OCR
Oxford Cambridge and RSA

Monday 05 October 2020 – Afternoon

A Level Further Mathematics B (MEI)

Y420/01 Core Pure

Time allowed: 2 hours 40 minutes

8178476396

You must have:
- the Printed Answer Booklet
- the Formulae Booklet for Further Mathematics B (MEI)
- a scientific or graphical calculator

INSTRUCTIONS
- Use black ink. You can use an HB pencil, but only for graphs and diagrams.
- Write your answer to each question in the space provided in the **Printed Answer Booklet**. If you need extra space use the lined pages at the end of the Printed Answer Booklet. The question numbers must be clearly shown.
- Fill in the boxes on the front of the Printed Answer Booklet.
- Answer **all** the questions.
- Where appropriate, your answer should be supported with working. Marks might be given for using a correct method, even if your answer is wrong.
- Give your final answers to a degree of accuracy that is appropriate to the context.
- Do **not** send this Question Paper for marking. Keep it in the centre or recycle it.

INFORMATION
- The total mark for this paper is **144**.
- The marks for each question are shown in brackets **[ ]**.
- This document has **8** pages.

ADVICE
- Read each question carefully before you start your answer.

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Section A (36 marks)

Answer **all** the questions.

1 Using standard summation of series formulae, determine the sum of the first n terms of the series

$(1\times2\times4)+(2\times3\times5)+(3\times4\times6)+\ldots$,

where n is a positive integer. Give your answer in fully factorised form. **[6]**

2 **(a)** The matrices $\mathbf{M}=\begin{pmatrix}0 & 1 & a\\ 1 & b & 0\end{pmatrix}$ and $\mathbf{N}=\begin{pmatrix}b & -5\\ -1 & c\\ -1 & 1\end{pmatrix}$ are such that $\mathbf{MN}=\mathbf{I}$.

Find a, b and c. **[5]**

(b) State with a reason whether or not **N** is the inverse of **M**. **[1]**

3 **In this question you must show detailed reasoning.**

Find $\displaystyle\int_0^{\frac{1}{3}}\frac{1}{\sqrt{4-9x^2}}\,\mathrm{d}x$, expressing your answer in terms of π. **[4]**

4 The roots of the equation $2x^3-5x+7=0$ are α, β and γ.

(a) Find $\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma}$. **[4]**

(b) Find an equation with integer coefficients whose roots are $2\alpha-1$, $2\beta-1$ and $2\gamma-1$. **[4]**

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5 Fig. 5 shows the curve with polar equation $r = a(3+2\cos\theta)$ for $-\pi \leqslant \theta \leqslant \pi$, where a is a constant.

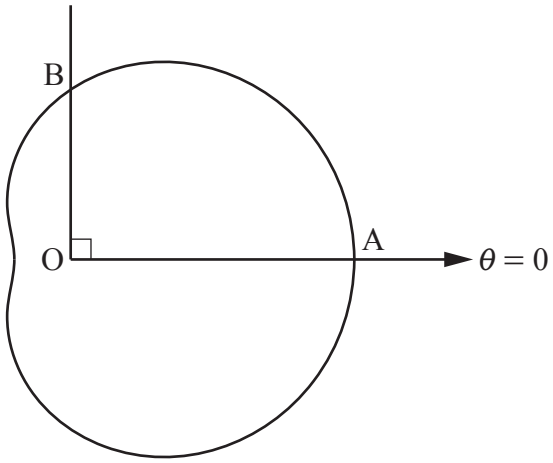


Fig. 5

(a) Write down the polar coordinates of the points A and B. **[2]**

(b) Explain why the curve is symmetrical about the initial line. **[2]**

(c) **In this question you must show detailed reasoning.**

Find in terms of a the exact area of the region enclosed by the curve. **[4]**

6 The complex number z satisfies the equation $z^2 - 4\mathrm{i}z^* + 11 = 0$.

Given that $\mathrm{Re}(z) > 0$, find z in the form $a + b\mathrm{i}$, where a and b are real numbers. **[4]**

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Section B (108 marks)

Answer **all** the questions.

7 Prove by mathematical induction that $\sum_{r=1}^{n}(r \times r!) = (n+1)! - 1$ for all positive integers n. **[6]**

8 **(a)** Given that the lines $\mathbf{r} = \begin{pmatrix} 0 \\ 2 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 1 \\ 3 \end{pmatrix}$ and $\mathbf{r} = \begin{pmatrix} -1 \\ 2 \\ k \end{pmatrix} + \mu \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix}$ meet, determine k. **[5]**

(b) **In this question you must show detailed reasoning.**

Find the acute angle between the two lines. **[4]**

9 A linear transformation of the plane is represented by the matrix $\mathbf{M} = \begin{pmatrix} 1 & -2 \\ \lambda & 3 \end{pmatrix}$, where λ is a constant.

(a) Find the set of values of λ for which the linear transformation has no invariant lines through the origin. **[5]**

(b) Given that the transformation multiplies areas by 5 and reverses orientation, find the invariant lines. **[3]**

10 **In this question you must show detailed reasoning.**

The region in the first quadrant bounded by curve $y = \cosh \frac{1}{2}x^2$, the y-axis, and the line $y = 2$ is rotated through 360° about the y-axis.

Find the exact volume of revolution generated, expressing your answer in a form involving a logarithm. **[7]**

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11 In this question you must show detailed reasoning.

In Fig. 11, the points A, B, C, D, E and F represent the complex sixth roots of 64 on an Argand diagram. The midpoints of AB, BC, CD, DE, EF and FA are G, H, I, J, K and L respectively.

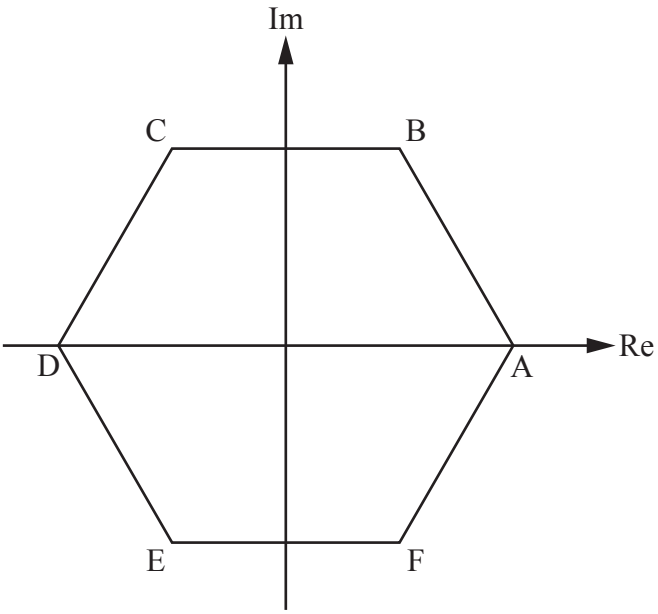


Fig. 11

(a) Write down, in exponential ($re^{i\theta}$) form, the complex numbers represented by the points A, B, C, D, E and F. **[2]**

(b) When these complex numbers are multiplied by the complex number w, the resulting complex numbers are represented by the points G, H, I, J, K and L.

Find w in exponential form. **[4]**

(c) You are given that G, H, I, J, K and L represent roots of the equation $z^6 = p$.

Find p. **[2]**

12 (a) Given that $z = \cos\theta + i\sin\theta$, express $z^n + \frac{1}{z^n}$ and $z^n - \frac{1}{z^n}$ in simplified trigonometric form. **[2]**

(b) By considering $\left(z + \frac{1}{z}\right)^3\left(z - \frac{1}{z}\right)^3$, find constants A and B such that

$\sin^3\theta\cos^3\theta = A\sin 6\theta + B\sin 2\theta$. **[6]**

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13 **(a)** Using exponentials, prove that $\sinh 2x = 2\cosh x \sinh x$. **[2]**

(b) Hence show that if $f(x) = \sinh^2 x$, then $f''(x) = 2\cosh 2x$. **[2]**

(c) Explain why the coefficients of odd powers in the Maclaurin series for $\sinh^2 x$ are all zero. **[2]**

(d) Find the coefficient of x^n in this series when n is a positive even number. **[3]**

14 Solve the simultaneous differential equations

$$\frac{dx}{dt} + 2x = 4y, \qquad \frac{dy}{dt} + 3x = 5y,$$

given that when $t = 0$, $x = 0$ and $y = 1$. **[11]**

15 **(a)** Show that the three planes with equations

$$\begin{aligned} x + \lambda y + 3z &= -12 \\ 2x + y + 5z &= -11 \\ x - 2y + 2z &= -9 \end{aligned}$$

where λ is a constant, meet at a unique point except for one value of λ which is to be determined. **[3]**

(b) In the case $\lambda = -2$, use matrices to find the point of intersection P of the planes, showing your method clearly. **[3]**

The line l has equation $\frac{x-1}{2} = \frac{y-1}{-1} = \frac{z+2}{-2}$.

(c) Find a vector equation of l. **[2]**

(d) Find the shortest distance between the point P and l. **[4]**

(e) **(i)** Show that l is parallel to the plane $x - 2y + 2z = -9$. **[3]**

(ii) Find the distance between l and the plane $x - 2y + 2z = -9$. **[2]**

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16 The population density P, in suitable units, of a certain bacterium at time t hours is to be modelled by a differential equation. Initially, the population density is zero, and its long-term value is A.

(a) One simple model is to assume that the rate of change of population density is directly proportional to $A - P$.

(i) Formulate a differential equation for this model. **[1]**

(ii) Verify that $P = A(1 - \mathrm{e}^{-kt})$, where k is a positive constant, satisfies

- this differential equation,
- the initial condition,
- the long-term condition. **[3]**

An alternative model uses the differential equation

$$\frac{\mathrm{d}P}{\mathrm{d}t} - \frac{P}{t(1+t^2)} = \mathrm{Q}(t),$$

where $\mathrm{Q}(t)$ is a function of t.

(b) Find the integrating factor for this differential equation, showing that it can be written in the form $\dfrac{\sqrt{1+t^2}}{t}$. **[8]**

(c) Suppose that $\mathrm{Q}(t) = 0$.

(i) Show that $P = \dfrac{At}{\sqrt{1+t^2}}$. **[4]**

(ii) Find the time predicted by this model for the population density to reach half its long-term value. Give your answer correct to the nearest minute. **[2]**

(d) Now suppose that $\mathrm{Q}(t) = \dfrac{t\mathrm{e}^{-t}}{\sqrt{1+t^2}}$.

Show that $P = \dfrac{At - t\mathrm{e}^{-t}}{\sqrt{1+t^2}}$. [You may assume that $\lim\limits_{t\to\infty} t\mathrm{e}^{-t} = 0$.] **[5]**

It is found that the long-term value of P is 10, and P reaches half this value after 37 minutes.

(e) Determine which of the models proposed in parts **(c)** and **(d)** is more consistent with these data. **[2]**

END OF QUESTION PAPER

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Surname	StemUp Model Solution
Answer booklet	Write the question number in the left-hand margin at the start of each answer.

Question number

1

rth term $= r(r+1)(r+3) = r^3 + 4r^2 + 3r$

$$\sum_{r=1}^{n} r(r+1)(r+3) = \sum r^3 + 4\sum r^2 + 3\sum r$$

$$= \frac{1}{4}n^2(n+1)^2 + \frac{4}{6}n(n+1)(2n+1) + \frac{3}{2}n(n+1)$$

$$= \frac{1}{12}n(n+1)[3n(n+1) + 8(2n+1) + 18]$$

$$= \frac{1}{12}n(n+1)(3n^2 + 3n + 16n + 8 + 18)$$

$$= \frac{1}{12}n(n+1)(3n^2 + 19n + 26)$$

$$= \frac{1}{12}n(n+1)(n+2)(3n+13)$$

2 (a)

$$MN = \begin{pmatrix} 0 & 1 & a \\ 1 & b & 0 \end{pmatrix}\begin{pmatrix} b & -5 \\ -1 & c \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} -1-a & c+a \\ b-b & -5+bc \end{pmatrix}$$

$$= \begin{pmatrix} -1-a & c+a \\ 0 & bc-5 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$-1 - a = 1 \Rightarrow a = -2$

$c + a = 0 \Rightarrow c = 2$

$bc - 5 = 1 \Rightarrow 2b = 6 \Rightarrow b = 3$

$a = -2,\ b = 3,\ c = 2$

2 (b)

N is not the inverse of M: M is a 2 × 3 matrix, which is not

Question number

square, so M has no inverse (NM is 3 × 3, so NM ≠ I).

3 $\int_0^{\frac{1}{3}} \frac{1}{\sqrt{4-9x^2}}\,dx = \int_0^{\frac{1}{3}} \frac{1}{3\sqrt{\frac{4}{9}-x^2}}\,dx$

$= \frac{1}{3}\left[\arcsin\left(\frac{3x}{2}\right)\right]_0^{\frac{1}{3}}$

$= \frac{1}{3}\left(\arcsin\frac{1}{2} - \arcsin 0\right)$

$= \frac{1}{3} \times \frac{\pi}{6} = \frac{\pi}{18}$

4 (a) $\alpha + \beta + \gamma = 0, \quad \alpha\beta + \beta\gamma + \gamma\alpha = \frac{-5}{2}, \quad \alpha\beta\gamma = -\frac{7}{2}$

$\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} = \frac{\alpha\beta + \beta\gamma + \gamma\alpha}{\alpha\beta\gamma}$

$= \frac{-\frac{5}{2}}{-\frac{7}{2}} = \frac{5}{7}$

4 (b) Let $y = 2x - 1$, so $x = \frac{y+1}{2}$

$2\left(\frac{y+1}{2}\right)^3 - 5\left(\frac{y+1}{2}\right) + 7 = 0$

$\frac{(y+1)^3}{4} - \frac{5(y+1)}{2} + 7 = 0$

$(y+1)^3 - 10(y+1) + 28 = 0$

$y^3 + 3y^2 + 3y + 1 - 10y - 10 + 28 = 0$

Question number

$y^3 + 3y^2 - 7y + 19 = 0$

Required equation: $x^3 + 3x^2 - 7x + 19 = 0$

5 (a) A: $\theta = 0$, $r = a(3 + 2) = 5a$, A is $(5a, 0)$

B: $\theta = \frac{\pi}{2}$, $r = a(3 + 0) = 3a$, B is $(3a, \frac{\pi}{2})$

5 (b) $\cos(-\theta) = \cos\theta$, so $r(-\theta) = a(3 + 2\cos\theta) = r(\theta)$.

So the point $(r, -\theta)$ is on the curve whenever (r, θ) is, and

$(r, -\theta)$ is the reflection of (r, θ) in the initial line.

5 (c) Area $= 2 \times \frac{1}{2}\int_0^{\pi} a^2(3 + 2\cos\theta)^2 \, d\theta$ (by symmetry)

$= a^2\int_0^{\pi} (9 + 12\cos\theta + 4\cos^2\theta) \, d\theta$

$4\cos^2\theta = 2(1 + \cos 2\theta) = 2 + 2\cos 2\theta$

$= a^2\int_0^{\pi} (11 + 12\cos\theta + 2\cos 2\theta) \, d\theta$

$= a^2[11\theta + 12\sin\theta + \sin 2\theta]_0^{\pi}$

$= a^2[(11\pi + 0 + 0) - 0]$

$= 11\pi a^2$

6 Let $z = a + bi$, so $z^* = a - bi$

$(a + bi)^2 - 4i(a - bi) + 11 = 0$

$a^2 - b^2 + 2abi - 4ai - 4b + 11 = 0$

Imaginary: $2ab - 4a = 0 \Rightarrow 2a(b - 2) = 0$

$\mathrm{Re}(z) > 0$ so $a \neq 0 \Rightarrow b = 2$

Real: $a^2 - b^2 - 4b + 11 = 0 \Rightarrow a^2 - 4 - 8 + 11 = 0$

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$a^2 = 1$, and $a > 0 \Rightarrow a = 1$

$z = 1 + 2i$

7 When $n = 1$: LHS $= 1 \times 1! = 1$, RHS $= 2! - 1 = 1$

So the result is true for $n = 1$.

Assume true for $n = k$: $\sum_{r=1}^{k} (r \times r!) = (k + 1)! - 1$

Then $\sum_{r=1}^{k+1} (r \times r!) = (k + 1)! - 1 + (k + 1)(k + 1)!$

$= (k + 1)!(1 + k + 1) - 1$

$= (k + 2)(k + 1)! - 1$

$= (k + 2)! - 1 = ((k + 1) + 1)! - 1$

which is the result for $n = k + 1$.

So if the result is true for $n = k$, it is true for $n = k + 1$.

Since it is true for $n = 1$, by induction it is true for all positive integers n.

8 (a) At the meeting point: $\begin{pmatrix} -\lambda \\ 2 + \lambda \\ 2 + 3\lambda \end{pmatrix} = \begin{pmatrix} -1 + 2\mu \\ 2 + 3\mu \\ k + 4\mu \end{pmatrix}$

y: $2 + \lambda = 2 + 3\mu \Rightarrow \lambda = 3\mu$

x: $-\lambda = -1 + 2\mu \Rightarrow -3\mu = -1 + 2\mu$

$\Rightarrow \mu = \frac{1}{5}, \ \lambda = \frac{3}{5}$

z: $2 + 3(\frac{3}{5}) = k + 4(\frac{1}{5})$

$\frac{19}{5} = k + \frac{4}{5} \Rightarrow k = 3$

Question number

8 (b) Directions: $\begin{pmatrix} -1 \\ 1 \\ 3 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix}$

Dot product $= -2 + 3 + 12 = 13$

Magnitudes: $\sqrt{(1 + 1 + 9)} = \sqrt{11}$, $\sqrt{(4 + 9 + 16)} = \sqrt{29}$

$\cos\theta = \frac{13}{\sqrt{11}\sqrt{29}} = \frac{13}{\sqrt{319}} = 0.7279...$

$\theta = 43.3°$ (acute, since $\cos\theta > 0$)

9 (a) Suppose the line $y = mx$ is invariant.

$$\begin{pmatrix} 1 & -2 \\ \lambda & 3 \end{pmatrix}\begin{pmatrix} x \\ mx \end{pmatrix} = \begin{pmatrix} x - 2mx \\ \lambda x + 3mx \end{pmatrix}$$

Image lies on $y = mx$: $\lambda x + 3mx = m(x - 2mx)$

$\lambda + 3m = m - 2m^2$

$2m^2 + 2m + \lambda = 0$

The line $x = 0$: $\begin{pmatrix} 1 & -2 \\ \lambda & 3 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -2 \\ 3 \end{pmatrix}$, not on $x = 0$,

so $x = 0$ is never invariant.

No invariant lines $\Leftrightarrow$ no real m $\Leftrightarrow$ discriminant < 0

$2^2 - 4(2)(\lambda) < 0 \Rightarrow 4 - 8\lambda < 0$

$\lambda > \frac{1}{2}$

9 (b) Area $\times 5$ and orientation reversed $\Rightarrow \det M = -5$

$\det M = 3 + 2\lambda = -5 \Rightarrow \lambda = -4$

$2m^2 + 2m - 4 = 0 \Rightarrow m^2 + m - 2 = 0$

$(m + 2)(m - 1) = 0 \Rightarrow m = 1$ or $m = -2$

Question number

Invariant lines: $y = x$ and $y = -2x$

10

$y = \cosh \frac{1}{2}x^2 \Rightarrow \frac{1}{2}x^2 = \operatorname{arcosh} y \Rightarrow x^2 = 2\operatorname{arcosh} y$

Region: from $y = 1$ (at $x = 0$) up to $y = 2$

$V = \pi\int_1^2 x^2\,dy = 2\pi\int_1^2 \operatorname{arcosh} y\,dy$

By parts: $u = \operatorname{arcosh} y$, $u' = \frac{1}{\sqrt{y^2-1}}$, $v' = 1$, $v = y$

$\int_1^2 \operatorname{arcosh} y\,dy = [y \operatorname{arcosh} y]_1^2 - \int_1^2 \frac{y}{\sqrt{y^2-1}}\,dy$

$= (2\operatorname{arcosh} 2 - \operatorname{arcosh} 1) - [\sqrt{y^2-1}]_1^2$

$= 2\operatorname{arcosh} 2 - 0 - (\sqrt{3} - 0)$

$\operatorname{arcosh} 2 = \ln(2 + \sqrt{(2^2-1)}) = \ln(2+\sqrt{3})$

$V = 2\pi(2\ln(2+\sqrt{3}) - \sqrt{3})$

11 (a)

$z^6 = 64 = 64e^{2k\pi i} \Rightarrow z = 2e^{k\pi i/3}$

$A = 2e^{0i} = 2$, $B = 2e^{i\pi/3}$, $C = 2e^{2i\pi/3}$

$D = 2e^{i\pi}$, $E = 2e^{4i\pi/3}$, $F = 2e^{5i\pi/3}$

11 (b)

$G = \frac{1}{2}(A + B) = \frac{1}{2}(2 + 1 + \sqrt{3}i) = \frac{3}{2} + \frac{\sqrt{3}}{2}i$

$|G| = \sqrt{\frac{9}{4} + \frac{3}{4}} = \sqrt{3}$, $\arg G = \arctan\left(\frac{\sqrt{3}/2}{3/2}\right) = \frac{\pi}{6}$

$G = \sqrt{3}e^{i\pi/6}$ and $A = 2$, with $Aw = G$

$w = \frac{G}{A} = \frac{\sqrt{3}}{2}e^{i\pi/6}$

(Each of A to F has modulus 2 and is rotated by $\pi/6$ and scaled

Question number

by $\sqrt{3}/2$ to give the corresponding midpoint, so this w works for all.)

11 (c) $p = G^6 = (\sqrt{3}e^{i\pi/6})^6$

$= (\sqrt{3})^6 e^{i\pi} = 27 \times (-1)$

$p = -27$

12 (a) $z^n + \frac{1}{z^n} = 2\cos n\theta$

$z^n - \frac{1}{z^n} = 2i \sin n\theta$

12 (b) $(z + \frac{1}{z})^3(z - \frac{1}{z})^3 = (2\cos\theta)^3(2i \sin\theta)^3 = -64i \cos^3\theta \sin^3\theta$

Also $(z + \frac{1}{z})^3(z - \frac{1}{z})^3 = (z^2 - \frac{1}{z^2})^3$

$= z^6 - 3z^2 + \frac{3}{z^2} - \frac{1}{z^6}$

$= (z^6 - \frac{1}{z^6}) - 3(z^2 - \frac{1}{z^2})$

$= 2i \sin 6\theta - 6i \sin 2\theta$

So $-64i \cos^3\theta \sin^3\theta = 2i \sin 6\theta - 6i \sin 2\theta$

$\sin^3\theta \cos^3\theta = -\frac{1}{32}\sin 6\theta + \frac{3}{32}\sin 2\theta$

$A = -\frac{1}{32}, \quad B = \frac{3}{32}$

13 (a) $2\cosh x \sinh x = 2(\frac{e^x + e^{-x}}{2})(\frac{e^x - e^{-x}}{2})$

Question number

$= \frac{e^{2x} - e^{-2x}}{2}$

$= \sinh 2x$ as required

13 (b) $f(x) = \sinh^2 x$

$f'(x) = 2\sinh x \cosh x = \sinh 2x$ (using (a))

$f''(x) = 2\cosh 2x$ as required

13 (c) $f'''(x) = 4\sinh 2x$, $f^{(4)}(x) = 8\cosh 2x$, $f^{(5)}(x) = 16\sinh 2x$, ...

so every odd derivative is a multiple of sinh 2x

(and $f'(x) = \sinh 2x$ too).

$\sinh 0 = 0$, so $f'(0) = f'''(0) = f^{(5)}(0) = \ldots = 0$

The coefficient of x^n is $\frac{f^{(n)}(0)}{n!}$, so all odd powers

have coefficient zero.

13 (d) For even $n \geq 2$: $f^{(n)}(x) = 2^{n-1}\cosh 2x$

($f'' = 2\cosh 2x$, $f^{(4)} = 8\cosh 2x$, $f^{(6)} = 32\cosh 2x$, ...)

$f^{(n)}(0) = 2^{n-1}\cosh 0 = 2^{n-1}$

Coefficient of $x^n = \frac{2^{n-1}}{n!}$

14 From the first equation: $y = \frac{1}{4}\left(\frac{dx}{dt} + 2x\right)$

$\frac{dy}{dt} = \frac{1}{4}\left(\frac{d^2x}{dt^2} + 2\frac{dx}{dt}\right)$

Substitute into the second: $\frac{dy}{dt} + 3x = 5y$

Question number

$\frac{1}{4}\left(\frac{d^2x}{dt^2} + 2\frac{dx}{dt}\right) + 3x = \frac{5}{4}\left(\frac{dx}{dt} + 2x\right)$

$\frac{d^2x}{dt^2} + 2\frac{dx}{dt} + 12x = 5\frac{dx}{dt} + 10x$

$\frac{d^2x}{dt^2} - 3\frac{dx}{dt} + 2x = 0$

Auxiliary equation: $\lambda^2 - 3\lambda + 2 = 0 \Rightarrow (\lambda - 1)(\lambda - 2) = 0$

$\lambda = 1$ or $\lambda = 2$

$x = Ae^t + Be^{2t}$

$\frac{dx}{dt} = Ae^t + 2Be^{2t}$

$y = \frac{1}{4}(Ae^t + 2Be^{2t} + 2Ae^t + 2Be^{2t}) = \frac{3}{4}Ae^t + Be^{2t}$

$t = 0, x = 0$: $A + B = 0$

$t = 0, y = 1$: $\frac{3}{4}A + B = 1$

Subtract: $-\frac{1}{4}A = 1 \Rightarrow A = -4, \quad B = 4$

$x = 4e^{2t} - 4e^t$

$y = 4e^{2t} - 3e^t$

15 (a) $\det\begin{pmatrix} 1 & \lambda & 3 \\ 2 & 1 & 5 \\ 1 & -2 & 2 \end{pmatrix} = 1(2 + 10) - \lambda(4 - 5) + 3(-4 - 1)$

$= 12 + \lambda - 15 = \lambda - 3$

The planes meet at a unique point when the matrix is

non-singular, ie det $\neq 0$, ie $\lambda \neq 3$.

So there is a unique point except when $\lambda = 3$.

Question number

15 (b)

$$\begin{pmatrix} 1 & -2 & 3 \\ 2 & 1 & 5 \\ 1 & -2 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -12 \\ -11 \\ -9 \end{pmatrix}$$

$\det = -2 - 3 = -5$

$$\text{Inverse} = -\frac{1}{5}\begin{pmatrix} 12 & -2 & -13 \\ 1 & -1 & 1 \\ -5 & 0 & 5 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = -\frac{1}{5}\begin{pmatrix} 12 & -2 & -13 \\ 1 & -1 & 1 \\ -5 & 0 & 5 \end{pmatrix}\begin{pmatrix} -12 \\ -11 \\ -9 \end{pmatrix}$$

$$= -\frac{1}{5}\begin{pmatrix} -144 + 22 + 117 \\ -12 + 11 - 9 \\ 60 + 0 - 45 \end{pmatrix} = -\frac{1}{5}\begin{pmatrix} -5 \\ -10 \\ 15 \end{pmatrix}$$

P is $(1, 2, -3)$

15 (c)

$$r = \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} + t\begin{pmatrix} 2 \\ -1 \\ -2 \end{pmatrix}$$

15 (d)

$A(1, 1, -2)$ on l: $\overrightarrow{AP} = \begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix} - \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$

$$\overrightarrow{AP} \times d = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \times \begin{pmatrix} 2 \\ -1 \\ -2 \end{pmatrix} = \begin{pmatrix} -2 - 1 \\ -2 - 0 \\ 0 - 2 \end{pmatrix} = \begin{pmatrix} -3 \\ -2 \\ -2 \end{pmatrix}$$

$|AP \times d| = \sqrt{(9 + 4 + 4)} = \sqrt{17}, \quad |d| = \sqrt{(4 + 1 + 4)} = 3$

Question number

Shortest distance $= \frac{|AP \times d|}{|d|} = \frac{\sqrt{17}}{3} = 1.37$ (3 s.f.)

15(e)(i) Normal to plane $n = \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix}$, direction of l is $d = \begin{pmatrix} 2 \\ -1 \\ -2 \end{pmatrix}$

$d \cdot n = 2 + 2 - 4 = 0$

so l is perpendicular to the normal, hence parallel to the plane.

(1, 1, −2): $1 - 2 - 4 = -5 \neq -9$, so l does not lie in the plane.

15(e)(ii) Distance from (1, 1, −2) to $x - 2y + 2z + 9 = 0$:

$= \frac{|1 - 2 - 4 + 9|}{\sqrt{1 + 4 + 4}} = \frac{4}{3}$

16(a)(i) $\frac{dP}{dt} = k(A - P)$, where k is a positive constant

16(a)(ii) $P = A - Ae^{-kt} \Rightarrow \frac{dP}{dt} = Ake^{-kt}$

$k(A - P) = k(A - A + Ae^{-kt}) = Ake^{-kt} = \frac{dP}{dt}$ ✓

$t = 0$: $P = A(1 - e^0) = A(1 - 1) = 0$ ✓

As $t \to \infty$, $e^{-kt} \to 0$ $(k > 0)$, so $P \to A$ ✓

16 (b) $IF = \exp(-\int \frac{1}{t(1 + t^2)}\, dt)$

$\frac{1}{t(1 + t^2)} = \frac{A}{t} + \frac{Bt + C}{1 + t^2}$

Question number

$1 = A(1 + t^2) + (Bt + C)t$

$t = 0$: $A = 1$

t^2: $0 = A + B \Rightarrow B = -1$; t: $0 = C$

$$\int \frac{1}{t(1 + t^2)}\, dt = \int \left(\frac{1}{t} - \frac{t}{1 + t^2}\right) dt$$

$$= \ln t - \frac{1}{2}\ln(1 + t^2)$$

$$IF = \exp\left(-\ln t + \frac{1}{2}\ln(1 + t^2)\right) = \exp\left(\ln \frac{\sqrt{1 + t^2}}{t}\right)$$

$$= \frac{\sqrt{1 + t^2}}{t} \quad \text{as required}$$

16(c)(i) $$\frac{d}{dt}\left(P\frac{\sqrt{1 + t^2}}{t}\right) = 0$$

$$P\frac{\sqrt{1 + t^2}}{t} = c \Rightarrow P = \frac{ct}{\sqrt{1 + t^2}}$$

As $t \to \infty$, $\frac{t}{\sqrt{1 + t^2}} = \frac{1}{\sqrt{\frac{1}{t^2} + 1}} \to 1$, so $P \to c$

Long-term value is A, so $c = A$

$$P = \frac{At}{\sqrt{1 + t^2}} \quad \text{as required} \quad (\text{and } P = 0 \text{ at } t = 0)$$

16(c)(ii) $$\frac{At}{\sqrt{1 + t^2}} = \frac{1}{2}A \Rightarrow 2t = \sqrt{1 + t^2}$$

$4t^2 = 1 + t^2 \Rightarrow 3t^2 = 1 \Rightarrow t = \frac{1}{\sqrt{3}} = 0.5774$ hours

= 34.6 minutes, so 35 minutes to the nearest minute

Question number

16 (d)

$$\frac{d}{dt}\left(P\frac{\sqrt{1+t^2}}{t}\right) = \frac{te^{-t}}{\sqrt{1+t^2}} \times \frac{\sqrt{1+t^2}}{t} = e^{-t}$$

$$P\frac{\sqrt{1+t^2}}{t} = \int e^{-t}\,dt = -e^{-t} + c$$

$$P = \frac{ct - te^{-t}}{\sqrt{1+t^2}}$$

As $t \to \infty$, $te^{-t} \to 0$ and $\frac{t}{\sqrt{1+t^2}} \to 1$,

so $P \to c$, the long-term value, so $c = A$

$$P = \frac{At - te^{-t}}{\sqrt{1+t^2}}$$ as required

16 (e)

$A = 10$, $t = \frac{37}{60} = 0.6167$ hours

Model (c): $P = \frac{10(0.6167)}{\sqrt{1+0.6167^2}} = 5.25$

Model (d): $P = \frac{10(0.6167) - 0.6167e^{-0.6167}}{\sqrt{1+0.6167^2}} = 4.97$

Data: $P = 5$ at 37 minutes. 4.97 is closer to 5 than 5.25,

so the model in part (d) is more consistent with the data.