Please check the examination details below before entering your candidate information

Candidate surname StemUp Model Solution | Other names

Centre Number | Candidate Number

# Pearson Edexcel International Advanced Level

**Time** 1 hour 30 minutes

**Paper reference** **WFM03/01**

# Mathematics

## International Advanced Subsidiary/Advanced Level
## Further Pure Mathematics F3

**You must have:**
Mathematical Formulae and Statistical Tables (Yellow), calculator

Total Marks

**Candidates may use any calculator permitted by Pearson regulations. Calculators must not have the facility for symbolic algebra manipulation, differentiation and integration, or have retrievable mathematical formulae stored in them.**

## Instructions

- Use **black** ink or ball-point pen.
- If pencil is used for diagrams/sketches/graphs it must be dark (HB or B).
- **Fill in the boxes** at the top of this page with your name, centre number and candidate number.
- Answer **all** questions and ensure that your answers to parts of questions are clearly labelled.
- Answer the questions in the spaces provided
  – *there may be more space than you need.*
- You should show sufficient working to make your methods clear. Answers without working may not gain full credit.
- Inexact answers should be given to three significant figures unless otherwise stated.

## Information

- A booklet 'Mathematical Formulae and Statistical Tables' is provided.
- There are 9 questions in this question paper. The total mark for this paper is 75.
- The marks for **each** question are shown in brackets
  – *use this as a guide as to how much time to spend on each question.*

## Advice

- Read each question carefully before you start to answer it.
- Try to answer every question.
- Check your answers if you have time at the end.
- If you change your mind about an answer, cross it out and put your new answer and any working underneath.

*Turn over* ▶

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Q:1/1/1/1/

P 7 2 4 0 3 A 0 1 3 6

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**1.** (a) Use the definitions of $\sinh x$ and $\cosh x$ in terms of exponentials to show that

$$\cosh A \cosh B + \sinh A \sinh B \equiv \cosh(A + B)$$

**(2)**

(b) Hence find the value of $x$ for which

$$\cosh(x + \ln 2) = 5 \sinh x$$

giving your answer in the form $\frac{1}{2} \ln k$, where $k$ is a rational number to be determined.

**(5)**

(a) $\cosh A \cosh B + \sinh A \sinh B$

$= \left(\frac{e^A + e^{-A}}{2}\right)\left(\frac{e^B + e^{-B}}{2}\right) + \left(\frac{e^A - e^{-A}}{2}\right)\left(\frac{e^B - e^{-B}}{2}\right)$

$= \frac{e^{A+B} + e^{A-B} + e^{B-A} + e^{-A-B} + e^{A+B} - e^{A-B} - e^{B-A} + e^{-A-B}}{4}$

$= \frac{2e^{A+B} + 2e^{-(A+B)}}{4} = \frac{e^{A+B} + e^{-(A+B)}}{2} = \cosh(A + B)$

$\therefore \cosh A \cosh B + \sinh A \sinh B \equiv \cosh(A + B)$

(b) $\cosh(x + \ln 2) = \cosh x \cosh(\ln 2) + \sinh x \sinh(\ln 2)$

$\cosh(\ln 2) = \frac{2 + \frac{1}{2}}{2} = \frac{5}{4}, \quad \sinh(\ln 2) = \frac{2 - \frac{1}{2}}{2} = \frac{3}{4}$

$\frac{5}{4} \cosh x + \frac{3}{4} \sinh x = 5 \sinh x$

$\Rightarrow \frac{5}{4} \cosh x = \frac{17}{4} \sinh x \Rightarrow 5 \cosh x = 17 \sinh x$

$\Rightarrow \tanh x = \frac{5}{17}$

$x = \operatorname{artanh} \frac{5}{17} = \frac{1}{2} \ln\left(\frac{1 + 5/17}{1 - 5/17}\right) = \frac{1}{2} \ln\left(\frac{22}{12}\right)$

$x = \frac{1}{2} \ln\left(\frac{11}{6}\right) \quad \left(k = \frac{11}{6}\right)$

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**Question 1 continued**

**Question 1 continued**

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**Question 1 continued**

**(Total for Question 1 is 7 marks)**

**2.** **In this question you must show all stages of your working.**

**Solutions relying entirely on calculator technology are not acceptable.**

(i) Determine

$$\int \frac{1}{\sqrt{5 + 4x - x^2}} \, dx$$

**(3)**

(ii) Use the substitution $x = 3 \sec\theta$ to determine the exact value of

$$\int_{2\sqrt{3}}^{6} \frac{18}{(x^2 - 9)^{\frac{3}{2}}} \, dx$$

Give your answer in the form $A + B\sqrt{3}$ where $A$ and $B$ are constants to be found.

**(6)**

(i) $5 + 4x - x^2 = 9 - (x^2 - 4x + 4) = 9 - (x - 2)^2$

$$\int \frac{1}{\sqrt{5 + 4x - x^2}} \, dx = \int \frac{1}{\sqrt{9 - (x-2)^2}} \, dx = \arcsin\left(\frac{x-2}{3}\right) + c$$

(ii) $x = 3\sec\theta \Rightarrow \frac{dx}{d\theta} = 3\sec\theta\tan\theta$

$x = 6$: $\sec\theta = 2 \Rightarrow \theta = \frac{\pi}{3}$; $\quad x = 2\sqrt{3}$: $\sec\theta = \frac{2\sqrt{3}}{3} \Rightarrow \theta = \frac{\pi}{6}$

$$\int \frac{18}{(x^2-9)^{3/2}} \, dx = \int \frac{18 \times 3\sec\theta\tan\theta}{(9\sec^2\theta - 9)^{3/2}} \, d\theta = \int \frac{54\sec\theta\tan\theta}{27\tan^3\theta} \, d\theta$$

$$= 2\int \frac{\sec\theta}{\tan^2\theta} \, d\theta = 2\int \frac{\cos\theta}{\sin^2\theta} \, d\theta = 2\int \operatorname{cosec}\theta\cot\theta \, d\theta$$

$$= -2\operatorname{cosec}\theta + c$$

$$\int \frac{18}{(x^2-9)^{3/2}} \, dx = \left[-2\operatorname{cosec}\theta\right]_{\pi/6}^{\pi/3} = -2\operatorname{cosec}\frac{\pi}{3} + 2\operatorname{cosec}\frac{\pi}{6}$$

$$= -2 \times \frac{2}{\sqrt{3}} + 2 \times 2 = 4 - \frac{4}{\sqrt{3}}$$

$$= 4 - \frac{4}{3}\sqrt{3} \qquad (A = 4, \; B = -\tfrac{4}{3})$$

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**Question 2 continued**

**Question 2 continued**

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**Question 2 continued**

**(Total for Question 2 is 9 marks)**

**3.** $$\mathbf{M} = \begin{pmatrix} -2 & 5 & 0 \\ 5 & 1 & -3 \\ 0 & -3 & 6 \end{pmatrix}$$

Given that $\mathbf{i} + \mathbf{j} + \mathbf{k}$ is an eigenvector of $\mathbf{M}$,

(a) determine the corresponding eigenvalue.

**(1)**

Given that 8 is an eigenvalue of $\mathbf{M}$,

(b) determine a corresponding eigenvector.

**(2)**

(c) Determine a diagonal matrix $\mathbf{D}$ and an orthogonal matrix $\mathbf{P}$ such that

$$\mathbf{D} = \mathbf{P}^{\mathrm{T}}\mathbf{MP}$$

**(5)**

(a) $\begin{pmatrix} -2 & 5 & 0 \\ 5 & 1 & -3 \\ 0 & -3 & 6 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \\ 3 \end{pmatrix} = 3\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ eigenvalue = 3

(b) $\begin{pmatrix} -2 & 5 & 0 \\ 5 & 1 & -3 \\ 0 & -3 & 6 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 8x \\ 8y \\ 8z \end{pmatrix} \Rightarrow -2x + 5y = 8x \Rightarrow y = 2x$

$-3y + 6z = 8z \Rightarrow z = -1.5y = -3x$

check: $5x + y - 3z = 5x + 2x + 9x = 16x = 8y$ ✓

Eigenvector: $\begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix}$

(c) $|M - \lambda I| = \begin{vmatrix} -2-\lambda & 5 & 0 \\ 5 & 1-\lambda & -3 \\ 0 & -3 & 6-\lambda \end{vmatrix} = 0$

$(-2 - \lambda)[(1 - \lambda)(6 - \lambda) - 9] - 5[5(6 - \lambda)] = 0$

$\Rightarrow \lambda^3 - 5\lambda^2 - 42\lambda + 144 = 0 \Rightarrow (\lambda - 3)(\lambda - 8)(\lambda + 6) = 0 \Rightarrow \lambda = -6$

$\lambda = -6$: $\begin{pmatrix} -2 & 5 & 0 \\ 5 & 1 & -3 \\ 0 & -3 & 6 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -6x \\ -6y \\ -6z \end{pmatrix} \Rightarrow 4x + 5y = 0, \ -3y + 12z = 0$

$\Rightarrow y = 4z, \ x = -5z$: eigenvector $\begin{pmatrix} -5 \\ 4 \\ 1 \end{pmatrix}$

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**Question 3 continued**

Normalise: $|i + j + k| = \sqrt{3}$, $|(1, 2, -3)| = \sqrt{14}$, $|(-5, 4, 1)| = \sqrt{42}$

$$P = \begin{pmatrix} 1/\sqrt{3} & 1/\sqrt{14} & -5/\sqrt{42} \\ 1/\sqrt{3} & 2/\sqrt{14} & 4/\sqrt{42} \\ 1/\sqrt{3} & -3/\sqrt{14} & 1/\sqrt{42} \end{pmatrix} \qquad D = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & -6 \end{pmatrix}$$

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**Question 3 continued**

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**Question 3 continued**

**(Total for Question 3 is 8 marks)**

**4.** $$y = \operatorname{artanh}\left(\frac{\cos x + a}{\cos x - a}\right)$$

where $a$ is a non-zero constant.

Show that

$$\frac{\mathrm{d}y}{\mathrm{d}x} = k \tan x$$

where $k$ is a constant to be determined.

**(4)**

Let $u = \dfrac{\cos x + a}{\cos x - a}$, $\dfrac{du}{dx} = \dfrac{(\cos x - a)(-\sin x) - (\cos x + a)(-\sin x)}{(\cos x - a)^2}$

$= \dfrac{-\sin x \cos x + a \sin x + \sin x \cos x + a \sin x}{(\cos x - a)^2} = \dfrac{2a \sin x}{(\cos x - a)^2}$

$\dfrac{dy}{dx} = \dfrac{1}{1 - u^2} \times \dfrac{du}{dx} = \dfrac{1}{1 - (\cos x + a)^2/(\cos x - a)^2} \times \dfrac{2a \sin x}{(\cos x - a)^2}$

$= \dfrac{(\cos x - a)^2}{(\cos x - a)^2 - (\cos x + a)^2} \times \dfrac{2a \sin x}{(\cos x - a)^2} = \dfrac{2a \sin x}{-4a \cos x}$

$= -\dfrac{1}{2} \dfrac{\sin x}{\cos x} = -\dfrac{1}{2} \tan x$

$\therefore \dfrac{dy}{dx} = k \tan x$ with $k = -\dfrac{1}{2}$

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**Question 4 continued**

**(Total for Question 4 is 4 marks)**

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**5.** A curve has parametric equations

$$x = 4e^{\frac{1}{2}t} \qquad y = e^t - t \qquad 0 \leqslant t \leqslant 4$$

The curve is rotated through $2\pi$ radians about the $x$-axis.

Show that the area of the curved surface generated is

$$\pi(e^8 + Ae^4 + B)$$

where $A$ and $B$ are constants to be determined.

**(7)**

$\frac{dx}{dt} = 2e^{t/2}, \quad \frac{dy}{dt} = e^t - 1$

$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = 4e^t + e^{2t} - 2e^t + 1 = e^{2t} + 2e^t + 1 = (e^t + 1)^2$

$S = 2\pi \int_0^4 y\sqrt{(dx/dt)^2 + (dy/dt)^2}\, dt = 2\pi \int_0^4 (e^t - t)(e^t + 1)\, dt$

$= 2\pi \int_0^4 (e^{2t} + e^t - te^t - t)\, dt$

$\int te^t\, dt = te^t - \int e^t\, dt = te^t - e^t$

$S = 2\pi\left[\frac{1}{2}e^{2t} + e^t - te^t + e^t - \frac{1}{2}t^2\right]_0^4 = 2\pi\left[\frac{1}{2}e^{2t} + 2e^t - te^t - \frac{1}{2}t^2\right]_0^4$

$= 2\pi\left[\left(\frac{1}{2}e^8 + 2e^4 - 4e^4 - 8\right) - \left(\frac{1}{2} + 2\right)\right]$

$= 2\pi\left(\frac{1}{2}e^8 - 2e^4 - \frac{21}{2}\right) = \pi(e^8 - 4e^4 - 21)$

$\therefore A = -4, \quad B = -21$

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**Question 5 continued**

**Question 5 continued**

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**Question 5 continued**

**(Total for Question 5 is 7 marks)**

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**6.** $\mathbf{A} = \begin{pmatrix} x & 1 & 3 \\ 2 & 4 & x \\ -4 & -2 & -1 \end{pmatrix}$

(a) Show that **A** is non-singular for all real values of $x$.

**(4)**

(b) Determine, in terms of $x$, $\mathbf{A}^{-1}$

**(4)**

(a) $\det A = x(-4 + 2x) - 1(-2 + 4x) + 3(-4 + 16)$

$= x(2x - 4) - (4x - 2) + 36 = 2x^2 - 8x + 38$

$= 2(x^2 - 4x + 19) = 2(x - 2)^2 + 30$

$(x - 2)^2 \geq 0 \Rightarrow \det A \geq 30 > 0$ for all real $x$

$\det A \neq 0$, so A is non-singular for all real values of $x$

(b) Minors: $\begin{pmatrix} 2x - 4 & -2 + 4x & 12 \\ -1 + 6 & -x + 12 & -2x + 4 \\ x - 12 & x^2 - 6 & 4x - 2 \end{pmatrix}$ Cofactors: $\begin{pmatrix} 2x - 4 & 2 - 4x & 12 \\ -5 & 12 - x & 2x - 4 \\ x - 12 & 6 - x^2 & 4x - 2 \end{pmatrix}$

$\text{adj } A = \begin{pmatrix} 2x - 4 & -5 & x - 12 \\ 2 - 4x & 12 - x & 6 - x^2 \\ 12 & 2x - 4 & 4x - 2 \end{pmatrix}$

$A^{-1} = \frac{1}{2x^2 - 8x + 38} \begin{pmatrix} 2x - 4 & -5 & x - 12 \\ 2 - 4x & 12 - x & 6 - x^2 \\ 12 & 2x - 4 & 4x - 2 \end{pmatrix}$

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**Question 6 continued**

**Question 6 continued**

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**Question 6 continued**

**(Total for Question 6 is 8 marks)**

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7. $$I_n = \int \frac{x^n}{\sqrt{10 - x^2}}\,\mathrm{d}x \qquad n \in \mathbb{N} \qquad |x| < \sqrt{10}$$

(a) Show that

$$nI_n = 10(n-1)I_{n-2} - x^{n-1}(10 - x^2)^{\frac{1}{2}} \qquad n \geqslant 2$$

**(6)**

(b) Hence find the exact value of

$$\int_0^1 \frac{x^5}{\sqrt{10 - x^2}}\,\mathrm{d}x$$

giving your answer in the form $\frac{1}{15}\left(p\sqrt{10} + q\right)$ where $p$ and $q$ are integers to be determined.

**(4)**

(a) $I_n = \int x^{n-1} \times \frac{x}{\sqrt{10 - x^2}}\,dx; \qquad \int \frac{x}{\sqrt{10 - x^2}}\,dx = -(10 - x^2)^{1/2}$

$I_n = -x^{n-1}(10 - x^2)^{1/2} + (n-1)\int x^{n-2}(10 - x^2)^{1/2}\,dx$

$= -x^{n-1}(10 - x^2)^{1/2} + (n-1)\int \frac{x^{n-2}(10 - x^2)}{\sqrt{10 - x^2}}\,dx$

$= -x^{n-1}(10 - x^2)^{1/2} + 10(n-1)\int \frac{x^{n-2}}{\sqrt{10 - x^2}}\,dx - (n-1)\int \frac{x^n}{\sqrt{10 - x^2}}\,dx$

$I_n = -x^{n-1}(10 - x^2)^{1/2} + 10(n-1)I_{n-2} - (n-1)I_n$

$I_n + (n-1)I_n = 10(n-1)I_{n-2} - x^{n-1}(10 - x^2)^{1/2}$

$\therefore \; nI_n = 10(n-1)I_{n-2} - x^{n-1}(10 - x^2)^{1/2}$

(b) $I_1 = \int_0^1 \frac{x}{\sqrt{10 - x^2}}\,dx = \left[-(10 - x^2)^{1/2}\right]_0^1 = -3 + \sqrt{10}$

With limits 0 to 1: $nI_n = 10(n-1)I_{n-2} - \left[x^{n-1}(10 - x^2)^{1/2}\right]_0^1 = 10(n-1)I_{n-2} - 3$

$3I_3 = 20I_1 - 3 \;\Rightarrow\; I_3 = \frac{20}{3}I_1 - 1$

$5I_5 = 40I_3 - 3 \;\Rightarrow\; I_5 = 8I_3 - \frac{3}{5} = 8\left(\frac{20}{3}I_1 - 1\right) - \frac{3}{5} = \frac{160}{3}I_1 - \frac{43}{5}$

$I_5 = \frac{160}{3}(\sqrt{10} - 3) - \frac{43}{5} = \frac{800\sqrt{10} - 2400 - 129}{15}$

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**Question 7 continued**

$$\int_0^1 \frac{x^5}{\sqrt{10 - x^2}}\, dx = \frac{1}{15}(800\sqrt{10} - 2529) \qquad (p = 800,\ q = -2529)$$

**Question 7 continued**

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**Question 7 continued**

**(Total for Question 7 is 10 marks)**

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**8.** The plane $\Pi$ has equation

$$3x + 4y - z = 17$$

The line $l_1$ is perpendicular to $\Pi$ and passes through the point $P(-4, -5, 3)$

The line $l_1$ intersects $\Pi$ at the point $Q$

(a) Determine the coordinates of $Q$

**(4)**

Given that the point $R(-1, 6, 4)$ lies on $\Pi$

(b) determine a Cartesian equation of the plane containing $PQR$

**(4)**

The line $l_2$ passes through $P$ and $R$

The line $l_3$ is the reflection of $l_2$ in $\Pi$

(c) Determine a vector equation for $l_3$

**(4)**

(a) $l_1$: $\mathbf{r} = \begin{pmatrix} -4 \\ -5 \\ 3 \end{pmatrix} + t\begin{pmatrix} 3 \\ 4 \\ -1 \end{pmatrix} = \begin{pmatrix} 3t-4 \\ 4t-5 \\ 3-t \end{pmatrix}$

$3(3t-4) + 4(4t-5) - (3-t) = 17 \Rightarrow 26t - 35 = 17 \Rightarrow t = 2$

$\overrightarrow{OQ} = \begin{pmatrix} -4 \\ -5 \\ 3 \end{pmatrix} + 2\begin{pmatrix} 3 \\ 4 \\ -1 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}$ $\quad Q(2, 3, 1)$

(b) $\overrightarrow{PQ} = \begin{pmatrix} 6 \\ 8 \\ -2 \end{pmatrix}$, $\overrightarrow{PR} = \begin{pmatrix} 3 \\ 11 \\ 1 \end{pmatrix}$, $\overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix} i & j & k \\ 6 & 8 & -2 \\ 3 & 11 & 1 \end{vmatrix} = \begin{pmatrix} 30 \\ -12 \\ 42 \end{pmatrix}$

$\mathbf{n} = \begin{pmatrix} 5 \\ -2 \\ 7 \end{pmatrix}$: $\begin{pmatrix} 5 \\ -2 \\ 7 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} = 10 - 6 + 7 = 11$

Plane $PQR$: $5x - 2y + 7z = 11$

(c) $R$ lies on $\Pi$, so $R$ lies on $l_3$. Reflection $P'$ of $P$ in $\Pi$ ($Q$ is the midpoint of $PP'$):

$\overrightarrow{OP'} = \begin{pmatrix} -4 \\ -5 \\ 3 \end{pmatrix} + 2 \times 2\begin{pmatrix} 3 \\ 4 \\ -1 \end{pmatrix} = \begin{pmatrix} 8 \\ 11 \\ -1 \end{pmatrix}$ $\quad \overrightarrow{RP'} = \begin{pmatrix} 8 \\ 11 \\ -1 \end{pmatrix} - \begin{pmatrix} -1 \\ 6 \\ 4 \end{pmatrix} = \begin{pmatrix} 9 \\ 5 \\ -5 \end{pmatrix}$

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**Question 8 continued**

$l_3$: $\mathbf{r} = \begin{pmatrix} -1 \\ 6 \\ 4 \end{pmatrix} + \lambda \begin{pmatrix} 9 \\ 5 \\ -5 \end{pmatrix}$

**Question 8 continued**

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**Question 8 continued**

**(Total for Question 8 is 12 marks)**

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**9.** The ellipse $E$ has equation

$$\frac{x^2}{9} + \frac{y^2}{4} = 1$$

The line $l$ has equation $y = kx - 3$, where $k$ is a constant.

Given that $E$ and $l$ meet at 2 distinct points $P$ and $Q$

(a) show that the $x$ coordinates of $P$ and $Q$ are solutions of the equation

$$(9k^2 + 4)x^2 - 54kx + 45 = 0$$

**(2)**

The point $M$ is the midpoint of $PQ$

(b) Determine, in simplest form in terms of $k$, the coordinates of $M$

**(3)**

(c) Hence show that, as $k$ varies, $M$ lies on the curve with equation

$$x^2 + py^2 = qy$$

where $p$ and $q$ are constants to be determined.

**(5)**

(a) $\frac{x^2}{9} + \frac{(kx-3)^2}{4} = 1 \Rightarrow 4x^2 + 9(k^2x^2 - 6kx + 9) = 36$

$4x^2 + 9k^2x^2 - 54kx + 81 - 36 = 0 \Rightarrow (9k^2 + 4)x^2 - 54kx + 45 = 0$

(b) $x_M = \frac{1}{2}\left(\frac{54k}{9k^2+4}\right) = \frac{27k}{9k^2+4}$

$y_M = k\left(\frac{27k}{9k^2+4}\right) - 3 = \frac{27k^2 - 27k^2 - 12}{9k^2+4} = -\frac{12}{9k^2+4}$

$M\left(\frac{27k}{9k^2+4}, -\frac{12}{9k^2+4}\right)$

(c) $x^2 + py^2 = \frac{729k^2}{(9k^2+4)^2} + \frac{144p}{(9k^2+4)^2} = \frac{729k^2 + 144p}{(9k^2+4)^2}$

$qy = -\frac{12q}{9k^2+4} = -\frac{12q(9k^2+4)}{(9k^2+4)^2} \Rightarrow 729k^2 + 144p = -108qk^2 - 48q$

$k^2$: $729 = -108q \Rightarrow q = -\frac{27}{4}$

constant: $144p = -48q = 324 \Rightarrow p = \frac{9}{4}$

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**Question 9 continued**

M lies on $x^2 + \frac{9}{4}y^2 = -\frac{27}{4}y$ $\quad (p = \frac{9}{4},\ q = -\frac{27}{4})$

**Question 9 continued**

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**Question 9 continued**

**Question 9 continued**

**(Total for Question 9 is 10 marks)**

**TOTAL FOR PAPER IS 75 MARKS**

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