OCR
Oxford Cambridge and RSA

Monday 24 June 2019 – Morning

A Level Further Mathematics B (MEI)

Y435/01 Extra Pure

Time allowed: 1 hour 15 minutes

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You must have:
- Printed Answer Booklet
- Formulae Further Mathematics B (MEI)

You may use:
- a scientific or graphical calculator

INSTRUCTIONS
- Use black ink. HB pencil may be used for graphs and diagrams only.
- Answer **all** the questions.
- **Write your answer to each question in the space provided in the Printed Answer Booklet.** If additional space is required, you should use the lined page(s) at the end of the Printed Answer Booklet. The question number(s) must be clearly shown.
- You are permitted to use a scientific or graphical calculator in this paper.
- Final answers should be given to a degree of accuracy appropriate to the context.

INFORMATION
- The total mark for this paper is **60**.
- The marks for each question are shown in brackets **[ ]**.
- You are advised that an answer may receive **no marks** unless you show sufficient detail of the working to indicate that a correct method is used. You should communicate your method with correct reasoning.
- The Printed Answer Booklet consists of **12** pages. The Question Paper consists of **4** pages.

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Answer **all** the questions.

1 The matrix **A** is $\begin{pmatrix} 0.6 & 0.8 \\ 0.8 & -0.6 \end{pmatrix}$.

(a) Given that **A** represents a reflection, write down the eigenvalues of **A**. **[1]**

(b) Hence find the eigenvectors of **A**. **[3]**

(c) Write down the equation of the mirror line of the reflection represented by **A**. **[1]**

2 A surface S is defined by $z = 4x^2 + 4y^2 - 4x + 8y + 11$.

(a) Show that the point P (0.5, −1, 6) is the only stationary point on S. **[2]**

(b) **(i)** On the axes in the Printed Answer Booklet, draw a sketch of the contour of the surface corresponding to $z = 42$. **[2]**

(ii) By using the sketch in part **(b)(i)**, deduce that P must be a minimum point on S. **[3]**

(c) In the section of S corresponding to $y = c$, the minimum value of z occurs at the point where $x = a$ and $z = 22$.
Find all possible values of a and c. **[4]**

3 The matrix **A** is $\begin{pmatrix} -1 & 2 & 4 \\ 0 & -1 & -25 \\ -3 & 5 & -1 \end{pmatrix}$.

Use the Cayley-Hamilton theorem to find $\mathbf{A}^{-1}$. **[8]**

4 T is the set $\{1, 2, 3, 4\}$. A binary operation • is defined on T such that $a \bullet a = 2$ for all $a \in T$. It is given that $(T, \bullet)$ is a group.

(a) Deduce the identity element in T, giving a reason for your answer. **[2]**

(b) Find the value of $1 \bullet 3$, showing how the result is obtained. **[3]**

(c) **(i)** Complete a group table for $(T, \bullet)$. **[2]**

(ii) State with a reason whether the group is abelian. **[1]**

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5 A financial institution models the repayment of a loan to a client in the following way.

- An amount, £C, is loaned to the client at the start of the repayment period.
- The amount owed n years after the start of the repayment period is £L_n, so that $L_0 = C$.
- At the end of each year, interest of $\alpha\%$ ($\alpha > 0$) of the amount owed at the start of that year is added to the amount owed.
- Immediately after interest has been added to the amount owed a repayment of £R is made by the client.
- Once L_n becomes negative the repayment is finished and the overpayment is refunded to the client.

(a) Show that during the repayment period, $L_{n+1} = aL_n + b$, giving a and b in terms of α and R. **[2]**

(b) Find the solution of the recurrence relation $L_{n+1} = aL_n + b$ with $L_0 = C$, giving your solution in terms of a, b, C and n. **[5]**

(c) Deduce from parts **(a)** and **(b)** that, for the repayment scheme to terminate, $R > \frac{\alpha C}{100}$. **[2]**

A client takes out a £30 000 loan at 8% interest and agrees to repay £3000 at the end of each year.

(d) **(i)** Use an algebraic method to find the number of years it will take for the loan to be repaid. **[3]**

(ii) Taking into account the refund of overpayment, find the total amount that the client repays over the lifetime of the loan. **[3]**

6 **(a)** Given that $\sqrt{7}$ is an irrational number, prove that $a^2 - 7b^2 \neq 0$ for all $a, b \in \mathbb{Q}$ where a and b are not both 0. **[2]**

(b) A set G is defined by $G = \{a + b\sqrt{7} : a, b \in \mathbb{Q}, a \text{ and } b \text{ not both } 0\}$.
Prove that G is a group under multiplication. (You may assume that multiplication is associative.) **[7]**

(c) A subset H of G is defined by $H = \{1 + c\sqrt{7} : c \in \mathbb{Q}\}$.
Determine whether or not H is a subgroup of $(G, \times)$. **[2]**

(d) Using $(G, \times)$, prove by counter-example that the statement 'An infinite group cannot have a non-trivial subgroup of finite order' is false. **[2]**

END OF QUESTION PAPER

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Surname	StemUp Model Solution
Answer booklet	Write the question number in the left-hand margin at the start of each answer.

Question number

1 (a) A reflection leaves the mirror line fixed and reverses the normal direction,

so the eigenvalues are $\lambda = 1$ and $\lambda = -1$

1 (b) $\lambda = 1$: $\begin{pmatrix} 0.6 & 0.8 \\ 0.8 & -0.6 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$

$0.6x + 0.8y = x \Rightarrow 0.8y = 0.4x \Rightarrow x = 2y$

eigenvector $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$

$\lambda = -1$: $\begin{pmatrix} 0.6 & 0.8 \\ 0.8 & -0.6 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = -\begin{pmatrix} x \\ y \end{pmatrix}$

$0.6x + 0.8y = -x \Rightarrow 0.8y = -1.6x \Rightarrow y = -2x$

eigenvector $\begin{pmatrix} 1 \\ -2 \end{pmatrix}$

(any non-zero multiples of these)

1 (c) The mirror line is the line of invariant points, along the eigenvector for $\lambda = 1$:

$y = \frac{1}{2}x$

2 (a) $z = 4x^2 + 4y^2 - 4x + 8y + 11$

$\frac{\partial z}{\partial x} = 8x - 4, \quad \frac{\partial z}{\partial y} = 8y + 8$

Stationary points: $8x - 4 = 0$ and $8y + 8 = 0 \Rightarrow x = 0.5, y = -1$ only

$z = 4(0.5)^2 + 4(-1)^2 - 4(0.5) + 8(-1) + 11 = 1 + 4 - 2 - 8 + 11 = 6$

So $P(0.5, -1, 6)$ is the only stationary point on S, as required

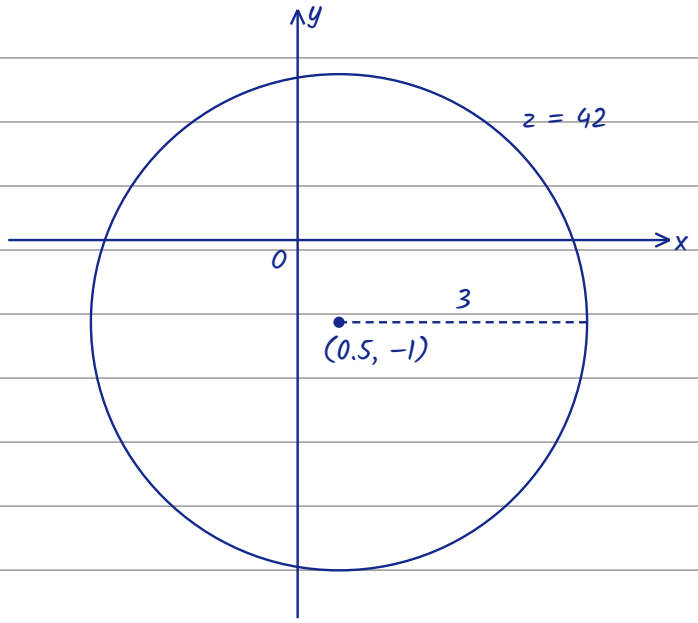
Question number

2 (b)(i) $42 = 4x^2 + 4y^2 - 4x + 8y + 11$

$42 = 4(x - 0.5)^2 - 1 + 4(y + 1)^2 - 4 + 11$

$4(x - 0.5)^2 + 4(y + 1)^2 = 36 \Rightarrow (x - 0.5)^2 + (y + 1)^2 = 9$

Circle, centre $(0.5, -1)$, radius 3



2 (b)(ii) The point $(0.5, -1)$ directly below P lies inside the contour $z = 42$.

At P, $z = 6$, so P is below the contour where $z = 42$.

P is the only stationary point, so moving from P towards the contour in every direction, z must increase from 6 up to 42 (it is going upwards).

Hence P must be a minimum point on S.

2 (c) Section $y = c$: $z = 4x^2 + 4c^2 - 4x + 8c + 11$

$\frac{dz}{dx} = 8x - 4 = 0 \Rightarrow x = 0.5$, so $a = 0.5$

$z = 22$ at $x = 0.5$: $22 = 4(0.5)^2 + 4c^2 - 4(0.5) + 8c + 11$

$22 = 4c^2 + 8c + 10 \Rightarrow 4c^2 + 8c - 12 = 0 \Rightarrow c^2 + 2c - 3 = 0$

$(c + 3)(c - 1) = 0 \Rightarrow c = 1$ or $c = -3$

Question number

$a = 0.5$, $c = 1$ or $c = -3$

3

$$\det(A - \lambda I) = \det\begin{pmatrix} -1-\lambda & 2 & 4 \\ 0 & -1-\lambda & -25 \\ -3 & 5 & -1-\lambda \end{pmatrix}$$

$$= (-1-\lambda)[(-1-\lambda)^2 + 125] - 2[0 - 75] + 4[0 - 3(-1-\lambda)]$$

$$= (-1-\lambda)(\lambda^2 + 2\lambda + 126) + 150 + 12 + 12\lambda$$

$$= -\lambda^3 - 3\lambda^2 - 128\lambda - 126 + 162 + 12\lambda$$

$$= -\lambda^3 - 3\lambda^2 - 140\lambda + 12$$

Characteristic equation: $-\lambda^3 - 3\lambda^2 - 140\lambda + 12 = 0$

By the Cayley-Hamilton theorem: $-A^3 - 3A^2 - 140A + 12I = 0$

Multiply by A^{-1}: $-A^2 - 3A - 140I + 12A^{-1} = 0$

$$A^{-1} = \frac{1}{12}(A^2 + 3A + 140I)$$

$$A^2 = \begin{pmatrix} -11 & 16 & -58 \\ 75 & -124 & 50 \\ 6 & -16 & -136 \end{pmatrix} \qquad 3A = \begin{pmatrix} -3 & 6 & 12 \\ 0 & -3 & -75 \\ -9 & 15 & -3 \end{pmatrix}$$

$$A^2 + 3A + 140I = \begin{pmatrix} 126 & 22 & -46 \\ 75 & 13 & -25 \\ -3 & -1 & 1 \end{pmatrix}$$

$$A^{-1} = \frac{1}{12}\begin{pmatrix} 126 & 22 & -46 \\ 75 & 13 & -25 \\ -3 & -1 & 1 \end{pmatrix}$$

4 (a)

$2 \cdot 2 = 2$. Combining both sides with 2^{-1}: $2 = e$

(in a group only the identity satisfies $x \cdot x = x$). So the identity is 2.

4 (b)

$1 \cdot 3 = 1$ would mean 3 is the identity: false since 2 is.

Question number

$1 \cdot 3 = 3$ would mean 1 is the identity: false.

$1 \cdot 3 = 2 = 1 \cdot 1 \Rightarrow 3 = 1$ (cancellation): contradiction.

Hence $1 \cdot 3 = 4$

4 (c)(i)

$\cdot$	1	2	3	4
1	2	1	4	3
2	1	2	3	4
3	4	3	2	1
4	3	4	1	2

4 (c)(ii) The group is abelian, since the table is symmetrical about the leading diagonal.

5 (a) $L_{n+1} = L_n + \frac{\alpha}{100} L_n - R = (1 + \frac{\alpha}{100}) L_n - R$

so $a = 1 + \frac{\alpha}{100}$ and $b = -R$

5 (b) Complementary function: try $L_n = kp^n \Rightarrow kp^{n+1} = akp^n \Rightarrow p = a$

so $L_n = ka^n$

Particular solution: try $L_n = c \Rightarrow c = ac + b \Rightarrow c = \frac{b}{1-a}$

General solution: $L_n = ka^n + \frac{b}{1-a}$

$L_0 = C \Rightarrow C = k + \frac{b}{1-a} \Rightarrow k = C - \frac{b}{1-a}$

$L_n = (C - \frac{b}{1-a}) a^n + \frac{b}{1-a}$

Question number

5 (c) $a > 1$, so $a^n \to \infty$; the scheme terminates only if L_n eventually < 0,

i.e. if $C - \frac{b}{1-a} < 0$

$\frac{b}{1-a} = \frac{-R}{-\alpha/100} = \frac{100R}{\alpha} \Rightarrow C - \frac{100R}{\alpha} < 0$

$\frac{100R}{\alpha} > C \Rightarrow R > \frac{\alpha C}{100}$, as required

5 (d)(i) $a = 1.08$, $b = -3000$, $C = 30\,000$, $\frac{b}{1-a} = \frac{-3000}{-0.08} = 37\,500$

$L_n = (30\,000 - 37\,500)(1.08)^n + 37\,500 = 37\,500 - 7500(1.08)^n$

$L_n < 0 \Rightarrow 7500(1.08)^n > 37\,500 \Rightarrow 1.08^n > 5$

$n > \frac{\ln 5}{\ln 1.08} = 20.91\ldots$

So the loan takes 21 years to repay

5 (d)(ii) $L_{21} = 37\,500 - 7500(1.08)^{21} = -253.75$

So £253.75 is refunded.

Total repaid $= 21 \times 3000 - 253.75 =$ £62 746.25 ($\approx$ £62 746)

6 (a) Suppose $a^2 - 7b^2 = 0$ for some $a, b \in \mathbb{Q}$, not both 0.

If $b = 0$ then $a^2 = 0$ so $a = 0$: contradiction. So $b \neq 0$, and similarly $a \neq 0$.

Write $a = \frac{p}{q}$, $b = \frac{r}{s}$ with p, q, r, s non-zero integers.

Then $(\frac{a}{b})^2 = 7 \Rightarrow \pm\sqrt{7} = \frac{a}{b} = \frac{ps}{qr}$, which is rational.

This contradicts $\sqrt{7}$ being irrational, so $a^2 - 7b^2 \neq 0$, as required.

Question number

6 (b)

Closure: $(a + b\sqrt{7})(c + d\sqrt{7}) = (ac + 7bd) + (ad + bc)\sqrt{7}$

$ac + 7bd$ and $ad + bc$ are rational, as $\mathbb{Q}$ is closed under $+$ and $\times$.

$a + b\sqrt{7} \neq 0$ and $c + d\sqrt{7} \neq 0$ (by part (a), as $a^2 - 7b^2 \neq 0$ means $a \neq \pm b\sqrt{7}$), so the product $\neq 0$, so $ac + 7bd$, $ad + bc$ are not both 0.

So the product is in G: closed.

Identity: $1 = 1 + 0\sqrt{7} \in G$, and $1 \times (a + b\sqrt{7}) = (a + b\sqrt{7}) \times 1 = a + b\sqrt{7}$

Inverse: $\dfrac{1}{a + b\sqrt{7}} = \dfrac{a - b\sqrt{7}}{(a + b\sqrt{7})(a - b\sqrt{7})} = \dfrac{a - b\sqrt{7}}{a^2 - 7b^2}$

$= \dfrac{a}{a^2 - 7b^2} + \dfrac{-b}{a^2 - 7b^2}\sqrt{7}$

$a^2 - 7b^2 \neq 0$ by part (a), so this exists;

$\dfrac{a}{a^2 - 7b^2}$ and $\dfrac{-b}{a^2 - 7b^2}$ are rational and not both 0 (as a, b not both 0)

so every element has an inverse in G.

Associativity: multiplication is associative (given).

Closure, identity, inverses and associativity all hold, so G is a group under $\times$.

6 (c)

Take $c = 1$: $1 + \sqrt{7} \in H$

$(1 + \sqrt{7})(1 + \sqrt{7}) = 1 + 2\sqrt{7} + 7 = 8 + 2\sqrt{7}$

$= 8(1 + \frac{1}{4}\sqrt{7})$, which is not of the form $1 + c\sqrt{7}$, so not in H.

H is not closed under $\times$, so H is not a subgroup of $(G, \times)$.

6 (d)

$(G, \times)$ has infinite order (eg it contains every non-zero rational $a + 0\sqrt{7}$).

Consider $\{1, -1\} \subset G$: $1 \times 1 = 1$, $1 \times (-1) = -1$, $(-1) \times (-1) = 1$, so closed;

identity 1 is in it, each element is its own inverse, associativity is inherited.

So $\{1, -1\}$ is a non-trivial subgroup of finite order (order 2) of an infinite group.

This is a counter-example, so the statement is false.