OCR
Oxford Cambridge and RSA

Monday 27 June 2022 – Afternoon

A Level Further Mathematics B (MEI)

Y436/01 Further Pure with Technology

Time allowed: 1 hour 45 minutes

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You must have:
- the Printed Answer Booklet
- the Formulae Booklet for Further Mathematics B (MEI)
- a computer with appropriate software
- a scientific or graphical calculator

INSTRUCTIONS
- Use black ink. You can use an HB pencil, but only for graphs and diagrams.
- Write your answer to each question in the space provided in the **Printed Answer Booklet**. If you need extra space use the lined pages at the end of the Printed Answer Booklet. The question numbers must be clearly shown.
- Fill in the boxes on the front of the Printed Answer Booklet.
- Answer **all** the questions.
- Where appropriate, your answer should be supported with working. Marks might be given for using a correct method, even if your answer is wrong.
- Give your final answers to a degree of accuracy that is appropriate to the context.
- Do **not** send this Question Paper for marking. Keep it in the centre or recycle it.

INFORMATION
- The total mark for this paper is **60**.
- The marks for each question are shown in brackets **[ ]**.
- This document has **8** pages.

ADVICE
- Read each question carefully before you start your answer.

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Answer **all** the questions.

1 **(a)** A family of curves is given by the equation

$x^2 + y^2 + 2axy = 1$ (*)

where the parameter a is a real number.
You may find it helpful to use a slider (for a) to investigate this family of curves.

(i) On the axes in the Printed Answer Booklet, sketch the curve in each of the cases

- $a = 0$
- $a = 0.5$
- $a = 2$

[3]

(ii) State a feature of the curve for the cases $a = 0$, $a = 0.5$ that is **not** a feature of the curve in the case $a = 2$. **[1]**

(iii) In the case $a = 1$, the curve consists of two straight lines. Determine the equations of these lines. **[2]**

(b) **(i)** Find an equation of the curve (*) in polar form. **[3]**

(ii) Hence, or otherwise, find, in exact form, the area bounded by the curve, the positive part of the x-axis and the positive part of the y-axis, in the case $a = 2$. **[2]**

(c) In this part of the question m is any real number.

Describing all possible cases, determine the pairs of values a and m for which the curve with equation (*) intersects the straight line given by $y = mx$. **[9]**

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2 **(a)** In this part of the question n is an integer greater than 1.

An integer q, where $0 < q < n$ is said to be a quadratic residue modulo n if there exists an integer x such that $x^2 \equiv q \pmod{n}$.

Note that under this definition 0 is not considered to be a quadratic residue modulo n.

(i) Find all the integers x, where $0 \leqslant x < 1000$ which satisfy $x^2 \equiv 481 \pmod{1000}$. **[1]**

(ii) Explain why 481 is a quadratic residue modulo 1000. **[1]**

(iii) Determine the quadratic residues modulo 11. **[2]**

(iv) Determine the quadratic residues modulo 13. **[2]**

(v) Show that, for any integer m, $m^2 \equiv (n-m)^2 \pmod{n}$. **[2]**

(vi) Prove that if p is prime, where $p > 2$, then the number of quadratic residues modulo p is $\frac{p-1}{2}$. **[4]**

(b) Fermat's little theorem states that if p is prime and a is an integer which is co-prime to p, then $a^{p-1} \equiv 1 \pmod{p}$.

(i) Use Fermat's little theorem to show that 91 is not prime. **[2]**

(ii) Create a program to find an integer n between 500 and 600 which is not prime and such that $a^{n-1} \equiv 1 \pmod{n}$ for all integers a which are co-prime to n.
In the Printed Answer Booklet

- Write down your program in full.
- State the integer found by your program.

[6]

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3 In this question you are required to consider the family of differential equations

$$\frac{dy}{dx} = \frac{y^a}{x+1} - \frac{1}{y} \quad (*)$$

and its solutions. The parameter a is a real number.

You should assume that $x \geqslant 0$ and $y > 0$ throughout this question.

(a) In this part of the question $a = 1$.

(i) On the axes in the Printed Answer Booklet

- Sketch the isocline defined by $\frac{dy}{dx} = 0$.
- Shade and label the region in which $\frac{dy}{dx} > 0$.
- Shade and label the region in which $\frac{dy}{dx} < 0$.

[3]

(ii) For $b > 0$, find, in terms of b, the solution to (*) which passes through the point $(0, b)$.

[1]

(iii) Determine

- The values of $b > 0$ for which the solution in **(ii)** has a turning point.
- The corresponding maximum value of y.

[4]

(b) **Fig. 3.1** and **Fig. 3.2** show tangent fields for two distinct but unspecified values of a. In each case a sketch of the solution curve $y = g(x)$ which passes through $(0, 2)$ is shown for $0 \leqslant x \leqslant 0.5$.

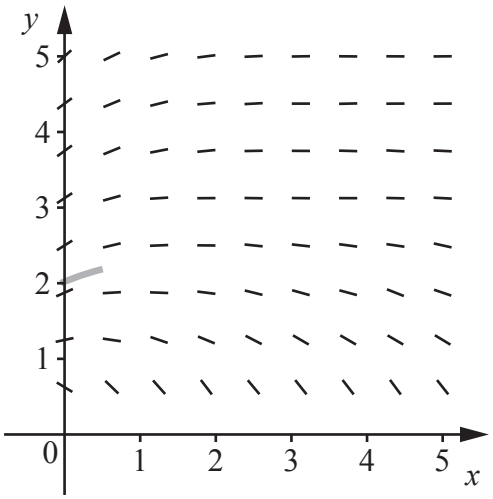


Fig. 3.1

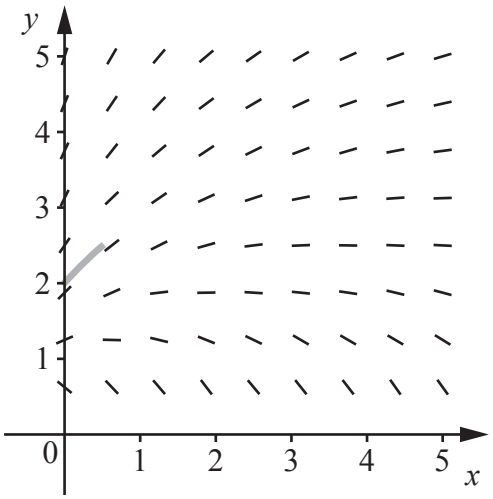


Fig. 3.2

(i) For the case in **Fig. 3.1** suggest a possible value of a. **[1]**

(ii) For the case in **Fig. 3.2** suggest a possible value of a. **[1]**

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(iii) In each case, continue the sketch of the solution curves for $0.5 \leqslant x \leqslant 5$ in the Printed Answer Booklet. **[2]**

(iv) State a feature which is present in one of the curves in part **(iii)** for $0.5 \leqslant x \leqslant 5$ but not in the other. **[1]**

(c) (i) The Euler method for the solution of the differential equation $\frac{dy}{dx} = f(x, y)$ is as follows

$$y_{n+1} = y_n + hf(x_n, y_n).$$

It is given that $x_0 = 0$ and $y_0 = 2$.

- Construct a spreadsheet to solve (*) using the Euler method so that the value of a and the value of h can be varied, in the case $x_0 = 0$ and $y_0 = 2$.
- State the formulae you have used in your spreadsheet.

[3]

(ii) In this part of the question $a = 0.1$.

Use your spreadsheet with $h = 0.1$ to approximate the value of y when $x = 3$ for the solution to (*) in which $y = 2$ when $x = 0$. **[1]**

(iii) In this part of the question $a = -0.2$.

Use your spreadsheet to approximate, to **1** decimal place, the x-coordinate of the local maximum for the solution to (*) in which $y = 2$ when $x = 0$. **[3]**

END OF QUESTION PAPER

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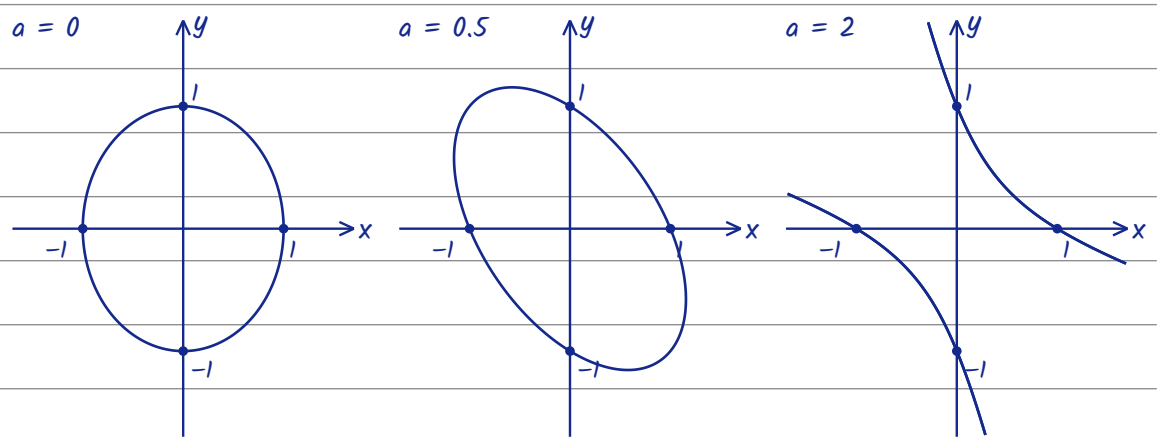
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Surname	StemUp Model Solution
Answer booklet	Write the question number in the left-hand margin at the start of each answer.

Question number

1 (a)(i) $x^2 + y^2 + 2axy = 1$



$a = 0$: circle, centre O, radius 1

$a = 0.5$: ellipse through $(\pm 1, 0)$ and $(0, \pm 1)$, long axis along $y = -x$

$a = 2$: hyperbola through $(\pm 1, 0)$ and $(0, \pm 1)$, asymptotes $y = (-2 \pm \sqrt{3})x$

1 (a)(ii) For $a = 0$ and $a = 0.5$ the curve is a closed (bounded) curve;

for $a = 2$ it is not closed (it goes off to infinity).

1 (a)(iii) $a = 1$: $x^2 + y^2 + 2xy = 1$

$(x + y)^2 = 1$

$x + y = 1$ or $x + y = -1$

The lines are $y = 1 - x$ and $y = -1 - x$

1 (b)(i) $x = r\cos\theta$, $y = r\sin\theta$

$r^2\cos^2\theta + r^2\sin^2\theta + 2a\, r^2 \sin\theta\cos\theta = 1$

$r^2(1 + a\sin 2\theta) = 1$

Question number

$r^2 = \dfrac{1}{1 + a \sin 2\theta}$

1 (b)(ii) $a = 2$: $r^2 = \dfrac{1}{1 + 2 \sin 2\theta}$, region is $0 \leqslant \theta \leqslant \dfrac{\pi}{2}$

$\text{Area} = \dfrac{1}{2}\displaystyle\int_0^{\pi/2} r^2 \, d\theta = \dfrac{1}{2}\int_0^{\pi/2} \dfrac{1}{1 + 2 \sin 2\theta} \, d\theta$

Let $t = \tan \theta$: $\sin 2\theta = \dfrac{2t}{1 + t^2}$, $d\theta = \dfrac{dt}{1 + t^2}$

$\displaystyle\int_0^{\pi/2} \frac{d\theta}{1 + 2 \sin 2\theta} = \int_0^{\infty} \frac{dt}{t^2 + 4t + 1} = \int_0^{\infty} \frac{dt}{(t + 2)^2 - 3}$

$= \dfrac{1}{2\sqrt{3}} \left[\ln \dfrac{t + 2 - \sqrt{3}}{t + 2 + \sqrt{3}}\right]_0^{\infty} = \dfrac{1}{2\sqrt{3}} \left(0 - \ln \dfrac{2 - \sqrt{3}}{2 + \sqrt{3}}\right)$

$= \dfrac{1}{2\sqrt{3}} \ln (2 + \sqrt{3})^2 = \dfrac{1}{2\sqrt{3}} \ln (7 + 4\sqrt{3})$

$\text{Area} = \dfrac{1}{4\sqrt{3}} \ln (7 + 4\sqrt{3}) = \dfrac{\sqrt{3}}{12} \ln (7 + 4\sqrt{3}) \quad (\approx 0.380)$

1 (c) Substitute $y = mx$ into $x^2 + y^2 + 2axy = 1$:

$x^2 + m^2x^2 + 2amx^2 = 1$

$x^2(m^2 + 2am + 1) = 1$

So $x = \pm\dfrac{1}{\sqrt{m^2 + 2am + 1}}$, $y = mx$

These two (distinct) points exist if and only if $m^2 + 2am + 1 > 0$

(if $m^2 + 2am + 1 \leqslant 0$ there is no solution, so no intersection)

$m^2 + 2am + 1 > 0 \Leftrightarrow (m + a)^2 + 1 - a^2 > 0 \Leftrightarrow (m + a)^2 > a^2 - 1$

Case $-1 < a < 1$: $a^2 - 1 < 0$, so this is true for all values of m.

Question number	
	Case $a = 1$: $(m + 1)^2 > 0$, true for all m except $m = -1$.
	Case $a = -1$: $(m - 1)^2 > 0$, true for all m except $m = 1$.
	Case $a < -1$ or $a > 1$: need $\lvert m + a \rvert > \sqrt{a^2 - 1}$, i.e.
	$m < -a - \sqrt{a^2 - 1}$ or $m > -a + \sqrt{a^2 - 1}$
	The line meets the curve (at two points) exactly in these cases.
2 (a)(i)	Program: `for x in range(0, 1000):`
	`    if (x*x) % 1000 == 481: print(x)`
	$x = 59, 191, 309, 441, 559, 691, 809, 941$
2 (a)(ii)	$0 < 481 < 1000$ and there is an integer x with $x^2 \equiv 481 \pmod{1000}$,
	for example $59^2 = 3481 \equiv 481 \pmod{1000}$, so 481 is a quadratic residue.
2 (a)(iii)	$1^2 \equiv 1$, $2^2 \equiv 4$, $3^2 \equiv 9$, $4^2 = 16 \equiv 5$, $5^2 = 25 \equiv 3 \pmod{11}$
	(6^2, 7^2, 8^2, 9^2, 10^2 give the same values as 5^2, 4^2, 3^2, 2^2, 1^2)
	Quadratic residues modulo 11: 1, 3, 4, 5, 9
2 (a)(iv)	$1^2 \equiv 1$, $2^2 \equiv 4$, $3^2 \equiv 9$, $4^2 \equiv 3$, $5^2 \equiv 12$, $6^2 \equiv 10 \pmod{13}$
	(7^2, ..., 12^2 give the same values as 6^2, ..., 1^2)
	Quadratic residues modulo 13: 1, 3, 4, 9, 10, 12
2 (a)(v)	$(n - m)^2 = n^2 - 2mn + m^2 = n(n - 2m) + m^2$
	So $(n - m)^2 - m^2 = n(n - 2m)$, which is a multiple of n
	$\therefore$ $(n - m)^2 \equiv m^2 \pmod{n}$ as required
2 (a)(vi)	Every non-zero quadratic residue is x^2 mod p for some x with $1 \leq x \leq p - 1$
	($x \equiv 0$ gives 0, which is not counted).
	By (v) with $n = p$: $(p - m)^2 \equiv m^2 \pmod{p}$, so

Question number

$1^2 \equiv (p-1)^2,\ 2^2 \equiv (p-2)^2,\ \ldots,\ \left(\frac{p-1}{2}\right)^2 \equiv \left(\frac{p+1}{2}\right)^2 \pmod{p}$

So the $p-1$ values pair up and there are at most $\frac{p-1}{2}$ residues,

namely the squares of $1, 2, \ldots, \frac{p-1}{2}$.

Suppose $1 \leq m \leq n \leq \frac{p-1}{2}$ and $m^2 \equiv n^2 \pmod{p}$.

Then p divides $n^2 - m^2 = (n-m)(n+m)$.

p is prime, so p divides $n-m$ or p divides $n+m$.

But $0 \leq n-m < n+m \leq p-1 < p$, so p cannot divide $n+m$,

hence $n-m = 0$, i.e. $m = n$.

So $1^2, 2^2, \ldots, \left(\frac{p-1}{2}\right)^2$ are all different mod p (and none is 0,

as p is prime and p does not divide x).

Therefore there are exactly $\frac{p-1}{2}$ quadratic residues modulo p,

as required.

2 (b)(i) Take $a = 2$, which is co-prime to 91.

If 91 were prime, Fermat gives $2^{90} \equiv 1 \pmod{91}$.

$2^6 = 64,\quad 2^{12} = 4096 = 45 \times 91 + 1 \equiv 1 \pmod{91}$

$2^{90} = (2^{12})^7 \times 2^6 \equiv 1 \times 64 = 64 \pmod{91}$

$64 \not\equiv 1 \pmod{91}$, so 91 is not prime.

2 (b)(ii) Program (Python):

```
from math import gcd
for n in range(500, 601):
```

Question number

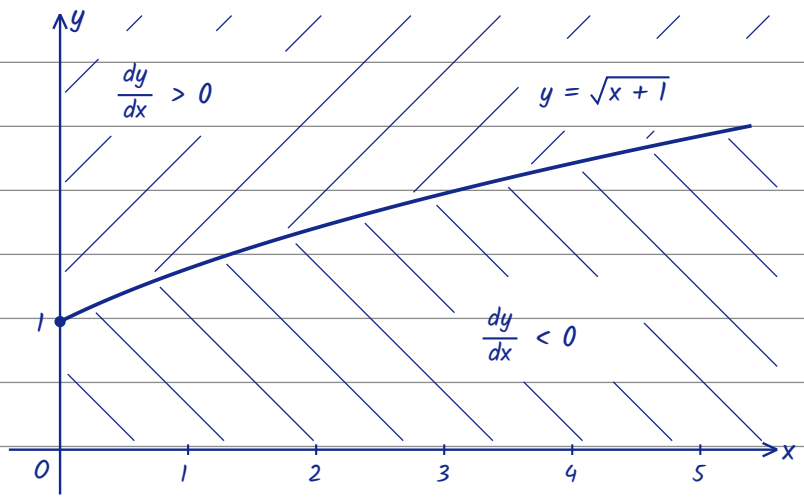
```
    prime = True
    for k in range(2, n):
        if n % k == 0:  prime = False
    if prime == False:
        ok = True
        for a in range(2, n):
            if gcd(a, n) == 1 and pow(a, n - 1, n) != 1:
                ok = False
        if ok == True:
            print(n)
```

Output: 561

$n = 561$ ($= 3 \times 11 \times 17$, not prime, but $a^{560} \equiv 1 \pmod{561}$ for all a co-prime to 561)

3 (a)(i) $a = 1$: $\frac{dy}{dx} = \frac{y}{x+1} - \frac{1}{y} = 0 \Leftrightarrow y^2 = x + 1 \Leftrightarrow y = \sqrt{x+1}$ $(y > 0)$

$\frac{dy}{dx} > 0 \Leftrightarrow y^2 > x + 1$ (above the isocline)



Question number

Isocline $\frac{dy}{dx} = 0$ is $y = \sqrt{x+1}$; region above: $\frac{dy}{dx} > 0$, below: $\frac{dy}{dx} < 0$

3 (a)(ii) $\frac{dy}{dx} = \frac{y}{x+1} - \frac{1}{y}$; let $u = y^2$: $\frac{du}{dx} = \frac{2u}{x+1} - 2$

Integrating factor $(x+1)^{-2}$: $\frac{d}{dx}\left(\frac{u}{(x+1)^2}\right) = -\frac{2}{(x+1)^2}$

$\frac{u}{(x+1)^2} = \frac{2}{x+1} + C$; $x = 0, y = b$: $b^2 = 2 + C \Rightarrow C = b^2 - 2$

$y = \sqrt{(b^2-2)(x+1)^2 + 2(x+1)}$

3 (a)(iii) From (*), $\frac{dy}{dx} = 0 \Leftrightarrow y^2 = x + 1$

$(b^2-2)(x+1)^2 + 2(x+1) = x + 1$

$(x+1)[(b^2-2)(x+1) + 1] = 0$, and $x + 1 \neq 0$

$x + 1 = \frac{1}{2-b^2} \Rightarrow x = \frac{1}{2-b^2} - 1 = \frac{b^2-1}{2-b^2}$

Need $x \geq 0$: $\frac{b^2-1}{2-b^2} \geq 0 \Leftrightarrow 1 \leq b^2 < 2$

So there is a turning point when $1 \leq b < \sqrt{2}$

(y^2 is then a quadratic in $(x+1)$ with negative leading coefficient, so the turning point is a maximum.)

Maximum value: $y = \sqrt{x+1} = \frac{1}{\sqrt{2-b^2}}$

3 (b)(i) $a = 0$ (approximately)

3 (b)(ii) $a = 0.6$ (approximately)

Question number

3 (b)(iii)

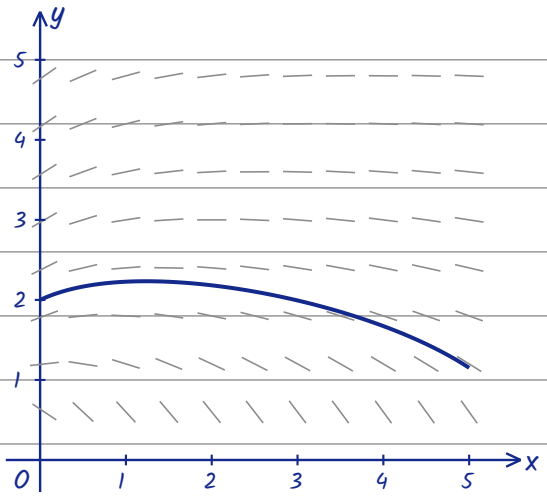


Fig. 3.1

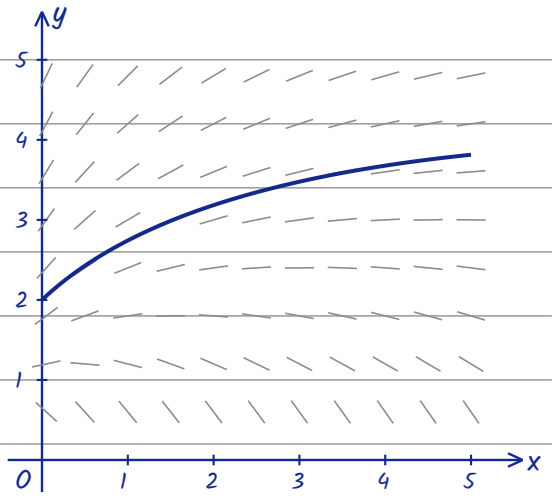


Fig. 3.2

Fig. 3.1: rises to a maximum (near $x = 1$) then decreases, concave down.

Fig. 3.2: increases throughout with no turning point, concave down.

3 (b)(iv) The curve in Fig. 3.1 has a turning point (a local maximum) for $0.5 \leq x \leq 5$; the curve in Fig. 3.2 has no turning point.

3 (c)(i) Cell I1: h (e.g. 0.1) Cell I2: a (e.g. 0.1)

A1: 0 (x_0) B1: 2 (y_0)

C1: =(B1^I2)/(A1 + 1) − 1/B1 (this is $f(x_n, y_n)$)

A2: = A1 + I1

B2: = B1 + I1*C1

Fill C1 down, and A2, B2 down (then C2 down etc.)

3 (c)(ii) $a = 0.1$, $h = 0.1$: reading the row with $x = 3$

$y \approx 2.2225$ ($y \approx 2.22$)

3 (c)(iii) $a = -0.2$, $h = 0.01$ (part of spreadsheet):

x	y	f(x, y)

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0.80	2.125993	0.00740
0.81	2.126067	0.00477
0.82	2.126115	0.00217
0.83	2.126136	−0.00041
0.84	2.126132	−0.00296
0.85	2.126103	−0.00550

y increases then decreases, and dy/dx changes from positive to negative between x = 0.82 and x = 0.84, so there is a local maximum here.

Repeating with smaller h (e.g. h = 0.001) gives the same position.

x-coordinate of local maximum ≈ 0.8 (1 d.p.)