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Thursday 22 October 2020 – Afternoon

A Level Further Mathematics B (MEI)

Y435/01 Extra Pure

Time allowed: 1 hour 15 minutes

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You must have:
- the Printed Answer Booklet
- the Formulae Booklet for Further Mathematics B (MEI)
- a scientific or graphical calculator

INSTRUCTIONS
- Use black ink. You can use an HB pencil, but only for graphs and diagrams.
- Write your answer to each question in the space provided in the **Printed Answer Booklet**. If you need extra space use the lined pages at the end of the Printed Answer Booklet. The question numbers must be clearly shown.
- Fill in the boxes on the front of the Printed Answer Booklet.
- Answer **all** the questions.
- Where appropriate, your answer should be supported with working. Marks might be given for using a correct method, even if your answer is wrong.
- Give your final answers to a degree of accuracy that is appropriate to the context.
- Do **not** send this Question Paper for marking. Keep it in the centre or recycle it.

INFORMATION
- The total mark for this paper is **60**.
- The marks for each question are shown in brackets **[ ]**.
- This document has **4** pages.

ADVICE
- Read each question carefully before you start your answer.

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Answer **all** the questions.

1 The matrix $\mathbf{A}$ is given by $\mathbf{A} = \begin{pmatrix} 0 & 2 \\ 3 & -1 \end{pmatrix}$.

Find

- the eigenvalues of $\mathbf{A}$,
- an eigenvector associated with each eigenvalue. **[5]**

2 A sequence is defined by the recurrence relation $t_{n+1} = \dfrac{t_n}{n+3}$ for $n \geqslant 1$, with $t_1 = 8$.

Verify that the particular solution to the recurrence relation is given by $t_n = \dfrac{a}{(n+b)!}$ where a and b are constants whose values are to be determined. **[5]**

3 A sequence is defined by the recurrence relation $u_{n+2} = 4u_{n+1} - 5u_n$ for $n \geqslant 0$, with $u_0 = 0$ and $u_1 = 1$.

(a) Find an exact real expression for u_n in terms of n and θ, where $\tan\theta = \frac{1}{2}$. **[7]**

A sequence is defined by $v_n = a^{\frac{1}{2}n} u_n$ for $n \geqslant 0$, where a is a positive constant.

(b) In each of the following cases, describe the behaviour of v_n as $n \to \infty$.

- $a = 0.1$
- $a = 0.2$
- $a = 1$ **[5]**

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4 **(a)** In each of the following cases, a set G and a binary operation $\circ$ are given. The operation $\circ$ may be assumed to be associative on G.

Determine which, if any, of the other three group axioms are satisfied by $(G, \circ)$ and which, if any, are not satisfied.

(i) $G = \{2n+1 : n \in \mathbb{Z}\}$ and $\circ$ is addition. **[3]**

(ii) $G = \{a + b\sqrt{2} : a, b \in \mathbb{Z}\}$ and $\circ$ is multiplication. **[3]**

(iii) G is the set of all real numbers and $\circ$ is multiplication. **[3]**

(b) A group M consists of eight 2×2 matrices under the operation of matrix multiplication. Five of the eight elements of M are as follows.

$$\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & \mathrm{i} \\ \mathrm{i} & 1 \end{pmatrix} \quad \frac{1}{\sqrt{2}}\begin{pmatrix} -1 & \mathrm{i} \\ \mathrm{i} & -1 \end{pmatrix} \quad \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -\mathrm{i} \\ -\mathrm{i} & 1 \end{pmatrix} \quad \begin{pmatrix} 0 & \mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix} \quad \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

(i) Find the other three elements of M. **[3]**

$(N, *)$ is another group of order 8, with identity element e. You are given that $N = \langle a, b, c \rangle$ where $a * a = b * b = c * c = e$.

(ii) State whether M and N are isomorphic to each other, giving a reason for your answer. **[1]**

5 **In this question you must show detailed reasoning.**

The matrix **A** is given by $\begin{pmatrix} \frac{1}{3} & -\frac{2}{3} & -\frac{2}{3} \\ -\frac{2}{3} & \frac{1}{3} & -\frac{2}{3} \\ -\frac{2}{3} & -\frac{2}{3} & \frac{1}{3} \end{pmatrix}$ and the vector **e** is given by $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$. You are given that **e** is an eigenvector of **A** with an associated eigenvalue of -1.

f is any vector which is perpendicular to **e**.

(a) Show that **f** is also an eigenvector of **A**. **[4]**

(b) State the eigenvalue associated with **f**. **[1]**

You are now given that **A** represents a reflection in 3-D space.

(c) Explain the significance of **e** and **f** in relation to the transformation that **A** represents. **[2]**

(d) State the cartesian equation of the plane of reflection of the transformation represented by **A**. **[1]**

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6 A surface S is defined by $z = \mathrm{f}(x, y) = 4x^4 + 4y^4 - 17x^2y^2$.

(a) (i) Show that there is only one stationary point on S. **[5]**

The value of z at the stationary point is denoted by s.

(ii) State the value of s. **[1]**

(iii) By factorising $\mathrm{f}(x, y)$, sketch the contour lines of the surface for $z = s$. **[3]**

(iv) Hence explain whether the stationary point is a maximum point, a minimum point or a saddle point. **[1]**

C is a point on S with coordinates $(a, a, \mathrm{f}(a, a))$ where a is a constant and $a \neq 0$. Π is the tangent plane to S at C.

(b) (i) Find the equation of Π in the form $\mathbf{r.n} = p$. **[3]**

(ii) The shortest distance from the origin to Π is denoted by d. Show that $\frac{d}{a} \to \frac{3\sqrt{2}}{4}$ as $a \to \infty$. **[3]**

(iii) Explain whether the origin lies above or below Π. **[1]**

END OF QUESTION PAPER

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Surname	StemUp Model Solution
Answer booklet	Write the question number in the left-hand margin at the start of each answer.

Question number

1

$\det(A - \lambda I) = \begin{pmatrix} -\lambda & 2 \\ 3 & -1-\lambda \end{pmatrix} = -\lambda(-1-\lambda) - 6$

$\lambda^2 + \lambda - 6 = 0 \Rightarrow (\lambda - 2)(\lambda + 3) = 0$

Eigenvalues: $\lambda = 2$ and $\lambda = -3$

$\lambda = 2$: $\begin{pmatrix} 0 & 2 \\ 3 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 2\begin{pmatrix} x \\ y \end{pmatrix} \Rightarrow 2y = 2x \Rightarrow x = y$

eigenvector $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$\lambda = -3$: $\begin{pmatrix} 0 & 2 \\ 3 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = -3\begin{pmatrix} x \\ y \end{pmatrix} \Rightarrow 2y = -3x$

eigenvector $\begin{pmatrix} 2 \\ -3 \end{pmatrix}$

2

$t_1 = \frac{a}{(1+b)!} = 8$

$t_{n+1} = \frac{t_n}{n+3} = \frac{a}{(n+b)!(n+3)}$

For the solution we need $t_{n+1} = \frac{a}{(n+1+b)!} = \frac{a}{(n+1+b)(n+b)!}$

Comparing: $n + 3 = n + 1 + b \Rightarrow b = 2$

Then $a = 8 \times (1+2)! = 8 \times 6 = 48$

So $t_n = \frac{48}{(n+2)!}$ satisfies the recurrence relation and $t_1 = \frac{48}{3!} = 8$

$a = 48,\ b = 2$

Question number

3 (a) Auxiliary equation: $p^2 - 4p + 5 = 0$

$p = \frac{4 \pm \sqrt{16 - 20}}{2} = 2 \pm i$

$|2 \pm i| = \sqrt{5}$, $\arg(2 + i) = \theta$ where $\tan\theta = \frac{1}{2}$

$2 \pm i = \sqrt{5}(\cos\theta \pm i \sin\theta)$

$u_n = A(2 + i)^n + B(2 - i)^n$

$= 5^{n/2}[A(\cos n\theta + i \sin n\theta) + B(\cos n\theta - i \sin n\theta)]$

$u_n = 5^{n/2}(\alpha \cos n\theta + \beta \sin n\theta)$

$u_0 = 0$: $0 = \alpha \Rightarrow \alpha = 0$

$u_1 = 1$: $1 = \sqrt{5}\,\beta \sin\theta$, $\sin\theta = \frac{1}{\sqrt{5}} \Rightarrow \beta = 1$

$u_n = 5^{n/2} \sin n\theta$

3 (b) $v_n = a^{n/2} \times 5^{n/2} \sin n\theta = (5a)^{n/2} \sin n\theta$

$a = 0.1$: $v_n = (0.5)^{n/2} \sin n\theta$, $(0.5)^{n/2} \to 0$

so v_n converges to 0 (oscillating in sign).

$a = 0.2$: $v_n = \sin n\theta$, which does not converge;

it is bounded (between -1 and 1) and oscillates between positive and negative.

$a = 1$: $v_n = 5^{n/2} \sin n\theta$, $5^{n/2} \to \infty$

so v_n diverges; it oscillates in sign with terms getting bigger in size.

4 (a)(i) $(2n + 1) + (2m + 1) = 2(n + m + 1)$, which is even, so not in G:

closure is not satisfied.

The identity for addition would be 0, which is even so $0 \notin G$:

identity is not satisfied.

With no identity, inverses cannot exist: inverse is not satisfied.

Question number

4 (a)(ii) $(a + b\sqrt{2})(c + d\sqrt{2}) = (ac + 2bd) + (ad + bc)\sqrt{2}$

$ac + 2bd, ad + bc \in \mathbb{Z}$, so closure is satisfied.

$1 = 1 + 0\sqrt{2} \in G$ $(a = 1, b = 0)$ and $1 \times x = x \times 1 = x$: identity is satisfied.

Eg $\frac{1}{2 + \sqrt{2}} = \frac{2 - \sqrt{2}}{(2 + \sqrt{2})(2 - \sqrt{2})} = \frac{2 - \sqrt{2}}{2} = 1 - \frac{1}{2}\sqrt{2} \notin G$

(also $0 \in G$ has no inverse), so inverse is not satisfied.

4 (a)(iii) $a, b \in \mathbb{R} \Rightarrow ab \in \mathbb{R}$, so closure is satisfied.

$1 \in \mathbb{R}$ and $1 \times a = a \times 1 = a$ for all $a \in \mathbb{R}$: identity is satisfied.

$0 \in \mathbb{R}$ but $0 \times x = 0 \neq 1$ for every x, so 0^{-1} does not exist:

inverse is not satisfied.

4 (b)(i) Let $R = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}$. $R^2 = \frac{1}{2}\begin{pmatrix} 1 - 1 & 2i \\ 2i & -1 + 1 \end{pmatrix} = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$

$R^3 = R^2R = \frac{1}{\sqrt{2}}\begin{pmatrix} -1 & i \\ i & -1 \end{pmatrix}$, $R^4 = (R^2)^2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = -1$

$R^5 = -R$, $R^6 = -R^2$, $R^7 = -R^3 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix}$, $R^8 = 1$

So $M = \{1, R, R^2, \ldots, R^7\}$ and the other three elements are

$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$, $\frac{1}{\sqrt{2}}\begin{pmatrix} -1 & -i \\ -i & -1 \end{pmatrix}$, $\begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}$

4 (b)(ii) Not isomorphic. M is cyclic (generated by R, order 8), so it has only one element of order 2 (namely -1), while N has at least three elements of order 2 (a, b and c).

Question number

5 (a) Let $\mathbf{f} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$. $\mathbf{f} \perp \mathbf{e} \Rightarrow \mathbf{e}.\mathbf{f} = 0 \Rightarrow a + b + c = 0$

$$A\mathbf{f} = \frac{1}{3}\begin{pmatrix} 1 & -2 & -2 \\ -2 & 1 & -2 \\ -2 & -2 & 1 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \frac{1}{3}\begin{pmatrix} a - 2b - 2c \\ -2a + b - 2c \\ -2a - 2b + c \end{pmatrix}$$

Using $b + c = -a$, $a + c = -b$, $a + b = -c$:

$$A\mathbf{f} = \frac{1}{3}\begin{pmatrix} a - 2(b + c) \\ b - 2(a + c) \\ c - 2(a + b) \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 3a \\ 3b \\ 3c \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \mathbf{f}$$

So $A\mathbf{f} = \mathbf{f}$ with $\mathbf{f} \neq 0$, hence $\mathbf{f}$ is an eigenvector of A, as required

5 (b) $\lambda = 1$

5 (c) $\mathbf{f}$ has eigenvalue 1, so every $\mathbf{f}$ is invariant: $\mathbf{f}$ is parallel to (lies in) the mirror plane.

$\mathbf{e}$ has eigenvalue -1, so $\mathbf{e}$ is reversed: $\mathbf{e}$ is perpendicular to the mirror plane (it is a normal to the plane).

5 (d) $\mathbf{e}$ is normal to the plane and O lies in the plane (A is linear):

$x + y + z = 0$

6 (a)(i) $\frac{\partial f}{\partial x} = 16x^3 - 34xy^2$, $\frac{\partial f}{\partial y} = 16y^3 - 34x^2y$

Stationary points: $16x^3 - 34xy^2 = 0$ and $16y^3 - 34x^2y = 0$

First: $2x(8x^2 - 17y^2) = 0 \Rightarrow x = 0$ or $8x^2 = 17y^2$

If $x = 0$: second gives $16y^3 = 0 \Rightarrow y = 0$

Question number

If $8x^2 = 17y^2$: $x^2 = \frac{17}{8}y^2$; second: $2y(8y^2 - 17x^2) = 0$

$2y(8y^2 - \frac{289}{8}y^2) = 0 \Rightarrow -\frac{225}{4}y^3 = 0 \Rightarrow y = 0$, so $x = 0$

So $x = 0$, $y = 0$ is the only solution: only one stationary point,

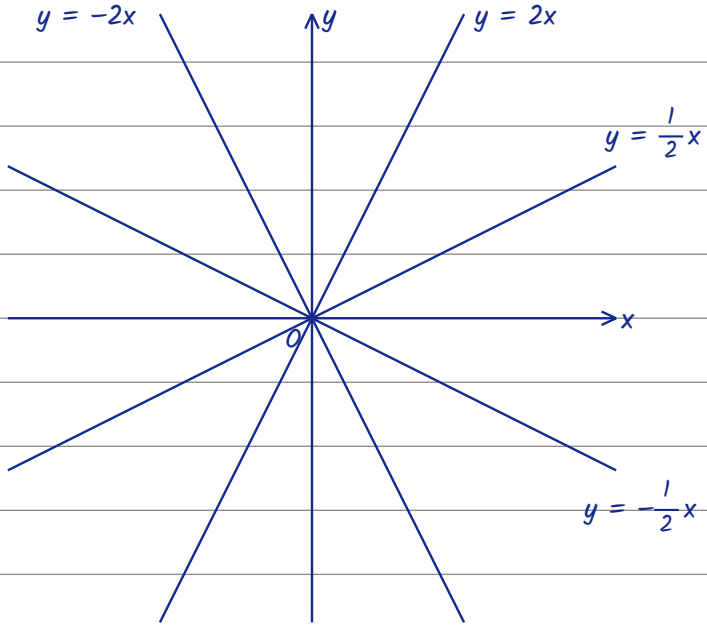
at $(0, 0, 0)$, as required

6 (a)(ii) $s = f(0, 0) = 0$

6 (a)(iii) $4x^4 + 4y^4 - 17x^2y^2 = (4x^2 - y^2)(x^2 - 4y^2)$

$= (2x - y)(2x + y)(x - 2y)(x + 2y)$

$z = 0 \Rightarrow y = 2x,\ y = -2x,\ y = \frac{1}{2}x,\ y = -\frac{1}{2}x$



6 (a)(iv) The contour $z = 0$ splits the plane into wedges where z is positive and negative,

eg along the x-axis $z = 4x^4 > 0$, but along $y = x$, $z = -9x^4 < 0$.

So near the stationary point z takes both positive and negative values:

it is neither a maximum nor a minimum, so it is a saddle point.

Question number

6 (b)(i) $f(a, a) = 4a^4 + 4a^4 - 17a^4 = -9a^4$

At C: $f_x = 16a^3 - 34a^3 = -18a^3$, $f_y = 16a^3 - 34a^3 = -18a^3$

Normal: $n = \begin{pmatrix} f_x \\ f_y \\ -1 \end{pmatrix} = \begin{pmatrix} -18a^3 \\ -18a^3 \\ -1 \end{pmatrix}$

$p = \begin{pmatrix} a \\ a \\ -9a^4 \end{pmatrix} . \begin{pmatrix} -18a^3 \\ -18a^3 \\ -1 \end{pmatrix} = -18a^4 - 18a^4 + 9a^4 = -27a^4$

Π: $r . \begin{pmatrix} 18a^3 \\ 18a^3 \\ 1 \end{pmatrix} = 27a^4$

6 (b)(ii) $d = \frac{27a^4}{|n|}$, $|n| = \sqrt{(18a^3)^2 + (18a^3)^2 + 1} = \sqrt{648a^6 + 1}$

$d/a = 27a^3 / (648a^6 + 1)^{1/2}$ ($a > 0$ as $a \to \infty$)

$d/a = 27 / (648 + a^{-6})^{1/2}$ (dividing top and bottom by a^3)

As $a \to \infty$, $a^{-6} \to 0$:

$\frac{d}{a} \to \frac{27}{\sqrt{648}} = \frac{27}{18\sqrt{2}} = \frac{3}{2\sqrt{2}} = \frac{3\sqrt{2}}{4}$, as required

6 (b)(iii) Put $x = y = 0$ in the equation of Π: $z = 27a^4 > 0$

So Π meets the z-axis above 0: the origin lies below Π.