OCR
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Thursday 21 October 2021 – Afternoon

A Level Further Mathematics B (MEI)

Y435/01 Extra Pure

Time allowed: 1 hour 15 minutes

9051866266

You must have:
- the Printed Answer Booklet
- the Formulae Booklet for Further Mathematics B (MEI)
- a scientific or graphical calculator

INSTRUCTIONS
- Use black ink. You can use an HB pencil, but only for graphs and diagrams.
- Write your answer to each question in the space provided in the **Printed Answer Booklet**. If you need extra space use the lined pages at the end of the Printed Answer Booklet. The question numbers must be clearly shown.
- Fill in the boxes on the front of the Printed Answer Booklet.
- Answer **all** the questions.
- Where appropriate, your answer should be supported with working. Marks might be given for using a correct method, even if your answer is wrong.
- Give your final answers to a degree of accuracy that is appropriate to the context.
- Do **not** send this Question Paper for marking. Keep it in the centre or recycle it.

INFORMATION
- The total mark for this paper is **60**.
- The marks for each question are shown in brackets **[ ]**.
- This document has **4** pages.

ADVICE
- Read each question carefully before you start your answer.

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Answer **all** the questions.

1 **In this question you must show detailed reasoning.**

A surface S is defined by $z = \mathrm{f}(x, y)$ where $\mathrm{f}(x, y) = x^3 + x^2y - 2y^2$.

(a) On the coordinate axes in the Printed Answer Booklet, sketch the section $z = \mathrm{f}(2, y)$ giving the coordinates of any turning points and any points of intersection with the axes. **[4]**

(b) Find the stationary points on S. **[7]**

2 G is a group of order 8.

(a) Explain why there is no subgroup of G of order 6. **[1]**

You are now given that G is a cyclic group with the following features:

- e is the identity element of G,
- g is a generator of G,
- H is the subgroup of G of order 4.

(b) Write down the possible generators of H. **[2]**

M is the group $(\{0, 1, 2, 3, 4, 5, 6, 7\}, +_8)$ where $+_8$ denotes the binary operation of addition modulo 8. You are given that M is isomorphic to G.

(c) Specify all possible isomorphisms between M and G. **[4]**

3 The matrix $\mathbf{A}$ is given by $\mathbf{A} = \begin{pmatrix} 3 & 3 & 0 \\ 0 & 2 & 2 \\ 1 & 3 & 4 \end{pmatrix}$.

(a) Determine the characteristic equation of $\mathbf{A}$. **[3]**

(b) Hence verify that the eigenvalues of $\mathbf{A}$ are 1, 2 and 6. **[1]**

(c) For each eigenvalue of $\mathbf{A}$ determine an associated eigenvector. **[4]**

(d) Use the results of parts **(b)** and **(c)** to find $\mathbf{A}^n$ as a single matrix, where n is a positive integer. **[6]**

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4 The sequence $u_0, u_1, u_2, \ldots$ satisfies the recurrence relation $u_{n+2} - 3u_{n+1} - 10u_n = 24n - 10$.

(a) Determine the general solution of the recurrence relation. **[6]**

(b) Hence determine the particular solution of the recurrence relation for which $u_0 = 6$ and $u_1 = 10$. **[3]**

(c) Show, by direct calculation, that your solution in part **(b)** gives the correct value for u_2. **[1]**

The sequence $v_0, v_1, v_2, \ldots$ is defined by $v_n = \dfrac{u_n}{p^n}$ for some constant p, where u_n denotes the particular solution found in part **(b)**.

You are given that v_n converges to a finite non-zero limit, q, as $n \to \infty$.

(d) Determine p and q. **[4]**

5 A surface S is defined for $z \geqslant 0$ by $x^2 + y^2 + 2z^2 = 126$. C is the set of points on S for which the tangent plane to S at that point intersects the x-y plane at an angle of $\frac{1}{3}\pi$ radians.

Show that C lies in a plane, Π, whose equation should be determined. **[6]**

6 You are given that $q \in \mathbb{Z}$ with $q \geqslant 1$ and that

$$S = \frac{1}{(q+1)} + \frac{1}{(q+1)(q+2)} + \frac{1}{(q+1)(q+2)(q+3)} + \ldots.$$

(a) By considering a suitable geometric series show that $S < \dfrac{1}{q}$. **[3]**

(b) Deduce that $S \notin \mathbb{Z}$. **[2]**

You are also given that $\mathrm{e} = \displaystyle\sum_{r=0}^{\infty} \frac{1}{r!}$.

(c) Assume that $\mathrm{e} = \dfrac{p}{q}$, where p and q are positive integers. By writing the infinite series for e in a form using q and S and using the result from part **(b)**, prove by contradiction that e is irrational. **[3]**

END OF QUESTION PAPER

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Surname	StemUp Model Solution
Answer booklet	Write the question number in the left-hand margin at the start of each answer.

Question number

1 (a)

$z = f(2, y) = 2^3 + 2^2y - 2y^2 = 8 + 4y - 2y^2$

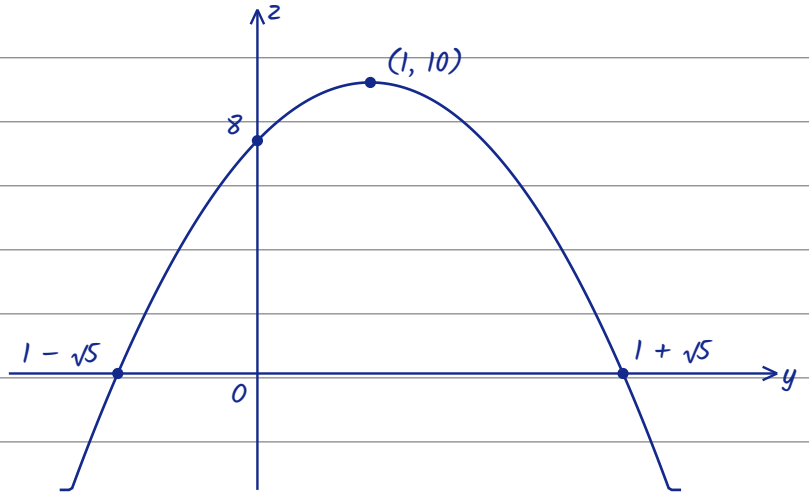
$= -2(y^2 - 2y) + 8 = -2(y - 1)^2 + 2 + 8 = 10 - 2(y - 1)^2$

so the turning point is a maximum at $y = 1$, $z = 10$, ie $(1, 10)$

$y = 0 \Rightarrow z = 8$, so it crosses the z-axis at $(0, 8)$

$z = 0 \Rightarrow (y - 1)^2 = 5 \Rightarrow y = 1 \pm \sqrt{5}$

so it crosses the y-axis at $(1 - \sqrt{5}, 0)$ and $(1 + \sqrt{5}, 0)$



1 (b)

$\frac{\partial z}{\partial x} = 3x^2 + 2xy, \quad \frac{\partial z}{\partial y} = x^2 - 4y$

Stationary points where both partial derivatives are 0:

$\frac{\partial z}{\partial y} = 0 \Rightarrow y = \frac{x^2}{4}$

$\frac{\partial z}{\partial x} = 0 \Rightarrow x(3x + 2y) = 0 \Rightarrow x = 0$ or $y = -\frac{3x}{2}$

$x = 0 \Rightarrow y = 0 \Rightarrow z = 0$, giving $(0, 0, 0)$

$y = -\frac{3x}{2}$: $\frac{x^2}{4} = -\frac{3x}{2} \Rightarrow x^2 + 6x = 0 \Rightarrow x = -6$ ($x \neq 0$ here)

Question number

$y = \frac{(-6)^2}{4} = 9$

$z = (-6)^3 + (-6)^2(9) - 2(9)^2 = -216 + 324 - 162 = -54$

Stationary points are $(0, 0, 0)$ and $(-6, 9, -54)$

2 (a) By Lagrange's theorem the order of any subgroup of G must be a factor of $|G| = 8$. 6 is not a factor of 8, so G has no subgroup of order 6.

2 (b) $H = \{e, g^2, g^4, g^6\}$

Generators of H are g^2 and g^6

2 (c) An isomorphism must send the identity to the identity: $e \leftrightarrow 0$,

and the generator g to a generator of M, ie 1, 3, 5 or 7.

Then $g^k \leftrightarrow (k \times \text{image of } g) \bmod 8$. The four isomorphisms are:

$e\leftrightarrow0, g\leftrightarrow1, g^2\leftrightarrow2, g^3\leftrightarrow3, g^4\leftrightarrow4, g^5\leftrightarrow5, g^6\leftrightarrow6, g^7\leftrightarrow7$

$e\leftrightarrow0, g\leftrightarrow3, g^2\leftrightarrow6, g^3\leftrightarrow1, g^4\leftrightarrow4, g^5\leftrightarrow7, g^6\leftrightarrow2, g^7\leftrightarrow5$

$e\leftrightarrow0, g\leftrightarrow5, g^2\leftrightarrow2, g^3\leftrightarrow7, g^4\leftrightarrow4, g^5\leftrightarrow1, g^6\leftrightarrow6, g^7\leftrightarrow3$

$e\leftrightarrow0, g\leftrightarrow7, g^2\leftrightarrow6, g^3\leftrightarrow5, g^4\leftrightarrow4, g^5\leftrightarrow3, g^6\leftrightarrow2, g^7\leftrightarrow1$

3 (a) $\det(A - \lambda I) = \det\begin{pmatrix} 3-\lambda & 3 & 0 \\ 0 & 2-\lambda & 2 \\ 1 & 3 & 4-\lambda \end{pmatrix}$

$= (3 - \lambda)[(2 - \lambda)(4 - \lambda) - 2\times3] - 3[0\times(4 - \lambda) - 2\times1] + 0$

$= (3 - \lambda)(\lambda^2 - 6\lambda + 2) + 6$

$= 3\lambda^2 - 18\lambda + 6 - \lambda^3 + 6\lambda^2 - 2\lambda + 6$

$= -\lambda^3 + 9\lambda^2 - 20\lambda + 12$

Characteristic equation: $\lambda^3 - 9\lambda^2 + 20\lambda - 12 = 0$

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3 (b)

$\lambda = 1$: $1 - 9 + 20 - 12 = 0$

$\lambda = 2$: $8 - 36 + 40 - 12 = 0$

$\lambda = 6$: $216 - 324 + 120 - 12 = 0$

All satisfy the equation, so the eigenvalues are 1, 2 and 6

3 (c)

Let $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$ be an eigenvector: $3a + 3b = \lambda a$, $2b + 2c = \lambda b$, $a + 3b + 4c = \lambda c$

$\lambda = 1$: $2a + 3b = 0$, $b + 2c = 0$. Let $c = 1 \Rightarrow b = -2$, $a = 3$

(check: $3 + 3(-2) + 4(1) = 1 = c$) eigenvector $\begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix}$

$\lambda = 2$: $a + 3b = 0$, $2c = 0$. Let $b = -1 \Rightarrow a = 3$, $c = 0$

(check: $3 - 3 + 0 = 0 = 2c$) eigenvector $\begin{pmatrix} 3 \\ -1 \\ 0 \end{pmatrix}$

$\lambda = 6$: $-3a + 3b = 0$, $-4b + 2c = 0$. Let $a = 1 \Rightarrow b = 1$, $c = 2$

(check: $1 + 3 + 8 = 12 = 6c$) eigenvector $\begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$

3 (d)

$A = EDE^{-1}$ with $E = \begin{pmatrix} 3 & 3 & 1 \\ -2 & -1 & 1 \\ 1 & 0 & 2 \end{pmatrix}$, $D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 6 \end{pmatrix}$

$\det E = 3(-2 - 0) - 3(-4 - 1) + 1(0 + 1) = 10$, $E^{-1} = \frac{1}{10}\begin{pmatrix} -2 & -6 & 4 \\ 5 & 5 & -5 \\ 1 & 3 & 3 \end{pmatrix}$

Question number

$$A^n = ED^nE^{-1}, \quad D^n = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2^n & 0 \\ 0 & 0 & 6^n \end{pmatrix}$$

$$ED^n = \begin{pmatrix} 3 & 3\times2^n & 6^n \\ -2 & -2^n & 6^n \\ 1 & 0 & 2\times6^n \end{pmatrix}$$

$$A^n = \frac{1}{10}\begin{pmatrix} 3 & 3\times2^n & 6^n \\ -2 & -2^n & 6^n \\ 1 & 0 & 2\times6^n \end{pmatrix}\begin{pmatrix} -2 & -6 & 4 \\ 5 & 5 & -5 \\ 1 & 3 & 3 \end{pmatrix}$$

$$A^n = \frac{1}{10}\begin{pmatrix} -6 + 15\times2^n + 6^n & -18 + 15\times2^n + 3\times6^n & 12 - 15\times2^n + 3\times6^n \\ 4 - 5\times2^n + 6^n & 12 - 5\times2^n + 3\times6^n & -8 + 5\times2^n + 3\times6^n \\ -2 + 2\times6^n & -6 + 6\times6^n & 4 + 6\times6^n \end{pmatrix}$$

4 (a)

Complementary function: $u_{n+2} - 3u_{n+1} - 10u_n = 0$

Auxiliary equation: $r^2 - 3r - 10 = 0 \Rightarrow (r - 5)(r + 2) = 0 \Rightarrow r = 5$ or $r = -2$

CF: $u_n = \alpha 5^n + \beta(-2)^n$

Particular solution: try $u_n = an + b$

$a(n + 2) + b - 3[a(n + 1) + b] - 10(an + b) = 24n - 10$

n terms: $a - 3a - 10a = 24 \Rightarrow -12a = 24 \Rightarrow a = -2$

constants: $2a + b - 3a - 3b - 10b = -10 \Rightarrow -a - 12b = -10$

$\Rightarrow 2 - 12b = -10 \Rightarrow b = 1$

General solution: $u_n = 1 - 2n + \alpha 5^n + \beta(-2)^n$

4 (b)

$n = 0$: $1 + \alpha + \beta = 6 \Rightarrow \alpha + \beta = 5$

$n = 1$: $1 - 2 + 5\alpha - 2\beta = 10 \Rightarrow 5\alpha - 2\beta = 11$

$2\alpha + 2\beta = 10$, adding: $7\alpha = 21 \Rightarrow \alpha = 3$, $\beta = 2$

$u_n = 1 - 2n + 3\times5^n + 2\times(-2)^n$

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4 (c)

From the recurrence relation ($n = 0$): $u_2 = 3u_1 + 10u_0 + 24\times0 - 10$

$= 3\times10 + 10\times6 - 10 = 80$

From the solution: $u_2 = 1 - 2\times2 + 3\times5^2 + 2\times(-2)^2 = 1 - 4 + 75 + 8 = 80$

Both give $u_2 = 80$, so the solution gives the correct value

4 (d)

$v_n = \frac{u_n}{p^n} = \frac{1 - 2n}{p^n} + 3(\frac{5}{p})^n + 2(\frac{-2}{p})^n$

If $|p| < 5$ then $|5/p| > 1$, so $3(5/p)^n$ dominates and v_n does not converge.

If $|p| > 5$ then every term $\to 0$, so $v_n \to 0$, but the limit is non-zero.

So $p = 5$: $v_n = \frac{1 - 2n}{5^n} + 3 + 2(-\frac{2}{5})^n \to 0 + 3 + 0 = 3$

$p = 5$, $q = 3$

5

Let $g(x, y, z) = x^2 + y^2 + 2z^2$, so S is $g = 126$

$\frac{\partial g}{\partial x} = 2x$, $\frac{\partial g}{\partial y} = 2y$, $\frac{\partial g}{\partial z} = 4z$

Normal to the tangent plane: $\nabla g = \begin{pmatrix} 2x \\ 2y \\ 4z \end{pmatrix}$. Normal to the x-y plane: $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

Angle between the planes = angle between the normals, so

$\begin{pmatrix} 2x \\ 2y \\ 4z \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \sqrt{4x^2 + 4y^2 + 16z^2} \times 1 \times \cos\frac{\pi}{3}$

$4z = \frac{1}{2}\sqrt{4x^2 + 4y^2 + 16z^2}$ $(z \geq 0)$

$64z^2 = 4x^2 + 4y^2 + 16z^2 = 4(x^2 + y^2 + 2z^2) + 8z^2$

$64z^2 = 4\times126 + 8z^2$ (since $x^2 + y^2 + 2z^2 = 126$ on S)

$56z^2 = 504 \Rightarrow z^2 = 9 \Rightarrow z = \pm3$, and $z \geq 0$ so $z = 3$

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So every point of C lies in the plane $\Pi: z = 3$

6 (a) For $k \geqslant 2$, $(q+1)(q+2)\ldots(q+k) > (q+1)^k$, so

$$S < \frac{1}{q+1} + \frac{1}{(q+1)^2} + \frac{1}{(q+1)^3} + \ldots$$

This is a geometric series with $a = \frac{1}{q+1}$, $r = \frac{1}{q+1}$, $|r| < 1$

$$\text{Sum to infinity} = \frac{\frac{1}{q+1}}{1 - \frac{1}{q+1}} = \frac{1}{(q+1) - 1} = \frac{1}{q}$$

$\therefore S < \frac{1}{q}$ as required

6 (b) $q \geqslant 1 \Rightarrow \frac{1}{q} \leqslant 1$, so $S < \frac{1}{q} \leqslant 1$

Every term of S is positive, so $S > 0$

So $0 < S < 1$ and there is no integer strictly between 0 and 1, $\therefore S \notin \mathbb{Z}$

6 (c) $e = \frac{p}{q} \Rightarrow q!\, e = q! \times \frac{p}{q} = p(q-1)!$

$$q!\, e = \sum_{r=0}^{\infty} \frac{q!}{r!} = \sum_{r=0}^{q} \frac{q!}{r!} + \sum_{r=q+1}^{\infty} \frac{q!}{r!}$$

$$\sum_{r=q+1}^{\infty} \frac{q!}{r!} = \frac{1}{q+1} + \frac{1}{(q+1)(q+2)} + \ldots = S$$

So $S = p(q-1)! - \sum_{r=0}^{q} \frac{q!}{r!}$

$p(q-1)!$ is an integer, and for $r \leqslant q$, $\frac{q!}{r!} = q(q-1)\ldots(r+1)$ is an integer,

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so the right-hand side is an integer, ie $S \in \mathbb{Z}$.

This contradicts part (b), so the assumption is false: e is irrational.