OCR
Oxford Cambridge and RSA

Thursday 15 October 2020 – Afternoon

A Level Further Mathematics B (MEI)

Y421/01 Mechanics Major

Time allowed: 2 hours 15 minutes

8312950094

You must have:
- the Printed Answer Booklet
- the Formulae Booklet for Further Mathematics B (MEI)
- a scientific or graphical calculator

INSTRUCTIONS
- Use black ink. You can use an HB pencil, but only for graphs and diagrams.
- Write your answer to each question in the space provided in the **Printed Answer Booklet**. If you need extra space use the lined pages at the end of the Printed Answer Booklet. The question numbers must be clearly shown.
- Fill in the boxes on the front of the Printed Answer Booklet.
- Answer **all** the questions.
- Where appropriate, your answer should be supported with working. Marks might be given for using a correct method, even if your answer is wrong.
- Give your final answers to a degree of accuracy that is appropriate to the context.
- The acceleration due to gravity is denoted by $g\,\mathrm{m\,s^{-2}}$. When a numerical value is needed use $g = 9.8$ unless a different value is specified in the question.
- Do **not** send this Question Paper for marking. Keep it in the centre or recycle it.

INFORMATION
- The total mark for this paper is **120**.
- The marks for each question are shown in brackets **[ ]**.
- This document has **12** pages.

ADVICE
- Read each question carefully before you start your answer.

© OCR 2020 [D/508/5593]
DC (NF/FC) 199483/3
OCR is an exempt Charity

stemup.co.uk

Section A (35 marks)

Answer **all** the questions.

1 A particle P of mass 0.5 kg is attached to a fixed point O by a light elastic string of natural length 3 m and modulus of elasticity 75 N.

P is released from rest at O and is allowed to fall freely.

Determine the length of the string when P is at its lowest point in the subsequent motion. **[5]**

2 A student conducts an experiment by first stretching a length of wire and fixing its ends. The student then plucks the wire causing it to vibrate. The frequency of these vibrations, f, is modelled by the formula

$$f = kC^{\alpha} l^{\beta} \sigma^{\gamma},$$

where C is the tension in the wire,
l is the length of the stretched wire,
σ is the mass per unit length of the stretched wire and
k is a dimensionless constant.

Use dimensional analysis to find α, β and γ. **[5]**

3 The vertices of a triangular lamina, which is in the x–y plane, are at the origin O and the points $\mathrm{A}(2,3)$ and $\mathrm{B}(-2,1)$.

Forces $2\mathbf{i}+\mathbf{j}$ and $-3\mathbf{i}+2\mathbf{j}$ are applied to the lamina at A and B, respectively, and a force $\mathbf{F}$, whose line of action is in the x–y plane, is applied at O.

The three forces form a couple.

(a) Determine the magnitude and the direction of $\mathbf{F}$. **[4]**

(b) Determine the magnitude and direction of the additional couple that must be applied to the lamina in order to keep it in equilibrium. **[3]**

© OCR 2020

stemup.co.uk

4 A particle P moves so that its position vector **r** at time t is given by

$\mathbf{r} = (5+20t)\mathbf{i} + (95+10t-5t^2)\mathbf{j}$.

(a) Determine the initial velocity of P. **[3]**

At time $t = T$, P is moving in a direction perpendicular to its initial direction of motion.

(b) Determine the value of T. **[3]**

(c) Determine the distance of P from its initial position at time T. **[4]**

5 A car of mass 900 kg moves along a straight level road. The power developed by the car is constant and equal to 60 kW. The resistance to the motion of the car is constant and equal to 1500 N.

At time t seconds the velocity of the car is denoted by $v\,\mathrm{m\,s^{-1}}$. Initially the car is at rest.

(a) Show that $\frac{3v}{5}\frac{\mathrm{d}v}{\mathrm{d}t} = 40 - v$. **[3]**

(b) Verify that $t = 24\ln\left(\frac{40}{40-v}\right) - \frac{3}{5}v$. **[5]**

stemup.co.uk

Section B (85 marks)

6 A small ball of mass m kg is held at a height of 78.4 m above horizontal ground. The ball is released from rest, falls vertically and rebounds from the ground. The coefficient of restitution between the ball and ground is e.

The ball continues to bounce until it comes to rest after 6 seconds.

(a) Determine the value of e. **[8]**

(b) Given that the magnitude of the impulse that the ground exerts on the ball at the first bounce is 23.52 Ns, determine the value of m. **[2]**

7

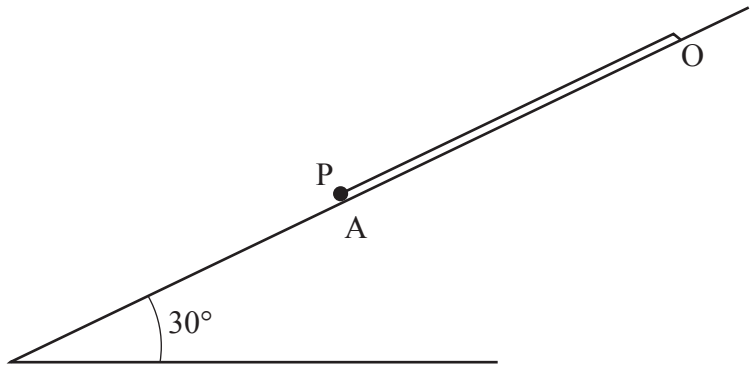


Fig. 7

A particle P of mass m is attached to one end of a light elastic string of natural length $6a$ and modulus of elasticity $3mg$. The other end of the string is fixed to a point O on a smooth plane, which is inclined at an angle of 30° to the horizontal. The string lies along a line of greatest slope of the plane and P rests in equilibrium on the inclined plane at a point A, as shown in Fig. 7.

P is now pulled a further distance $2a$ down the line of greatest slope through A and released from rest. At time t later, the displacement of P from A is x, where the positive direction of x is down the plane.

(a) Show that, until the string slackens, x satisfies the differential equation

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + \frac{gx}{2a} = 0.$$ **[6]**

(b) Determine, in terms of a and g, the time at which the string slackens. **[5]**

(c) Find, in terms of a and g, the speed of P when the string slackens. **[2]**

© OCR 2020

stemup.co.uk

8 [In this question, you may use the fact that the volume of a right circular cone of base radius r and height h is $\frac{1}{3}\pi r^2 h$.]

(a) By using integration, show that the centre of mass of a uniform solid right circular cone of height h and base radius r is at a distance $\frac{3}{4}h$ from the vertex. **[5]**

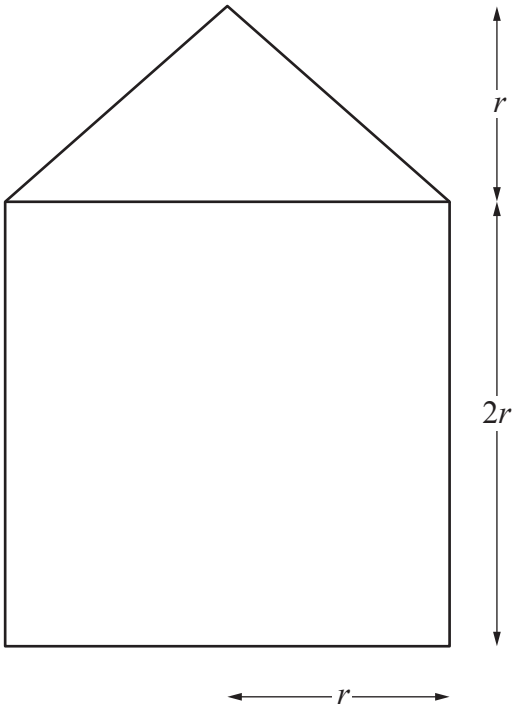


Fig. 8

Fig. 8 shows the side view of a toy formed by joining a uniform solid circular cylinder of radius r and height $2r$ to a uniform solid right circular cone, made of the same material as the cylinder, of radius r and height r.

The toy is placed on a horizontal floor with the curved surface of the cone in contact with the floor.

(b) Determine whether the toy will topple. **[7]**

(c) Explain why it is not necessary to know whether the floor is rough or smooth in answering part **(b)**. **[1]**

© OCR 2020

stemup.co.uk

9

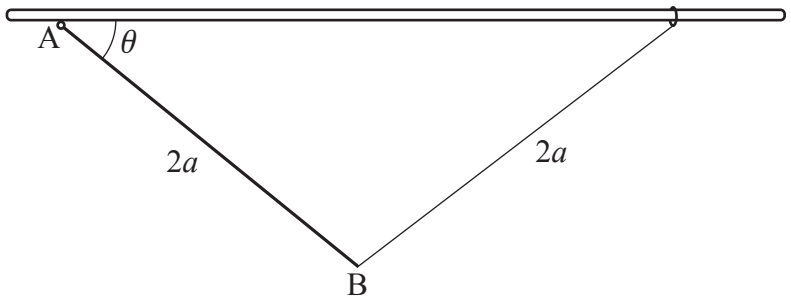


Fig. 9

Fig. 9 shows a uniform rod AB of length $2a$ and weight $8W$ which is smoothly hinged at the end A to a point on a fixed horizontal rough bar. A small ring of weight W is threaded on the bar and is connected to the rod at B by a light inextensible string of length $2a$. The system is in equilibrium with the rod inclined at an angle θ to the horizontal.

(a) Determine, in terms of W and θ, the tension in the string. **[4]**

It is given that, for equilibrium to be possible, the greatest distance the ring can be from A is $2.4a$.

(b) Determine the coefficient of friction between the bar and the ring. **[6]**

© OCR 2020

stemup.co.uk

10

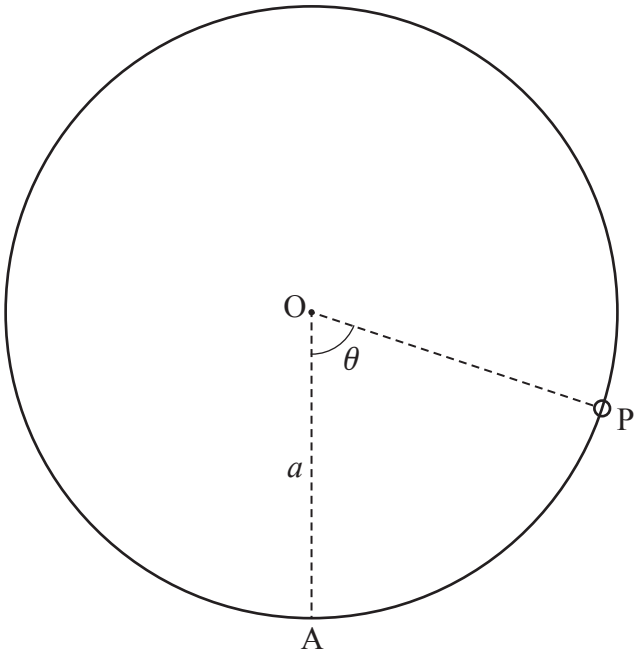


Fig. 10

Fig. 10 shows a small bead P of mass m which is threaded on a smooth thin wire. The wire is in the form of a circle of radius a and centre O. The wire is fixed in a vertical plane.

The bead is initially at the lowest point A of the wire and is projected along the wire with a velocity which is just sufficient to carry it to the highest point on the wire. The angle between OP and the downward vertical is denoted by θ.

(a) Determine the value of θ when the magnitude of the reaction of the wire on the bead is $\frac{7}{2}mg$. **[7]**

(b) Show that the angular velocity of P when OP makes an angle θ with the downward vertical is given by $k\sqrt{\frac{g}{a}}\cos\left(\frac{\theta}{2}\right)$, stating the value of the constant k. **[4]**

(c) Hence determine, in terms of g and a, the angular acceleration of P when θ takes the value found in part **(a)**. **[3]**

© OCR 2020

stemup.co.uk

11 Two uniform small smooth spheres A and B have equal radii and equal masses. The spheres are on a smooth horizontal surface. Sphere A is moving at an acute angle α to the line of centres, when it collides with B, which is stationary.

After the impact A is moving at an acute angle β to the line of centres. The coefficient of restitution between A and B is $\frac{1}{3}$.

(a) Show that $\tan\beta = 3\tan\alpha$. **[5]**

(b) Explain why the assumption that the contact between the spheres is smooth is needed in answering part **(a)**. **[1]**

It is given that A is deflected through an angle γ.

(c) Determine, in terms of α, an expression for $\tan\gamma$. **[2]**

(d) Determine the maximum value of γ. You do not need to justify that this value is a maximum. **[5]**

© OCR 2020

stemup.co.uk

12

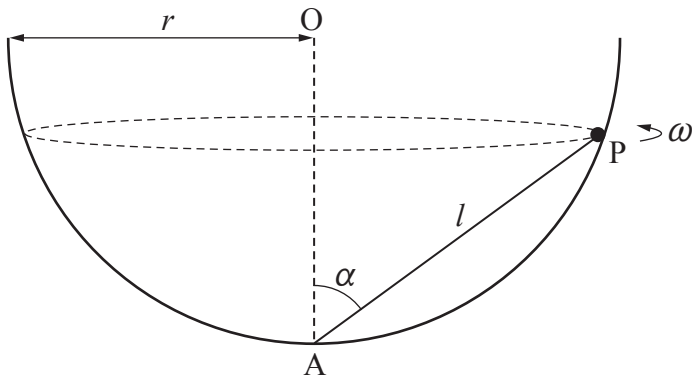


Fig. 12

Fig. 12 shows a hemispherical bowl. The rim of this bowl is a circle with centre O and radius r. The bowl is fixed with its rim horizontal and uppermost.

A particle P, of mass m, is connected by a light inextensible string of length l to the lowest point A on the bowl and describes a horizontal circle with constant angular speed ω on the smooth inner surface of the bowl.

The string is taut, and AP makes an angle α with the vertical.

(a) Show that the normal contact force between P and the bowl is of magnitude $mg + 2mr\omega^2 \cos^2\alpha$. **[9]**

(b) Deduce that $g < r\omega^2\left(k_1 + k_2\cos^2\alpha\right)$, stating the value of the constants k_1 and k_2. **[3]**

END OF QUESTION PAPER

© OCR 2020

stemup.co.uk

BLANK PAGE

© OCR 2020

stemup.co.uk

BLANK PAGE

© OCR 2020

stemup.co.uk

OCR
Oxford Cambridge and RSA

Copyright Information

OCR is committed to seeking permission to reproduce all third-party content that it uses in its assessment materials. OCR has attempted to identify and contact all copyright holders whose work is used in this paper. To avoid the issue of disclosure of answer-related information to candidates, all copyright acknowledgements are reproduced in the OCR Copyright Acknowledgements Booklet. This is produced for each series of examinations and is freely available to download from our public website (www.ocr.org.uk) after the live examination series.

If OCR has unwittingly failed to correctly acknowledge or clear any third-party content in this assessment material, OCR will be happy to correct its mistake at the earliest possible opportunity.

For queries or further information please contact The OCR Copyright Team, The Triangle Building, Shaftesbury Road, Cambridge CB2 8EA.

OCR is part of the Cambridge Assessment Group; Cambridge Assessment is the brand name of University of Cambridge Local Examinations Syndicate (UCLES), which is itself a department of the University of Cambridge.

© OCR 2020

stemup.co.uk

Surname	StemUp Model Solution
Answer booklet	Write the question number in the left-hand margin at the start of each answer.

Question number

1

Let x m be the extension of the string at the lowest point.

P falls a total distance (3 + x) m and is at rest at O and at the lowest point.

Loss of GPE = $0.5g(3 + x) = 4.9(3 + x)$

Gain in EPE = $\frac{\lambda x^2}{2l} = \frac{75x^2}{2(3)} = 12.5x^2$

Conservation of energy: $12.5x^2 = 4.9(3 + x)$

$12.5x^2 - 4.9x - 14.7 = 0$

$x = \frac{4.9 \pm \sqrt{4.9^2 + 4(12.5)(14.7)}}{25}$

$x = 1.298...$ or $x = -0.906...$ (reject, since $x > 0$)

Length of string = 3 + 1.298... = 4.30 m (3 s.f.)

2

$[f] = T^{-1}$, $[C] = MLT^{-2}$, $[l] = L$, $[\sigma] = ML^{-1}$

$T^{-1} = (MLT^{-2})^{\alpha} L^{\beta} (ML^{-1})^{\gamma}$

M: $0 = \alpha + \gamma$

L: $0 = \alpha + \beta - \gamma$

T: $-1 = -2\alpha$

$\Rightarrow \alpha = \frac{1}{2}$, $\gamma = -\frac{1}{2}$, $\beta = \gamma - \alpha = -1$

$\alpha = \frac{1}{2}$, $\beta = -1$, $\gamma = -\frac{1}{2}$

3 (a)

Let $F = ai + bj$. The forces form a couple, so their resultant is zero:

$(2i + j) + (-3i + 2j) + (ai + bj) = 0$

i: $2 - 3 + a = 0 \Rightarrow a = 1$

Question number

j: $1 + 2 + b = 0 \Rightarrow b = -3$

$F = i - 3j$

$|F| = \sqrt{1^2 + 3^2} = \sqrt{10} = 3.16$ N (3 s.f.)

Direction: $\tan^{-1}(3) = 71.6°$ below the positive x-direction (i direction)

3 (b) Moments about O (anticlockwise +), moment of (X, Y) at (x, y) is $xY - yX$:

Force at A(2, 3): $2(1) - 3(2) = -4$

Force at B(−2, 1): $(-2)(2) - 1(-3) = -1$

F acts at O, so has no moment about O.

Total moment of the couple = −5, i.e. 5 Nm clockwise

Additional couple needed: magnitude 5 Nm, anticlockwise

4 (a) $v = \frac{dr}{dt} = 20i + (10 - 10t)j$

At $t = 0$: $v = 20i + 10j$ m s^{-1}

4 (b) Perpendicular to initial direction ⇒ scalar product is zero:

$$\begin{pmatrix} 20 \\ 10 \end{pmatrix} \cdot \begin{pmatrix} 20 \\ 10 - 10T \end{pmatrix} = 0$$

$400 + 10(10 - 10T) = 0$

$500 - 100T = 0$

$T = 5$

4 (c) $t = 0$: $r = 5i + 95j$

$t = 5$: $r = (5 + 100)i + (95 + 50 - 125)j = 105i + 20j$

Displacement = $(105i + 20j) - (5i + 95j) = 100i - 75j$

Distance = $\sqrt{100^2 + 75^2} = 125$ m

Question number

5 (a) Driving force $= \frac{P}{v} = \frac{60000}{v}$

N2L along the road: $\frac{60000}{v} - 1500 = 900\frac{dv}{dt}$

Divide by 300: $\frac{200}{v} - 5 = 3\frac{dv}{dt}$

Multiply by v: $200 - 5v = 3v\frac{dv}{dt}$

Divide by 5: $\frac{3v}{5}\frac{dv}{dt} = 40 - v$ as required

5 (b) Initial condition: when $v = 0$, $t = 24\ln(\frac{40}{40}) - 0 = 24\ln 1 = 0$ ✓

$t = 24\ln 40 - 24\ln(40 - v) - \frac{3}{5}v$

$\frac{dt}{dv} = -24 \times \frac{-1}{40 - v} - \frac{3}{5} = \frac{24}{40 - v} - \frac{3}{5}$

$\frac{dt}{dv} = \frac{120 - 3(40 - v)}{5(40 - v)} = \frac{3v}{5(40 - v)}$

$\Rightarrow \frac{dv}{dt} = \frac{5(40 - v)}{3v}$

$\Rightarrow \frac{3v}{5}\frac{dv}{dt} = 40 - v$, which is the equation from part (a)

So $t = 24\ln(\frac{40}{40 - v}) - \frac{3}{5}v$ satisfies the equation and $v = 0$ at $t = 0$, as required.

Question number

6 (a) Time to first bounce: $78.4 = \frac{1}{2}(9.8)t_1^2 \Rightarrow t_1^2 = 16 \Rightarrow t_1 = 4$ s

Speed at first impact: $v_1 = 9.8 \times 4 = 39.2$ m s^{-1}

Speed after first bounce = $39.2e$

Time in the air until next bounce ($s = 0$): $0 = 39.2e\,t - 4.9t^2$

$\Rightarrow t = \frac{39.2e}{4.9} = 8e$

Each bounce multiplies the speed by e, so the times in the air are $8e$, $8e^2$, $8e^3$, ...

Total time: $4 + 8e + 8e^2 + 8e^3 + \ldots = 6$

$4 + \frac{8e}{1-e} = 6$ (sum to infinity of a GP, $0 < e < 1$)

$8e = 2(1 - e)$

$10e = 2 \Rightarrow e = 0.2$

6 (b) Impulse = change in momentum (upwards +):

$23.52 = m(39.2 \times 0.2) - m(-39.2) = m(7.84 + 39.2) = 47.04m$

$m = \frac{23.52}{47.04} = 0.5$

7 (a) Let e be the extension at A. Hooke: $T = \frac{3mg\,e}{6a} = \frac{mge}{2a}$

Equilibrium at A, resolving along the plane: $T = mg \sin 30° = \frac{mg}{2}$

$\frac{mge}{2a} = \frac{mg}{2} \Rightarrow e = a$

At displacement x down the plane from A, extension = $a + x$, so

$T = \frac{3mg(a+x)}{6a} = \frac{mg(a+x)}{2a}$

Question number

N2L down the plane: $mg \sin 30° - T = m\frac{d^2x}{dt^2}$

$\frac{mg}{2} - \frac{mg(a + x)}{2a} = m\frac{d^2x}{dt^2}$

$\frac{mg}{2} - \frac{mg}{2} - \frac{mgx}{2a} = m\frac{d^2x}{dt^2}$

$\frac{d^2x}{dt^2} + \frac{gx}{2a} = 0$ as required

7 (b) SHM with $\omega^2 = \frac{g}{2a}$, released from rest at $x = 2a$ (amplitude $2a$):

$x = 2a\cos\left(\sqrt{\frac{g}{2a}}\ t\right)$

The string slackens when the extension is zero, i.e. $x = -a$:

$-a = 2a\cos\left(\sqrt{\frac{g}{2a}}\ t\right) \Rightarrow \cos\left(\sqrt{\frac{g}{2a}}\ t\right) = -\frac{1}{2}$

$\sqrt{\frac{g}{2a}}\ t = \frac{2\pi}{3}$ (first time)

$t = \frac{2\pi}{3}\sqrt{\frac{2a}{g}}$

7 (c) $v^2 = \omega^2(A^2 - x^2) = \frac{g}{2a}((2a)^2 - (-a)^2) = \frac{g}{2a} \times 3a^2$

$v^2 = \frac{3ga}{2} \Rightarrow$ speed $= \sqrt{\frac{3ga}{2}}$

8 (a) Place the vertex at the origin with the axis along the x-axis.

Question number

The slant edge is the line $y = \frac{r}{h}x$, for $0 \leq x \leq h$, rotated about the x-axis.

$$V\bar{x} = \pi\int_0^h xy^2\,dx = \pi\int_0^h x(\tfrac{r}{h}x)^2\,dx = \frac{\pi r^2}{h^2}\int_0^h x^3\,dx$$

$$V\bar{x} = \frac{\pi r^2}{h^2}\left[\frac{x^4}{4}\right]_0^h = \frac{\pi r^2}{h^2} \times \frac{h^4}{4} = \frac{\pi r^2 h^2}{4}$$

$$V = \frac{1}{3}\pi r^2 h, \text{ so } \frac{1}{3}\pi r^2 h\bar{x} = \frac{\pi r^2 h^2}{4}$$

$\bar{x} = \frac{3h}{4}$ from the vertex, as required

8 (b) Distances measured from the vertex of the cone along the axis of symmetry:

Cone: volume $\frac{1}{3}\pi r^3$, centre of mass at $\frac{3}{4}r$

Cylinder: volume $\pi r^2(2r) = 2\pi r^3$, centre of mass at $r + r = 2r$

Toy: volume $\frac{1}{3}\pi r^3 + 2\pi r^3 = \frac{7}{3}\pi r^3$, centre of mass at $\bar{x}$

$$\frac{7}{3}\pi r^3\bar{x} = \frac{1}{3}\pi r^3(\tfrac{3r}{4}) + 2\pi r^3(2r) = \frac{17}{4}\pi r^4$$

$$\bar{x} = \frac{17}{4} \times \frac{3}{7}r = \frac{51}{28}r$$

The cone has height r and radius r, so its semi-vertical angle is 45°.

The cone touches the floor along a slant edge of length $\sqrt{r^2 + r^2} = r\sqrt{2}$,

so the axis of the toy makes an angle of 45° with the floor.

Horizontal distance of G from the vertex (along the line of contact):

$$\frac{51}{28}r\cos 45° = \frac{51\sqrt{2}}{56}r = 1.29r \text{ (3 s.f.)}$$

Slant edge in contact with floor $= r\sqrt{2} = 1.41r$

Question number

$\frac{51\sqrt{2}}{56}r < r\sqrt{2}$, so the vertical through G passes through the line of contact with the floor. The toy does not topple.

8 (c) Any friction acts at the points of contact with the floor, so its moment about the line of contact is zero; it cannot affect whether the toy topples. (Only the weight has a turning effect about the edge of contact.)

9 (a) AB = BC = 2a, so the string BC also makes angle θ with the horizontal.

Taking moments about A for the rod (T along BC):

$8W \times a\cos\theta = (2a\cos\theta)(T\sin\theta) + (2a\sin\theta)(T\cos\theta)$

$8Wa\cos\theta = 4aT\sin\theta\cos\theta$

$T = \frac{2W}{\sin\theta} = 2W\operatorname{cosec}\theta$

9 (b) For the ring, R = normal reaction of bar, F = friction.

Resolving vertically: $R = W + T\sin\theta = W + 2W = 3W$

Resolving horizontally: $F = T\cos\theta = 2W\cot\theta$

For equilibrium: $F \leq \mu R \Rightarrow 2W\cot\theta \leq 3\mu W$

$\cot\theta \leq \frac{3\mu}{2}$, i.e. $\tan\theta \geq \frac{2}{3\mu}$

Distance of ring from A = AC = $4a\cos\theta$, which is greatest when θ is least,

i.e. when $\tan\theta$ takes its least value, $\tan\theta = \frac{2}{3\mu}$ (limiting friction).

$4a\cos\theta = 2.4a \Rightarrow \cos\theta = 0.6,\ \sin\theta = 0.8,\ \tan\theta = \frac{4}{3}$

$\frac{2}{3\mu} = \frac{4}{3} \Rightarrow \mu = \frac{1}{2}$

Question number

10 (a) Taking A as the zero level of GPE. Just sufficient to reach the top,

so at the highest point $v = 0$: energy $= mg(2a)$

At angle θ: height above A $= a(1 - \cos\theta)$

Conservation of energy: $2mga = mga(1 - \cos\theta) + \frac{1}{2}mv^2$

$v^2 = 2ga(1 + \cos\theta)$

N2L towards O (R towards O): $R - mg\cos\theta = \frac{mv^2}{a}$

$R = mg\cos\theta + 2mg(1 + \cos\theta) = mg(2 + 3\cos\theta)$

$mg(2 + 3\cos\theta) = \frac{7}{2}mg \Rightarrow \cos\theta = \frac{1}{2}$

$\theta = \frac{\pi}{3}$ $(60°)$

10 (b) $v = a\omega \Rightarrow a^2\omega^2 = 2ga(1 + \cos\theta)$

Using $1 + \cos\theta = 2\cos^2(\frac{\theta}{2})$: $a^2\omega^2 = 4ga\cos^2(\frac{\theta}{2})$

$\omega^2 = \frac{4g}{a}\cos^2(\frac{\theta}{2})$

$\omega = 2\sqrt{\frac{g}{a}}\cos(\frac{\theta}{2})$, so $k = 2$

10 (c) Angular acceleration $= \frac{d\omega}{dt} = \frac{d\omega}{d\theta} \times \frac{d\theta}{dt} = \omega\frac{d\omega}{d\theta}$

$\frac{d\omega}{d\theta} = -\sqrt{\frac{g}{a}}\sin(\frac{\theta}{2})$

Question number

$\frac{d\omega}{dt} = -\sqrt{\frac{g}{a}}\sin\left(\frac{\theta}{2}\right) \times 2\sqrt{\frac{g}{a}}\cos\left(\frac{\theta}{2}\right) = -\frac{g}{a}\sin\theta$

At $\theta = \frac{\pi}{3}$: $\frac{d\omega}{dt} = -\frac{g}{a} \times \frac{\sqrt{3}}{2} = -\frac{\sqrt{3}g}{2a}$

Angular acceleration $= -\frac{\sqrt{3}g}{2a}$ (a deceleration of $\frac{\sqrt{3}g}{2a}$ rad s^{-2})

11 (a) Let both masses be m, A has initial speed u. Along the line of centres,

v_1 and v_2 are the velocity components of A and B after impact.

Conservation of momentum: $mu\cos\alpha = mv_1 + mv_2$

Newton's experimental law: $v_2 - v_1 = \frac{1}{3}u\cos\alpha$

Subtracting: $2v_1 = \frac{2}{3}u\cos\alpha \Rightarrow v_1 = \frac{1}{3}u\cos\alpha$

Perpendicular to the line of centres, A's component is unchanged: $u\sin\alpha$

$\tan\beta = (u\sin\alpha) \div v_1 = (u\sin\alpha) \div (\frac{1}{3}u\cos\alpha)$

$\tan\beta = 3\tan\alpha$ as required

11 (b) Smooth contact means the impulse acts only along the line of centres, so

the component of A's velocity perpendicular to the line of centres is unchanged.

11 (c) Since $\tan\beta > \tan\alpha$, $\beta > \alpha$ and the deflection is $\gamma = \beta - \alpha$.

$\tan\gamma = \tan(\beta - \alpha) = \frac{\tan\beta - \tan\alpha}{1 + \tan\beta\tan\alpha} = \frac{3\tan\alpha - \tan\alpha}{1 + 3\tan^2\alpha}$

$\tan\gamma = \frac{2\tan\alpha}{1 + 3\tan^2\alpha}$

Question number

11 (d)

Differentiating with respect to α:

$$\sec^2\gamma\frac{d\gamma}{d\alpha} = \frac{(1+3\tan^2\alpha)(2\sec^2\alpha) - (2\tan\alpha)(6\tan\alpha\sec^2\alpha)}{(1+3\tan^2\alpha)^2}$$

For maximum γ, $\frac{d\gamma}{d\alpha} = 0$: $2\sec^2\alpha(1+3\tan^2\alpha - 6\tan^2\alpha) = 0$

$\sec^2\alpha \neq 0$, so $1 - 3\tan^2\alpha = 0 \Rightarrow \tan\alpha = \frac{1}{\sqrt{3}}$ (α acute), $\alpha = \frac{\pi}{6}$

$$\tan\gamma = \frac{2/\sqrt{3}}{1+3(1/3)} = \frac{2/\sqrt{3}}{2} = \frac{1}{\sqrt{3}}$$

Maximum value of $\gamma = \frac{\pi}{6}$ ($30°$)

12 (a)

Triangle OAP is isosceles (OA = OP = r), so $\angle OPA = \angle OAP = \alpha$ and

$l = 2r\cos\alpha$

Radius of the horizontal circle: $x = l\sin\alpha = 2r\sin\alpha\cos\alpha$

Let R act along PO, making angle θ with the horizontal; T along PA.

Height of P above A $= l\cos\alpha = 2r\cos^2\alpha$, and O is r above A, so

$$\sin\theta = \frac{r - 2r\cos^2\alpha}{r} = 1 - 2\cos^2\alpha, \quad \cos\theta = \frac{x}{r} = 2\sin\alpha\cos\alpha$$

Resolving vertically: $R\sin\theta = T\cos\alpha + mg$

$\Rightarrow R(1 - 2\cos^2\alpha) = T\cos\alpha + mg$ ...(1)

N2L horizontally (towards the axis): $R\cos\theta + T\sin\alpha = mx\omega^2$

$\Rightarrow 2R\sin\alpha\cos\alpha + T\sin\alpha = m(2r\sin\alpha\cos\alpha)\omega^2$

Divide by $\sin\alpha$: $T = 2mr\omega^2\cos\alpha - 2R\cos\alpha$ ...(2)

Substitute (2) into (1):

$R - 2R\cos^2\alpha = (2mr\omega^2\cos\alpha - 2R\cos\alpha)\cos\alpha + mg$

$R - 2R\cos^2\alpha = 2mr\omega^2\cos^2\alpha - 2R\cos^2\alpha + mg$

$R = mg + 2mr\omega^2\cos^2\alpha$ as required

Question number

12 (b)

From (2): $T = 2\cos\alpha(mr\omega^2 - R) = 2\cos\alpha(mr\omega^2 - mg - 2mr\omega^2\cos^2\alpha)$

The string is taut, so $T > 0$, and $\cos\alpha > 0$:

$mr\omega^2 - mg - 2mr\omega^2\cos^2\alpha > 0$

$g < r\omega^2(1 - 2\cos^2\alpha)$

$k_1 = 1, \quad k_2 = -2$