Please check the examination details below before entering your candidate information

Candidate surname StemUp Model Solution | Other names

Centre Number | Candidate Number

Pearson Edexcel International Advanced Level

Time 1 hour 30 minutes

Paper reference **WFM02/01**

Mathematics

International Advanced Subsidiary/Advanced Level
Further Pure Mathematics F2

You must have:
Mathematical Formulae and Statistical Tables (Yellow), calculator

Total Marks

Candidates may use any calculator allowed by Pearson regulations. Calculators must not have the facility for symbolic algebra manipulation, differentiation and integration, or have retrievable mathematical formulae stored in them.

Instructions

- Use **black** ink or ball-point pen.
- If pencil is used for diagrams/sketches/graphs it must be dark (HB or B).
- **Fill in the boxes** at the top of this page with your name, centre number and candidate number.
- Answer **all** questions and ensure that your answers to parts of questions are clearly labelled.
- Answer the questions in the spaces provided
  – *there may be more space than you need.*
- You should show sufficient working to make your methods clear.
  Answers without working may not gain full credit.
- Inexact answers should be given to three significant figures unless otherwise stated.

Information

- A booklet 'Mathematical Formulae and Statistical Tables' is provided.
- There are 8 questions in this question paper. The total mark for this paper is 75.
- The marks for **each** question are shown in brackets
  – *use this as a guide as to how much time to spend on each question.*

Advice

- Read each question carefully before you start to answer it.
- Try to answer every question.
- Check your answers if you have time at the end.
- If you change your mind about an answer, cross it out and put your new answer and any working underneath.

Turn over ▶

P71101A

©2022 Pearson Education Ltd.

L:1/1/1/1/

P 7 1 1 0 1 A 0 1 3 2

Pearson

Leave blank

1. **In this question you must show all stages of your working.**

Solutions relying entirely on calculator technology are not acceptable.

(a) Express the complex number

$$-4 - 4\sqrt{3}\mathrm{i}$$

in the form $r(\cos\theta + \mathrm{i}\sin\theta)$, where $r > 0$ and $-\pi < \theta \leqslant \pi$

(3)

(b) Solve the equation

$$z^3 + 4 + 4\sqrt{3}\mathrm{i} = 0$$

giving your answers in the form $re^{i\theta}$, where $r > 0$ and $-\pi < \theta \leqslant \pi$

(4)

(a) $r = \sqrt{(-4)^2 + (-4\sqrt{3})^2} = \sqrt{16 + 48} = \sqrt{64} = 8$

$\tan\alpha = \frac{4\sqrt{3}}{4} = \sqrt{3} \Rightarrow \alpha = \frac{\pi}{3}$

$-4 - 4\sqrt{3}\,i$ is in the third quadrant, so $\theta = -\pi + \frac{\pi}{3} = -\frac{2\pi}{3}$

$-4 - 4\sqrt{3}\,i = 8\left(\cos\left(-\frac{2\pi}{3}\right) + i\sin\left(-\frac{2\pi}{3}\right)\right)$

(b) $z^3 = -4 - 4\sqrt{3}\,i = 8e^{-2\pi i/3}$

Let $z = re^{i\theta}$: $r^3e^{3i\theta} = 8e^{i(-2\pi/3 + 2k\pi)}$

$r^3 = 8 \Rightarrow r = \sqrt[3]{8} = 2$

$3\theta = -\frac{2\pi}{3} + 2k\pi \Rightarrow \theta = -\frac{2\pi}{9} + \frac{2k\pi}{3}$

$k = 0$: $\theta = -\frac{2\pi}{9}$ $\quad k = 1$: $\theta = \frac{4\pi}{9}$ $\quad k = -1$: $\theta = -\frac{8\pi}{9}$

$z = 2e^{-8\pi i/9}$, $\quad z = 2e^{-2\pi i/9}$, $\quad z = 2e^{4\pi i/9}$

DO NOT WRITE IN THIS AREA
DO NOT WRITE IN THIS AREA
DO NOT WRITE IN THIS AREA

DO NOT WRITE IN THIS AREA
DO NOT WRITE IN THIS AREA
DO NOT WRITE IN THIS AREA

Leave blank

Question 1 continued

Leave blank

Question 1 continued

DO NOT WRITE IN THIS AREA

DO NOT WRITE IN THIS AREA

DO NOT WRITE IN THIS AREA

DO NOT WRITE IN THIS AREA
DO NOT WRITE IN THIS AREA
DO NOT WRITE IN THIS AREA

Leave blank

Question 1 continued

Q1

(Total 7 marks)

Leave blank

2. Determine the general solution of the differential equation

$$2\frac{d^2y}{dx^2} - 5\frac{dy}{dx} - 3y = 2e^{3x}$$

(6)

Auxiliary equation: $2m^2 - 5m - 3 = 0$

$(2m + 1)(m - 3) = 0 \Rightarrow m = -\frac{1}{2}, \quad m = 3$

C.F.: $y = Ae^{-x/2} + Be^{3x}$

3 is a root of the auxiliary equation, so try P.I. $y = axe^{3x}$

$\frac{dy}{dx} = ae^{3x} + 3axe^{3x}$

$\frac{d^2y}{dx^2} = 3ae^{3x} + 3ae^{3x} + 9axe^{3x} = 6ae^{3x} + 9axe^{3x}$

$2(6a + 9ax)e^{3x} - 5(a + 3ax)e^{3x} - 3axe^{3x} = 2e^{3x}$

$12a + 18ax - 5a - 15ax - 3ax = 2$

$7a = 2 \Rightarrow a = \frac{2}{7}$

General solution: $y = Ae^{-x/2} + Be^{3x} + \frac{2}{7}xe^{3x}$

DO NOT WRITE IN THIS AREA

DO NOT WRITE IN THIS AREA

DO NOT WRITE IN THIS AREA

DO NOT WRITE IN THIS AREA
DO NOT WRITE IN THIS AREA
DO NOT WRITE IN THIS AREA

Leave blank

Question 2 continued

Q2

(Total 6 marks)

Leave blank

3.

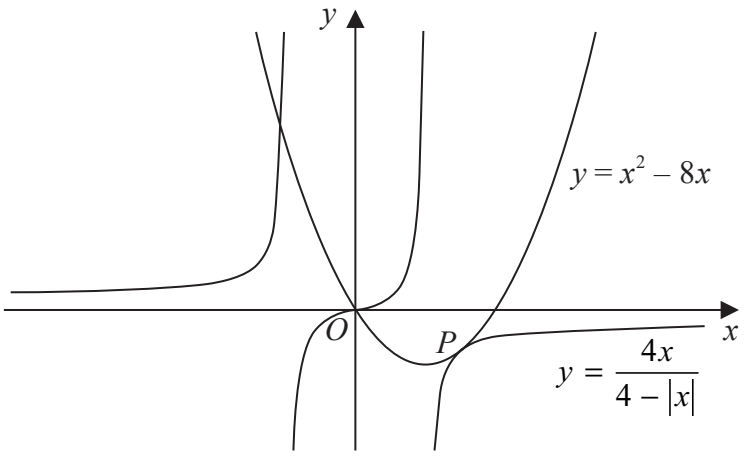


Figure 1

Figure 1 shows a sketch of the curve C_1 with equation

$$y = \frac{4x}{4 - |x|}$$

and the curve C_2 with equation

$$y = x^2 - 8x$$

For $x > 0$, C_1 has equation $y = \frac{4x}{4 - x}$

(a) Use algebra to show that C_1 touches C_2 at a point P, stating the coordinates of P

(5)

(b) Hence or otherwise, using algebra, solve the inequality

$$x^2 - 8x > \frac{4x}{4 - |x|}$$

(6)

(a) $x^2 - 8x = \frac{4x}{4 - x}$

$(x^2 - 8x)(4 - x) = 4x$

$4x^2 - x^3 - 32x + 8x^2 - 4x = 0$

$x^3 - 12x^2 + 36x = 0$

$x(x^2 - 12x + 36) = 0$

$x(x - 6)^2 = 0 \Rightarrow x = 0$ or $x = 6$

$x = 6$: $y = 36 - 48 = -12$ (check: $\frac{24}{4 - 6} = -12$)

$x = 6$ is a repeated root, so C_1 touches C_2 at $P(6, -12)$

DO NOT WRITE IN THIS AREA
DO NOT WRITE IN THIS AREA
DO NOT WRITE IN THIS AREA

DO NOT WRITE IN THIS AREA
DO NOT WRITE IN THIS AREA
DO NOT WRITE IN THIS AREA

Leave blank

Question 3 continued

(b) For $x < 0$, C_1 is $y = \frac{4x}{4 + x}$

$\frac{4x}{4 + x} = x^2 - 8x \Rightarrow 4x = (x^2 - 8x)(4 + x)$

$4x = 4x^2 + x^3 - 32x - 8x^2$

$x^3 - 4x^2 - 36x = 0 \Rightarrow x(x^2 - 4x - 36) = 0$

$x = \frac{4 \pm \sqrt{(16 + 144)}}{2} = 2 \pm 2\sqrt{10}$

$x < 0$, so the curves meet at $x = 2 - 2\sqrt{10}$ (≈ -4.32)

Critical values: $x = 2 - 2\sqrt{10}$, -4, 0, 4, 6

($x = \pm 4$ are the vertical asymptotes of C_1)

Solution: $x < 2 - 2\sqrt{10}$, $-4 < x < 0$, $4 < x < 6$, $x > 6$

Leave blank

Question 3 continued

DO NOT WRITE IN THIS AREA

DO NOT WRITE IN THIS AREA

DO NOT WRITE IN THIS AREA

DO NOT WRITE IN THIS AREA
DO NOT WRITE IN THIS AREA
DO NOT WRITE IN THIS AREA

Leave blank

Question 3 continued

(Total 11 marks)

Q3

Leave blank

4.

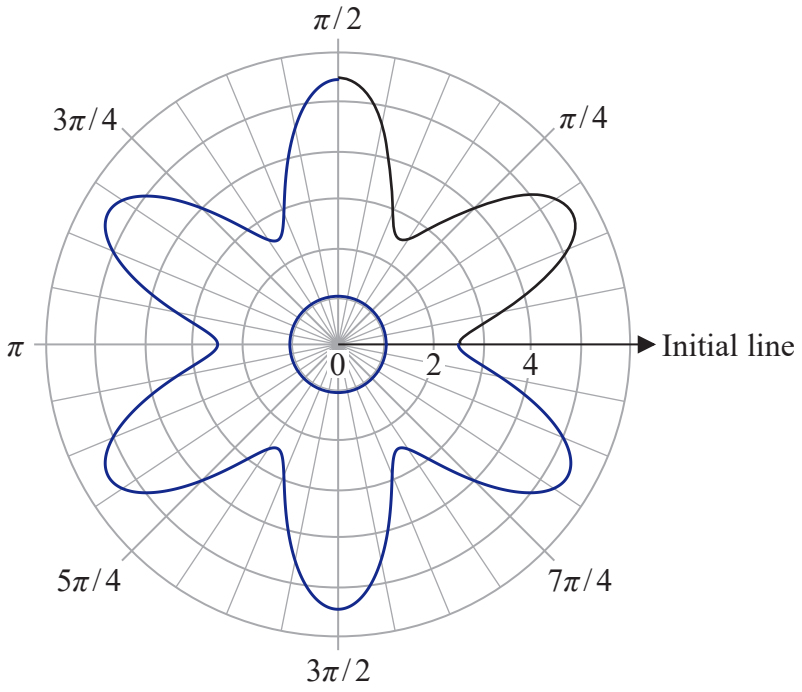


Figure 2

Figure 2 shows part of the curve with polar equation

$$r = 4 - \frac{3}{2}\cos 6\theta \qquad 0 \leqslant \theta < 2\pi$$

(a) Sketch, on the polar grid in Figure 2,

(i) the rest of the curve with equation $r = 4 - \frac{3}{2}\cos 6\theta \qquad 0 \leqslant \theta < 2\pi$

(ii) the polar curve with equation $r = 1 \qquad 0 \leqslant \theta < 2\pi$

A spare copy of the grid is given on page 15.

(3)

In part (b) you must show all stages of your working.

Solutions relying entirely on calculator technology are not acceptable.

(b) Determine the exact area enclosed between the two curves defined in part (a).

(7)

(b) Area $= \frac{1}{2}\int_0^{2\pi} r^2\, d\theta - \pi(1)^2$

$r^2 = \left(4 - \frac{3}{2}\cos 6\theta\right)^2 = 16 - 12\cos 6\theta + \frac{9}{4}\cos^2 6\theta$

$= 16 - 12\cos 6\theta + \frac{9}{8}(1 + \cos 12\theta)$

DO NOT WRITE IN THIS AREA

DO NOT WRITE IN THIS AREA

DO NOT WRITE IN THIS AREA

DO NOT WRITE IN THIS AREA
DO NOT WRITE IN THIS AREA
DO NOT WRITE IN THIS AREA

Leave blank

Question 4 continued

$$\frac{1}{2}\int_0^{2\pi} r^2\,d\theta = \frac{1}{2}\int_0^{2\pi}\left(16 - 12\cos 6\theta + \frac{9}{8} + \frac{9}{8}\cos 12\theta\right)d\theta$$

$$= \frac{1}{2}\left[16\theta - 2\sin 6\theta + \frac{9}{8}\left(\theta + \frac{1}{12}\sin 12\theta\right)\right]_0^{2\pi}$$

$$= \frac{1}{2}\left(32\pi - 0 + \frac{9}{8}(2\pi + 0) - 0\right) = \frac{1}{2}\left(32\pi + \frac{9\pi}{4}\right) = \frac{137\pi}{8}$$

Area between the curves $= \frac{137\pi}{8} - \pi$

$$= \frac{129\pi}{8}$$

Leave blank

Question 4 continued

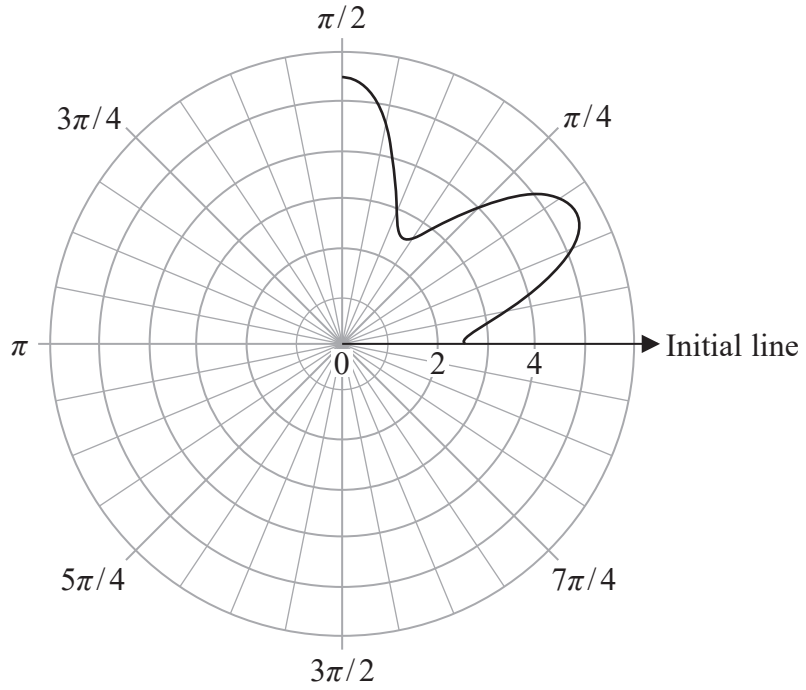
DO NOT WRITE IN THIS AREA
DO NOT WRITE IN THIS AREA
DO NOT WRITE IN THIS AREA

DO NOT WRITE IN THIS AREA
DO NOT WRITE IN THIS AREA
DO NOT WRITE IN THIS AREA

Leave blank

Question 4 continued

Only use this grid if you need to redraw your answer to part (a)



Copy of Figure 2

(Total 10 marks)

Q4

Leave blank

5. $y = \sqrt{4 + \ln x} \qquad x > \frac{1}{2}$

(a) Show that

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = -\frac{9 + 2\ln x}{4x^2(4 + \ln x)^{\frac{3}{2}}}$$

(5)

(b) Hence, or otherwise, determine the Taylor series expansion about $x = 1$ for y, in ascending powers of $(x - 1)$, up to and including the term in $(x - 1)^2$, giving each coefficient in simplest form.

(3)

(a) $y = (4 + \ln x)^{1/2}$

$\frac{dy}{dx} = \frac{1}{2}(4 + \ln x)^{-1/2} \times \frac{1}{x} = \frac{1}{2}x^{-1}(4 + \ln x)^{-1/2}$

$\frac{d^2y}{dx^2} = -\frac{1}{2}x^{-2}(4 + \ln x)^{-1/2} + \frac{1}{2}x^{-1} \times \left(-\frac{1}{2}\right)(4 + \ln x)^{-3/2} \times \frac{1}{x}$

$= -\frac{1}{2x^2(4 + \ln x)^{1/2}} - \frac{1}{4x^2(4 + \ln x)^{3/2}}$

$= -\frac{2(4 + \ln x) + 1}{4x^2(4 + \ln x)^{3/2}}$

$= -\frac{9 + 2\ln x}{4x^2(4 + \ln x)^{3/2}}$

(b) At $x = 1$: $y = \sqrt{4} = 2$

$\frac{dy}{dx} = \frac{1}{2 \times 1 \times \sqrt{4}} = \frac{1}{4}$

$\frac{d^2y}{dx^2} = -\frac{9 + 0}{4 \times 1 \times 4^{3/2}} = -\frac{9}{32}$

$y = y(1) + y'(1)(x - 1) + \frac{y''(1)}{2!}(x - 1)^2 + \ldots$

$y = 2 + \frac{1}{4}(x - 1) + \frac{1}{2} \times \left(-\frac{9}{32}\right)(x - 1)^2 + \ldots$

$y = 2 + \frac{1}{4}(x - 1) - \frac{9}{64}(x - 1)^2 + \ldots$

DO NOT WRITE IN THIS AREA
DO NOT WRITE IN THIS AREA
DO NOT WRITE IN THIS AREA

DO NOT WRITE IN THIS AREA
DO NOT WRITE IN THIS AREA
DO NOT WRITE IN THIS AREA

Leave blank

Question 5 continued

Leave blank

Question 5 continued

DO NOT WRITE IN THIS AREA

DO NOT WRITE IN THIS AREA

DO NOT WRITE IN THIS AREA

DO NOT WRITE IN THIS AREA
DO NOT WRITE IN THIS AREA
DO NOT WRITE IN THIS AREA

Leave blank

Question 5 continued

Q5

(Total 8 marks)

Leave blank

6. Given that $A > B > 0$, by letting $x = \arctan A$ and $y = \arctan B$

(a) prove that

$$\arctan A - \arctan B = \arctan\left(\frac{A - B}{1 + AB}\right)$$

(3)

(b) Show that when $A = r + 2$ and $B = r$

$$\frac{A - B}{1 + AB} = \frac{2}{(1 + r)^2}$$

(2)

(c) Hence, using the method of differences, show that

$$\sum_{r=1}^{n} \arctan\left(\frac{2}{(1 + r)^2}\right) = \arctan(n + p) + \arctan(n + q) - \arctan 2 - \frac{\pi}{4}$$

where p and q are integers to be determined.

(4)

(d) Hence, making your reasoning clear, determine

$$\sum_{r=1}^{\infty} \arctan\left(\frac{2}{(1 + r)^2}\right)$$

giving the answer in the form $k\pi - \arctan 2$, where k is a constant.

(2)

(a) $x = \arctan A \Rightarrow \tan x = A$; $y = \arctan B \Rightarrow \tan y = B$

$\tan(x - y) = \dfrac{\tan x - \tan y}{1 + \tan x \tan y} = \dfrac{A - B}{1 + AB}$

$\Rightarrow x - y = \arctan\left(\dfrac{A - B}{1 + AB}\right)$

$\therefore \arctan A - \arctan B = \arctan\left(\dfrac{A - B}{1 + AB}\right)$

(b) $\dfrac{A - B}{1 + AB} = \dfrac{(r + 2) - r}{1 + r(r + 2)} = \dfrac{2}{1 + 2r + r^2} = \dfrac{2}{(1 + r)^2}$

DO NOT WRITE IN THIS AREA

DO NOT WRITE IN THIS AREA

DO NOT WRITE IN THIS AREA

DO NOT WRITE IN THIS AREA
DO NOT WRITE IN THIS AREA
DO NOT WRITE IN THIS AREA

Leave blank

Question 6 continued

(c) $\sum_{r=1}^{n} \arctan\left(\frac{2}{(1+r)^2}\right) = \sum_{r=1}^{n} \left[\arctan(r+2) - \arctan r\right]$

$r = 1$: $\arctan 3 - \arctan 1$

$r = 2$: $\arctan 4 - \arctan 2$

$r = 3$: $\arctan 5 - \arctan 3$

$\vdots$

$r = n - 1$: $\arctan(n+1) - \arctan(n-1)$

$r = n$: $\arctan(n+2) - \arctan n$

Sum $= \arctan(n+2) + \arctan(n+1) - \arctan 2 - \arctan 1$

$= \arctan(n+1) + \arctan(n+2) - \arctan 2 - \frac{\pi}{4}$

$p = 1$, $q = 2$

(d) As $n \to \infty$, $\arctan(n+1) \to \frac{\pi}{2}$ and $\arctan(n+2) \to \frac{\pi}{2}$

$\sum_{r=1}^{\infty} \arctan\left(\frac{2}{(1+r)^2}\right) = \frac{\pi}{2} + \frac{\pi}{2} - \arctan 2 - \frac{\pi}{4}$

$= \frac{3\pi}{4} - \arctan 2 \qquad \left(k = \frac{3}{4}\right)$

Leave blank

Question 6 continued

DO NOT WRITE IN THIS AREA

DO NOT WRITE IN THIS AREA

DO NOT WRITE IN THIS AREA

DO NOT WRITE IN THIS AREA
DO NOT WRITE IN THIS AREA
DO NOT WRITE IN THIS AREA

Leave blank

Question 6 continued

(Total 11 marks)

Q6

Leave blank

7. A transformation from the z-plane to the w-plane is given by

$$w = \frac{(1 + \mathrm{i})z + 2(1 - \mathrm{i})}{z - \mathrm{i}} \qquad z \neq \mathrm{i}$$

The transformation maps points on the imaginary axis in the z-plane onto a line in the w-plane.

(a) Find an equation for this line.

(2)

The transformation maps points on the real axis in the z-plane onto a circle in the w-plane.

(b) Find the centre and radius of this circle.

(6)

(a) $z = iy$: $w = \dfrac{(1+i)iy + 2(1-i)}{iy - i} = \dfrac{(2-y) + i(y-2)}{i(y-1)}$

$= \dfrac{((2-y) + i(y-2))(-i)}{y-1} = \dfrac{(y-2) + i(y-2)}{y-1}$

So $u = v$: the line is $v = u$ (Im w = Re w)

(b) $w(z - i) = (1 + i)z + 2(1 - i)$

$z(w - 1 - i) = iw + 2(1 - i) \Rightarrow z = \dfrac{iw + 2 - 2i}{w - 1 - i}$

$w = u + iv$: $z = \dfrac{(2-v) + i(u-2)}{(u-1) + i(v-1)} \times \dfrac{(u-1) - i(v-1)}{(u-1) - i(v-1)}$

$\text{Im } z = \dfrac{(u-2)(u-1) - (2-v)(v-1)}{(u-1)^2 + (v-1)^2}$

z on the real axis $\Rightarrow$ Im $z = 0$:

$(u - 1)(u - 2) - (2 - v)(v - 1) = 0$

$u^2 - 3u + 2 + v^2 - 3v + 2 = 0$

$u^2 - 3u + v^2 - 3v + 2 = 0$

$\left(u - \frac{3}{2}\right)^2 + \left(v - \frac{3}{2}\right)^2 = \frac{9}{4} + \frac{9}{4} - 2 = \frac{1}{2}$

Centre $\frac{3}{2} + \frac{3}{2}i$, radius $\frac{\sqrt{2}}{2}$

DO NOT WRITE IN THIS AREA

DO NOT WRITE IN THIS AREA

DO NOT WRITE IN THIS AREA

DO NOT WRITE IN THIS AREA
DO NOT WRITE IN THIS AREA
DO NOT WRITE IN THIS AREA

Leave blank

Question 7 continued

Leave blank

Question 7 continued

DO NOT WRITE IN THIS AREA

DO NOT WRITE IN THIS AREA

DO NOT WRITE IN THIS AREA

DO NOT WRITE IN THIS AREA
DO NOT WRITE IN THIS AREA
DO NOT WRITE IN THIS AREA

Leave blank

Question 7 continued

Q7

(Total 8 marks)

Leave blank

8. (a) Show that the transformation $v = y - 2x$ transforms the differential equation

$$\frac{dy}{dx} + 2yx(y - 4x) = 2 - 8x^3 \qquad \text{(I)}$$

into the differential equation

$$\frac{dv}{dx} = -2xv^2 \qquad \text{(II)}$$

(4)

(b) Solve the differential equation (II) to determine v as a function of x

(4)

(c) Hence obtain the general solution of the differential equation (I).

(1)

(d) Sketch the solution curve that passes through the point $(-1, -1)$.

On your sketch show clearly the equation of any horizontal or vertical asymptotes.

You do **not** need to find the coordinates of any intercepts with the coordinate axes or the coordinates of any stationary points.

(5)

(a) $v = y - 2x \Rightarrow y = v + 2x$, $\frac{dy}{dx} = \frac{dv}{dx} + 2$

$\frac{dv}{dx} + 2 + 2x(v + 2x)(v + 2x - 4x) = 2 - 8x^3$

$\frac{dv}{dx} + 2 + 2x(v + 2x)(v - 2x) = 2 - 8x^3$

$\frac{dv}{dx} + 2 + 2x(v^2 - 4x^2) = 2 - 8x^3$

$\frac{dv}{dx} + 2 + 2xv^2 - 8x^3 = 2 - 8x^3 \Rightarrow \frac{dv}{dx} = -2xv^2$

(b) $\int v^{-2}\, dv = \int -2x\, dx$

$-v^{-1} = -x^2 - c \Rightarrow \frac{1}{v} = x^2 + c$

$v = \frac{1}{x^2 + c}$

(c) $y = 2x + \frac{1}{x^2 + c}$

DO NOT WRITE IN THIS AREA
DO NOT WRITE IN THIS AREA
DO NOT WRITE IN THIS AREA

DO NOT WRITE IN THIS AREA
DO NOT WRITE IN THIS AREA
DO NOT WRITE IN THIS AREA

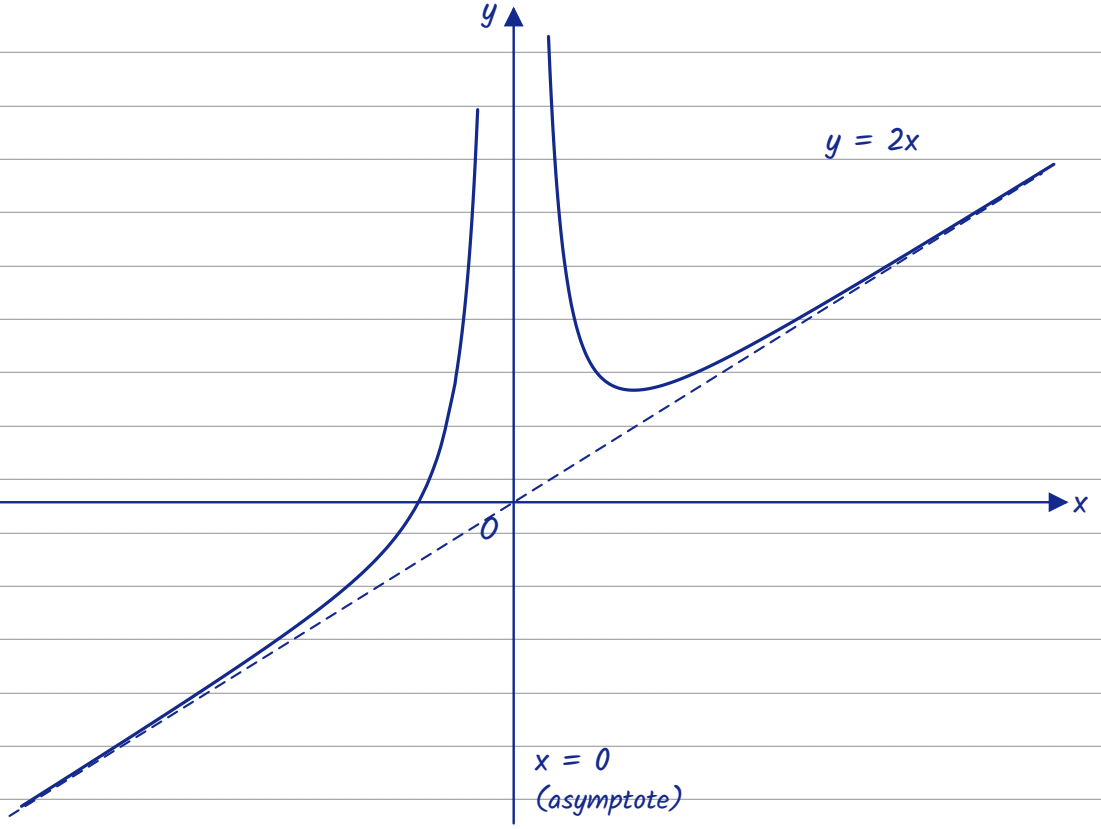
Leave blank

Question 8 continued

(d) $(-1, -1)$: $-1 = -2 + \frac{1}{1+c} \Rightarrow 1 + c = 1 \Rightarrow c = 0$

$y = 2x + \frac{1}{x^2}$

As $x \to \pm\infty$, $y \to 2x$ (oblique asymptote); vertical asymptote $x = 0$



Leave blank

Question 8 continued

DO NOT WRITE IN THIS AREA
DO NOT WRITE IN THIS AREA
DO NOT WRITE IN THIS AREA

DO NOT WRITE IN THIS AREA
DO NOT WRITE IN THIS AREA
DO NOT WRITE IN THIS AREA

Leave blank

Question 8 continued

Leave blank

Question 8 continued

Q8

(Total 14 marks)

TOTAL FOR PAPER: 75 MARKS

END

DO NOT WRITE IN THIS AREA

DO NOT WRITE IN THIS AREA

DO NOT WRITE IN THIS AREA