OCR
Oxford Cambridge and RSA

Tuesday 25 June 2019 – Morning

A Level Further Mathematics B (MEI)

Y436/01 Further Pure with Technology

Time allowed: 1 hour 45 minutes

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You must have:
- Printed Answer Booklet
- Formulae Further Mathematics B (MEI)
- Computer with appropriate software

You may use:
- a scientific or graphical calculator

INSTRUCTIONS
- Use black ink. HB pencil may be used for graphs and diagrams only.
- Answer **all** the questions.
- **Write your answer to each question in the space provided in the Printed Answer Booklet.** Additional paper may be used if necessary but you must clearly show your candidate number, centre number and question number(s).
- You are permitted to use a scientific or graphical calculator in this paper.
- Final answers should be given to a degree of accuracy appropriate to the context.

COMPUTING RESOURCES
- Candidates will require access to a computer with a computer algebra system, a spreadsheet, a programming language and graph-plotting software throughout the examination.

INFORMATION
- The total number of marks for this paper is **60**.
- The marks for each question are shown in brackets **[ ]**.
- You are advised that an answer may receive **no marks** unless you show sufficient detail of the working to indicate that a correct method is used. You should communicate your method with correct reasoning.
- The Printed Answer Booklet consists of **12** pages. The Question Paper consists of **8** pages.

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Answer **all** the questions.

1 A family of curves is given by the parametric equations

$$x(t) = \cos(t) - \frac{\cos((m+1)t)}{m+1} \text{ and } y(t) = \sin(t) - \frac{\sin((m+1)t)}{m+1}$$

where $0 \leqslant t < 2\pi$ and m is a positive integer.

(a) (i) Sketch the curves in the cases $m = 3$, $m = 4$ and $m = 5$ on separate axes in the Printed Answer Booklet. **[3]**

(ii) State one common feature of these three curves. **[1]**

(iii) State a feature for the case $m = 4$ which is absent in the cases $m = 3$ and $m = 5$. **[1]**

(b) (i) Determine, in terms of m, the values of t for which $\frac{dx}{dt} = 0$ but $\frac{dy}{dt} \neq 0$. **[4]**

(ii) Describe the tangent to the curve at the points corresponding to such values of t. **[1]**

(c) (i) Show that the curve lies between the circle centred at the origin with radius

$$1 - \frac{1}{m+1}$$

and the circle centred at the origin with radius

$$1 + \frac{1}{m+1}.$$ **[2]**

(ii) Hence, or otherwise, show that the area A bounded by the curve satisfies

$$\frac{m^2\pi}{(m+1)^2} < A < \frac{(m+2)^2\pi}{(m+1)^2}.$$ **[1]**

(iii) Find the limit of the area bounded by the curve as m tends to infinity. **[1]**

(d) The arc length of a curve defined by parametric equations $x(t)$ and $y(t)$ between points corresponding to $t = c$ and $t = d$, where $c < d$, is

$$\int_c^d \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt.$$

Use this to show that the length of the curve is independent of m. **[6]**

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2 **(a)** Prove that if x and y are integers which satisfy $x^2 - 2y^2 = 1$, then x is odd and y is even. **[3]**

(b) Create a program to find, for a fixed positive integer s, all the positive integer solutions (x, y) to the equation $x^2 - 2y^2 = 1$ where $x \leqslant s$ and $y \leqslant s$. Write out your program in the Printed Answer Booklet. **[3]**

(c) Use your program to find all the positive integer solutions (x, y) to the equation $x^2 - 2y^2 = 1$ where $x \leqslant 600$ and $y \leqslant 600$. Give the solutions in ascending order of the value of x. **[1]**

(d) By writing the equation $x^2 - 2y^2 = 1$ in the form $(x + \sqrt{2}y)(x - \sqrt{2}y) = 1$ show how the first solution (the one with the lowest value of x) in your answer to part **(c)** can be used to generate the other solutions you found in part **(c)**. **[4]**

(e) What can you deduce about the number of positive integer solutions (x, y) to the equation $x^2 - 2y^2 = 1$? **[1]**

In the remainder of this question T_m is the m^{th} triangular number, the sum of the first m positive integers, so that $T_m = \dfrac{m(m+1)}{2}$.

(f) Create a program to find, for a fixed positive integer t, all pairs of positive integers m and n which satisfy $T_m = n^2$ where $m \leqslant t$ and $n \leqslant t$. Write out your program in the Printed Answer Booklet. **[2]**

(g) Use your program to find all pairs of positive integers m and n which satisfy $T_m = n^2$ where $m \leqslant 300$ and $n \leqslant 300$. Give the pairs in ascending order of the value of m. **[1]**

(h) By comparing your answers to part **(c)** and part **(g)**, or otherwise, prove that there are infinitely many triangular numbers which are perfect squares. **[5]**

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3 This question concerns the family of differential equations

$$\frac{dy}{dx} = 1 - x^a y \quad (*)$$

where a is –1, 0 or 1.

(a) Determine and describe geometrically the isoclines of (*) when

(i) $a = -1$, **[2]**

(ii) $a = 0$, **[2]**

(iii) $a = 1$. **[2]**

(b) In this part of the question $a = 0$.

(i) Write down the solution to (*) which passes through the point $(0, b)$ where $b \neq 1$. **[1]**

(ii) Write down the equation of the asymptote to this solution. **[1]**

(c) In this part of the question $a = -1$.

(i) Write down the solution to (*) which passes through the point (c, d) where $c \neq 0$. **[1]**

(ii) Describe the relationship between c and d when the solution in part **(i)** has a stationary point. **[4]**

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(d) In this part of the question $a = 1$.

(i) The standard Runge-Kutta method of order 4 for the solution of the differential equation $\mathrm{f}(x, y) = \dfrac{\mathrm{d}y}{\mathrm{d}x}$ is as follows.

$$k_1 = h\mathrm{f}(x_n, y_n)$$

$$k_2 = h\mathrm{f}\left(x_n + \frac{h}{2}, y_n + \frac{k_1}{2}\right)$$

$$k_3 = h\mathrm{f}\left(x_n + \frac{h}{2}, y_n + \frac{k_2}{2}\right)$$

$$k_4 = h\mathrm{f}(x_n + h, y_n + k_3)$$

$$y_{n+1} = y_n + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4).$$

Construct a spreadsheet to solve (*) in the case $x_0 = 0$ and $y_0 = 1.5$. State the formulae you have used in your spreadsheet. **[4]**

(ii) Use your spreadsheet with $h = 0.05$ to find an approximation to the value of y when $x = 1$. **[1]**

(iii) The solution to (*) in which $x_0 = 0$ and $y_0 = 1.5$ has a maximum point (r, s) with $0 < r < 1$. Use your spreadsheet with suitable values of h to estimate r to two decimal places. Justify your answer. **[2]**

END OF QUESTION PAPER

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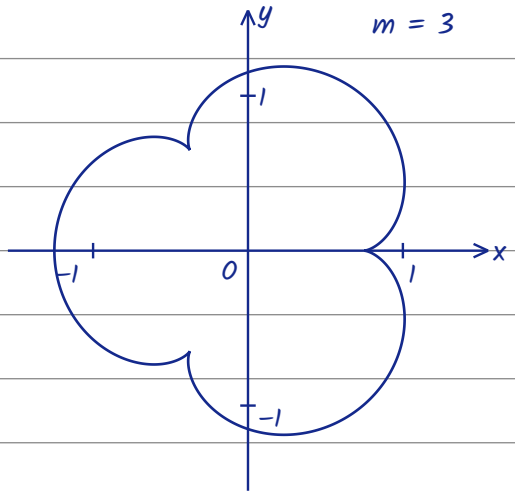
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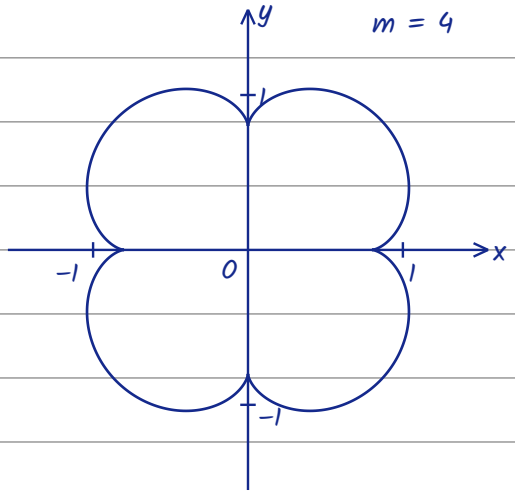
Surname	StemUp Model Solution
Answer booklet	Write the question number in the left-hand margin at the start of each answer.

Question number

1 (a)(i) $m = 3$ (3 cusps, at $t = 0, 2\pi/3, 4\pi/3$)

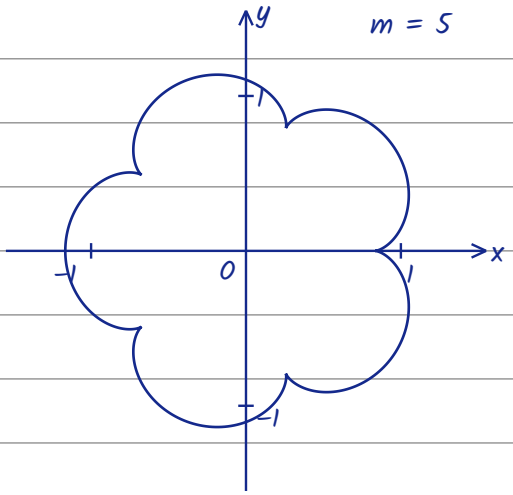


$m = 4$ (4 cusps, on the axes)



$m = 5$ (5 cusps)

Question number



1 (a)(ii) All three curves are closed curves with reflective symmetry in the x-axis.

1 (a)(iii) For $m = 4$ the curve has reflective symmetry in the y-axis;

the curves for $m = 3$ and $m = 5$ do not.

1 (b)(i) $\frac{dx}{dt} = -\sin t + \sin((m+1)t)$, $\frac{dy}{dt} = \cos t - \cos((m+1)t)$

$\frac{dx}{dt} = 0 \Rightarrow \sin((m+1)t) = \sin t$

$\Rightarrow (m+1)t = t + 2k\pi$ or $(m+1)t = \pi - t + 2l\pi$ (k, l integers)

$\Rightarrow t = \frac{2k\pi}{m}$ or $t = \frac{(2l+1)\pi}{m+2}$

$\frac{dy}{dt} = 0 \Rightarrow \cos((m+1)t) = \cos t$

$\Rightarrow (m+1)t = t + 2k\pi$ or $(m+1)t = -t + 2l\pi$

$\Rightarrow t = \frac{2k\pi}{m}$ or $t = \frac{2l\pi}{m+2}$

So $\frac{dx}{dt} = 0$ but $\frac{dy}{dt} \neq 0$ for the odd multiples of $\frac{\pi}{m+2}$ in $0 \leq t < 2\pi$:

Question number

$t = \frac{\pi}{m+2}, \frac{3\pi}{m+2}, \frac{5\pi}{m+2}, \ldots, \frac{(2m+3)\pi}{m+2}$

(leaving out any of these which also equal $\frac{2k\pi}{m}$, eg $t = \frac{\pi}{2}$ when $m = 4$, since there both derivatives are zero: these are cusps)

1 (b)(ii) At these points the tangent to the curve is vertical (parallel to the y-axis).

1 (c)(i) $r^2 = x^2 + y^2 = (\cos t - \frac{\cos((m+1)t)}{m+1})^2 + (\sin t - \frac{\sin((m+1)t)}{m+1})^2$

$= \cos^2 t + \sin^2 t + \frac{\cos^2((m+1)t) + \sin^2((m+1)t)}{(m+1)^2}$

$- \frac{2(\cos t \cos((m+1)t) + \sin t \sin((m+1)t))}{m+1}$

$= 1 + \frac{1}{(m+1)^2} - \frac{2\cos(mt)}{m+1}$

Since $-1 \leq \cos(mt) \leq 1$:

$1 - \frac{2}{m+1} + \frac{1}{(m+1)^2} \leq r^2 \leq 1 + \frac{2}{m+1} + \frac{1}{(m+1)^2}$

$(1 - \frac{1}{m+1})^2 \leq r^2 \leq (1 + \frac{1}{m+1})^2$

$1 - \frac{1}{m+1} \leq r \leq 1 + \frac{1}{m+1}$

so the curve lies between the two circles, as required

1 (c)(ii) Inner radius $1 - \frac{1}{m+1} = \frac{m}{m+1}$, outer radius $1 + \frac{1}{m+1} = \frac{m+2}{m+1}$

The curve only touches each circle at isolated points, so the region bounded

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by the curve contains the inner disc and lies inside the outer disc:

$$\pi\left(\frac{m}{m+1}\right)^2 < A < \pi\left(\frac{m+2}{m+1}\right)^2$$

$$\frac{m^2\pi}{(m+1)^2} < A < \frac{(m+2)^2\pi}{(m+1)^2} \quad \text{as required}$$

1 (c)(iii) As $m \to \infty$, $\frac{m^2}{(m+1)^2} \to 1$ and $\frac{(m+2)^2}{(m+1)^2} \to 1$

so both bounds tend to π, and A is squeezed between them:

the limit of the area is π

1 (d) $\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = (\sin((m+1)t) - \sin t)^2 + (\cos t - \cos((m+1)t))^2$

$= \sin^2((m+1)t) + \cos^2((m+1)t) + \sin^2 t + \cos^2 t$

$\quad - 2(\cos t \cos((m+1)t) + \sin t \sin((m+1)t))$

$= 2 - 2\cos(mt)$

$= 4\sin^2\left(\frac{mt}{2}\right)$

Length $L = \int_0^{2\pi} \sqrt{4\sin^2\left(\frac{mt}{2}\right)}\, dt = \int_0^{2\pi} 2\left|\sin\left(\frac{mt}{2}\right)\right| dt$

$\left|\sin\left(\frac{mt}{2}\right)\right|$ has period $\frac{2\pi}{m}$, so $[0, 2\pi]$ splits into m equal parts:

$$L = m\int_0^{2\pi/m} 2\sin\left(\frac{mt}{2}\right) dt$$

$$= m\left[-\frac{4}{m}\cos\left(\frac{mt}{2}\right)\right]_0^{2\pi/m}$$

$$= m \times \frac{4}{m}(-\cos\pi + \cos 0) = m \times \frac{8}{m}$$

Question number

$= 8$

$L = 8$ for every m, so the length of the curve is independent of m

2 (a)

Suppose $x^2 - 2y^2 = 1$ with x, y integers.

Then $x^2 = 2y^2 + 1$, which is odd, so x is odd.

Write $x = 2t + 1$ for some integer t:

$2y^2 = x^2 - 1 = (2t + 1)^2 - 1 = 4t^2 + 4t$

$y^2 = 2t^2 + 2t = 2(t^2 + t)$, which is even, so y is even.

Hence x is odd and y is even.

2 (b)

Python:

```
s = int(input("s = "))
for x in range(1, s + 1):
    for y in range(1, s + 1):
        if x*x - 2*y*y == 1:
            print(x, y)
```

2 (c)

With $s = 600$:

$(3, 2)$, $(17, 12)$, $(99, 70)$, $(577, 408)$

2 (d)

$(x + y\sqrt{2})(x - y\sqrt{2}) = 1$

First solution $x = 3$, $y = 2$: $(3 + 2\sqrt{2})(3 - 2\sqrt{2}) = 9 - 8 = 1$

Square both sides: $(3 + 2\sqrt{2})^2(3 - 2\sqrt{2})^2 = 1$

$(9 + 12\sqrt{2} + 8)(9 - 12\sqrt{2} + 8) = 1$

$(17 + 12\sqrt{2})(17 - 12\sqrt{2}) = 1 \quad \Rightarrow \quad 17^2 - 2 \times 12^2 = 1$

so $x = 17$, $y = 12$ is the next solution.

Cubing: $(3 + 2\sqrt{2})^3 = (17 + 12\sqrt{2})(3 + 2\sqrt{2})$

$= 51 + 34\sqrt{2} + 36\sqrt{2} + 48 = 99 + 70\sqrt{2}$

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and $(3 - 2\sqrt{2})^3 = 99 - 70\sqrt{2}$, so $(99 + 70\sqrt{2})(99 - 70\sqrt{2}) = 1$

gives $x = 99$, $y = 70$

Fourth power: $(3 + 2\sqrt{2})^4 = (17 + 12\sqrt{2})^2 = 289 + 408\sqrt{2} + 288$

$= 577 + 408\sqrt{2}$

so $(577 + 408\sqrt{2})(577 - 408\sqrt{2}) = 1$ gives $x = 577$, $y = 408$

2 (e) $(3 + 2\sqrt{2})^n(3 - 2\sqrt{2})^n = 1$ for every positive integer n, and writing

$(3 + 2\sqrt{2})^n = x + y\sqrt{2}$ gives a new positive integer solution each time.

So there are infinitely many positive integer solutions (x, y).

2 (f) Python:

```
t = int(input("t = "))
for m in range(1, t + 1):
    for n in range(1, t + 1):
        if m*(m + 1) == 2*n*n:
            print(m, n)
```

2 (g) With $t = 300$:

$(m, n) = (1, 1),\ (8, 6),\ (49, 35),\ (288, 204)$

2 (h) Comparing (c) and (g): $m = 1, 8, 49, 288 \leftrightarrow x = 3, 17, 99, 577$ and

$n = 1, 6, 35, 204 \leftrightarrow y = 2, 12, 70, 408$, so it appears that

$$m = \frac{x - 1}{2}, \quad n = \frac{y}{2}$$

Let (x, y) be any positive integer solution of $x^2 - 2y^2 = 1$.

By (a), x is odd so $m = (x - 1)/2$ is a positive integer $(x \geq 3)$,

and y is even so $n = y/2$ is a positive integer.

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$T_m = \frac{m(m+1)}{2} = \frac{1}{2} \times \frac{x-1}{2} \times \frac{x+1}{2} = \frac{x^2-1}{8}$

$n^2 = \frac{y^2}{4} = \frac{2y^2}{8}$

Since $x^2 - 1 = 2y^2$, $T_m = (x^2 - 1)/8 = 2y^2/8 = n^2$

So every positive integer solution (x, y) gives a triangular number $T_m = n^2$, and different x give different m.

By (e) there are infinitely many solutions (x, y), so there are infinitely many triangular numbers which are perfect squares.

3 (a)(i) $a = -1$: $\frac{dy}{dx} = 1 - \frac{y}{x}$. Isocline for gradient k: $1 - \frac{y}{x} = k$

$\Rightarrow y = (1 - k)x \quad (x \neq 0)$

The isoclines are straight lines through the origin.

3 (a)(ii) $a = 0$: $\frac{dy}{dx} = 1 - y$. Isocline for gradient k: $1 - y = k$

$\Rightarrow y = 1 - k$

The isoclines are horizontal straight lines.

3 (a)(iii) $a = 1$: $\frac{dy}{dx} = 1 - xy$. Isocline for gradient k: $1 - xy = k$

$\Rightarrow y = \frac{1-k}{x}$

For $k \neq 1$ these are rectangular hyperbolae (stretches of $y = \frac{1}{x}$) with the x- and y-axes as asymptotes; for $k = 1$ the isocline is the axes ($y = 0$).

Question number	
3 (b)(i)	$\frac{dy}{dx} = 1 - y \Rightarrow y = 1 + Ae^{-x}$, and $y = b$ at $x = 0$ gives $A = b - 1$
	$y = 1 + (b - 1)e^{-x}$
3 (b)(ii)	$y = 1$
3 (c)(i)	$\frac{dy}{dx} + \frac{y}{x} = 1 \Rightarrow \frac{d}{dx}(xy) = x \Rightarrow xy = \frac{x^2}{2} + C$
	Through (c, d): $C = cd - \frac{c^2}{2}$
	$y = \frac{x^2 + 2cd - c^2}{2x}$
3 (c)(ii)	$y = \frac{x}{2} + \frac{2cd - c^2}{2x} \Rightarrow \frac{dy}{dx} = \frac{1}{2} - \frac{2cd - c^2}{2x^2}$
	$\frac{dy}{dx} = 0 \Rightarrow x^2 = 2cd - c^2 = c(2d - c)$
	This has a solution with $x \neq 0$ if and only if $c(2d - c) > 0$
	(and then $x = \pm\sqrt{(c(2d - c))}$, taking the sign of c to stay on the same branch).
	So there is a stationary point if and only if
	$c > 0$ and $d > c/2$, or $c < 0$ and $d < c/2$
3 (d)(i)	$f(x, y) = 1 - xy$. Step length h in cell H1.
	A2 = 0 (x_0), B2 = 1.5 (y_0), H1 = 0.05
	C2 (k_1) = $H$1*(1 - A2*B2)
	D2 (k_2) = $H$1*(1 - (A2 + $H$1/2)*(B2 + C2/2))
	E2 (k_3) = $H$1*(1 - (A2 + $H$1/2)*(B2 + D2/2))
	F2 (k_4) = $H$1*(1 - (A2 + $H$1)*(B2 + E2))

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A3 = A2 + $H$1

B3 = B2 + (C2 + 2*D2 + 2*E2 + F2)/6

Copy C2:F2 down to row 3, then fill A3:F3 down to row 22 (x = 1).

First rows (h = 0.05):

x	y	k_1	k_2	k_3	k_4
0	1.5	0.05	0.048094	0.048095	0.046130
0.05	1.5480845	0.046130	0.044108	0.044112	0.042039
0.1	1.5921860	0.042039	0.039917	0.039924	0.037759

3 (d)(ii) h = 0.05, after 20 steps: x = 1, y = 1.63457442

y ≈ 1.6346 (4 d.p.)

3 (d)(iii) h = 0.05: y increases up to x = 0.55 (1.78721) then falls (1.78583 at x = 0.6), so r is near 0.55. Using h = 0.005:

x	0.550	0.555	0.560	0.565	0.570
y	1.7872058	1.7872685	1.7872865	1.7872598	1.7871885

y increases up to x = 0.560 and then decreases, so the maximum is between x = 0.555 and x = 0.565 (h = 0.0025 gives the same values to 7 d.p.).

r = 0.56 (2 d.p.)