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Friday 23 June 2023 – Afternoon

A Level Further Mathematics B (MEI)

Y435/01 Extra Pure

Time allowed: 1 hour 15 minutes

9973233460

You must have:
- the Printed Answer Booklet
- the Formulae Booklet for Further Mathematics B (MEI)
- a scientific or graphical calculator

QP

INSTRUCTIONS
- Use black ink. You can use an HB pencil, but only for graphs and diagrams.
- Write your answer to each question in the space provided in the **Printed Answer Booklet**. If you need extra space use the lined pages at the end of the Printed Answer Booklet. The question numbers must be clearly shown.
- Fill in the boxes on the front of the Printed Answer Booklet.
- Answer **all** the questions.
- Where appropriate, your answer should be supported with working. Marks might be given for using a correct method, even if your answer is wrong.
- Give your final answers to a degree of accuracy that is appropriate to the context.
- Do **not** send this Question Paper for marking. Keep it in the centre or recycle it.

INFORMATION
- The total mark for this paper is **60**.
- The marks for each question are shown in brackets **[ ]**.
- This document has **4** pages.

ADVICE
- Read each question carefully before you start your answer.

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1 A surface is defined in 3-D by $z = 3x^3 + 6xy + y^2$.

Determine the coordinates of any stationary points on the surface. **[7]**

2 A sequence is defined by the recurrence relation $4t_{n+1} - t_n = 15n + 17$ for $n \geqslant 1$, with $t_1 = 2$.

(a) Solve the recurrence relation to find the particular solution for t_n. **[7]**

Another sequence is defined by the recurrence relation $(n+1)u_{n+1} - u_n^2 = 2n - \frac{1}{n^2}$ for $n \geqslant 1$, with $u_1 = 2$.

(b) (i) Explain why the recurrence relation for u_n **cannot** be solved using standard techniques for non-homogeneous first order recurrence relations. **[1]**

(ii) Verify that the particular solution to this recurrence relation is given by $u_n = an + \frac{b}{n}$ where a and b are constants whose values are to be determined. **[5]**

A third sequence is defined by $v_n = \frac{t_n}{u_n}$ for $n \geqslant 1$.

(c) Determine $\lim_{n\to\infty} v_n$. **[2]**

3 A surface, S, is defined by $g(x, y, z) = 0$ where $g(x, y, z) = 2x^3 - x^2y + 2xy^2 + 27z$. The normal to S at the point $\left(1, 1, -\frac{1}{9}\right)$ and the tangent plane to S at the point $(3, 3, -3)$ intersect at P.

Determine the position vector of P. **[8]**

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4 The set G is given by $G = \{\mathbf{M}: \mathbf{M} \text{ is a real } 2\times 2 \text{ matrix and } \det \mathbf{M} = 1\}$.

(a) Show that G forms a group under matrix multiplication, $\times$. You may assume that matrix multiplication is associative. **[5]**

(b) The matrix $\mathbf{A}_n$ is defined by $\mathbf{A}_n = \begin{pmatrix} 1 & 0 \\ n & 1 \end{pmatrix}$ for any integer n. The set S is defined by $S = \{\mathbf{A}_n : n \in \mathbb{Z}, n \geqslant 0\}$.

(i) Determine whether S is closed under $\times$. **[2]**

(ii) Determine whether S is a subgroup of $(G, \times)$. **[2]**

(c) **(i)** Find a subgroup of $(G, \times)$ of order 2. **[2]**

(ii) By considering the inverse of the non-identity element in any such subgroup, or otherwise, show that this is the only subgroup of $(G, \times)$ of order 2. **[2]**

The set of all real 2×2 matrices is denoted by H.

(d) With the help of an example, explain why $(H, \times)$ is **not** a group. **[2]**

5 The matrix $\mathbf{P}$ is given by $\mathbf{P} = \begin{pmatrix} a & 0 \\ 2 & 3 \end{pmatrix}$ where a is a constant and $a \neq 3$.

(a) Given that the acute angle between the directions of the eigenvectors of $\mathbf{P}$ is $\frac{1}{4}\pi$ radians, determine the possible values of a. **[8]**

(b) You are given instead that $\mathbf{P}$ satisfies the matrix equation $\mathbf{I} = \mathbf{P}^2 + r\mathbf{P}$ for some rational number r.

(i) Use the Cayley-Hamilton theorem to determine the value of a and the corresponding value of r. **[4]**

(ii) Hence show that $\mathbf{P}^4 = s\mathbf{I} + t\mathbf{P}$ where s and t are rational numbers to be determined. You should **not** calculate $\mathbf{P}^4$. **[3]**

END OF QUESTION PAPER

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Surname	StemUp Model Solution
Answer booklet	Write the question number in the left-hand margin at the start of each answer.

Question number

1

$z = 3x^3 + 6xy + y^2$

$\frac{\partial z}{\partial x} = 9x^2 + 6y \qquad \frac{\partial z}{\partial y} = 6x + 2y$

Stationary points where $\frac{\partial z}{\partial x} = 0$ and $\frac{\partial z}{\partial y} = 0$

$6x + 2y = 0 \Rightarrow y = -3x$

$9x^2 + 6(-3x) = 0 \Rightarrow 9x^2 - 18x = 0 \Rightarrow 9x(x - 2) = 0$

$x = 0 \Rightarrow y = 0, \; z = 0$

$x = 2 \Rightarrow y = -6, \; z = 3(8) + 6(2)(-6) + 36 = 24 - 72 + 36 = -12$

Stationary points are $(0, 0, 0)$ and $(2, -6, -12)$ and no others

2 (a)

Complementary function: $4t_{n+1} - t_n = 0 \Rightarrow t_n = \alpha \times 4^{-n}$

Particular solution: try $t_n = pn + q$

$4(p(n + 1) + q) - (pn + q) = 15n + 17$

$3pn + 4p + 3q = 15n + 17$

n: $3p = 15 \Rightarrow p = 5$ constant: $20 + 3q = 17 \Rightarrow q = -1$

General solution: $t_n = 5n - 1 + \alpha \times 4^{-n}$

$t_1 = 2$: $2 = 5 - 1 + \frac{\alpha}{4} \Rightarrow \alpha = -8$

$t_n = 5n - 1 - 8 \times 4^{-n}$

2 (b)(i)

The recurrence relation is non-linear (it contains u_n^2), and the coefficient of u_{n+1} is not constant, so it cannot be written in the form $u_{n+1} = cu_n + f(n)$ with c constant.

Question number

2 (b)(ii) $u_n = an + \frac{b}{n}$. $n = 1$: $u_1 = a + b = 2$

$(n+1)u_{n+1} - u_n^2 = (n+1)(a(n+1) + \frac{b}{n+1}) - (an + \frac{b}{n})^2$

$= a(n+1)^2 + b - a^2n^2 - 2ab - \frac{b^2}{n^2}$

$= an^2 + 2an + a + b - a^2n^2 - 2ab - \frac{b^2}{n^2}$

$= (a - a^2)n^2 + 2an + (a + b - 2ab) - \frac{b^2}{n^2}$

This must equal $2n - \frac{1}{n^2}$ for all n:

n terms: $2a = 2 \Rightarrow a = 1$

constant: $a + b - 2ab = 0 \Rightarrow 1 + b - 2b = 0 \Rightarrow b = 1$

Check: n^2 coefficient $a - a^2 = 1 - 1 = 0$ ✓

$\frac{1}{n^2}$ term: $-b^2 = -1$ ✓ $\quad u_1 = a + b = 2$ ✓

So $u_n = n + \frac{1}{n}$ is the particular solution, with $a = 1$, $b = 1$

2 (c) $v_n = \frac{t_n}{u_n} = \frac{Sn - 1 - 8 \times 4^{-n}}{n + \frac{1}{n}}$

$= \frac{S - \frac{1}{n} - \frac{8 \times 4^{-n}}{n}}{1 + \frac{1}{n^2}}$ (dividing top and bottom by n)

As $n \to \infty$: $\frac{1}{n} \to 0$, $\frac{8 \times 4^{-n}}{n} \to 0$, $\frac{1}{n^2} \to 0$

Question number

$\lim_{n\to\infty} v_n = \frac{5}{1} = 5$

3 $\frac{\partial g}{\partial x} = 6x^2 - 2xy + 2y^2 \qquad \frac{\partial g}{\partial y} = -x^2 + 4xy \qquad \frac{\partial g}{\partial z} = 27$

$$\nabla g = \begin{pmatrix} 6x^2 - 2xy + 2y^2 \\ -x^2 + 4xy \\ 27 \end{pmatrix}$$

At $(1, 1, -\frac{1}{4})$: $\nabla g = \begin{pmatrix} 6 - 2 + 2 \\ -1 + 4 \\ 27 \end{pmatrix} = \begin{pmatrix} 6 \\ 3 \\ 27 \end{pmatrix} = 3\begin{pmatrix} 2 \\ 1 \\ 9 \end{pmatrix}$

Normal line: $r = (i + j - \frac{1}{4}k) + \lambda(2i + j + 9k)$

At $(3, 3, -3)$: $\nabla g = \begin{pmatrix} 54 - 18 + 18 \\ -9 + 36 \\ 27 \end{pmatrix} = \begin{pmatrix} 54 \\ 27 \\ 27 \end{pmatrix} = 27\begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$

Tangent plane: $r.\begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \\ -3 \end{pmatrix}.\begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} = 6 + 3 - 3 = 6$

i.e. $2x + y + z = 6$

Substitute the normal line: $2(1 + 2\lambda) + (1 + \lambda) + (-\frac{1}{4} + 9\lambda) = 6$

$\frac{26}{4} + 14\lambda = 6 \Rightarrow 14\lambda = \frac{28}{4} \Rightarrow \lambda = \frac{2}{4}$

$OP = (i + j - \frac{1}{4}k) + \frac{2}{4}(2i + j + 9k)$

$= (1 + \frac{4}{4})i + (1 + \frac{2}{4})j + (-\frac{1}{4} + 2)k = \frac{13}{4}i + \frac{11}{4}j + \frac{17}{4}k$

Question number

Position vector of P: $\frac{1}{9}\begin{pmatrix}13\\11\\17\end{pmatrix}$

4 (a) Identity: $I = \begin{pmatrix}1 & 0\\0 & 1\end{pmatrix} \in G$ since $\det I = 1$,

and $IA = AI = A$ for all $A \in G$, so the identity property holds.

Closure: if $A, B \in G$ then AB is a real 2×2 matrix and

$\det(AB) = \det A \times \det B = 1 \times 1 = 1$, so $AB \in G$: G is closed.

Inverse: for $A \in G$, $\det A = 1 \neq 0$ so A^{-1} exists (real 2×2),

and $AA^{-1} = I \Rightarrow \det A \times \det A^{-1} = \det I = 1 \Rightarrow \det A^{-1} = 1$

so $A^{-1} \in G$ and the inverse property holds.

Associativity: given (matrix multiplication is associative).

All four group axioms are satisfied, so $(G, \times)$ is a group.

4 (b)(i) For $m, n \geq 0$: $A_m \times A_n = \begin{pmatrix}1 & 0\\m & 1\end{pmatrix}\begin{pmatrix}1 & 0\\n & 1\end{pmatrix} = \begin{pmatrix}1 & 0\\m+n & 1\end{pmatrix} = A_{m+n}$

$m + n$ is a non-negative integer, so $A_{m+n} \in S$.

So S is closed under $\times$.

4 (b)(ii) For $n > 0$: $A_n^{-1} = \begin{pmatrix}1 & 0\\-n & 1\end{pmatrix} = A_{-n}$ $(\det A_n = 1)$

$-n$ is a negative integer, so $A_n^{-1} \notin S$.

The inverse axiom fails, so S is not a subgroup of $(G, \times)$.

4 (c)(i) $\begin{pmatrix}-1 & 0\\0 & -1\end{pmatrix} \times \begin{pmatrix}-1 & 0\\0 & -1\end{pmatrix} = \begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}$, and $\det(-1) = 1$

Question number

Subgroup of order 2: $\left(\left\{\begin{pmatrix}1&0\\0&1\end{pmatrix}, \begin{pmatrix}-1&0\\0&-1\end{pmatrix}\right\}, \times\right)$

4 (c)(ii) A subgroup of order 2 is $\{I, M\}$ with $M \neq I$, and it must be cyclic,

so $M^2 = I$, i.e. $M^{-1} = M$.

Let $M = \begin{pmatrix}a&b\\c&d\end{pmatrix}$ with $ad - bc = 1$. Then $M^{-1} = \begin{pmatrix}d&-b\\-c&a\end{pmatrix}$

$M^{-1} = M \Rightarrow d = a,\ -b = b,\ -c = c \Rightarrow b = c = 0,\ a = d$

$\det M = a^2 = 1 \Rightarrow a = \pm 1$

$a = 1$ gives $M = I$, which is not allowed, so $a = d = -1$ and $M = -I$.

So $\{I, -I\}$ is the only subgroup of $(G, \times)$ of order 2.

4 (d) Not every element of H has an inverse. For example $\begin{pmatrix}1&0\\1&0\end{pmatrix} \in H$

but $\det\begin{pmatrix}1&0\\1&0\end{pmatrix} = 0$, so it has no inverse in H.

The inverse axiom fails, so $(H, \times)$ is not a group.

5 (a) $\det(P - \lambda I) = \det\begin{pmatrix}a-\lambda&0\\2&3-\lambda\end{pmatrix} = (a - \lambda)(3 - \lambda) = 0$

Eigenvalues are $\lambda = 3$ and $\lambda = a$

$\lambda = 3$: $\begin{pmatrix}a&0\\2&3\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}3x\\3y\end{pmatrix} \Rightarrow ax = 3x \Rightarrow x = 0\ (a \neq 3)$

$e_1 = \begin{pmatrix}0\\1\end{pmatrix}$

$\lambda = a$: $\begin{pmatrix}a&0\\2&3\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}ax\\ay\end{pmatrix} \Rightarrow 2x + 3y = ay \Rightarrow 2x = (a - 3)y$

Question number

$\mathbf{e}_2 = \begin{pmatrix} a-3 \\ 2 \end{pmatrix}$

$\mathbf{e}_1.\mathbf{e}_2 = 0 \times (a-3) + 1 \times 2 = 2, \quad |\mathbf{e}_1| = 1, \quad |\mathbf{e}_2| = \sqrt{(a-3)^2 + 4}$

$\mathbf{e}_1.\mathbf{e}_2 = |\mathbf{e}_1||\mathbf{e}_2| \cos \frac{\pi}{4}$

$2 = 1 \times \sqrt{(a-3)^2 + 4} \times \frac{1}{\sqrt{2}}$

$8 = (a-3)^2 + 4 \Rightarrow (a-3)^2 = 4 \Rightarrow a - 3 = \pm 2$

$a = 1$ or $a = 5$

5 (b)(i) Characteristic equation: $(a - \lambda)(3 - \lambda) = 0 \Rightarrow \lambda^2 - (a+3)\lambda + 3a = 0$

By the Cayley-Hamilton theorem: $P^2 - (a+3)P + 3aI = 0$

$\Rightarrow P^2 - (a+3)P = -3aI$

Compare with $P^2 + rP = I$: subtracting gives $(r + a + 3)P = (1 + 3a)I$

P is not a multiple of I, so $r + a + 3 = 0$ and $1 + 3a = 0$

$-3a = 1 \Rightarrow a = -\frac{1}{3}$

$r = -(a+3) = -(-\frac{1}{3} + 3) = -\frac{8}{3}$

5 (b)(ii) $P^2 - \frac{8}{3}P = I \Rightarrow P^2 = I + \frac{8}{3}P$

$P^4 = (P^2)^2 = (I + \frac{8}{3}P)^2 = I + \frac{16}{3}P + \frac{64}{9}P^2$

$= I + \frac{16}{3}P + \frac{64}{9}(I + \frac{8}{3}P)$

$= (1 + \frac{64}{9})I + (\frac{16}{3} + \frac{512}{27})P$

Question number

$P^4 = \frac{73}{9}I + \frac{656}{27}P$

so $s = \frac{73}{9}$ and $t = \frac{656}{27}$