OCR
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Monday 26 June 2023 – Afternoon

A Level Further Mathematics B (MEI)

Y436/01 Further Pure with Technology

Time allowed: 1 hour 45 minutes

9052252857

You must have:
- the Printed Answer Booklet
- the Formulae Booklet for Further Mathematics B (MEI)
- a computer with appropriate software
- a scientific or graphical calculator

QP

INSTRUCTIONS
- Use black ink. You can use an HB pencil, but only for graphs and diagrams.
- Write your answer to each question in the space provided in the **Printed Answer Booklet**. If you need extra space use the lined pages at the end of the Printed Answer Booklet. The question numbers must be clearly shown.
- Fill in the boxes on the front of the Printed Answer Booklet.
- Answer **all** the questions.
- Where appropriate, your answer should be supported with working. Marks might be given for using a correct method, even if your answer is wrong.
- Give your final answers to a degree of accuracy that is appropriate to the context.
- Do **not** send this Question Paper for marking. Keep it in the centre or recycle it.

INFORMATION
- The total mark for this paper is **60**.
- The marks for each question are shown in brackets **[ ]**.
- This document has **8** pages.

ADVICE
- Read each question carefully before you start your answer.

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1 A family of functions is defined as

$$f(x) = ax + \frac{x^2}{1+x}, \quad x \neq -1$$

where the parameter a is a real number. You may find it helpful to use a slider (for a) to investigate the family of curves $y = f(x)$.

(a) (i) On the axes in the Printed Answer Booklet, sketch the curve $y = f(x)$ in each of the following cases.

- $a = -2$
- $a = -1$
- $a = 0$

[3]

(ii) State a feature which is common to the curve in all three cases, $a = -2$, $a = -1$ and $a = 0$. **[1]**

(iii) State a feature of the curve for the cases $a = -2$, $a = -1$ that is **not** a feature of the curve in the case $a = 0$. **[1]**

(b) (i) Determine the equation of the oblique asymptote to the curve $y = f(x)$ in terms of a. **[3]**

(ii) For $b \neq -1, 0, 1$ let A be the point with coordinates $(-b, f(-b))$ and let B be the point with coordinates $(b, f(b))$.

Show that the y-coordinate of the point at which the chord to the curve $y = f(x)$ between A and B meets the y-axis is independent of a. **[3]**

(iii) With $y = f(x)$, determine the range of values of a for which

- $y \geqslant 0$ for all $x \geqslant 0$
- $y \leqslant 0$ for all $x \geqslant 0$

[5]

(c) In the case of $a = 0$, the curve $y = \sqrt[4]{f(x)}$ has a cusp.

Find its coordinates and fully justify that it is a cusp. **[5]**

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2 Throughout this question (a, b, c) is a Pythagorean triple with the positive integers a, b, c ordered such that $a \leqslant b \leqslant c$.

(a) Show that $a^2 = b + c$ if and only if $c = b + 1$. **[4]**

(b) Create a program to find all the Pythagorean triples (a, b, c) such that $a^2 = b + c$ and $c \leqslant 1000$. Write out your program in full in the Printed Answer Booklet. **[3]**

(c) Write down the number of Pythagorean triples found by your program in **(b)**. **[1]**

(d) Prove that there is no Pythagorean triple, (a, b, c), in which $b^2 = a + c$. **[3]**

3 Wilson's theorem states that an integer $p > 1$ is prime if and only if $(p-1)! \equiv -1 \pmod{p}$.

(a) Use Wilson's theorem to show that $17! \equiv 1 \pmod{19}$. **[2]**

(b) A prime number p is called a Wilson prime if $(p-1)! \equiv -1 \pmod{p^2}$.
For example, 5 is a Wilson prime because $(5-1)! \equiv 24 \equiv -1 \pmod{25}$.
At the time of writing all known Wilson primes are less than 1000.

(i) Create a program to find all the known Wilson primes. Write out your program in full in the Printed Answer Booklet. **[4]**

(ii) Use your program to find and write down all the known Wilson primes. **[1]**

(iii) Prove that if there is an integer solution m to the equation $(p-1)! + 1 = m^2$ where p is prime, then p is a Wilson prime. **[3]**

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4 In this question you are required to consider the family of differential equations

$$\frac{dP}{dt} = rP\left(1 - \frac{P}{K}\right), \quad t \geqslant 0, \quad P(t) \geqslant 0 \ (*)$$

where r and K are positive constants. This differential equation can be used as a model for the size of a population P as a function of time t.

(a) (i) Determine the values of P for which

- $\frac{dP}{dt} = 0$
- $\frac{dP}{dt} > 0$
- $\frac{dP}{dt} < 0$ **[4]**

(ii) Solve the equation (*) subject to the initial condition that $P = P_0$ when $t = 0$. **[1]**

(iii) Find a property common to your solution in **(ii)** in the cases $P_0 > K$ and $P_0 < K$. **[1]**

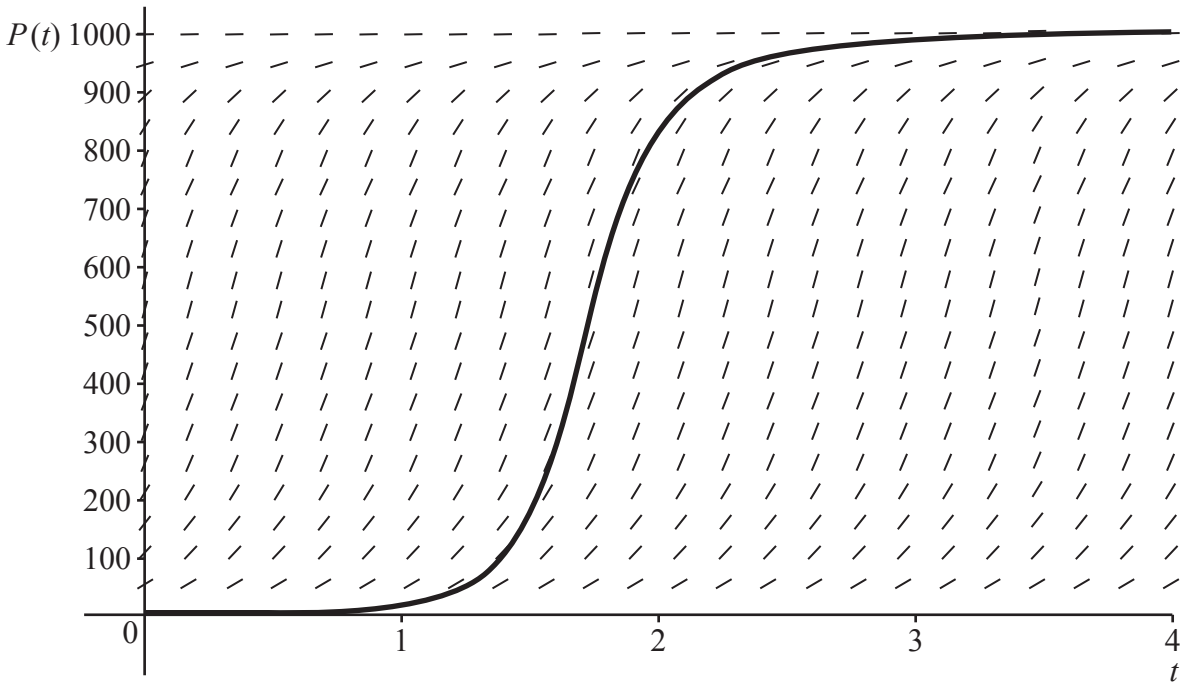
(iv) State a feature of your solution in **(iii)** for the case $P_0 > K$ which is different to the case $P_0 < K$. **[1]**

(v) Interpret the value K when $P(t)$ is the size of a population at time t. **[1]**

(b) In this question you will explore the limitations of using the Euler method to approximate solutions to the differential equation

$$\frac{dP}{dt} = 2P^{1.25}\left(1 - \frac{P}{1000}\right)^{1.5}, \ t \geqslant 0, \ P(t) \geqslant 0 \ (**)$$

The diagram shows the tangent field to (**), and a solution in which $P = 1$ when $t = 0$, produced using a much more accurate numerical method.



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(i) The Euler method for the solution of the differential equation $\mathrm{f}(t, P) = \dfrac{\mathrm{d}P}{\mathrm{d}t}$ is as follows

$P_{n+1} = P_n + h\mathrm{f}(t_n, P_n)$.

It is given that $t_0 = 0$ and $P_0 = 1$.

- Construct a spreadsheet to solve (**) using the Euler method so that the value of h can be varied.
- State the formulae you have used in your spreadsheet.

[4]

(ii) Use your spreadsheet with $h = 0.1$ to approximate

- $P(1)$
- $P(2)$
- $P(3)$

[1]

(iii) Use your spreadsheet with $h = 0.05$ to approximate

- $P(1)$
- $P(2)$
- $P(3)$

[1]

(iv) State, with reasons, whether the estimates to $P(t)$ given in your spreadsheet are likely to be overestimates or underestimates to the exact values. **[2]**

(v) With reference to the diagram, explain any noticeable feature identified when comparing the approximations given to $P(2)$ in **(ii)** and **(iii)**. **[2]**

END OF QUESTION PAPER

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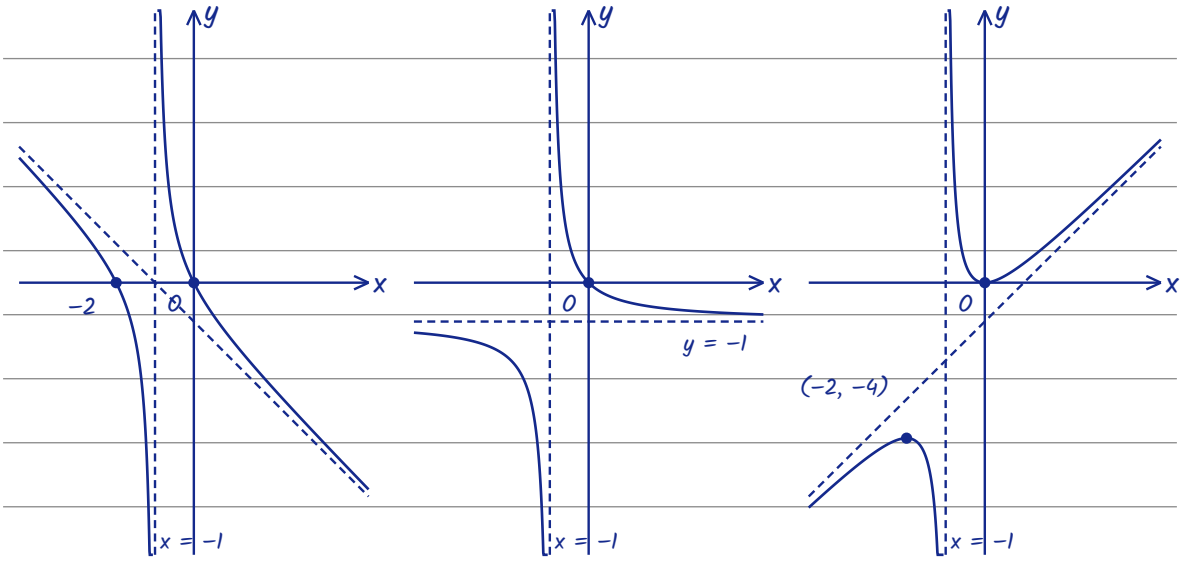
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Surname	StemUp Model Solution
Answer booklet	Write the question number in the left-hand margin at the start of each answer.

Question number

1 (a)(i)

$a = -2$ $\qquad$ $a = -1$ $\qquad$ $a = 0$



1 (a)(ii) In all three cases the curve has the vertical asymptote $x = -1$ (and every curve passes through the origin).

1 (a)(iii) For $a = -2$ and $a = -1$ the curve has no stationary points (it is always decreasing), whereas for $a = 0$ the curve has stationary points: a minimum at $(0, 0)$ and a maximum at $(-2, -4)$.

1 (b)(i) $y = ax + \dfrac{x^2}{1 + x} = ax + \dfrac{(x + 1)(x - 1) + 1}{x + 1}$

$= ax + x - 1 + \dfrac{1}{x + 1} = (a + 1)x - 1 + \dfrac{1}{x + 1}$

As $x \to \pm\infty$, $\dfrac{1}{x + 1} \to 0$

so the oblique asymptote is $y = (a + 1)x - 1$

Question number

1 (b)(ii) $f(b) = ab + \frac{b^2}{1+b}$, $\quad f(-b) = -ab + \frac{b^2}{1-b}$

A and B have x-coordinates $-b$ and b, so the chord meets the y-axis ($x = 0$) at the midpoint of AB. Its y-coordinate is

$$y = \frac{f(b) + f(-b)}{2} = \frac{1}{2}\left(\frac{b^2}{1+b} + \frac{b^2}{1-b}\right)$$
$$= \frac{1}{2} \times \frac{b^2(1-b) + b^2(1+b)}{1-b^2} = \frac{b^2}{1-b^2}$$

The ab terms cancel, so this y-coordinate does not contain a, i.e. it is independent of a, as required.

1 (b)(iii) $y = ax + \frac{x^2}{1+x} = \frac{ax(1+x) + x^2}{1+x} = \frac{x((a+1)x + a)}{x+1}$

For $x \geq 0$: $x \geq 0$ and $x + 1 > 0$, so the sign of y is the sign of $(a+1)x + a$.

If $a \geq 0$: $(a+1)x + a \geq 0$ for all $x \geq 0$, so $y \geq 0$.

If $a \leq -1$: $(a+1)x \leq 0$ and $a < 0$, so $(a+1)x + a < 0$, so $y \leq 0$.

If $-1 < a < 0$: $(a+1)x + a < 0$ for $0 < x < \frac{-a}{a+1}$

and > 0 for $x > \frac{-a}{a+1}$, so y takes both signs.

$\therefore$ $y \geq 0$ for all $x \geq 0$ precisely when $a \geq 0$

$y \leq 0$ for all $x \geq 0$ precisely when $a \leq -1$

1 (c) $a = 0$: $y = \left(\frac{x^2}{1+x}\right)^{1/4}$, defined for $x > -1$

Plotting the curve shows the cusp is at the origin.

Question number

When $x = 0$, $y = 0$, so $(0, 0)$ is on the curve (and $y \geq 0$ elsewhere).

$dy/dx = (1/4)(x^2/(1 + x))^{-3/4} \times (2x(1 + x) - x^2)/(1 + x)^2$

$= x(x + 2) / [4|x|^{3/2}(1 + x)^{5/4}]$

Near $x = 0$ this behaves like $\frac{2x}{4|x|^{3/2}} = \frac{x}{2|x|^{3/2}}$

so $\frac{dy}{dx} \to -\infty$ as $x \to 0^-$ and $\frac{dy}{dx} \to +\infty$ as $x \to 0^+$

So the tangent becomes vertical from both sides, with the curve coming down steeply to $(0, 0)$ on the left and going up steeply on the right (both branches above the x-axis, meeting at 0).

$\therefore$ the curve has a cusp at $(0, 0)$.

2 (a)

$a^2 + b^2 = c^2 \quad \Rightarrow \quad a^2 = c^2 - b^2 = (c - b)(c + b)$

($\Leftarrow$) If $c = b + 1$, then $c - b = 1$, so $a^2 = (1)(c + b) = b + c$.

($\Rightarrow$) If $a^2 = b + c$, then $(c - b)(c + b) = c + b$

$\Rightarrow (c + b)(c - b - 1) = 0$

Since $b, c > 0$, $c + b \neq 0$, so $c - b - 1 = 0$, i.e. $c = b + 1$.

$\therefore$ $a^2 = b + c$ if and only if $c = b + 1$, as required.

2 (b)

(Python)

```
count = 0
for a in range(1, 1001):
    for b in range(a, 1001):
        c = b + 1
        if c <= 1000 and a*a + b*b == c*c and a*a == b + c:
            print(a, b, c)
            count = count + 1
print(count)
```

Question number	
2 (c)	21 Pythagorean triples
	$(3, 4, 5), (5, 12, 13), (7, 24, 25), \ldots, (43, 924, 925)$
2 (d)	Suppose (a, b, c) is a Pythagorean triple with $b^2 = a + c$.
	$b^2 = c^2 - a^2 = (c - a)(c + a) = c + a$
	$\Rightarrow (c + a)(c - a - 1) = 0$, and $c + a > 0$, so $c = a + 1$.
	Now $a \leq b \leq c = a + 1$ with b an integer, so $b = a$ or $b = a + 1$.
	$b = a + 1 = c$ would give $a^2 = c^2 - b^2 = 0$, impossible, so $b = a$.
	Then $c^2 = a^2 + b^2 = 2a^2 \Rightarrow (c/a)^2 = 2 \Rightarrow \sqrt{2} = c/a$,
	a ratio of integers, which is impossible since $\sqrt{2}$ is irrational.
	$\therefore$ there is no Pythagorean triple with $b^2 = a + c$.
3 (a)	19 is prime, so by Wilson's theorem
	$18! \equiv -1 \equiv 18 \pmod{19}$
	$18! = 18 \times 17!$, so $18 \times 17! \equiv 18 \pmod{19}$
	18 and 19 are coprime (19 is prime), so we can divide by 18:
	$17! \equiv 1 \pmod{19}$, as required.
3 (b)(i)	(Python)

```
def is_prime(n):
    if n < 2:
        return False
    for i in range(2, int(n**0.5) + 1):
        if n % i == 0:
            return False
    return True
```

Question number

```
for p in range(2, 1000):
    if is_prime(p):
        f = 1
        for k in range(1, p):
            f = (f * k) % (p * p)        # (p - 1)! mod p²
        if f == p * p - 1:
            print(p)
```

3 (b)(ii) Wilson primes: 5, 13, 563

3 (b)(iii) Suppose m is an integer with $(p-1)! + 1 = m^2$, p prime.

By Wilson's theorem, $(p-1)! \equiv -1 \pmod{p}$, so $p \mid (p-1)! + 1$,

i.e. $p \mid m^2$.

Since p is prime, $p \mid m^2 \Rightarrow p \mid m$ (each prime appears an even

number of times in the prime factorisation of m^2), so $p^2 \mid m^2$.

$\therefore (p-1)! + 1 = m^2 \equiv 0 \pmod{p^2}$

$\Rightarrow (p-1)! \equiv -1 \pmod{p^2}$, so p is a Wilson prime.

4 (a)(i) $dP/dt = rP(1 - P/K)$

$dP/dt = 0 \Leftrightarrow P = 0$ or $P = K$

$r > 0$, so consider the signs of P and $1 - P/K$ for $P > 0$:

$dP/dt > 0$ when $0 < P < K$

$dP/dt < 0$ when $P > K$

4 (a)(ii) $$P(t) = \frac{KP_0e^{rt}}{K + P_0(e^{rt} - 1)}$$

4 (a)(iii) In both cases $P(t) \to K$ as $t \to \infty$.

Question number

4 (a)(iv) If $P_0 > K$, $P(t)$ is decreasing (approaches K from above);
if $P_0 < K$, $P(t)$ is increasing (approaches K from below).

4 (a)(v) K is the long-term size of the population (the carrying capacity).

4 (b)(i) Spreadsheet: h is entered in cell G1 so it can be varied.

A1: 0 (t_0)

B1: 1 (P_0)

C1: =2*(B1^1.25)*((1-B1/1000)^1.5) (dP/dt)

A2: =A1+$G$1

B2: =B1+$G$1*C1

C2: =2*(B2^1.25)*((1-B2/1000)^1.5)

then fill A2:C2 down the columns.

First rows with $h = 0.1$:

t	P	dP/dt
0	1	1.997001
0.1	1.199700	2.506622
0.2	1.450362	3.176363
0.3	1.767999	4.066583

4 (b)(ii) $h = 0.1$: $P(1) \approx 9.568$, $P(2) \approx 421.7$, $P(3) \approx 979.6$

4 (b)(iii) $h = 0.05$: $P(1) \approx 11.77$, $P(2) \approx 672.3$, $P(3) \approx 981.1$

4 (b)(iv) Halving h increases all three approximations, and a smaller h generally gives a better estimate, so the exact values are larger.
Also, before the steep part the gradient increases with t, so each

Question number

tangent step lies below the curve and these errors accumulate.

$\therefore$ the estimates are likely to be underestimates.

4 (b)(v) The two approximations to $P(2)$ differ greatly (421.7 and 672.3), much more than for $P(1)$ or $P(3)$. From the diagram, $P(2) \approx 820$. From the tangent field, for small t the Euler steps underestimate P, so the very steep part of the approximate curve is pushed to the right (later t). Near $t = 2$ the gradient is very large, so a small shift in t gives a big difference in P, so the estimates of $P(2)$ are very inaccurate and very sensitive to h.