | (i) | F(5) = 9, the corresponding fractions are $\frac{1}{5},\frac{1}{4},\frac{1}{3},\frac{2}{5},\frac{1}{2},\frac{3}{5},\frac{2}{3},\frac{3}{4},\frac{4}{5}$.<br>F(6) = 11, the corresponding fractions are $\frac{1}{6},\frac{1}{5},\frac{1}{4},\frac{1}{3},\frac{2}{5},\frac{1}{2},\frac{3}{5},\frac{2}{3},\frac{3}{4},\frac{4}{5},\frac{5}{6}$ | **B1**<br><br><br><br>**B1**<br>**[2]** | **1.1**<br><br><br><br>**1.1** | | |
| | | (ii) | Adding to the distinct fractions between 0 and 1 with denominator $k$, the only 'new' fractions with denominator $k+1$ have numerators which are coprime to $k+1$. Therefore there are $\varphi(k+1)$ of these. | **B1**<br><br><br>**B1**<br>**[2]** | **3.1a**<br><br><br>**2.4** | | |
| | | (iii) | By (c)(ii) required value is $\sum_{k=1}^{100}\varphi(k)$ . By adapting previous program this is 3043. | **M1**<br><br><br>**A1**<br>**[2]** | **3.1a**<br><br><br>**1.1** | By adding code such as<br>def fracs(k):<br>count = 0<br>for i in range(1,k+1):<br>count = count + phi(i)<br>return count<br>print(fracs(100)) | |

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 4 | (a) | (i) | Solution is $y = \ln(x+1) - \frac{x}{2}$ | B1<br>[1] | 1.1a | | |
| | | (ii) |  | B1<br>[1] | 1.1 | | |
| | | (iii) | With $y = \ln(x+1) - \frac{x}{2}$, $\frac{dy}{dx} = \frac{1}{x+1} - \frac{1}{2}$.<br>Solving $\frac{dy}{dx} = 0$ gives $x = 1$. When $x = 1$,<br>$y = \ln(2) - \frac{1}{2}$ | M1<br>A1<br>[2] | 1.1a<br>1.1 | | |
| | (b) | (i) | Fig 4.1, $a$ is around 0.5 | B1<br>[1] | 1.1 | | |

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| | | (ii) | Fig 4.2. $a$ is around 1 | **B1 [1]** | **1.1** | | |
| | | (iii) | Fig 4.1   | **B1** | **1.1** | Sufficient to see slight downturn as $x$ increases and only stationary pt is the maximum | |

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| | | (iv) | Fig 3.2<br><br> | **B1**<br><br><br><br><br><br>**[2]** | **1.1** | Sufficient to be increasing. | |
| | | (v) | One is increasing for the values of $x$ shown. The other has a stationary point (local maximum). | **B1**<br>**[1]** | **1.2** | Either comment will do. Allow 'one intersects the x-axis (eventually), the other doesn't.' | |

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| | (c) | (i) | A1 contains 0<br>B1 contains 0<br>I1 contains 0.1 (the value of $h$)<br>K1 contains a (the value of $a$)<br>C1 =$I$1*((1-A1)/(2*(A1+1))+$K$1*ATAN(B1))<br>D1 =$I$1*((1-(A1+$I$1))/(2*(A1+$I$1+1))+$K$1*ATAN(B1+C1))<br>A2 =A1+$I$1<br>B2 = =B1+0.5*(C1+D1)<br>copy down | **B1**<br>**B1**<br>**B1**<br>**B1**<br>**[4]** | **3.1a**<br>**3.1a**<br>**3.1a**<br>**2.5** | Give reasonable BOD on possible transcription errors and consider correct answers to 4(c)(ii), 4(c)(iii), 4(c)(iv) as evidence of correct formulae in the spreadsheet.<br>Allows for $a$ and $h$ to be varied.<br>Cols for $x$ and $y$<br>Cols for $k_1$ and $k_2$<br>Formulae for $x_{n+1}$ and $y_{n+1}$ | |
| | | (ii) | Approximation to $y$ when $x = 5.0$ with $a = 0.5$, using $h = 0.1$ is -0.249889 (to 6 d.p.) | **B1**<br>**[1]** | **1.1** | Correct answer to at least 3 s.f. Must for correct for the number of significant figures given. | |
| | | (iii) | Approximation to $y$ when $x = 5.0$ with $a = 1$, using $h = 0.1$ is 3.160809 (to 6 d.p.) | **B1**<br>**[1]** | **1.1** | Correct answer to at least 3 s.f. Must for correct for the number of significant figures given. | |

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| | | (iv) | Using $a = 0.645$ (with $h = 0.1$) produces an approximate solution which increases initially but then peaks and descreases.<br>Using $a = 0.655$ (with $h = 0.1$) produces an approximate solution which increases indefinitely.<br><br>This is evidence that $a^*$ is between 0.645 and 0.655.<br><br>This suggests that $a^* = 0.65$ to 2 d.p. | **M1**<br><br>**M1**<br><br>**M1**<br><br>**A1**<br>**[4]** | **3.1a**<br><br>**3.1a**<br><br>**2.2a**<br><br>**2.2b** | Further investigation of other values of $h$ and graphing software might be used to confirm this. This should be awarded credit appropriately.<br><br>E.g here is solution with $a = 0.645$<br><br>And here is solution with $a = 0.655$<br> | |

| Question | A01 | A02 | A03 | E | C | A | |
|---|---|---|---|---|---|---|---|
| 1(i)(A) | 1 | | | 1 | | | C1, C4 |
| 1(i)(B) | 1 | 1 | | 2 | | | C4 |
| 1(i)(C) | 1 | 1 | | 1 | 1 | | C4 |
| 1(i)(D) | 2 | | | 2 | | | C9 |
| 1(i)(E) | 2 | | | 2 | | | C4, C9 |
| 1(ii)(A) | 2 | | | 2 | | | C4 |
| 1(ii)(B) | | 2 | 1 | 1 | 2 | | C4 |
| 1(ii)(C) | 2 | | | 1 | 1 | | C9 |
| 1(ii)(D) | 2 | 2 | 1 | | | 5 | C9 |
| 2(i)(A) | | 2 | 1 | 2 | 1 | | T1, T5 |
| 2(i)(B) | 1 | | | 1 | | | T5 |
| 2(ii)(A) | | 2 | 1 | 2 | 1 | | T6 |
| 2(ii)(B) | 2 | | | 1 | 1 | | T5, T6 |
| 2(ii)(C) | | 1 | 1 | 1 | 1 | | T5, T6 |
| 2(ii)(D) | | 2 | 1 | 1 | 1 | 1 | T5, T6 |
| 2(iii)(A) | 2 | | | 2 | | | T5 |
| 2(iii)(B) | | 1 | 1 | | | 2 | T5, T6 |
| 2(iii)(C) | 1 | | 1 | | | 2 | T5, T6 |
| 3(i)(A) | 1 | | | 1 | | | C1 |
| 3(i)(B) | 1 | | | 1 | | | C1 |
| 3(i)(C) | 2 | | | 2 | | | C5 |
| 3(ii)(A) | 2 | | | 2 | | | C2, C6 |
| 3(ii)(B) | 2 | | | 2 | | | C6 |
| 3(ii)(C) | 1 | | | | 1 | | C6 |
| 3(iii)(A) | | 1 | 3 | | 3 | 1 | C7 |
| 3(iii)(B) | 1 | | | | 1 | | C7 |
| 3(iii)(C) | 1 | | | | 1 | | C7 |
| 3(iii)(D) | | 2 | 2 | | | 4 | C6, C7, C8 |
| Total | 30 | 17 | 13 | 30 | 15 | 15 | 0.00 |

S&C marks: 1(ii)D 5 marks

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