OCR
Oxford Cambridge and RSA

Monday 4 October 2021 – Afternoon

A Level Further Mathematics B (MEI)

Y420/01 Core Pure

Time allowed: 2 hours 40 minutes

9050479766

You must have:
- the Printed Answer Booklet
- the Formulae Booklet for Further Mathematics B (MEI)
- a scientific or graphical calculator

INSTRUCTIONS
- Use black ink. You can use an HB pencil, but only for graphs and diagrams.
- Write your answer to each question in the space provided in the **Printed Answer Booklet**. If you need extra space use the lined pages at the end of the Printed Answer Booklet. The question numbers must be clearly shown.
- Fill in the boxes on the front of the Printed Answer Booklet.
- Answer **all** the questions.
- Where appropriate, your answer should be supported with working. Marks might be given for using a correct method, even if your answer is wrong.
- Give your final answers to a degree of accuracy that is appropriate to the context.
- Do **not** send this Question Paper for marking. Keep it in the centre or recycle it.

INFORMATION
- The total mark for this paper is **144**.
- The marks for each question are shown in brackets **[ ]**.
- This document has **8** pages.

ADVICE
- Read each question carefully before you start your answer.

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Section A (31 marks)

Answer **all** the questions.

1 **(a)** Express $\frac{1}{(2r-1)(2r+1)}$ in partial fractions. **[3]**

(b) Hence find $\sum_{r=1}^{n}\frac{1}{(2r-1)(2r+1)}$, expressing the result as a single fraction. **[4]**

2 **In this question you must show detailed reasoning.**

Find the gradient of the curve $y = 6\arcsin(2x)$ at the point with x-coordinate $\frac{1}{4}$. Express the result in the form $m\sqrt{n}$, where m and n are integers. **[4]**

3 **In this question you must show detailed reasoning.**

The complex numbers z_1 and z_2 are given by $z_1 = -2+2\mathrm{i}$ and $z_2 = 2\left(\cos\frac{1}{6}\pi + \mathrm{i}\sin\frac{1}{6}\pi\right)$.

(a) Find the modulus and argument of z_1. **[2]**

(b) Hence express $\frac{z_1}{z_2}$ in exact modulus-argument form. **[4]**

4 **In this question you must show detailed reasoning.**

Determine the mean value of $\frac{1}{1+4x^2}$ between $x = -1$ and $x = 1$. Give your answer to **3** significant figures. **[4]**

5 **(a)** Use a Maclaurin series to find a quadratic approximation for $\ln(1+2x)$. **[1]**

(b) Find the percentage error in using the approximation in part **(a)** to calculate $\ln(1.2)$. **[3]**

(c) Jane uses the Maclaurin series in part **(a)** to try to calculate an approximation for $\ln 3$.

Explain whether her method is valid. **[2]**

6 Given that $y = mx$ is an invariant line of the transformation with matrix $\begin{pmatrix}1 & 2\\ 2 & -2\end{pmatrix}$, determine the possible values of m. **[4]**

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Section B (113 marks)

Answer **all** the questions.

7 Prove that $\sum_{r=1}^{n} \frac{r}{2^{r-1}} = 4 - \frac{n+2}{2^{n-1}}$ for all $n \geqslant 1$. **[6]**

8 The equation $4x^4 - 4x^3 + px^2 + qx - 9 = 0$, where p and q are constants, has roots α, $-\alpha$, β and $\frac{1}{\beta}$.

(a) Determine the exact roots of the equation. **[5]**

(b) Determine the values of p and q. **[4]**

9 The transformation T of the plane has associated matrix $\mathbf{M}$, where $\mathbf{M} = \begin{pmatrix} -1 & 0 \\ -2 & 1 \end{pmatrix}$.

(a) On the grid in the Printed Answer Booklet, plot the image OA′B′C′ of the unit square OABC under the transformation T. **[2]**

(b) (i) Calculate the value of $\det \mathbf{M}$. **[1]**

(ii) Explain the significance of the value of $\det \mathbf{M}$ in relation to the image OA′B′C′. **[2]**

(c) T is equivalent to a sequence of two transformations of the plane.

(i) Specify fully **two** transformations equivalent to T. **[3]**

(ii) Use matrices to verify your answer. **[3]**

10 (a) Show on an Argand diagram the points representing the three cube roots of unity. **[2]**

(b) (i) Find the exact roots of the equation $z^3 - 1 = \sqrt{3}\,\mathrm{i}$, expressing them in the form $re^{i\theta}$, where $r > 0$ and $-\pi < \theta < \pi$. **[5]**

(ii) The points representing the cube roots of unity form a triangle Δ_1. The points representing the roots of the equation $z^3 - 1 = \sqrt{3}\,\mathrm{i}$ form a triangle Δ_2.

State a sequence of two transformations that maps Δ_1 onto Δ_2. **[2]**

(iii) The three roots in part **(b)(i)** are z_1, z_2 and z_3.

By simplifying $z_1 + z_2 + z_3$, verify that the sum of these roots is zero. **[2]**

(iv) Hence show that $\sin 20° + \sin 140° = \sin 100°$. **[2]**

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11 **(a)** Given that $\mathbf{u} = \lambda\mathbf{i} + \mathbf{j} - 3\mathbf{k}$ and $\mathbf{v} = \mathbf{i} + 2\mathbf{j} - 2\mathbf{k}$, find the following, giving your answers in terms of λ.

(i) $\mathbf{u}.\mathbf{v}$ **[1]**

(ii) $\mathbf{u} \times \mathbf{v}$ **[2]**

(b) Hence determine

(i) the acute angle between the planes $2x+y-3z=10$ and $x+2y-2z=10$, **[3]**

(ii) the shortest distance between the lines $\frac{x-3}{3}=\frac{y}{1}=\frac{z-2}{-3}$ and $\frac{x}{1}=\frac{y-4}{2}=\frac{z+2}{-2}$, giving your answer as a multiple of $\sqrt{2}$. **[3]**

12 **Fig. 12** shows a rhombus OACB in an Argand diagram. The points A and B represent the complex numbers z and w respectively.

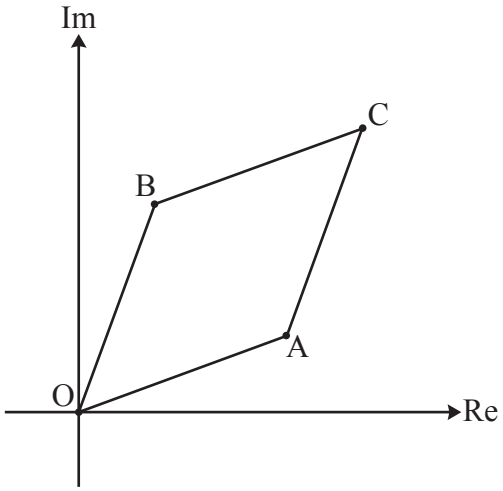


Fig. 12

Prove that $\arg(z+w) = \frac{1}{2}(\arg z + \arg w)$.

[A copy of **Fig. 12** is provided in the Printed Answer Booklet.] **[4]**

13 Find the general solution of the differential equation $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+2\frac{\mathrm{d}y}{\mathrm{d}x}-3y=2\mathrm{e}^x$. **[7]**

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14 A curve has polar equation $r = a(\cos\theta + 2\sin\theta)$, where a is a positive constant and $0 \leqslant \theta \leqslant \pi$.

(a) Determine the polar coordinates of the point on the curve which is furthest from the pole. **[7]**

(b) (i) Show that the curve is a circle whose radius should be specified. **[6]**

(ii) Write down the polar coordinates of the centre of the circle. **[1]**

15 The equations of three planes are

$$\begin{aligned} -4x + ky + 7z &= 4, \\ x - 2y + 5z &= l, \\ 2x + 3y + z &= 2. \end{aligned}$$

Given that the planes form a sheaf, determine the values of k and l. **[6]**

16 (a) Show using exponentials that $\cosh 2u = 1 + 2\sinh^2 u$. **[4]**

(b) Show that $\displaystyle\int_0^2 \frac{x^2}{\sqrt{4+x^2}}\,\mathrm{d}x = 2\sqrt{2} - 2\ln(1+\sqrt{2})$. **[10]**

17 In a chemical process, a vessel contains 1 litre of pure water. A liquid chemical is then passed into the top of the vessel at a constant rate of a litres per minute and thoroughly mixed with the water. At the same time, the resulting mixture is drawn from the bottom of the vessel at a constant rate of b litres per minute. You may assume that the chemical mixes instantly and uniformly with the water. After t minutes, the mixture in the vessel contains x litres of the chemical.

(a) (i) Show that the proportion of chemical present in the vessel after t minutes is $\dfrac{x}{1+(a-b)t}$. **[2]**

(ii) Hence show that $\dfrac{\mathrm{d}x}{\mathrm{d}t} + \dfrac{bx}{1+(a-b)t} = a$. **[2]**

(b) First, consider the case where $b = a$.

(i) Solve the differential equation to find x in terms of a and t. **[4]**

(ii) Given that after 1 minute the vessel contains equal amounts of water and chemical, find the rate of inflow of chemical. **[2]**

(c) Now consider the case where $b = 2a$.

(i) Explain why the differential equation in part **(a)(ii)** is now invalid for $t \geqslant \dfrac{1}{a}$. **[1]**

(ii) Find the maximum amount of chemical in the vessel. **[9]**

END OF QUESTION PAPER

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Surname	StemUp Model Solution
Answer booklet	Write the question number in the left-hand margin at the start of each answer.

Question number

1 (a)

$$\frac{1}{(2r-1)(2r+1)} = \frac{A}{2r-1} + \frac{B}{2r+1}$$

$$1 = A(2r+1) + B(2r-1)$$

$r = \frac{1}{2}$: $1 = 2A \Rightarrow A = \frac{1}{2}$

$r = -\frac{1}{2}$: $1 = -2B \Rightarrow B = -\frac{1}{2}$

$$\frac{1}{(2r-1)(2r+1)} = \frac{1}{2(2r-1)} - \frac{1}{2(2r+1)}$$

1 (b)

$$\sum_{r=1}^{n} \frac{1}{(2r-1)(2r+1)} = \frac{1}{2}\sum_{r=1}^{n}\left(\frac{1}{2r-1} - \frac{1}{2r+1}\right)$$

$$= \frac{1}{2}\left[\left(1 - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{5}\right) + \left(\frac{1}{5} - \frac{1}{7}\right) + \ldots\right.$$

$$\left. + \left(\frac{1}{2n-3} - \frac{1}{2n-1}\right) + \left(\frac{1}{2n-1} - \frac{1}{2n+1}\right)\right]$$

All the middle terms cancel:

$$= \frac{1}{2}\left(1 - \frac{1}{2n+1}\right) = \frac{1}{2} \times \frac{2n}{2n+1}$$

$$= \frac{n}{2n+1}$$

2

$$y = 6\arcsin(2x)$$

$$\frac{dy}{dx} = 6 \times \frac{1}{\sqrt{1-(2x)^2}} \times 2 = \frac{12}{\sqrt{1-4x^2}}$$

Question number

When $x = \frac{1}{4}$: $1 - 4(\frac{1}{4})^2 = 1 - \frac{1}{4} = \frac{3}{4}$

$\frac{dy}{dx} = \frac{12}{\sqrt{\frac{3}{4}}} = \frac{12}{\frac{\sqrt{3}}{2}} = \frac{24}{\sqrt{3}} = \frac{24\sqrt{3}}{3}$

Gradient $= 8\sqrt{3}$

3 (a) $z_1 = -2 + 2i$

$|z_1| = \sqrt{((-2)^2 + 2^2)} = \sqrt{8} = 2\sqrt{2}$

z_1 is in the second quadrant, so

$\arg z_1 = \pi - \arctan(\frac{2}{2}) = \pi - \frac{\pi}{4} = \frac{3\pi}{4}$

3 (b) $|\frac{z_1}{z_2}| = \frac{|z_1|}{|z_2|} = \frac{2\sqrt{2}}{2} = \sqrt{2}$

$\arg(\frac{z_1}{z_2}) = \arg z_1 - \arg z_2 = \frac{3\pi}{4} - \frac{\pi}{6} = \frac{9\pi}{12} - \frac{2\pi}{12} = \frac{7\pi}{12}$

$\frac{z_1}{z_2} = \sqrt{2}\,(\cos\frac{7\pi}{12} + i\sin\frac{7\pi}{12})$

4 Mean value $= \frac{1}{1-(-1)} \int_{-1}^{1} \frac{1}{1+4x^2}\,dx$

$= \frac{1}{2} \times \frac{1}{4} \int_{-1}^{1} \frac{1}{\frac{1}{4} + x^2}\,dx$

$= \frac{1}{8}\left[\frac{1}{\frac{1}{2}} \arctan(\frac{x}{\frac{1}{2}})\right]_{-1}^{1} = \frac{1}{8}\left[2\arctan(2x)\right]_{-1}^{1}$

Question number

$= \frac{1}{4}\left(\arctan 2 - \arctan(-2)\right) = \frac{1}{4} \times 2 \arctan 2$

$= \frac{1}{2} \arctan 2 = 0.554 \quad (3 \text{ s.f.})$

5 (a) $\ln(1 + x) = x - \frac{x^2}{2} + \ldots$ so, replacing x with 2x,

$\ln(1 + 2x) \approx 2x - \frac{(2x)^2}{2} = 2x - 2x^2$

5 (b) $1 + 2x = 1.2 \Rightarrow x = 0.1$

$\ln(1.2) \approx 2(0.1) - 2(0.1)^2 = 0.2 - 0.02 = 0.18$

True value: $\ln(1.2) = 0.182321\ldots$

$\%\text{ error} = \frac{0.18 - 0.182321\ldots}{0.182321\ldots} \times 100 = -1.27\%$

(the approximation is 1.27% too small)

5 (c) The series for $\ln(1 + 2x)$ is only valid for $-1 < 2x \leq 1$,

i.e. $-\frac{1}{2} < x \leq \frac{1}{2}$

For ln 3 she needs $1 + 2x = 3$, so $x = 1$, which is outside this range.

The series does not converge there, so her method is not valid.

6 $\begin{pmatrix} 1 & 2 \\ 2 & -2 \end{pmatrix} \begin{pmatrix} x \\ mx \end{pmatrix} = \begin{pmatrix} x + 2mx \\ 2x - 2mx \end{pmatrix}$

For $y = mx$ to be invariant, the image point must also lie on $y = mx$:

$2x - 2mx = m(x + 2mx)$

$2 - 2m = m + 2m^2$

Question number

$2m^2 + 3m - 2 = 0$

$(2m - 1)(m + 2) = 0$

$m = \frac{1}{2}$ or $m = -2$

7 $n = 1$: $LHS = \frac{1}{2^0} = 1$, $RHS = 4 - \frac{1+2}{2^0} = 4 - 3 = 1$

So the result is true for $n = 1$.

Assume true for $n = k$: $\sum_{r=1}^{k} \frac{r}{2^{r-1}} = 4 - \frac{k+2}{2^{k-1}}$

Then $\sum_{r=1}^{k+1} \frac{r}{2^{r-1}} = 4 - \frac{k+2}{2^{k-1}} + \frac{k+1}{2^k}$

$= 4 - \frac{2(k+2) - (k+1)}{2^k} = 4 - \frac{2k + 4 - k - 1}{2^k}$

$= 4 - \frac{k+3}{2^k} = 4 - \frac{(k+1)+2}{2^{(k+1)-1}}$

which is the result with $n = k + 1$, so it is true for $n = k + 1$.

The result is true for $n = 1$, and if it is true for $n = k$ then it is true for $n = k + 1$. Hence by induction it is true for all $n \geq 1$.

8 (a) Product of roots: $\alpha \times (-\alpha) \times \beta \times \frac{1}{\beta} = \frac{-9}{4}$

$-\alpha^2 = -\frac{9}{4} \Rightarrow \alpha = \pm\frac{3}{2}$

Sum of roots: $\alpha + (-\alpha) + \beta + \frac{1}{\beta} = -\frac{-4}{4} = 1$

$\beta + \frac{1}{\beta} = 1 \Rightarrow \beta^2 - \beta + 1 = 0$

Question number

$\beta = \frac{1 \pm \sqrt{1-4}}{2} = \frac{1 \pm i\sqrt{3}}{2}$

(and then $\frac{1}{\beta}$ is the other one of this conjugate pair)

Roots: $\frac{3}{2}, -\frac{3}{2}, \frac{1+i\sqrt{3}}{2}, \frac{1-i\sqrt{3}}{2}$

8 (b) Sum of pairs $= \frac{p}{4}$:

$\alpha(-\alpha) + \alpha\beta + \frac{\alpha}{\beta} - \alpha\beta - \frac{\alpha}{\beta} + \beta \times \frac{1}{\beta} = -\alpha^2 + 1$

$= -\frac{9}{4} + 1 = -\frac{5}{4}$ so $\frac{p}{4} = -\frac{5}{4} \Rightarrow p = -5$

Sum of triples $= -\frac{q}{4}$:

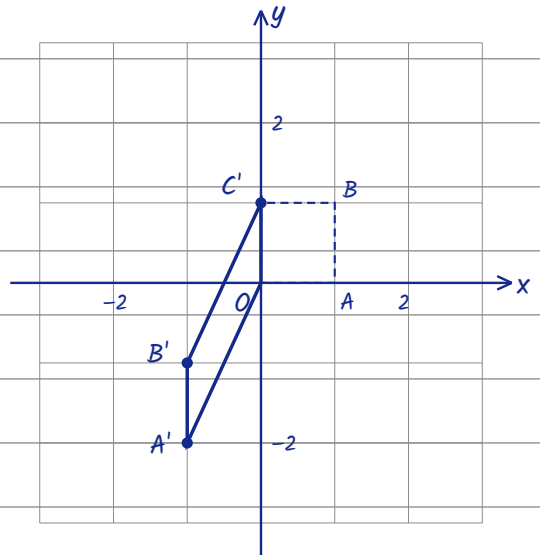
$\alpha(-\alpha)\beta + \alpha(-\alpha)\frac{1}{\beta} + \alpha\beta\frac{1}{\beta} + (-\alpha)\beta\frac{1}{\beta} = -\alpha^2(\beta + \frac{1}{\beta}) + \alpha - \alpha$

$= -\frac{9}{4} \times 1 = -\frac{9}{4}$ so $-\frac{q}{4} = -\frac{9}{4} \Rightarrow q = 9$

9 (a) $\begin{pmatrix} -1 & 0 \\ -2 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 0 & -1 & -1 & 0 \\ 0 & -2 & -1 & 1 \end{pmatrix}$

$O(0, 0)$, $A'(-1, -2)$, $B'(-1, -1)$, $C'(0, 1)$

Question number



9 (b)(i) $\det M = (-1)(1) - (0)(-2) = -1$

9 (b)(ii) $|\det M| = 1$, so the area of OA'B'C' is the same as the area of OABC (1 square unit): area is preserved.

$\det M$ is negative, so the orientation is reversed (OABC goes anticlockwise, OA'B'C' goes clockwise).

9 (c)(i) First: a reflection in the y-axis.

Then: a shear with the y-axis invariant, mapping (1, 0) to (1, 2) [equivalently (−1, 0) to (−1, −2)], i.e. $(x, y) \to (x, y + 2x)$.

9 (c)(ii) Reflection in the y-axis: $R = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$

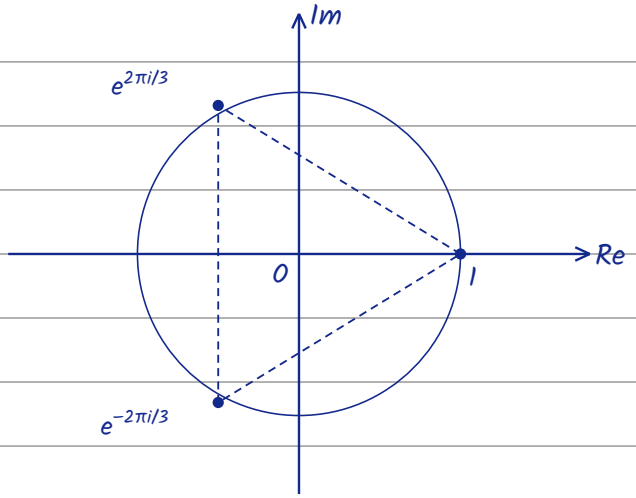
Shear: $S = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$

Reflection first, then shear: $SR = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$

Question number

$$= \begin{pmatrix} 1\times(-1) + 0\times0 & 1\times0 + 0\times1 \\ 2\times(-1) + 1\times0 & 2\times0 + 1\times1 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ -2 & 1 \end{pmatrix} = M$$

10 (a) Cube roots of unity: 1, $e^{2\pi i/3}$, $e^{-2\pi i/3}$



10 (b)(i) $z^3 = 1 + \sqrt{3}\,i$

$|1 + \sqrt{3}i| = \sqrt{1 + 3} = 2$, $\quad \arg(1 + \sqrt{3}i) = \arctan\sqrt{3} = \frac{\pi}{3}$

$z^3 = 2e^{i(\pi/3 + 2k\pi)} \quad \Rightarrow \quad z = \sqrt[3]{2}e^{i(\pi/9 + 2k\pi/3)}$

$k = 0$: $z = \sqrt[3]{2}\, e^{\pi i/9}$

$k = 1$: $z = \sqrt[3]{2}\, e^{7\pi i/9}$

$k = -1$: $z = \sqrt[3]{2}\, e^{-5\pi i/9}$

10 (b)(ii) Rotation about O through $\frac{\pi}{9}$ $(= 20°)$ anticlockwise,

then an enlargement, centre O, scale factor $\sqrt[3]{2}$.

10 (b)(iii) $z_1 + z_2 + z_3 = \sqrt[3]{2}\, e^{-5\pi i/9} + \sqrt[3]{2}\, e^{\pi i/9} + \sqrt[3]{2}\, e^{7\pi i/9}$

This is a geometric series with $a = \sqrt[3]{2}\, e^{-5\pi i/9}$, ratio $e^{2\pi i/3}$:

Sum $= a(1 - r^3) \div (1 - r)$:

Question number

$= \sqrt[3]{2}\, e^{-5\pi i/9} (1 - (e^{2\pi i/3})^3) \div (1 - e^{2\pi i/3})$

$= \sqrt[3]{2}\, e^{-5\pi i/9} (1 - e^{2\pi i}) \div (1 - e^{2\pi i/3})$

Since $e^{2\pi i} = 1$, the numerator is 0, so $z_1 + z_2 + z_3 = 0$

10 (b)(iv) The imaginary part of $z_1 + z_2 + z_3$ is zero:

$\sqrt[3]{2} \left(\sin \frac{\pi}{9} + \sin \frac{7\pi}{9} + \sin\left(-\frac{5\pi}{9}\right) \right) = 0$

$\sin 20° + \sin 140° - \sin 100° = 0$

$\sin 20° + \sin 140° = \sin 100°$ as required

11 (a)(i) $u.v = \lambda(1) + 1(2) + (-3)(-2) = \lambda + 2 + 6 = \lambda + 8$

11 (a)(ii) $u \times v = \begin{pmatrix} \lambda \\ 1 \\ -3 \end{pmatrix} \times \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix} = \begin{pmatrix} (1)(-2) - (-3)(2) \\ (-3)(1) - \lambda(-2) \\ \lambda(2) - (1)(1) \end{pmatrix} = \begin{pmatrix} 4 \\ 2\lambda - 3 \\ 2\lambda - 1 \end{pmatrix}$

11 (b)(i) Normals: $2i + j - 3k$ (u with $\lambda = 2$) and $i + 2j - 2k$ ($= v$)

$u.v = 2 + 8 = 10$, $|u| = \sqrt{(4 + 1 + 9)} = \sqrt{14}$, $|v| = \sqrt{(1 + 4 + 4)} = 3$

$\cos\theta = \frac{10}{3\sqrt{14}} = 0.8909...$

$\theta = 27.0°$ (0.472 rad)

11 (b)(ii) Directions: $3i + j - 3k$ (u with $\lambda = 3$) and $i + 2j - 2k$ ($= v$)

With $\lambda = 3$: $n = u \times v = 4i + 3j + 5k$, $|n| = \sqrt{(16 + 9 + 25)} = \sqrt{50} = 5\sqrt{2}$

Points on the lines: (3, 0, 2) and (0, 4, −2)

Vector between them: (3 − 0, 0 − 4, 2 − (−2)) = (3, −4, 4)

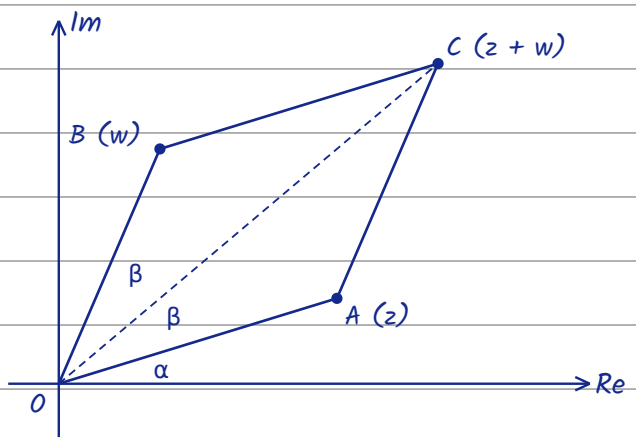
(3, −4, 4).(4, 3, 5) = 12 − 12 + 20 = 20

Question number

Distance $= \frac{|20|}{5\sqrt{2}}$

$= \frac{4}{\sqrt{2}} = 2\sqrt{2}$

12



Let arg z = α (angle between OA and the positive real axis)

and let ∠AOC = β.

C represents z + w (parallelogram law), so arg(z + w) = α + β.

In a rhombus the diagonal OC bisects ∠AOB, so ∠COB = β too.

So arg w = α + 2β.

arg z + arg w = α + (α + 2β) = 2(α + β) = 2 arg(z + w)

⇒ $\arg(z + w) = \frac{1}{2}(\arg z + \arg w)$ as required

13

Auxiliary equation: $\lambda^2 + 2\lambda - 3 = 0 \Rightarrow (\lambda + 3)(\lambda - 1) = 0$

$\lambda = -3$ or $\lambda = 1$

CF: $y = Ae^{-3x} + Be^{x}$

e^x is in the CF, so try PI: $y = Cxe^x$

$\frac{dy}{dx} = C(e^x + xe^x)$, $\frac{d^2y}{dx^2} = C(2e^x + xe^x)$

Substitute: $C(2e^x + xe^x) + 2C(e^x + xe^x) - 3Cxe^x = 2e^x$

Question number

$C(2 + x + 2 + 2x - 3x) = 2 \Rightarrow 4C = 2$

$C = \frac{1}{2}$

General solution: $y = Ae^{-3x} + Be^{x} + \frac{1}{2}xe^{x}$

14 (a) Write $\cos\theta + 2\sin\theta = R\cos(\theta - \alpha) = R\cos\theta\cos\alpha + R\sin\theta\sin\alpha$

$R\cos\alpha = 1, \quad R\sin\alpha = 2$

$R = \sqrt{(1^2 + 2^2)} = \sqrt{5}, \quad \tan\alpha = 2 \Rightarrow \alpha = 1.107$ rad

So $r = \sqrt{5}\,a\cos(\theta - 1.107)$

r is greatest when $\cos(\theta - 1.107) = 1$, i.e. $\theta = 1.107$ (in $0 \leq \theta \leq \pi$)

Then $r = \sqrt{5}\,a$

Furthest point: $(\sqrt{5}\,a, 1.107)$

14 (b)(i) Multiply by r: $r^2 = ar\cos\theta + 2ar\sin\theta$

$x^2 + y^2 = ax + 2ay$

$x^2 - ax + y^2 - 2ay = 0$

$(x - \frac{1}{2}a)^2 - \frac{1}{4}a^2 + (y - a)^2 - a^2 = 0$

$(x - \frac{1}{2}a)^2 + (y - a)^2 = \frac{5}{4}a^2$

This is a circle, centre $(\frac{1}{2}a, a)$, radius $\frac{\sqrt{5}}{2}a$

14 (b)(ii) Centre $(\frac{1}{2}a, a)$: distance from $O = \sqrt{\frac{1}{4}a^2 + a^2} = \frac{\sqrt{5}}{2}a$, angle $= \arctan 2$

Polar coordinates of centre: $(\frac{\sqrt{5}}{2}a, 1.107)$

Question number

15 For a sheaf the planes meet in a line, so the matrix is singular:

$$\det\begin{pmatrix} -4 & k & 7 \\ 1 & -2 & 5 \\ 2 & 3 & 1 \end{pmatrix} = -4(-2-15) - k(1-10) + 7(3+4)$$

$$= 68 + 9k + 49 = 117 + 9k$$

$117 + 9k = 0 \Rightarrow k = -13$

$-4x - 13y + 7z = 4 \quad (1)$

$x - 2y + 5z = 1 \quad (2)$

$2x + 3y + z = 2 \quad (3)$

$(1) + 2\times(3)$: $-7y + 9z = 8$

$(3) - 2\times(2)$: $7y - 9z = 2 - 2l$

For a sheaf these must be the same equation (consistent):

$2 - 2l = -8 \Rightarrow l = 5$

$k = -13, \quad l = 5$

16 (a) LHS: $\cosh 2u = \frac{e^{2u} + e^{-2u}}{2}$

RHS: $1 + 2\sinh^2 u = 1 + 2\left(\frac{e^u - e^{-u}}{2}\right)^2 = 1 + 2\left(\frac{e^{2u} - 2 + e^{-2u}}{4}\right)$

$$= 1 + \frac{e^{2u} - 2 + e^{-2u}}{2} = \frac{2 + e^{2u} - 2 + e^{-2u}}{2}$$

$$= \frac{e^{2u} + e^{-2u}}{2} = \cosh 2u = \text{LHS} \quad \text{as required}$$

16 (b) Let $x = 2\sinh u$, $\frac{dx}{du} = 2\cosh u$

$x = 0 \Rightarrow u = 0$; $\quad x = 2 \Rightarrow \sinh u = 1 \Rightarrow u = \operatorname{arsinh} 1$

Question number

$$\int_0^2 \frac{x^2}{\sqrt{4+x^2}}\,dx = \int_0^{\operatorname{arsinh} 1} \frac{4\sinh^2 u}{\sqrt{4+4\sinh^2 u}} \times 2\cosh u\,du$$

$$= \int_0^{\operatorname{arsinh} 1} \frac{4\sinh^2 u}{2\cosh u} \times 2\cosh u\,du \quad (\text{using } 1+\sinh^2 u = \cosh^2 u)$$

$$= \int_0^{\operatorname{arsinh} 1} 4\sinh^2 u\,du = \int_0^{\operatorname{arsinh} 1} (2\cosh 2u - 2)\,du$$

$$= [\sinh 2u - 2u]_0^{\operatorname{arsinh} 1}$$

$\operatorname{arsinh} 1 = \ln(1+\sqrt{(1+1)}) = \ln(1+\sqrt{2})$

At $u = \operatorname{arsinh} 1$: $\sinh u = 1$, $\cosh u = \sqrt{(1+\sinh^2 u)} = \sqrt{2}$

$\sinh 2u = 2\sinh u \cosh u = 2 \times 1 \times \sqrt{2} = 2\sqrt{2}$

Integral $= (2\sqrt{2} - 2\ln(1+\sqrt{2})) - (0-0)$

$= 2\sqrt{2} - 2\ln(1+\sqrt{2})$ as required

17 (a)(i) After t minutes: at litres of chemical in, bt litres of mixture out.

Volume of liquid in the vessel $= 1 + at - bt = 1 + (a-b)t$ litres

Proportion of chemical $= \frac{x}{1+(a-b)t}$ as required

17 (a)(ii) Rate chemical flows in $= a$ litres per minute

Rate chemical flows out $= b \times \frac{x}{1+(a-b)t} = \frac{bx}{1+(a-b)t}$

$$\frac{dx}{dt} = a - \frac{bx}{1+(a-b)t}$$

$$\Rightarrow \frac{dx}{dt} + \frac{bx}{1+(a-b)t} = a \quad \text{as required}$$

17 (b)(i) $b = a$: $\frac{dx}{dt} + ax = a \Rightarrow \frac{dx}{dt} = a(1-x)$

Question number

$\int \frac{1}{1-x}\,dx = \int a\,dt$

$-\ln(1-x) = at + c$

$t = 0, x = 0$: $-\ln 1 = c \Rightarrow c = 0$

$\ln(1-x) = -at \Rightarrow 1 - x = e^{-at}$

$x = 1 - e^{-at}$

17 (b)(ii) The volume stays at 1 litre, so equal amounts means $x = 0.5$ when $t = 1$:

$0.5 = 1 - e^{-a} \Rightarrow e^{-a} = 0.5 \Rightarrow a = \ln 2$

Rate of inflow = $\ln 2 = 0.693$ litres per minute

17 (c)(i) With $b = 2a$ the volume is $1 + (a - 2a)t = 1 - at$.

When $t = \frac{1}{a}$ the vessel has no liquid left (volume 0), and for

$t > \frac{1}{a}$ the volume would be negative, so the equation is invalid.

17 (c)(ii) $\frac{dx}{dt} + \frac{2ax}{1-at} = a$

Integrating factor = $e^{\int 2a/(1-at)\,dt} = e^{-2\ln(1-at)} = (1-at)^{-2}$

$\frac{d}{dt}\left(x(1-at)^{-2}\right) = a(1-at)^{-2}$

$x(1-at)^{-2} = (1-at)^{-1} + c$

$t = 0, x = 0$: $0 = 1 + c \Rightarrow c = -1$

$x = (1-at)^2\left((1-at)^{-1} - 1\right) = (1-at) - (1-at)^2$

$x = (1-at)(1-(1-at)) = at(1-at) = at - a^2t^2$

$\frac{dx}{dt} = a - 2a^2t = 0 \Rightarrow t = \frac{1}{2a}$

Question number

$\frac{d^2x}{dt^2} = -2a^2 < 0$, so this is a maximum

$x = a(\frac{1}{2a}) - a^2(\frac{1}{2a})^2 = \frac{1}{2} - \frac{1}{4} = \frac{1}{4}$

Maximum amount of chemical = 0.25 litres