Please check the examination details below before entering your candidate information

Candidate surname StemUp Model Solution | Other names

Centre Number | Candidate Number

Pearson Edexcel International Advanced Level

Monday 22 January 2024

Morning (Time: 1 hour 30 minutes)

Paper reference **WFM03/01**

Mathematics

International Advanced Subsidiary/ Advanced Level
Further Pure Mathematics F3

You must have:
Mathematical Formulae and Statistics Tables (Yellow), calculator

Total Marks

Candidates may use any calculator allowed by Pearson regulations. Calculators must not have the facility for symbolic algebra manipulation, differentiation and integration, or have retrievable mathematical formulae stored in them.

Instructions

- Use **black** ink or ball-point pen.
- If pencil is used for diagrams/sketches/graphs it must be dark (HB or B).
- **Fill in the boxes** at the top of this page with your name, centre number and candidate number.
- Answer **all** questions and ensure that your answers to parts of questions are clearly labelled.
- Answer the questions in the spaces provided
  – *there may be more space than you need.*
- You should show sufficient working to make your methods clear. Answers without working may not gain full credit.
- Inexact answers should be given to three significant figures unless otherwise stated.

Information

- A booklet 'Mathematical Formulae and Statistical Tables' is provided.
- There are 8 questions in this question paper. The total mark for this paper is 75.
- The marks for **each** question are shown in brackets
  – *use this as a guide as to how much time to spend on each question.*

Advice

- Read each question carefully before you start to answer it.
- Try to answer every question.
- Check your answers if you have time at the end.
- If you change your mind about an answer, cross it out and put your new answer and any working underneath.

Turn over ▶

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1. **In this question you must show all stages of your working.**

Solutions relying entirely on calculator technology are not acceptable.

(i) Show that

$$\int_{4}^{4\sqrt{3}} \frac{8}{16 + x^2}\,\mathrm{d}x = p\pi$$

where p is a rational number to be determined.

(3)

(ii) Determine the exact value of k for which

$$\int_{\frac{3}{4}}^{k} \frac{2}{\sqrt{9 - 4x^2}}\,\mathrm{d}x = \frac{\pi}{12}$$

(4)

(i) $\int \frac{8}{16 + x^2}\,dx = 8 \times \frac{1}{4} \arctan\left(\frac{x}{4}\right) = 2\arctan\left(\frac{x}{4}\right)$

$\int_{4}^{4\sqrt{3}} \frac{8}{16 + x^2}\,dx = \left[2\arctan\left(\frac{x}{4}\right)\right]_{4}^{4\sqrt{3}} = 2(\arctan\sqrt{3} - \arctan 1)$

$= 2\left(\frac{\pi}{3} - \frac{\pi}{4}\right) = 2 \times \frac{\pi}{12} = \frac{\pi}{6}$

$\therefore\ p = \frac{1}{6}$

(ii) $\frac{2}{\sqrt{9 - 4x^2}} = \frac{2}{2\sqrt{9/4 - x^2}} = \frac{1}{\sqrt{(3/2)^2 - x^2}}$

$\int \frac{2}{\sqrt{9 - 4x^2}}\,dx = \arcsin\left(\frac{x}{3/2}\right) = \arcsin\left(\frac{2x}{3}\right)$

$\left[\arcsin\left(\frac{2x}{3}\right)\right]_{3/4}^{k} = \arcsin\left(\frac{2k}{3}\right) - \arcsin\left(\frac{1}{2}\right) = \frac{\pi}{12}$

$\arcsin\left(\frac{2k}{3}\right) - \frac{\pi}{6} = \frac{\pi}{12} \Rightarrow \arcsin\left(\frac{2k}{3}\right) = \frac{\pi}{4}$

$\frac{2k}{3} = \sin\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$

$k = \frac{3\sqrt{2}}{4}$

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Question 1 continued

Question 1 continued

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Question 1 continued

(Total for Question 1 is 7 marks)

2.

$$\mathbf{T} = \begin{pmatrix} 2 & 3 & 7 \\ 3 & 2 & 6 \\ a & 4 & b \end{pmatrix} \qquad \mathbf{U} = \begin{pmatrix} 6 & -1 & -4 \\ 15 & c & -9 \\ -8 & a & 5 \end{pmatrix}$$

where a, b and c are constants.

Given that $\mathbf{TU} = \mathbf{I}$

(a) determine the value of a, the value of b and the value of c

(4)

The transformation represented by the matrix $\mathbf{T}$ transforms the line l_1 to the line l_2

Given that l_2 has equation

$$\frac{x-1}{3} = \frac{y}{-4} = z + 2$$

(b) determine a Cartesian equation for l_1

(4)

(a) $TU = I$:

row 3 of T × column 1 of U: $6a + 60 - 8b = 0 \Rightarrow 6a - 8b = -60$

row 3 of T × column 3 of U: $-4a - 36 + 5b = 1 \Rightarrow -4a + 5b = 37$

$\Rightarrow 12a - 16b = -120$ and $-12a + 15b = 111 \Rightarrow -b = -9$

$\Rightarrow b = 9, \quad a = \frac{-60 + 72}{6} = 2$

row 1 of T × column 2 of U: $-2 + 3c + 7a = 0 \Rightarrow 3c = 2 - 14 = -12$

$a = 2, \quad b = 9, \quad c = -4$

(b) l_2: $\mathbf{r} = \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ -4 \\ 1 \end{pmatrix}$ T maps l_1 to l_2, so $U = T^{-1}$ maps l_2 to l_1

$$\begin{pmatrix} 6 & -1 & -4 \\ 15 & -4 & -9 \\ -8 & 2 & 5 \end{pmatrix} \begin{pmatrix} 1 + 3\lambda \\ -4\lambda \\ -2 + \lambda \end{pmatrix} = \begin{pmatrix} 6 + 18\lambda + 4\lambda + 8 - 4\lambda \\ 15 + 45\lambda + 16\lambda + 18 - 9\lambda \\ -8 - 24\lambda - 8\lambda - 10 + 5\lambda \end{pmatrix} = \begin{pmatrix} 14 + 18\lambda \\ 33 + 52\lambda \\ -18 - 27\lambda \end{pmatrix}$$

l_1: $\mathbf{r} = \begin{pmatrix} 14 \\ 33 \\ -18 \end{pmatrix} + \lambda \begin{pmatrix} 18 \\ 52 \\ -27 \end{pmatrix}$

l_1: $\frac{x - 14}{18} = \frac{y - 33}{52} = \frac{z + 18}{-27}$

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Question 2 continued

Question 2 continued

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Question 2 continued

(Total for Question 2 is 8 marks)

3. The ellipse E has equation

$$\frac{x^2}{49} + \frac{y^2}{b^2} = 1$$

where b is a constant and $0 < b < 7$

The eccentricity of the ellipse is e

(a) Write down, in terms of e only,

(i) the coordinates of the foci of E

(ii) the equations of the directrices of E

(2)

Given that

- the point $P(x, y)$ lies on E where $x > 0$
- the point S is the focus of E on the positive x-axis
- the line l is the directrix of E which crosses the positive x-axis
- the point M lies on l such that the line through P and M is parallel to the x-axis

(b) determine an expression for

(i) PS^2 in terms of e, x and y

(ii) PM^2 in terms of e and x

(2)

(c) Hence show that

$$b^2 = 49(1 - e^2)$$

(2)

Given that E crosses the y-axis at the points with coordinates $\left(0, \pm 4\sqrt{3}\right)$

(d) determine the value of e

(2)

Given that the x coordinate of P is $\frac{7}{2}$

(e) determine the area of triangle OPM, where O is the origin.

(3)

(a)(i) foci $(\pm 7e, 0)$

(ii) directrices $x = \pm \frac{7}{e}$

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Question 3 continued

(b)(i) $PS^2 = (x - 7e)^2 + y^2$

(ii) $PM^2 = \left(\frac{7}{e} - x\right)^2$

(c) P lies on E, so $PS = e \times PM \Rightarrow PS^2 = e^2 PM^2$

$(x - 7e)^2 + y^2 = e^2\left(\frac{7}{e} - x\right)^2$

$x^2 - 14ex + 49e^2 + y^2 = 49 - 14ex + e^2x^2$

$x^2(1 - e^2) + y^2 = 49(1 - e^2)$

$\frac{x^2}{49} + \frac{y^2}{49(1 - e^2)} = 1$, compare with $\frac{x^2}{49} + \frac{y^2}{b^2} = 1$

$\therefore \quad b^2 = 49(1 - e^2)$

(d) $b^2 = (4\sqrt{3})^2 = 48$

$48 = 49(1 - e^2) \Rightarrow e^2 = 1 - \frac{48}{49} = \frac{1}{49}$

$e = \frac{1}{7}$

(e) $x = \frac{7}{2}$: $\frac{(7/2)^2}{49} + \frac{y^2}{48} = 1 \Rightarrow \frac{1}{4} + \frac{y^2}{48} = 1$

$y^2 = 36 \Rightarrow y = 6$ (taking $P\left(\frac{7}{2}, 6\right)$; $y = -6$ gives the same area)

l: $x = \frac{7}{e} = 49$, so $M(49, 6)$ and $PM = 49 - \frac{7}{2} = \frac{91}{2}$

Area $OPM = \frac{1}{2} \times PM \times 6 = \frac{1}{2} \times \frac{91}{2} \times 6$

$= \frac{273}{2}$

Question 3 continued

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Question 3 continued

(Total for Question 3 is 11 marks)

4.

$$\mathbf{M} = \begin{pmatrix} 0 & -1 & 3 \\ -1 & 4 & -1 \\ 3 & -1 & 0 \end{pmatrix}$$

Given that $\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$ is an eigenvector of $\mathbf{M}$

(a) determine its corresponding eigenvalue.

(2)

Given that –3 is an eigenvalue of $\mathbf{M}$

(b) determine a corresponding eigenvector.

(2)

Hence, given that $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ is also an eigenvector of $\mathbf{M}$

(c) determine a diagonal matrix $\mathbf{D}$ and an orthogonal matrix $\mathbf{P}$ such that $\mathbf{D} = \mathbf{P}^{\mathrm{T}}\mathbf{MP}$

(4)

(a) $\begin{pmatrix} 0 & -1 & 3 \\ -1 & 4 & -1 \\ 3 & -1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} = \begin{pmatrix} 0+2+3 \\ -1-8-1 \\ 3+2+0 \end{pmatrix} = \begin{pmatrix} 5 \\ -10 \\ 5 \end{pmatrix} = 5\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$

eigenvalue $\lambda = 5$

(b) $(M + 3I)x = 0$: $\begin{pmatrix} 3 & -1 & 3 \\ -1 & 7 & -1 \\ 3 & -1 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$3x - y + 3z = 0, \quad -x + 7y - z = 0$

7 × first + second: $20x + 20z = 0 \Rightarrow z = -x$, then $y = 3x + 3z = 0$

eigenvector $\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$

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Question 4 continued

(c) $\begin{pmatrix} 0 & -1 & 3 \\ -1 & 4 & -1 \\ 3 & -1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0-1+3 \\ -1+4-1 \\ 3-1+0 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix} = 2\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \Rightarrow \lambda = 2$

eigenvalues $-3,\ 2,\ 5$ with eigenvectors $(1, 0, -1),\ (1, 1, 1),\ (1, -2, 1)$

lengths: $\sqrt{1+0+1} = \sqrt{2}$, $\sqrt{1+1+1} = \sqrt{3}$, $\sqrt{1+4+1} = \sqrt{6}$

normalised: $\begin{pmatrix} 1/\sqrt{2} \\ 0 \\ -1/\sqrt{2} \end{pmatrix}$, $\begin{pmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{pmatrix}$, $\begin{pmatrix} 1/\sqrt{6} \\ -2/\sqrt{6} \\ 1/\sqrt{6} \end{pmatrix}$

$P = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{3} & 1/\sqrt{6} \\ 0 & 1/\sqrt{3} & -2/\sqrt{6} \\ -1/\sqrt{2} & 1/\sqrt{3} & 1/\sqrt{6} \end{pmatrix}$ $D = \begin{pmatrix} -3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 5 \end{pmatrix}$

Question 4 continued

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Question 4 continued

(Total for Question 4 is 8 marks)

5. (a) Use the definitions of hyperbolic functions in terms of exponentials to prove that

$$1 - \operatorname{sech}^2 x \equiv \tanh^2 x$$

(3)

$$I_n = \int_0^{\frac{1}{3}\ln 2} \tanh^n 3x \, \mathrm{d}x \qquad n \in \mathbb{Z} \qquad n \geqslant 0$$

(b) Show that

$$I_n = I_{n-2} - \frac{p^{n-1}}{3(n-1)} \qquad n \geqslant 2$$

where p is a rational number to be determined.

(4)

(c) Hence determine the exact value of

$$\int_0^{\frac{1}{3}\ln 2} \tanh^5 3x \, \mathrm{d}x$$

giving your answer in the form $a \ln b + c$ where a, b and c are rational numbers to be found.

(4)

(a) $1 - \operatorname{sech}^2 x = 1 - \left(\frac{2}{e^x + e^{-x}}\right)^2 = 1 - \frac{4}{(e^x + e^{-x})^2} = \frac{(e^x + e^{-x})^2 - 4}{(e^x + e^{-x})^2}$

$= \frac{e^{2x} + 2 + e^{-2x} - 4}{(e^x + e^{-x})^2} = \frac{e^{2x} - 2 + e^{-2x}}{(e^x + e^{-x})^2} = \frac{(e^x - e^{-x})^2}{(e^x + e^{-x})^2}$

$= \left(\frac{e^x - e^{-x}}{e^x + e^{-x}}\right)^2 = \tanh^2 x \qquad \therefore \quad 1 - \operatorname{sech}^2 x \equiv \tanh^2 x$

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Question 5 continued

(b) $I_n = \int_0^{\frac{1}{3}\ln 2} \tanh^{n-2}3x \tanh^2 3x \, dx = \int_0^{\frac{1}{3}\ln 2} \tanh^{n-2}3x \, (1 - \operatorname{sech}^2 3x) \, dx$

$= I_{n-2} - \int_0^{\frac{1}{3}\ln 2} \tanh^{n-2}3x \operatorname{sech}^2 3x \, dx$

$\frac{d}{dx}(\tanh^{n-1}3x) = 3(n-1)\tanh^{n-2}3x \operatorname{sech}^2 3x$

$\Rightarrow \int \tanh^{n-2}3x \operatorname{sech}^2 3x \, dx = \frac{1}{3(n-1)} \tanh^{n-1}3x$

$I_n = I_{n-2} - \frac{1}{3(n-1)} \left[\tanh^{n-1}3x\right]_0^{\frac{1}{3}\ln 2}$

$\tanh(\ln 2) = \frac{e^{2\ln 2} - 1}{e^{2\ln 2} + 1} = \frac{4-1}{4+1} = \frac{3}{5}, \quad \tanh 0 = 0$

$\therefore \quad I_n = I_{n-2} - \frac{(3/5)^{n-1}}{3(n-1)} \qquad p = \frac{3}{5}$

(c) $I_5 = I_3 - \frac{(3/5)^4}{3(4)} = I_3 - \frac{81}{7500} = I_3 - \frac{27}{2500}$

$I_3 = I_1 - \frac{(3/5)^2}{3(2)} = I_1 - \frac{9}{150} = I_1 - \frac{3}{50}$

$I_1 = \int_0^{\frac{1}{3}\ln 2} \tanh 3x \, dx = \left[\frac{1}{3}\ln(\cosh 3x)\right]_0^{\frac{1}{3}\ln 2} = \frac{1}{3}\ln(\cosh(\ln 2)) - \frac{1}{3}\ln 1$

$\cosh(\ln 2) = \frac{e^{\ln 2} + e^{-\ln 2}}{2} = \frac{2 + 1/2}{2} = \frac{5}{4} \quad \Rightarrow \quad I_1 = \frac{1}{3}\ln\left(\frac{5}{4}\right)$

$I_5 = \frac{1}{3}\ln\left(\frac{5}{4}\right) - \frac{3}{50} - \frac{27}{2500}$

$= \frac{1}{3}\ln\left(\frac{5}{4}\right) - \frac{177}{2500}$

Question 5 continued

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Question 5 continued

(Total for Question 5 is 11 marks)

6. The points A, B and C have coordinates (3, 2, 2), (–1, 1, 3) and (–2, 4, 2) respectively.

The plane Π_1 contains the points A, B and C

(a) Determine a Cartesian equation of Π_1

(4)

Given that

- point D has coordinates (–1, 1, –2)
- line l passes through D and is perpendicular to Π_1
- plane Π_2 has equation $\mathbf{r}.(14\mathbf{i} - \mathbf{j} - 17\mathbf{k}) = -66$
- l meets Π_2 at the point E

(b) show that $DE = p\sqrt{22}$ where p is a rational number to be determined.

(5)

The point F has coordinates (4, 3, q) where q is a constant.

Given that A, B, C and F are the vertices of a tetrahedron of volume 12

(c) determine the possible values of q

(3)

(a) $\overrightarrow{AB} = \begin{pmatrix}-1\\1\\3\end{pmatrix} - \begin{pmatrix}3\\2\\2\end{pmatrix} = \begin{pmatrix}-4\\-1\\1\end{pmatrix}$, $\overrightarrow{AC} = \begin{pmatrix}-2\\4\\2\end{pmatrix} - \begin{pmatrix}3\\2\\2\end{pmatrix} = \begin{pmatrix}-5\\2\\0\end{pmatrix}$

$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{pmatrix}-4\\-1\\1\end{pmatrix} \times \begin{pmatrix}-5\\2\\0\end{pmatrix} = \begin{pmatrix}(-1)(0) - (1)(2)\\(1)(-5) - (-4)(0)\\(-4)(2) - (-1)(-5)\end{pmatrix} = \begin{pmatrix}-2\\-5\\-13\end{pmatrix}$

$\mathbf{n} = \begin{pmatrix}2\\5\\13\end{pmatrix}$: $\begin{pmatrix}3\\2\\2\end{pmatrix} \cdot \begin{pmatrix}2\\5\\13\end{pmatrix} = 6 + 10 + 26 = 42$

Π_1: $2x + 5y + 13z = 42$

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Question 6 continued

(b) $l: \quad r = \begin{pmatrix} -1 \\ 1 \\ -2 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 5 \\ 13 \end{pmatrix}$

$\Pi_2: \quad 14(-1 + 2\lambda) - (1 + 5\lambda) - 17(-2 + 13\lambda) = -66$

$-14 + 28\lambda - 1 - 5\lambda + 34 - 221\lambda = -66 \quad \Rightarrow \quad 19 - 198\lambda = -66$

$\lambda = \frac{85}{198}$

$DE = |\lambda| \times |n| = \frac{85}{198}\sqrt{2^2 + 5^2 + 13^2} = \frac{85}{198}\sqrt{198} = \frac{85}{\sqrt{198}}$

$= \frac{85}{3\sqrt{22}} = \frac{85\sqrt{22}}{66} \qquad p = \frac{85}{66}$

(c) $AF = \begin{pmatrix} 4 \\ 3 \\ q \end{pmatrix} - \begin{pmatrix} 3 \\ 2 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ q - 2 \end{pmatrix}$

$AF \cdot (AB \times AC) = \begin{pmatrix} 1 \\ 1 \\ q - 2 \end{pmatrix} \cdot \begin{pmatrix} -2 \\ -5 \\ -13 \end{pmatrix} = -2 - 5 - 13q + 26 = 19 - 13q$

$\text{Volume} = \frac{1}{6}|19 - 13q| = 12 \quad \Rightarrow \quad 19 - 13q = \pm 72$

$q = -\frac{53}{13} \quad \text{or} \quad q = 7$

Question 6 continued

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Question 6 continued

(Total for Question 6 is 12 marks)

7.

$$y = \arccos(\operatorname{sech} x) \qquad x > 0$$

(a) Show that

$$\frac{dy}{dx} = \operatorname{sech} x$$

(3)

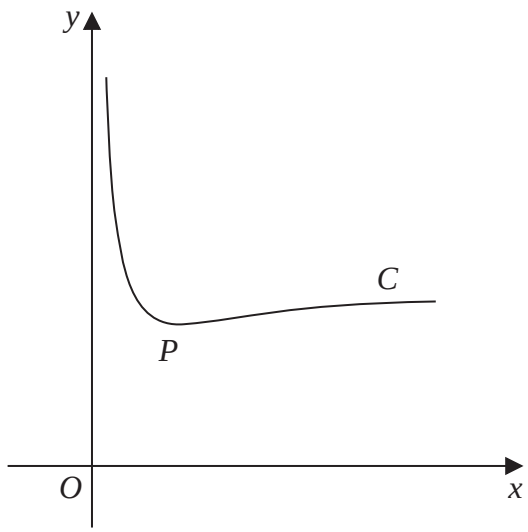


Figure 1

Figure 1 shows a sketch of part of the curve C with equation $y = f(x)$ where

$$f(x) = \arccos(\operatorname{sech} x) + \coth x \qquad x > 0$$

The point P is a minimum turning point of C

(b) Show that the x coordinate of P is $\ln\left(q + \sqrt{q}\right)$ where $q = \frac{1}{2}\left(1 + \sqrt{k}\right)$ and k is an integer to be determined.

(6)

(a) $\cos y = \operatorname{sech} x \Rightarrow -\sin y \frac{dy}{dx} = -\operatorname{sech} x \tanh x$

$\sin y = \sqrt{1 - \cos^2 y} = \sqrt{1 - \operatorname{sech}^2 x} = \sqrt{\tanh^2 x} = \tanh x \qquad (x > 0)$

$\Rightarrow \tanh x \frac{dy}{dx} = \operatorname{sech} x \tanh x \Rightarrow \frac{dy}{dx} = \operatorname{sech} x$

(b) $f'(x) = \operatorname{sech} x - \operatorname{cosech}^2 x = 0$

$\frac{1}{\cosh x} = \frac{1}{\sinh^2 x} \Rightarrow \sinh^2 x = \cosh x \Rightarrow \cosh^2 x - 1 = \cosh x$

$\cosh^2 x - \cosh x - 1 = 0$

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Question 7 continued

$\cosh x = \frac{1 \pm \sqrt{1+4}}{2} = \frac{1+\sqrt{5}}{2}$ $\left(\frac{1-\sqrt{5}}{2} < 1, \text{ rejected}\right)$

$x = \operatorname{arcosh} q = \ln\left(q + \sqrt{q^2-1}\right)$, where $q = \frac{1+\sqrt{5}}{2} = \frac{1}{2}(1+\sqrt{5})$

q satisfies $q^2 - q - 1 = 0$, so $q^2 - 1 = q \Rightarrow \sqrt{q^2-1} = \sqrt{q}$

$x = \ln\left(\frac{1}{2}(1+\sqrt{5}) + \sqrt{(\tfrac{1}{2}(1+\sqrt{5}))}\right)$

i.e. $x = \ln(q + \sqrt{q})$ where $q = \frac{1}{2}(1+\sqrt{k})$, $k = 5$

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Question 7 continued

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Question 7 continued

(Total for Question 7 is 9 marks)

8. **In this question you must show all stages of your working.**

Solutions relying entirely on calculator technology are not acceptable.

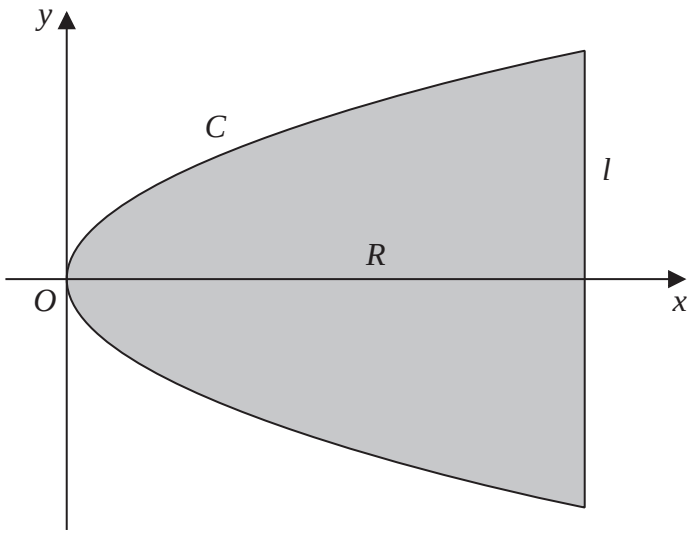


Figure 2

Figure 2 shows a sketch of part of the curve C with equation $y^2 = 8x$ and part of the line l with equation $x = 18$

The region R, shown shaded in Figure 2, is bounded by C and l

(a) Show that the perimeter of R is given by

$$\alpha + 2\int_0^{\beta} \sqrt{1 + \frac{y^2}{16}}\, dy$$

where α and β are positive constants to be determined.

(3)

(b) Use the substitution $y = 4\sinh u$ and algebraic integration to determine the exact perimeter of R, giving your answer in simplest form.

(6)

(a) $y^2 = 8x \Rightarrow x = \frac{y^2}{8}$, $\frac{dx}{dy} = \frac{y}{4}$

$x = 18$: $y^2 = 144$, $y = \pm 12$, so the length of l is 24

arc of $C = \int_{-12}^{12} \sqrt{1 + (dx/dy)^2}\, dy = 2\int_0^{12} \sqrt{1 + y^2/16}\, dy$ (symmetry)

perimeter $= 24 + 2\int_0^{12} \sqrt{1 + y^2/16}\, dy$, $\alpha = 24$, $\beta = 12$

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Question 8 continued

(b) $y = 4\sinh u \Rightarrow \frac{dy}{du} = 4\cosh u$

$\sqrt{1 + 16\sinh^2 u/16} = \sqrt{1 + \sinh^2 u} = \cosh u$

$\int \sqrt{1 + y^2/16}\, dy = \int \cosh u \times 4\cosh u\, du = 4\int \cosh^2 u\, du$

$= 4\int \frac{1}{2}(\cosh 2u + 1)\, du = 4\left(\frac{1}{4}\sinh 2u + \frac{1}{2}u\right) = \sinh 2u + 2u$

$= 2\sinh u \cosh u + 2u = 2\sinh u \sqrt{1 + \sinh^2 u} + 2u$

$y = 0 \Rightarrow u = 0; \quad y = 12 \Rightarrow \sinh u = 3 \Rightarrow u = \operatorname{arsinh} 3 = \ln(3 + \sqrt{10})$

$\int_0^{12} \sqrt{1 + y^2/16}\, dy = \left[2\sinh u \sqrt{1 + \sinh^2 u} + 2u\right]_0^{\operatorname{arsinh} 3}$

$= 2(3)\sqrt{10} + 2\ln(3 + \sqrt{10}) - 0 = 6\sqrt{10} + 2\ln(3 + \sqrt{10})$

perimeter $= 24 + 2(6\sqrt{10} + 2\ln(3 + \sqrt{10}))$

$= 24 + 12\sqrt{10} + 4\ln(3 + \sqrt{10})$

Question 8 continued

(Total for Question 8 is 9 marks)

TOTAL FOR PAPER IS 75 MARKS

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