OCR
Oxford Cambridge and RSA

Wednesday 14 June 2023 – Afternoon

A Level Further Mathematics B (MEI)

Y421/01 Mechanics Major

Time allowed: 2 hours 15 minutes

9966948315 7

You must have:
- the Printed Answer Booklet
- the Formulae Booklet for Further Mathematics B (MEI)
- a scientific or graphical calculator

QP

INSTRUCTIONS
- Use black ink. You can use an HB pencil, but only for graphs and diagrams.
- Write your answer to each question in the space provided in the **Printed Answer Booklet**. If you need extra space use the lined pages at the end of the Printed Answer Booklet. The question numbers must be clearly shown.
- Fill in the boxes on the front of the Printed Answer Booklet.
- Answer **all** the questions.
- Where appropriate, your answer should be supported with working. Marks might be given for using a correct method, even if your answer is wrong.
- Give your final answers to a degree of accuracy that is appropriate to the context.
- The acceleration due to gravity is denoted by $g\,\mathrm{m\,s^{-2}}$. When a numerical value is needed use $g = 9.8$ unless a different value is specified in the question.
- Do **not** send this Question Paper for marking. Keep it in the centre or recycle it.

INFORMATION
- The total mark for this paper is **120**.
- The marks for each question are shown in brackets **[ ]**.
- This document has **12** pages.

ADVICE
- Read each question carefully before you start your answer.

© OCR 2023 [D/508/5593]
DC (ST/SW) 329448/3

OCR is an exempt Charity

stemup.co.uk

Section A (26 marks)

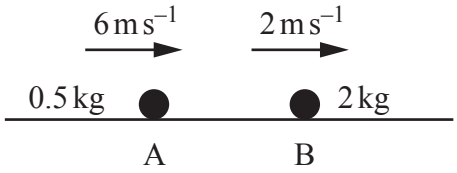
1 A car of mass 800 kg moves in a straight line along a horizontal road.

There is a constant resistance to the motion of the car of magnitude 600 N.

When the car is travelling at a speed of $15\,\text{m s}^{-1}$ the power developed by the car is 27 kW.

Determine the acceleration of the car when it is travelling at $15\,\text{m s}^{-1}$. **[4]**

2



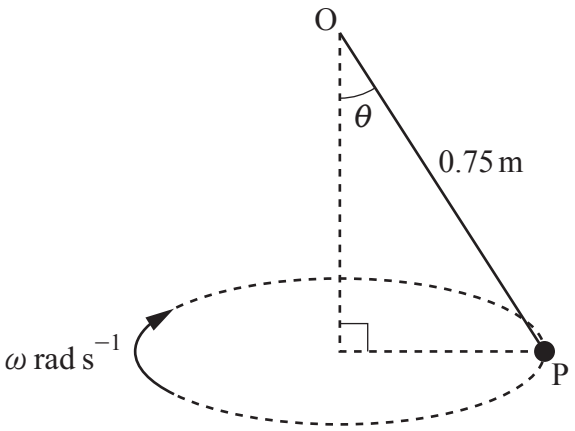
Two small uniform smooth spheres A and B have masses 0.5 kg and 2 kg respectively. The two spheres are travelling in the same direction in the same straight line on a smooth horizontal surface. Sphere A is moving towards B with speed $6\,\text{m s}^{-1}$ and B is moving away from A with speed $2\,\text{m s}^{-1}$ (see diagram). Spheres A and B collide. After this collision A moves with speed $0.2\,\text{m s}^{-1}$.

Determine the possible speeds with which B moves after the collision. **[4]**

© OCR 2023

stemup.co.uk

3



The diagram shows a particle P, of mass 0.2 kg, which is attached by a light inextensible string of length 0.75 m to a fixed point O.

Particle P moves with constant angular speed ω rad s^{-1} in a horizontal circle with centre vertically below O. The string is inclined at an angle θ to the vertical.

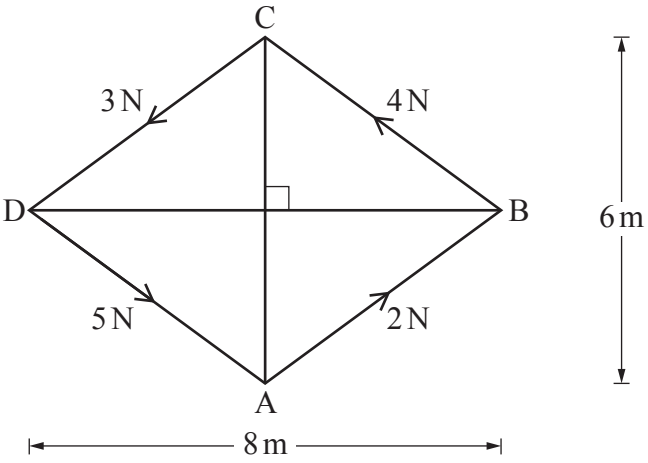
The greatest tension that the string can withstand without breaking is 15 N.

(a) Find the greatest possible value of θ, giving your answer to the nearest degree. **[2]**

(b) Determine the greatest possible value of ω. **[3]**

stemup.co.uk

4



A rigid lamina of negligible mass is in the form of a rhombus ABCD, where AC = 6 m and BD = 8 m. Forces of magnitude 2 N, 4 N, 3 N and 5 N act along its sides AB, BC, CD and DA, respectively, as shown in the diagram. A further force **F** N, acting at A, and a couple of magnitude G Nm are also applied to the lamina so that it is in equilibrium.

(a) Determine the magnitude and direction of **F**. **[4]**

(b) Determine the value of G. **[2]**

5 A particle P of mass m kg is projected with speed u m s^{-1} along a rough horizontal surface. During the motion of P, a constant frictional force of magnitude F N acts on P. When the velocity of P is v m s^{-1}, it experiences a force of magnitude kv N due to air resistance, where k is a constant.

(a) Determine the dimensions of k. **[3]**

At time T s after projection P comes to rest. A formula approximating the value of T is

$$T = \frac{mu}{F} - \frac{kmu^2}{2F^2} + \frac{1}{3}k^2 m^\alpha u^\beta F^\gamma .$$

(b) Use dimensional analysis to find α, β and γ. **[4]**

© OCR 2023

stemup.co.uk

Section B (94 marks)

6 At time t seconds, where $t \geqslant 0$, a particle P has position vector **r** metres, where

$\mathbf{r} = (2t^2 - 12t + 6)\mathbf{i} + (t^3 + 3t^2 - 8t)\mathbf{j}$.

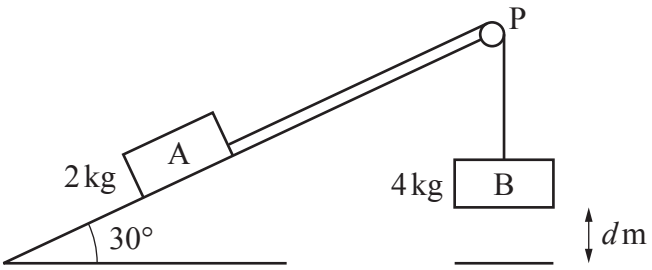
The velocity of P at time t seconds is $\mathbf{v}\,\text{m s}^{-1}$.

(a) Find **v** in terms of t. **[1]**

(b) Determine the speed of P at the instant when it is moving parallel to the vector $\mathbf{i} - 4\mathbf{j}$. **[5]**

(c) Determine the value of t when the magnitude of the acceleration of P is $20.2\,\text{m s}^{-2}$. **[3]**

7 One end of a rope is attached to a block A of mass 2 kg. The other end of the rope is attached to a second block B of mass 4 kg. Block A is held at rest on a fixed rough ramp inclined at 30° to the horizontal. The rope is taut and passes over a small smooth pulley P which is fixed at the top of the ramp. The part of the rope from A to P is parallel to a line of greatest slope of the ramp. Block B hangs vertically below P, at a distance d m above the ground, as shown in the diagram.



Block A is more than d m from P. The blocks are released from rest and A moves up the ramp. The coefficient of friction between A and the ramp is $\frac{1}{2\sqrt{3}}$.

The blocks are modelled as particles, the rope is modelled as light and inextensible, and air resistance can be ignored.

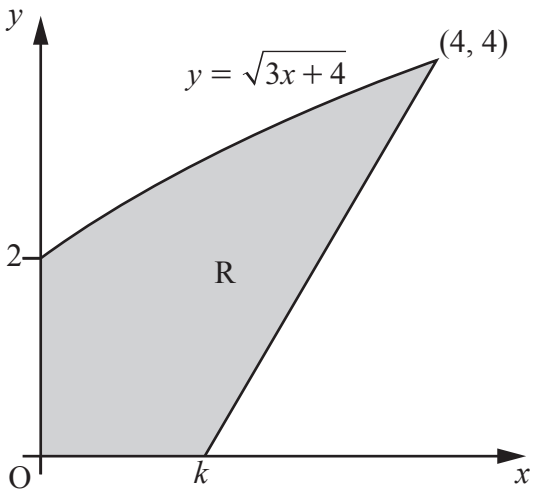
(a) Determine, in terms of g and d, the work done against friction as A moves d m up the ramp. **[3]**

(b) Given that the speed of B immediately before it hits the ground is $1.75\,\text{m s}^{-1}$, use the work–energy principle to determine the value of d. **[5]**

(c) Suggest one improvement, apart from including air resistance, that could be made to the model to make it more realistic. **[1]**

stemup.co.uk

8



The diagram shows the shaded region R bounded by the curve $y = \sqrt{3x+4}$, the x-axis, the y-axis, and the straight line that passes through the points $(k, 0)$ and $(4, 4)$, where $0 < k < 4$.

Region R is occupied by a uniform lamina.

(a) Determine, in terms of k, an expression for the y-coordinate of the centre of mass of the lamina. Give your answer in the form $\frac{a+bk}{c+dk}$, where a, b, c and d are integers to be determined. **[6]**

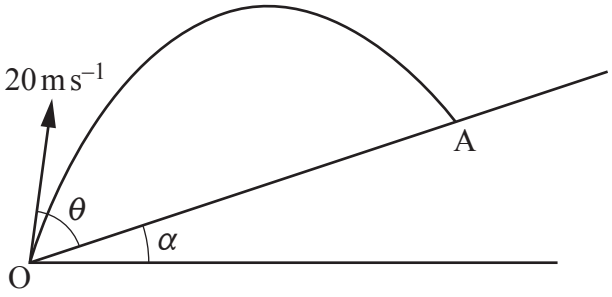
(b) Show that the y-coordinate of the centre of mass of the lamina cannot be $\frac{3}{2}$. **[2]**

© OCR 2023

stemup.co.uk

9 In this question take $g = 10$.

A small ball P is projected with speed $20\,\text{m s}^{-1}$ at an angle of elevation of $(\alpha + \theta)$ from a point O at the bottom of a smooth plane inclined at an angle α to the horizontal, where $\tan\alpha = \frac{5}{12}$ and $\tan\theta = \frac{3}{4}$. The ball subsequently hits the plane at a point A, where OA is a line of greatest slope of the plane, as shown in the diagram.



(a) Determine the following, in either order.

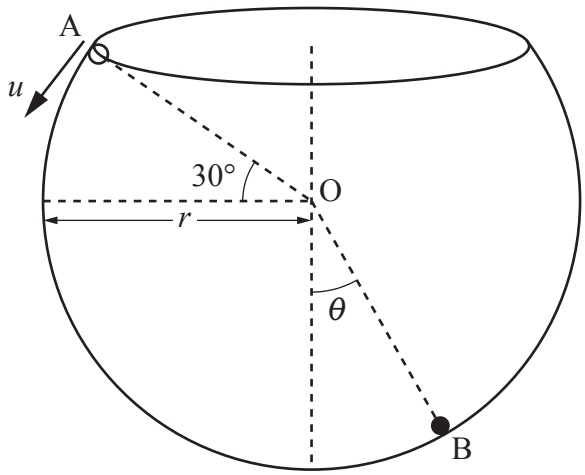
- The components of the velocity of P, parallel and perpendicular to the plane, immediately before P hits the plane at A.
- The distance OA. **[9]**

After P hits the plane at A it continues to move away from O. Immediately after hitting the plane at A the direction of motion of P makes an angle β with the horizontal.

(b) Determine the maximum possible value of β, giving your answer to the nearest degree. **[3]**

stemup.co.uk

10



A hollow sphere has centre O and internal radius r. A bowl is formed by removing part of the sphere. The bowl is fixed to a horizontal floor, with its circular rim horizontal and the centre of the rim vertically above O.

The point A lies on the rim of the bowl such that AO makes an angle of 30° with the horizontal (see diagram).

A particle P of mass m is projected from A, with speed u, where $u > \sqrt{\frac{gr}{2}}$, in a direction perpendicular to AO and moves on the smooth inner surface of the bowl.

The motion of P takes place in the vertical plane containing O and A. The particle P passes through a point B on the inner surface, where OB makes an acute angle θ with the vertical.

(a) Determine, in terms of m, g, u, r and θ, the magnitude of the force exerted on P by the bowl when P is at B. **[7]**

The difference between the magnitudes of the force exerted on P by the bowl when P is at points A and B is $4mg$.

(b) Determine, in terms of r, the vertical distance of B above the floor. **[4]**

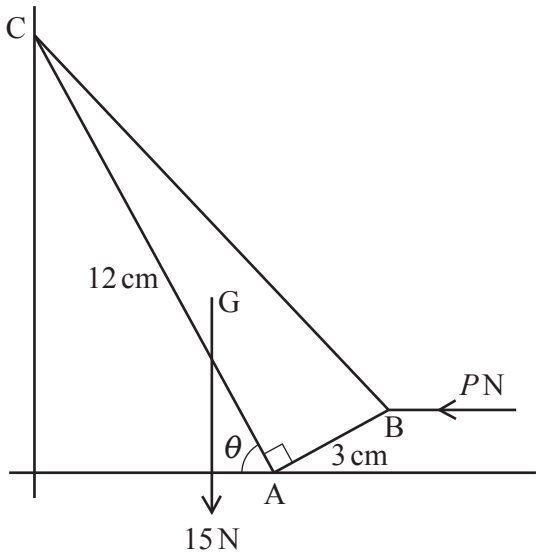
It is given that when P leaves the inner surface of the bowl it does not fall back into the bowl.

(c) Show that $u^2 > 2gr$. **[5]**

© OCR 2023

stemup.co.uk

11



The diagram shows the cross-section through the centre of mass of a uniform solid prism. The cross-section is a right-angled triangle ABC, with AB perpendicular to AC, which lies in a vertical plane. The length of AB is 3 cm, and the length of AC is 12 cm.

The prism is resting in equilibrium on a horizontal surface and against a vertical wall. The side AC of the prism makes an angle θ with the horizontal.

A horizontal force of magnitude P N is now applied to the prism at B. This force acts towards the wall in the vertical plane which passes through the centre of mass G of the prism and is perpendicular to the wall.

The weight of the prism is 15 N and the coefficients of friction between the prism and the surface, and between the prism and the wall, are each $\frac{1}{2}$.

(a) Show that the least value of P needed to move the prism is given by

$$P = \frac{40\cos\theta + 95\sin\theta}{16\sin\theta - 13\cos\theta}.$$ **[8]**

(b) Determine the range in which the value of θ must lie. **[4]**

stemup.co.uk

12 Two small uniform smooth spheres A and B are of equal radius and have masses m and λm respectively. The spheres are on a smooth horizontal surface.

Sphere A is moving on the surface with velocity $u_1\mathbf{i} + u_2\mathbf{j}$ towards B, which is at rest.
The spheres collide obliquely. When the spheres collide, the line joining their centres is parallel to **i**.

The coefficient of restitution between A and B is e.

(a) (i) Explain why, when the spheres collide, the impulse of A on B is in the direction of **i**. **[1]**

(ii) Determine this impulse in terms of λ, m, e and u_1. **[6]**

The loss in kinetic energy due to the collision between A and B is $\frac{1}{8}mu_1^2$.

(b) Determine the range of possible values of λ. **[6]**

© OCR 2023

stemup.co.uk

13 A particle P of mass m is fixed to one end of a light spring of natural length a and modulus of elasticity man^2, where $n > 0$. The other end of the spring is attached to the ceiling of a lift. The lift is at rest and P is hanging vertically in equilibrium.

(a) Find, in terms of g and n, the extension in the spring. **[3]**

At time $t = 0$ the lift begins to accelerate upwards from rest. At time t, the upward displacement of the lift from its initial position is y and the extension of the spring is x.

(b) Express, in terms of g, n, x and y, the upward displacement of P from its initial position at time t. **[2]**

(c) Given that $\ddot{y} = kt$, where k is a positive constant, express the upward acceleration of P in terms of $\ddot{x}$, k and t. **[1]**

(d) Show that x satisfies the differential equation

$$\ddot{x} + n^2 x = kt + g.$$ **[3]**

(e) Verify that $x = \frac{1}{n^3}(knt + gn - k\sin(nt))$. **[4]**

(f) By considering $\dot{x}$ comment on the motion of P relative to the ceiling of the lift for all times after the lift begins to move. **[2]**

END OF QUESTION PAPER

© OCR 2023

stemup.co.uk

OCR
Oxford Cambridge and RSA

Copyright Information

OCR is committed to seeking permission to reproduce all third-party content that it uses in its assessment materials. OCR has attempted to identify and contact all copyright holders whose work is used in this paper. To avoid the issue of disclosure of answer-related information to candidates, all copyright acknowledgements are reproduced in the OCR Copyright Acknowledgements Booklet. This is produced for each series of examinations and is freely available to download from our public website (www.ocr.org.uk) after the live examination series.

If OCR has unwittingly failed to correctly acknowledge or clear any third-party content in this assessment material, OCR will be happy to correct its mistake at the earliest possible opportunity.

For queries or further information please contact The OCR Copyright Team, The Triangle Building, Shaftesbury Road, Cambridge CB2 8EA.

OCR is part of Cambridge University Press & Assessment, which is itself a department of the University of Cambridge.

© OCR 2023

stemup.co.uk

Surname	StemUp Model Solution
Answer booklet	Write the question number in the left-hand margin at the start of each answer.

Question number

1

Driving force $D = \frac{P}{v} = \frac{27000}{15} = 1800$ N

N2L along the road: $D - 600 = 800a$

$1800 - 600 = 800a$

$a = \frac{1200}{800} = 1.5$ m s^{-2}

2

Take the original direction of motion as positive; v = velocity of B after.

Momentum before = $0.5(6) + 2(2) = 7$ N s

Case 1: A still moving forwards at 0.2 m s^{-1}

CLM: $7 = 0.5(0.2) + 2v \Rightarrow 2v = 6.9 \Rightarrow v = 3.45$

Case 2: A moving backwards at 0.2 m s^{-1}

CLM: $7 = 0.5(-0.2) + 2v \Rightarrow 2v = 7.1 \Rightarrow v = 3.55$

Possible speeds of B: 3.45 m s^{-1} or 3.55 m s^{-1}

3 (a)

Resolving vertically: $T \cos\theta = 0.2g$

Greatest θ when $T = 15$: $\cos\theta = \frac{0.2 \times 9.8}{15} = \frac{49}{375}$

$\theta = 82.49...°$ so greatest $\theta = 82°$ (nearest degree)

3 (b)

Radius of circle $r = 0.75 \sin\theta$

N2L horizontally (towards centre): $T \sin\theta = 0.2(0.75 \sin\theta)\omega^2$

$T = 0.15\omega^2$

Greatest ω when $T = 15$: $15 = 0.15\omega^2 \Rightarrow \omega^2 = 100$

Greatest $\omega = 10$ rad s^{-1}

Question number

4 (a) Axes: x along DB (towards B), y along AC (towards C), origin at centre.

Each side = $\sqrt{3^2 + 4^2} = 5$ m. Let α = angle each side makes with AC,

so $\sin\alpha = \frac{4}{5}$, $\cos\alpha = \frac{3}{5}$

Components: AB: $2(0.8, 0.6) = (1.6, 1.2)$

BC: $4(-0.8, 0.6) = (-3.2, 2.4)$ CD: $3(-0.8, -0.6) = (-2.4, -1.8)$

DA: $5(0.8, -0.6) = (4, -3)$

Sum x: $1.6 - 3.2 - 2.4 + 4 = 0$

Sum y: $1.2 + 2.4 - 1.8 - 3 = -1.2$

Equilibrium: $F = (0, 1.2)$, so the horizontal component of F is 0

Magnitude of F = 1.2 N, acting in the direction AC (vertically upwards in the diagram, from A towards C)

4 (b) Moments about A (forces along AB, DA and F pass through A):

4 N at B: perpendicular distance from A to BC = $6\sin\alpha = 4.8$ m

3 N at C: perpendicular distance from A to CD = $6\sin\alpha = 4.8$ m

Moment of forces = $4(4.8) + 3(4.8) = 33.6$ N m (anticlockwise)

The couple must balance this: $G = 33.6$

5 (a) $[\text{force}] = MLT^{-2}$, $[v] = LT^{-1}$

kv is a force, so $[k] = \frac{MLT^{-2}}{LT^{-1}}$

$[k] = MT^{-1}$

5 (b) Each term must have the dimensions of T:

$[k^2 m^\alpha u^\beta F^\gamma] = T$

$(MT^{-1})^2 M^\alpha (LT^{-1})^\beta (MLT^{-2})^\gamma = T$

Question number

$M^{2+\alpha+\gamma} L^{\beta+\gamma} T^{-2-\beta-2\gamma} = M^0 L^0 T^1$

M: $2 + \alpha + \gamma = 0$

L: $\beta + \gamma = 0$

T: $-2 - \beta - 2\gamma = 1$

From L: $\beta = -\gamma$, so T gives $-2 + \gamma - 2\gamma = 1 \Rightarrow \gamma = -3$

$\beta = 3$, $\alpha = -2 - \gamma = 1$

$\alpha = 1$, $\beta = 3$, $\gamma = -3$

6 (a) $v = \frac{dr}{dt} = (4t - 12)i + (3t^2 + 6t - 8)j$

6 (b) Parallel to $i - 4j \Rightarrow$ (j component) $= -4 \times$ (i component)

$3t^2 + 6t - 8 = -4(4t - 12)$

$3t^2 + 6t - 8 = -16t + 48$

$3t^2 + 22t - 56 = 0$

$(3t + 28)(t - 2) = 0 \quad \Rightarrow \quad t = 2 \quad (t \geq 0)$

$t = 2$: $v = (8 - 12)i + (12 + 12 - 8)j = -4i + 16j$

speed $= \sqrt{(-4)^2 + 16^2} = \sqrt{272} = 4\sqrt{17}$

speed $= 16.5 \text{ m s}^{-1}$ (3 s.f.)

6 (c) $a = \frac{dv}{dt} = 4i + (6t + 6)j$

$|a| = 20.2$: $4^2 + (6t + 6)^2 = 20.2^2$

$(6t + 6)^2 = 408.04 - 16 = 392.04$

$6t + 6 = 19.8$ (positive since $t \geq 0$)

$t = 2.3$

7 (a) Perpendicular to the ramp: $R = 2g \cos 30° = \sqrt{3}g$

Question number

Friction $F = \mu R = \frac{1}{2\sqrt{3}} \times \sqrt{3}g = \frac{1}{2}g$

Work done against friction = $Fd = \frac{1}{2}gd$ J

7 (b)

GPE lost by B = $4gd$

GPE gained by A = $2g(d \sin 30°) = gd$

KE gained by A and B = $\frac{1}{2}(2)(1.75^2) + \frac{1}{2}(4)(1.75^2) = 9.1875$ J

Work–energy principle:

KE gained = GPE lost − GPE gained − work against friction

$9.1875 = 4gd - gd - \frac{1}{2}gd = 2.5gd$

$d = \frac{9.1875}{2.5 \times 9.8}$

$d = 0.375$

7 (c)

Include the mass of the rope (or friction at the pulley).

8 (a)

Area under curve = $\int_0^4 \sqrt{3x + 4}\, dx = \left[\frac{2}{9}(3x + 4)^{3/2}\right]_0^4$

$= \frac{2}{9}(64 - 8) = \frac{112}{9}$

Moment about x-axis: $\frac{1}{2}\int_0^4 y^2\, dx = \frac{1}{2}\int_0^4 (3x + 4)\, dx$

$= \frac{1}{2}\left[\frac{3}{2}x^2 + 4x\right]_0^4 = \frac{1}{2}(24 + 16) = 20$

Triangle with vertices (k, 0), (4, 0), (4, 4) is removed:

Question number

area $= \frac{1}{2}(4-k)(4) = 2(4-k)$, centre of mass $\frac{4}{3}$ above x-axis

Area of R $= \frac{112}{9} - 2(4-k) = \frac{40}{9} + 2k$

Moments: $(\frac{40}{9} + 2k)\bar{y} = 20 - 2(4-k) \times \frac{4}{3}$

$(\frac{40}{9} + 2k)\bar{y} = \frac{28}{3} + \frac{8k}{3}$

Multiply by 9: $(40 + 18k)\bar{y} = 84 + 24k$

$\bar{y} = \frac{42 + 12k}{20 + 9k}$

$a = 42,\ b = 12,\ c = 20,\ d = 9$

8 (b)

Suppose $\frac{42 + 12k}{20 + 9k} = \frac{3}{2}$

$84 + 24k = 60 + 27k$

$k = 8$

But $0 < k < 4$, so $k = 8$ is impossible.

$\therefore$ the y-coordinate of the centre of mass cannot be $\frac{3}{2}$

9 (a)

$\sin\alpha = \frac{5}{13}, \cos\alpha = \frac{12}{13}$; $\sin\theta = \frac{3}{5}, \cos\theta = \frac{4}{5}$

Axes: x up the plane along OA, y perpendicular to the plane.

Initial velocity: x: $20\cos\theta = 16$, y: $20\sin\theta = 12$

Acceleration: x: $-g\sin\alpha = -\frac{50}{13}$, y: $-g\cos\alpha = -\frac{120}{13}$

Question number

Hits plane when $y = 0$: $12t - \frac{1}{2}\left(\frac{120}{13}\right)t^2 = 0$

$t\left(12 - \frac{60}{13}t\right) = 0 \quad \Rightarrow \quad t = \frac{12 \times 13}{60} = 2.6 \text{ s}$

Parallel to plane: $\dot{x} = 16 - \frac{50}{13}(2.6) = 16 - 10 = 6 \text{ m s}^{-1}$

Perpendicular: $\dot{y} = 12 - \frac{120}{13}(2.6) = 12 - 24 = -12 \text{ m s}^{-1}$

So just before A: 6 m s^{-1} up the plane, 12 m s^{-1} towards the plane.

$OA = 16(2.6) - \frac{1}{2}\left(\frac{50}{13}\right)(2.6)^2$

$OA = 41.6 - 13 = 28.6 \text{ m}$

9 (b)

Plane smooth: parallel component stays 6 m s^{-1}.

Perpendicular component after = 12e, where e is the coefficient of restitution.

Angle with the plane = $\tan^{-1}\left(\frac{12e}{6}\right) = \tan^{-1}(2e)$, so $\beta = \alpha + \tan^{-1}(2e)$

β is greatest when $e = 1$ (perfectly elastic):

$\beta = \tan^{-1}\left(\frac{5}{12}\right) + \tan^{-1}(2) = 22.62...^\circ + 63.43...^\circ = 86.05...^\circ$

Maximum $\beta = 86^\circ$ (nearest degree)

10 (a)

Take the level of O as zero GPE. A is r sin 30° = $\frac{1}{2}r$ above O;

B is r cos θ below O. Let v be the speed at B.

Conservation of energy:

$\frac{1}{2}mu^2 + mg\left(\frac{1}{2}r\right) = \frac{1}{2}mv^2 - mgr\cos\theta$

Question number

$v^2 = u^2 + gr + 2gr\cos\theta$

N2L towards O at B: $R - mg\cos\theta = \frac{mv^2}{r}$

$R = mg\cos\theta + \frac{m}{r}(u^2 + gr + 2gr\cos\theta)$

$R = 3mg\cos\theta + mg + \frac{mu^2}{r}$

10 (b) At A, AO is 30° below the horizontal; weight has component $mg\sin 30°$

towards O. N2L towards O: $R_A + mg\cos 60° = \frac{mu^2}{r}$

$R_A = \frac{mu^2}{r} - \frac{1}{2}mg$

$R_B - R_A = 4mg$:

$(3mg\cos\theta + mg + \frac{mu^2}{r}) - (\frac{mu^2}{r} - \frac{1}{2}mg) = 4mg$

$3\cos\theta + \frac{3}{2} = 4 \quad \Rightarrow \quad \cos\theta = \frac{5}{6}$

The floor is r below O, and B is $r\cos\theta = \frac{5}{6}r$ below O.

Vertical distance of B above floor $= r - \frac{5}{6}r = \frac{1}{6}r$

10 (c) P leaves the surface at the other end A′ of the rim diameter, at the same height as A, so by energy its speed there is u.

Its velocity is perpendicular to OA′, i.e. at 60° above the horizontal, directed back across the bowl.

Diameter of the rim $= 2r\cos 30° = r\sqrt{3}$

Question number

Horizontally: $r\sqrt{3} = (u \cos 60°)t \quad \Rightarrow \quad t = \frac{2r\sqrt{3}}{u}$

Vertically at this time: $y = (u \sin 60°)t - \frac{1}{2}gt^2$

$y = \frac{\sqrt{3}}{2}u \times \frac{2r\sqrt{3}}{u} - \frac{1}{2}g \times \frac{12r^2}{u^2} = 3r - \frac{6gr^2}{u^2}$

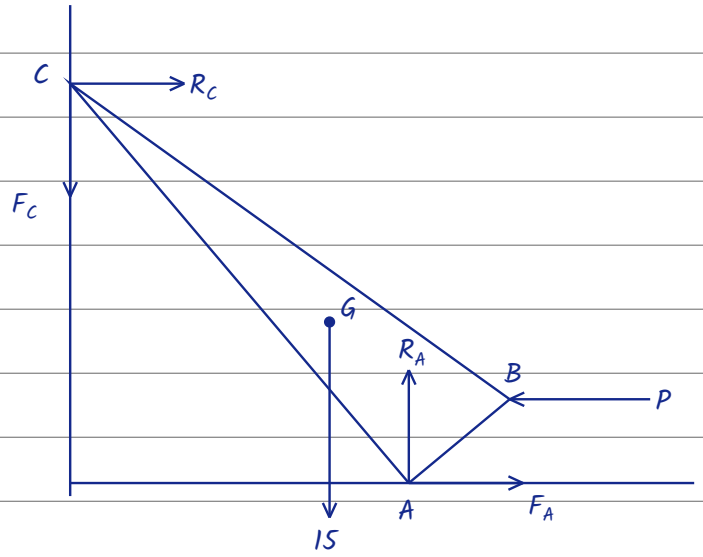
To not fall back in, P must still be above the rim level here: $y > 0$

$3r - \frac{6gr^2}{u^2} > 0$

$\frac{3r}{u^2}(u^2 - 2gr) > 0 \quad$ and $\frac{3r}{u^2} > 0$

$u^2 > 2gr \quad$ as required

11 (a)



When P is least, the prism is about to slip: A slides towards the wall and C slides up the wall, so friction is limiting at both contacts.

Resolve vertically: $R_A = 15 + F_C$

Resolve horizontally: $P = F_A + R_C$

Question number

Limiting friction: $F_A = \frac{1}{2}R_A, \quad F_C = \frac{1}{2}R_C$

Coordinates from A (x away from wall, y up):

$B = (3\sin\theta, 3\cos\theta), \quad C = (-12\cos\theta, 12\sin\theta)$

$G = \frac{1}{3}(A + B + C) = (\sin\theta - 4\cos\theta, \ \cos\theta + 4\sin\theta)$

Moments about A (anticlockwise +):

$P(3\cos\theta) - 15(\sin\theta - 4\cos\theta) - R_C(12\sin\theta) + F_C(12\cos\theta) = 0$

$3P\cos\theta + 60\cos\theta - 15\sin\theta = 12R_C\sin\theta - 6R_C\cos\theta \quad (1)$

From the resolving equations: $R_A = 15 + \frac{1}{2}R_C$

$P = \frac{1}{2}(15 + \frac{1}{2}R_C) + R_C = 7.5 + \frac{5}{4}R_C \quad \Rightarrow \quad R_C = \frac{4P - 30}{5}$

Sub into (1) and multiply by 5:

$15P\cos\theta + 300\cos\theta - 75\sin\theta = (4P - 30)(12\sin\theta - 6\cos\theta)$

$15P\cos\theta + 300\cos\theta - 75\sin\theta$

$= 48P\sin\theta - 24P\cos\theta - 360\sin\theta + 180\cos\theta$

$120\cos\theta + 285\sin\theta = 48P\sin\theta - 39P\cos\theta$

$P = \frac{120\cos\theta + 285\sin\theta}{48\sin\theta - 39\cos\theta}$

$P = \frac{40\cos\theta + 95\sin\theta}{16\sin\theta - 13\cos\theta}$ as required

11 (b) $P > 0$ and the numerator is positive for $0 < \theta < 90°$, so

$16\sin\theta - 13\cos\theta > 0 \quad \Rightarrow \quad \tan\theta > \frac{13}{16} \quad \Rightarrow \quad \theta > 39.09...°$

Before P is applied, the prism rests against the wall, so the weight must act on the wall side of A (or through A): x-coordinate of $G \leq 0$

Question number

$\sin\theta - 4\cos\theta \leq 0 \quad \Rightarrow \quad \tan\theta \leq 4 \quad \Rightarrow \quad \theta \leq 75.96...°$

$39.1° < \theta \leq 76.0°$ (3 s.f.)

12 (a)(i) The spheres are smooth, so the only force between them acts along the line of centres, which is parallel to i. So the impulse on B is in the i direction.

12 (a)(ii) Let v_A, v_B be the i components of velocity after impact.

CLM (i direction): $mu_1 = mv_A + \lambda mv_B \quad \Rightarrow \quad u_1 = v_A + \lambda v_B$

NEL: $v_B - v_A = eu_1$

Adding: $u_1 + eu_1 = v_B + \lambda v_B$

$$v_B = \frac{u_1(1+e)}{1+\lambda}$$

Impulse on B = change in momentum of B $= \lambda mv_B - 0$

$$\text{Impulse} = \frac{\lambda m(1+e)u_1}{1+\lambda}\,i$$

12 (b) The j component of A (u_2) is unchanged, so only the i motion loses KE.

$$v_A = v_B - eu_1 = \frac{u_1(1+e) - eu_1(1+\lambda)}{1+\lambda} = \frac{u_1(1-e\lambda)}{1+\lambda}$$

$$\text{Loss of KE} = \frac{1}{2}mu_1^2 - \frac{1}{2}mv_A^2 - \frac{1}{2}\lambda mv_B^2 = \frac{1}{8}mu_1^2$$

$$1 - \frac{(1-e\lambda)^2}{(1+\lambda)^2} - \frac{\lambda(1+e)^2}{(1+\lambda)^2} = \frac{1}{4}$$

$$(1+\lambda)^2 - (1-e\lambda)^2 - \lambda(1+e)^2 = \frac{1}{4}(1+\lambda)^2$$

$$LHS = 1 + 2\lambda + \lambda^2 - 1 + 2e\lambda - e^2\lambda^2 - \lambda - 2e\lambda - e^2\lambda$$

$$= \lambda + \lambda^2 - e^2\lambda - e^2\lambda^2 = \lambda(1+\lambda)(1-e^2)$$

Question number

So $\frac{\lambda(1-e^2)}{1+\lambda} = \frac{1}{4}$

$4\lambda - 4\lambda e^2 = 1 + \lambda \quad \Rightarrow \quad e^2 = \frac{3\lambda - 1}{4\lambda}$

Since $0 \leq e^2 \leq 1$:

$e^2 \geqslant 0 \quad \Rightarrow \quad 3\lambda - 1 \geqslant 0 \quad \Rightarrow \quad \lambda \geqslant \frac{1}{3}$

$e^2 \leqslant 1 \quad \Rightarrow \quad 3\lambda - 1 \leqslant 4\lambda \quad \Rightarrow \quad \lambda \geqslant -1$, true for all $\lambda > 0$

Range: $\lambda \geqslant \frac{1}{3}$

13 (a) Let e_0 be the extension. Hooke: $T = \frac{man^2e_0}{a} = mn^2e_0$

Equilibrium: $T = mg$

$mn^2e_0 = mg \quad \Rightarrow \quad e_0 = \frac{g}{n^2}$

13 (b) Let z be the upward displacement of P. Distance of P below the ceiling:

initially $a + \frac{g}{n^2}$, at time t: $a + x$

$z = y - (a + x) + (a + \frac{g}{n^2})$

$z = y + \frac{g}{n^2} - x$

13 (c) $\ddot{y} = kt$, so upward acceleration of P $= \ddot{z} = \ddot{y} - \ddot{x} = kt - \ddot{x}$

Question number

13 (d) Tension $T = \frac{ma n^2 x}{a} = mn^2x$

N2L upwards for P: $T - mg = m\ddot{z}$

$mn^2x - mg = m(kt - \ddot{x})$

$n^2x - g = kt - \ddot{x}$

$\ddot{x} + n^2x = kt + g$ as required

13 (e) $x = \frac{1}{n^3}(knt + gn - k\sin(nt))$

$\dot{x} = \frac{1}{n^3}(kn - kn\cos(nt))$

$\ddot{x} = \frac{1}{n^3}(kn^2\sin(nt)) = \frac{k\sin(nt)}{n}$

$\ddot{x} + n^2x = \frac{k\sin(nt)}{n} + \frac{knt + gn - k\sin(nt)}{n}$

$= \frac{knt + gn}{n} = kt + g$ so the differential equation is satisfied

Initial conditions: $t = 0$: $x = \frac{gn}{n^3} = \frac{g}{n^2}$ (equilibrium extension, part (a))

$\dot{x} = \frac{1}{n^3}(kn - kn\cos 0) = 0$ (P starts at rest relative to the lift)

Both conditions hold, so $x = \frac{1}{n^3}(knt + gn - k\sin(nt))$ is verified.

13 (f) $\dot{x} = \frac{k}{n^2}(1 - \cos(nt))$ and $\cos(nt) \leq 1$ for all t, so $\dot{x} \geq 0$

The extension never decreases, so P never moves towards the ceiling:

P moves away from (or is momentarily at rest relative to) the ceiling.