OCR
Oxford Cambridge and RSA

Thursday 25 May 2023 – Afternoon

A Level Further Mathematics B (MEI)

Y420/01 Core Pure

Time allowed: 2 hours 40 minutes

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You must have:
- the Printed Answer Booklet
- the Formulae Booklet for Further Mathematics B (MEI)
- a scientific or graphical calculator

QP

INSTRUCTIONS
- Use black ink. You can use an HB pencil, but only for graphs and diagrams.
- Write your answer to each question in the space provided in the **Printed Answer Booklet**. If you need extra space use the lined pages at the end of the Printed Answer Booklet. The question numbers must be clearly shown.
- Fill in the boxes on the front of the Printed Answer Booklet.
- Answer **all** the questions.
- Where appropriate, your answer should be supported with working. Marks might be given for using a correct method, even if your answer is wrong.
- Give your final answers to a degree of accuracy that is appropriate to the context.
- Do **not** send this Question Paper for marking. Keep it in the centre or recycle it.

INFORMATION
- The total mark for this paper is **144**.
- The marks for each question are shown in brackets **[ ]**.
- This document has **8** pages.

ADVICE
- Read each question carefully before you start your answer.

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Section A (40 marks)

1 **(a)** The complex number $a+\mathrm{i}b$ is denoted by z.

(i) Write down z^*. **[1]**

(ii) Find $\mathrm{Re}(\mathrm{i}z)$. **[2]**

(b) The complex number w is given by $w = \dfrac{5+\mathrm{i}\sqrt{3}}{2-\mathrm{i}\sqrt{3}}$.

(i) **In this question you must show detailed reasoning.**

Express w in the form $x+\mathrm{i}y$. **[2]**

(ii) Convert w to modulus-argument form. **[2]**

2 **In this question you must show detailed reasoning.**

Find the angle between the vector $3\mathbf{i}+2\mathbf{j}+\mathbf{k}$ and the plane $-x+3y+2z=8$. **[5]**

3 **(a)** Using partial fractions and the method of differences, show that

$$\frac{1}{1\times3}+\frac{1}{2\times4}+\frac{1}{3\times5}+\ldots+\frac{1}{n(n+2)}=\frac{3}{4}-\frac{an+b}{2(n+1)(n+2)},$$

where a and b are integers to be determined. **[5]**

(b) Deduce the sum to infinity of the series.

$$\frac{1}{1\times3}+\frac{1}{2\times4}+\frac{1}{3\times5}+\ldots.$$ **[1]**

4 **(a)** **(i)** Given that $\mathrm{f}(x)=\sqrt{1+2x}$, find $\mathrm{f}'(x)$ and $\mathrm{f}''(x)$. **[2]**

(ii) Hence, find the first three terms of the Maclaurin series for $\sqrt{1+2x}$. **[2]**

(b) Hence, using a suitable value for x, show that $\sqrt{5}\approx\dfrac{143}{64}$. **[2]**

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5 **(a)** **In this question you must show detailed reasoning.**

Determine the sixth roots of -64, expressed in $re^{i\theta}$ form. **[4]**

(b) Represent the roots on an Argand diagram. **[3]**

6 The matrices **M** and **N** are $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$ respectively.

(a) **In this question you must show detailed reasoning.**

Determine whether **M** and **N** commute under matrix multiplication. **[3]**

(b) Specify the transformation of the plane associated with each of the following matrices.

(i) **M** **[1]**

(ii) **N** **[2]**

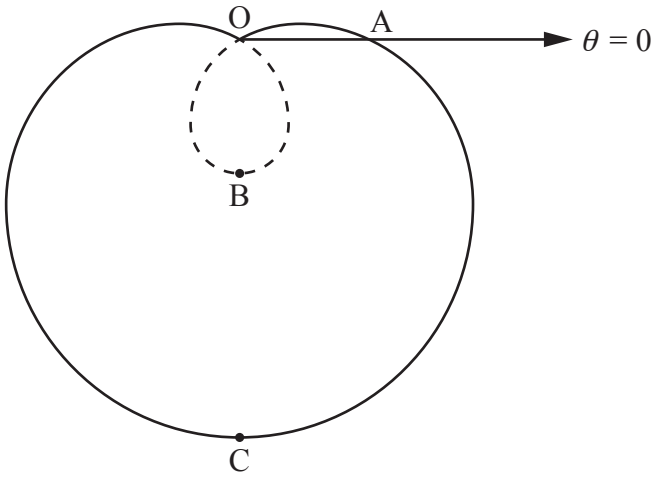
(c) State the significance of the result in part **(a)** for the transformations associated with **M** and **N**. **[1]**

(d) Use an algebraic method to show that all lines parallel to the x-axis are invariant lines of the transformation associated with **N**. **[2]**

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Section B (104 marks)

7 The diagram below shows the curve with polar equation $r = a(1 - 2\sin\theta)$ for $0 \leqslant \theta \leqslant 2\pi$, where a is a positive constant.



The curve crosses the initial line at A, and the points B and C are the lowest points on the two loops.

(a) Find the values of r and θ at the points A, B and C. **[3]**

(b) Find the set of values of θ for the points on the inner loop (shown in the diagram with a broken line). **[3]**

8 Prove by mathematical induction that $8^n - 3^n$ is divisible by 5 for all positive integers n. **[5]**

9 In an electrical circuit, the alternating current I amps is given by $I = a\sin nt$, where t is the time in seconds and a and n are positive constants. The RMS value of the current, in amps, is defined to be the square root of the mean value of I^2 over one complete period of $\frac{2\pi}{n}$ seconds.

Show that the RMS value of the current is $\frac{a}{\sqrt{2}}$ amps. **[6]**

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10 The equation $x^3 - 4x^2 + 7x + c = 0$, where c is a constant, has roots α, β and $\alpha + \beta$.

(a) Determine the roots of the equation. **[6]**

(b) Find c. **[1]**

11 Solve the differential equation $\cosh x \dfrac{\mathrm{d}y}{\mathrm{d}x} - 2y \sinh x = \cosh x$, given that $y = 1$ when $x = 0$. **[7]**

12 Show that $\sin^5\theta = a\sin 5\theta + b\sin 3\theta + c\sin\theta$, where a, b and c are constants to be determined. **[7]**

13 **(a)** On separate Argand diagrams, show the set of points representing each of the following inequalities.

(i) $|z| \leqslant \sqrt{5}$ **[3]**

(ii) $|z + 2 - 4\mathrm{i}| \geqslant |z - 2 - 6\mathrm{i}|$ **[3]**

(b) Show that there is a unique value of z, which should be determined, for which both $|z| \leqslant \sqrt{5}$ and $|z + 2 - 4\mathrm{i}| \geqslant |z - 2 - 6\mathrm{i}|$. **[8]**

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14 Three planes have equations

$$\begin{aligned} kx \quad\quad - \quad z &= 2, \\ -x + ky + 2z &= 1, \\ 2kx + 2y + 3z &= 0, \end{aligned}$$

where k is a constant.

(a) By considering a suitable determinant, show that the three planes meet at a point for all values of k. **[5]**

(b) Using a matrix method, find, in terms of k, the coordinates of the point of intersection of the planes. **[8]**

15 **In this question you must show detailed reasoning.**

Evaluate $\displaystyle\int_1^2 \frac{1}{\sqrt{1+2x-x^2}}\,\mathrm{d}x$, giving your answer in terms of π. **[5]**

16 The point P (4, 1, 0) is equidistant from the plane $2x+y+2z=0$ and the line $\dfrac{x-3}{2}=\dfrac{y-1}{b}=\dfrac{z+5}{3}$, where $b>0$.

Determine the value of b. **[10]**

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17 Two similar species, X and Y, of a small mammal compete for food and habitat. A model of this competition assumes, in a particular area, the following.

- In the absence of the other species, each species would increase at a rate proportional to the number present with the same constant of proportionality in each case.
- The competition reduces the rate of increase of each species by an amount proportional to the number of the other species present.

So if the numbers of species X and Y present at time t years are x and y respectively, the model gives the differential equations

$$\frac{\mathrm{d}x}{\mathrm{d}t} = kx - ay \quad \text{and} \quad \frac{\mathrm{d}y}{\mathrm{d}t} = ky - bx,$$

where k, a and b are positive constants.

(a) (i) Show that the general solution for x is $x = A\mathrm{e}^{(k+n)t} + B\mathrm{e}^{(k-n)t}$, where $n = \sqrt{ab}$ and A and B are arbitrary constants. **[6]**

(ii) Hence find the general solution for y in terms of A, B, k, n, a and t. **[2]**

Observations suggest that suitable values for the model are $k = 0.015$, $a = 0.04$ and $b = 0.01$. You should use these values in the rest of this question.

(b) When $t = 0$, the numbers present of species X and Y in this area are x_0 and y_0 respectively.

(i) Show that $x = \frac{1}{2}(x_0 - 2y_0)\mathrm{e}^{0.035t} + \frac{1}{2}(x_0 + 2y_0)\mathrm{e}^{-0.005t}$. **[3]**

(ii) Hence show that $y = \frac{1}{4}(x_0 + 2y_0)\mathrm{e}^{-0.005t} - \frac{1}{4}(x_0 - 2y_0)\mathrm{e}^{0.035t}$. **[1]**

(c) Use initial values $x_0 = 500$ and $y_0 = 300$ with the results in part **(b)** to determine what the model predicts for each of the following questions.

(i) What numbers of each species will be present after 25 years? **[2]**

(ii) **In this question you must show detailed reasoning.**

When will the numbers of the two species be equal? **[4]**

(iii) Does either species ever disappear from the area? Justify your answer. **[3]**

(d) Different initial values will apply in other areas where the two species compete, but previous studies indicate that one species or the other will eventually dominate in any given area.

(i) Identify a relationship between x_0 and y_0 where the model does **not** predict this outcome. **[1]**

(ii) Explain what the model predicts in the long term for this exceptional case. **[2]**

END OF QUESTION PAPER

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Surname	StemUp Model Solution
Answer booklet	Write the question number in the left-hand margin at the start of each answer.

Question number

1 (a)(i) $z^* = a - bi$

1 (a)(ii) $iz = i(a + bi) = ai + bi^2 = -b + ai$

$\mathrm{Re}(iz) = -b$

1 (b)(i) $w = \dfrac{5 + i\sqrt{3}}{2 - i\sqrt{3}} \times \dfrac{2 + i\sqrt{3}}{2 + i\sqrt{3}}$

$= \dfrac{10 + 5i\sqrt{3} + 2i\sqrt{3} + 3i^2}{4 - 3i^2} = \dfrac{10 - 3 + 7i\sqrt{3}}{4 + 3}$

$= \dfrac{7 + 7i\sqrt{3}}{7}$

$w = 1 + i\sqrt{3}$

1 (b)(ii) $|w| = \sqrt{1^2 + (\sqrt{3})^2} = \sqrt{4} = 2$

$\arg w = \arctan\dfrac{\sqrt{3}}{1} = \dfrac{\pi}{3}$ (w is in the first quadrant)

$w = 2\left(\cos\dfrac{\pi}{3} + i\sin\dfrac{\pi}{3}\right)$

2 Normal to the plane: $n = -i + 3j + 2k$, direction $d = 3i + 2j + k$

$d \cdot n = 3(-1) + 2(3) + 1(2) = -3 + 6 + 2 = 5$

$|d| = \sqrt{3^2 + 2^2 + 1^2} = \sqrt{14}$, $|n| = \sqrt{(-1)^2 + 3^2 + 2^2} = \sqrt{14}$

Angle φ between d and the normal: $\cos\varphi = \dfrac{5}{\sqrt{14}\sqrt{14}} = \dfrac{5}{14}$

Question number

$\varphi = 69.07...°$

Angle between the vector and the plane $= 90° - 69.07...°$

$= 20.9°$ (3 s.f.)

3 (a)

$$\frac{1}{r(r+2)} = \frac{A}{r} + \frac{B}{r+2} \quad \Rightarrow \quad 1 = A(r+2) + Br$$

$r = 0$: $A = \frac{1}{2}$ $\qquad$ $r = -2$: $B = -\frac{1}{2}$

$$\sum_{r=1}^{n} \frac{1}{r(r+2)} = \frac{1}{2}\sum_{r=1}^{n}\left(\frac{1}{r} - \frac{1}{r+2}\right)$$

$$= \frac{1}{2}\left[\left(1 - \frac{1}{3}\right) + \left(\frac{1}{2} - \frac{1}{4}\right) + \left(\frac{1}{3} - \frac{1}{5}\right) + \ldots\right.$$

$$\left. + \left(\frac{1}{n-1} - \frac{1}{n+1}\right) + \left(\frac{1}{n} - \frac{1}{n+2}\right)\right]$$

All terms cancel except the first two positive and last two negative:

$$= \frac{1}{2}\left[1 + \frac{1}{2} - \frac{1}{n+1} - \frac{1}{n+2}\right]$$

$$= \frac{3}{4} - \frac{1}{2}\left(\frac{(n+2)+(n+1)}{(n+1)(n+2)}\right)$$

$$= \frac{3}{4} - \frac{2n+3}{2(n+1)(n+2)}$$

so $a = 2$, $b = 3$

3 (b)

As $n \to \infty$, $\frac{2n+3}{2(n+1)(n+2)} \to 0$

Sum to infinity $= \frac{3}{4}$

Question number

4 (a)(i) $f(x) = (1 + 2x)^{\frac{1}{2}}$

$f'(x) = \frac{1}{2}(1 + 2x)^{-\frac{1}{2}} \times 2 = (1 + 2x)^{-\frac{1}{2}}$

$f''(x) = -\frac{1}{2}(1 + 2x)^{-\frac{3}{2}} \times 2 = -(1 + 2x)^{-\frac{3}{2}}$

4 (a)(ii) $f(0) = 1, \quad f'(0) = 1, \quad f''(0) = -1$

$f(x) \approx f(0) + f'(0)x + \frac{f''(0)}{2!}x^2$

$\sqrt{1 + 2x} \approx 1 + x - \frac{1}{2}x^2$

4 (b) Let $x = \frac{1}{8}$: $\sqrt{1 + \frac{2}{8}} = \sqrt{\frac{5}{4}} = \frac{\sqrt{5}}{2}$

$\frac{\sqrt{5}}{2} \approx 1 + \frac{1}{8} - \frac{1}{2}(\frac{1}{8})^2 = 1 + \frac{1}{8} - \frac{1}{128} = \frac{143}{128}$

$\sqrt{5} \approx 2 \times \frac{143}{128} = \frac{143}{64}$ as required

5 (a) Let $z = re^{i\theta}$. $-64 = 64e^{i\pi}$

$z^6 = r^6e^{6i\theta} = 64e^{i(\pi + 2k\pi)}$

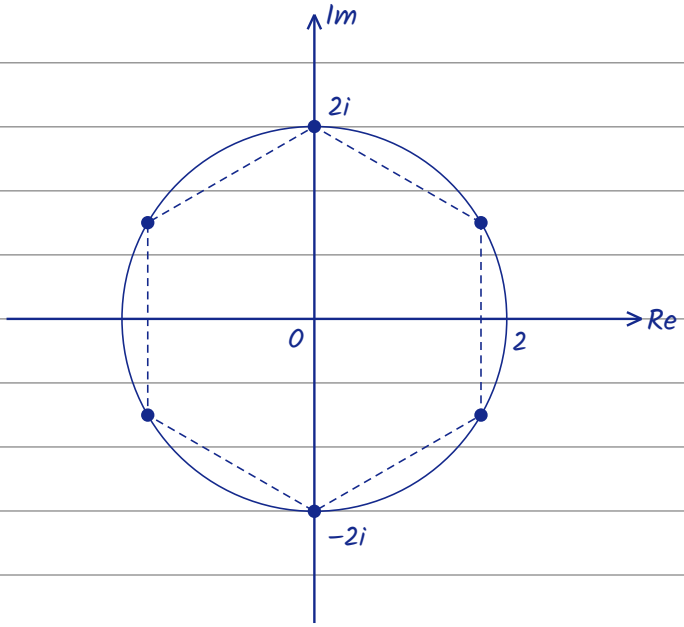
$r^6 = 64 \Rightarrow r = 2$

$6\theta = \pi + 2k\pi \Rightarrow \theta = \frac{\pi}{6} + \frac{k\pi}{3}$

$z = 2e^{\frac{\pi i}{6}}, \quad 2e^{\frac{\pi i}{2}}, \quad 2e^{\frac{5\pi i}{6}}, \quad 2e^{-\frac{\pi i}{6}}, \quad 2e^{-\frac{\pi i}{2}}, \quad 2e^{-\frac{5\pi i}{6}}$

Question number

5 (b)



Six points on the circle $|z| = 2$, equally spaced (angle $\pi/3$ apart), forming a regular hexagon, with roots at $2i$ and $-2i$.

6 (a)

$$MN = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix}$$

$$NM = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix}$$

$MN \neq NM$, so M and N do not commute.

6 (b)(i) M: reflection in the line $y = x$

6 (b)(ii) N: stretch parallel to the x-axis, scale factor 2 (y-axis invariant)

6 (c) The order matters: reflecting then stretching gives a different transformation from stretching then reflecting.

6 (d) A line parallel to the x-axis is $y = c$. Take a point (x, c) on it:

Question number

$\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ c \end{pmatrix} = \begin{pmatrix} 2x \\ c \end{pmatrix}$

The image $(2x, c)$ has the same y-coordinate c, so it lies on $y = c$.

So every line $y = c$ maps onto itself: all lines parallel to the

x-axis are invariant lines of the transformation.

7 (a)

A: on the initial line, $\theta = 0$, $r = a(1 - 0) = a$

B: $\theta = \frac{\pi}{2}$, $r = a(1 - 2) = -a$ (plotted a distance a below O)

C: $\theta = \frac{3\pi}{2}$, $r = a(1 - 2(-1)) = 3a$

7 (b)

The inner loop is where $r < 0$:

$a(1 - 2\sin\theta) < 0 \Rightarrow \sin\theta > \frac{1}{2}$

$\sin\theta = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{6}, \frac{5\pi}{6}$

$\frac{\pi}{6} < \theta < \frac{5\pi}{6}$

8

$n = 1$: $8^1 - 3^1 = 5$, which is divisible by 5, so true for $n = 1$.

Assume true for $n = k$: $8^k - 3^k = 5m$ for some integer m,

so $8^k = 3^k + 5m$.

$n = k + 1$: $8^{k+1} - 3^{k+1} = 8 \times 8^k - 3 \times 3^k$

$= 8(3^k + 5m) - 3 \times 3^k$

$= 5 \times 3^k + 40m$

$= 5(3^k + 8m)$, which is divisible by 5.

So if the result is true for $n = k$, it is true for $n = k + 1$.

Question number

Since it is true for $n = 1$, by induction $8^n - 3^n$ is divisible by 5 for all positive integers n.

9

$$\text{Mean of } I^2 = \frac{1}{2\pi/n}\int_0^{2\pi/n} a^2\sin^2 nt\, dt = \frac{na^2}{2\pi}\int_0^{2\pi/n} \sin^2 nt\, dt$$

$$\sin^2 nt = \frac{1}{2}(1 - \cos 2nt)$$

$$\int_0^{2\pi/n} \sin^2 nt\, dt = \frac{1}{2}\left[t - \frac{1}{2n}\sin 2nt\right]_0^{2\pi/n}$$

$$= \frac{1}{2}\left[\left(\frac{2\pi}{n} - \frac{1}{2n}\sin 4\pi\right) - (0 - 0)\right] = \frac{\pi}{n}$$

$$\text{Mean of } I^2 = \frac{na^2}{2\pi} \times \frac{\pi}{n} = \frac{a^2}{2}$$

$$\text{RMS value} = \sqrt{\frac{a^2}{2}} = \frac{a}{\sqrt{2}} \text{ amps} \quad \text{as required}$$

10 (a)

Sum of roots: $\alpha + \beta + (\alpha + \beta) = 4 \Rightarrow \alpha + \beta = 2$

So the third root is $\alpha + \beta = 2$.

Sum of products in pairs: $\alpha\beta + \alpha(\alpha + \beta) + \beta(\alpha + \beta) = 7$

$\alpha\beta + (\alpha + \beta)^2 = 7 \Rightarrow \alpha\beta + 4 = 7 \Rightarrow \alpha\beta = 3$

α and β are the roots of $x^2 - 2x + 3 = 0$

$$x = \frac{2 \pm \sqrt{4 - 12}}{2} = \frac{2 \pm \sqrt{-8}}{2} = 1 \pm i\sqrt{2}$$

Roots: $1 + i\sqrt{2}$, $1 - i\sqrt{2}$, 2

10 (b)

Product of roots: $\alpha\beta(\alpha + \beta) = -c$

$3 \times 2 = -c \Rightarrow c = -6$

Question number

11

Divide by cosh x: $\frac{dy}{dx} - 2y \tanh x = 1$

Integrating factor $= e^{\int -2 \tanh x \, dx} = e^{-2 \ln \cosh x} = \frac{1}{\cosh^2 x} = \operatorname{sech}^2 x$

$\operatorname{sech}^2 x \frac{dy}{dx} - 2y \operatorname{sech}^2 x \tanh x = \operatorname{sech}^2 x$

$\frac{d}{dx}(y \operatorname{sech}^2 x) = \operatorname{sech}^2 x$

$y \operatorname{sech}^2 x = \int \operatorname{sech}^2 x \, dx = \tanh x + c$

$x = 0, y = 1$: $1 \times 1 = 0 + c \Rightarrow c = 1$

$y \operatorname{sech}^2 x = \tanh x + 1$

$y = \cosh^2 x(\tanh x + 1) = \sinh x \cosh x + \cosh^2 x$

12

Let $z = \cos \theta + i \sin \theta$. Then $z^n - \frac{1}{z^n} = 2i \sin n\theta$

$(z - \frac{1}{z})^5 = (2i \sin \theta)^5 = 32i^5 \sin^5 \theta = 32i \sin^5 \theta$

$(z - \frac{1}{z})^5 = z^5 - 5z^3 + 10z - \frac{10}{z} + \frac{5}{z^3} - \frac{1}{z^5}$

$= (z^5 - \frac{1}{z^5}) - 5(z^3 - \frac{1}{z^3}) + 10(z - \frac{1}{z})$

$= 2i \sin 5\theta - 10i \sin 3\theta + 20i \sin \theta$

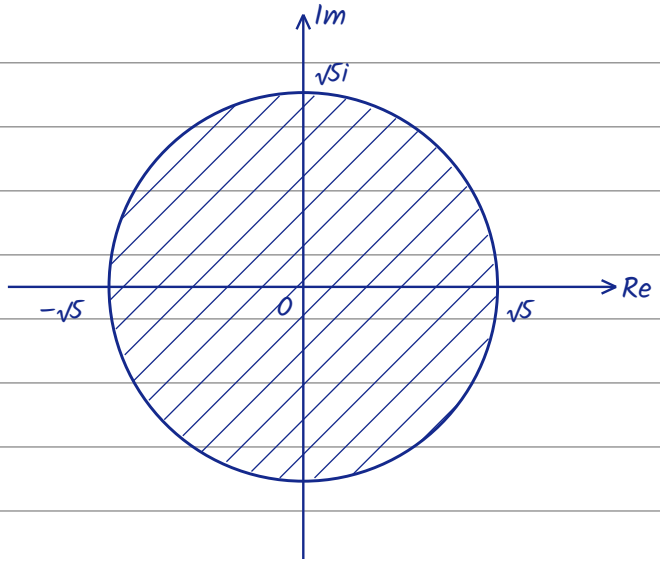
So $32i \sin^5 \theta = 2i \sin 5\theta - 10i \sin 3\theta + 20i \sin \theta$

$\sin^5 \theta = \frac{1}{16} \sin 5\theta - \frac{5}{16} \sin 3\theta + \frac{5}{8} \sin \theta$

$a = \frac{1}{16}, \quad b = -\frac{5}{16}, \quad c = \frac{5}{8}$

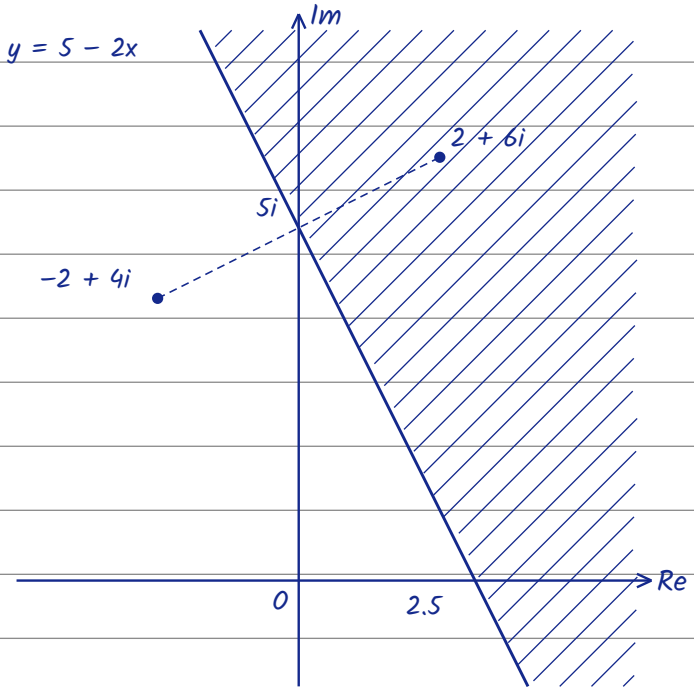
Question number

13 (a)(i)



Circle centre 0, radius √5; the shaded region (including the circle) is the required set.

13 (a)(ii)



Perpendicular bisector of −2 + 4i and 2 + 6i: midpoint 5i, gradient −2, so the line y = 5 − 2x. Shaded side (including the line) is the side nearer to 2 + 6i, i.e. y ≥ 5 − 2x.

13 (b) Gradient of the segment from (−2, 4) to (2, 6) = 2/4 = ½,

Question number

so the perpendicular bisector has gradient -2 and passes through $(0, 5)$:

$y = -2x + 5$

Circle: $x^2 + y^2 = 5$

Substitute: $x^2 + (5 - 2x)^2 = 5$

$x^2 + 25 - 20x + 4x^2 = 5 \Rightarrow 5x^2 - 20x + 20 = 0$

$x^2 - 4x + 4 = 0 \Rightarrow (x - 2)^2 = 0 \Rightarrow x = 2$ only, $y = 5 - 4 = 1$

A repeated root, so the line touches the circle at $(2, 1)$ only.

The disc lies on the side $y \leq 5 - 2x$ of this tangent, so the only point in both regions is the point of contact.

Unique value: $z = 2 + i$

14 (a)

$$M = \begin{pmatrix} k & 0 & -1 \\ -1 & k & 2 \\ 2k & 2 & 3 \end{pmatrix}$$

$\det M = k(3k - 4) - 0(-3 - 4k) + (-1)(-2 - 2k^2)$

$= 3k^2 - 4k + 2 + 2k^2 = 5k^2 - 4k + 2$

Discriminant of $5k^2 - 4k + 2$: $(-4)^2 - 4(5)(2) = 16 - 40 = -24 < 0$

So $5k^2 - 4k + 2$ has no real roots, and $\det M \neq 0$ for all values of k.

M is non-singular, so the three planes meet at a single point for all k.

14 (b)

Let $\Delta = \det M = 5k^2 - 4k + 2$

Matrix of cofactors:

$$\begin{pmatrix} 3k - 4 & 4k + 3 & -2 - 2k^2 \\ -2 & 5k & -2k \\ k & -2k + 1 & k^2 \end{pmatrix}$$

Transpose to get the adjugate, then divide by Δ:

Question number

$$M^{-1} = (1/\Delta)\begin{pmatrix} 3k-4 & -2 & k \\ 4k+3 & 5k & -2k+1 \\ -2-2k^2 & -2k & k^2 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = M^{-1}\begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} = (1/\Delta)\begin{pmatrix} 2(3k-4)-2 \\ 2(4k+3)+5k \\ 2(-2-2k^2)-2k \end{pmatrix} = (1/\Delta)\begin{pmatrix} 6k-10 \\ 13k+6 \\ -4k^2-2k-4 \end{pmatrix}$$

Point of intersection:

$$\left(\frac{6k-10}{5k^2-4k+2}, \frac{13k+6}{5k^2-4k+2}, \frac{-4k^2-2k-4}{5k^2-4k+2}\right)$$

15 $1 + 2x - x^2 = -(x^2 - 2x) + 1 = -[(x-1)^2 - 1] + 1 = 2 - (x-1)^2$

$$\int_1^2 \frac{1}{\sqrt{1+2x-x^2}}\,dx = \int_1^2 \frac{1}{\sqrt{2-(x-1)^2}}\,dx$$

$$= \left[\arcsin\left(\frac{x-1}{\sqrt{2}}\right)\right]_1^2$$

$$= \arcsin\left(\frac{1}{\sqrt{2}}\right) - \arcsin(0)$$

$$= \frac{\pi}{4} - 0 = \frac{\pi}{4}$$

16 Distance from P to plane $= \dfrac{|2(4)+1(1)+2(0)|}{\sqrt{2^2+1^2+2^2}} = \dfrac{9}{3} = 3$

Line: $r = \begin{pmatrix} 3 \\ 1 \\ -5 \end{pmatrix} + \lambda\begin{pmatrix} 2 \\ b \\ 3 \end{pmatrix}$, through $A(3, 1, -5)$, $d = 2i + bj + 3k$

Vector $AP = \begin{pmatrix} 4-3 \\ 1-1 \\ 0-(-5) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 5 \end{pmatrix}$

Question number

$$\overrightarrow{AP} \times \mathbf{d} = \begin{pmatrix} 1 \\ 0 \\ 5 \end{pmatrix} \times \begin{pmatrix} 2 \\ b \\ 3 \end{pmatrix} = \begin{pmatrix} 0 \times 3 - 5b \\ 5 \times 2 - 1 \times 3 \\ 1 \times b - 0 \times 2 \end{pmatrix} = \begin{pmatrix} -5b \\ 7 \\ b \end{pmatrix}$$

Distance from P to line $= \dfrac{|\overrightarrow{AP} \times \mathbf{d}|}{|\mathbf{d}|} = \dfrac{\sqrt{25b^2 + 49 + b^2}}{\sqrt{4 + b^2 + 9}} = \dfrac{\sqrt{26b^2 + 49}}{\sqrt{13 + b^2}}$

Equidistant, so:

$$\frac{26b^2 + 49}{13 + b^2} = 9$$

$$26b^2 + 49 = 117 + 9b^2$$

$$17b^2 = 68 \Rightarrow b^2 = 4$$

$b > 0$, so $b = 2$

17 (a)(i) $\dfrac{dx}{dt} = kx - ay \Rightarrow \dfrac{d^2x}{dt^2} = k\dfrac{dx}{dt} - a\dfrac{dy}{dt}$

$$= k\frac{dx}{dt} - a(ky - bx) = k\frac{dx}{dt} - k(ay) + abx$$

From the first equation, $ay = kx - \dfrac{dx}{dt}$, and $ab = n^2$:

$$\frac{d^2x}{dt^2} = k\frac{dx}{dt} - k\left(kx - \frac{dx}{dt}\right) + n^2x$$

$$\frac{d^2x}{dt^2} - 2k\frac{dx}{dt} + (k^2 - n^2)x = 0$$

Auxiliary equation: $m^2 - 2km + (k^2 - n^2) = 0$

$(m - k)^2 = n^2 \Rightarrow m = k \pm n$

$x = Ae^{(k+n)t} + Be^{(k-n)t}$ as required

17 (a)(ii) $\dfrac{dx}{dt} = A(k + n)e^{(k+n)t} + B(k - n)e^{(k-n)t}$

Question number

$y = \frac{1}{a}(kx - \frac{dx}{dt})$

$= \frac{1}{a}[kAe^{(k+n)t} + kBe^{(k-n)t} - A(k+n)e^{(k+n)t} - B(k-n)e^{(k-n)t}]$

$y = \frac{n}{a}(-Ae^{(k+n)t} + Be^{(k-n)t})$

17 (b)(i) $n = \sqrt{0.04 \times 0.01} = \sqrt{0.0004} = 0.02$

$k + n = 0.035, \quad k - n = -0.005, \quad \frac{n}{a} = \frac{0.02}{0.04} = \frac{1}{2}$

$t = 0: \quad x_0 = A + B, \quad y_0 = \frac{1}{2}(-A + B) \Rightarrow B - A = 2y_0$

Adding: $2B = x_0 + 2y_0 \Rightarrow B = \frac{1}{2}(x_0 + 2y_0)$

Subtracting: $2A = x_0 - 2y_0 \Rightarrow A = \frac{1}{2}(x_0 - 2y_0)$

$x = \frac{1}{2}(x_0 - 2y_0)e^{0.035t} + \frac{1}{2}(x_0 + 2y_0)e^{-0.005t}$ as required

17 (b)(ii) $y = \frac{1}{2}(-\frac{1}{2}(x_0 - 2y_0)e^{0.035t} + \frac{1}{2}(x_0 + 2y_0)e^{-0.005t})$

$= \frac{1}{4}(x_0 + 2y_0)e^{-0.005t} - \frac{1}{4}(x_0 - 2y_0)e^{0.035t}$ as required

17 (c)(i) $x_0 - 2y_0 = 500 - 600 = -100, \quad x_0 + 2y_0 = 500 + 600 = 1100$

$x = -50e^{0.035t} + 550e^{-0.005t}, \quad y = 275e^{-0.005t} + 25e^{0.035t}$

$t = 25: \quad x = -50e^{0.875} + 550e^{-0.125} = 365.4...$

$y = 275e^{-0.125} + 25e^{0.875} = 302.6...$

Question number

About 365 of species X and 302 of species Y.

17 (c)(ii) $-50e^{0.035t} + 550e^{-0.005t} = 275e^{-0.005t} + 25e^{0.035t}$

$275e^{-0.005t} = 75e^{0.035t}$

$e^{0.04t} = \frac{275}{75} = \frac{11}{3}$

$0.04t = \ln(\frac{11}{3}) \Rightarrow t = 25\ln(\frac{11}{3}) = 32.48...$

The numbers are equal after about 32.5 years.

17 (c)(iii) $x = 0$ when $550e^{-0.005t} = 50e^{0.035t} \Rightarrow e^{0.04t} = 11$

$\Rightarrow t = 25 \ln 11 = 59.9...$, which has a solution, since $-50e^{0.035t}$ grows in size while $550e^{-0.005t} \to 0$.

So species X disappears, after about 60 years.

$y = 275e^{-0.005t} + 25e^{0.035t}$ is the sum of two positive terms, so $y > 0$ for all t: species Y never disappears.

17 (d)(i) $x_0 = 2y_0$ (so that the $e^{0.035t}$ terms vanish)

17 (d)(ii) With $x_0 = 2y_0$: $x = 2y_0e^{-0.005t}$, $y = y_0e^{-0.005t}$

Both expressions $\to 0$ as $t \to \infty$, so neither species dominates: both populations decline and die out in the long term.