## Differentiation, Mixed Exercise 9

**1 a** $y = \ln x^2 = 2\ln x$

(using properties of logs)

$\therefore \frac{dy}{dx} = 2 \times \frac{1}{x} = \frac{2}{x}$

*Alternative method*:

When $y = 1\ln f(x)$, $\frac{dy}{dx} = \frac{f'(x)}{f(x)}$

(by the chain rule)

$\therefore y = \ln x^2 \Rightarrow \frac{dy}{dx} = \frac{2x}{x^2} = \frac{2}{x}$

**b** $y = x^2 \sin 3x$

Using the product rule,

$\frac{dy}{dx} = x^2(3\cos 3x) + (\sin 3x) \times 2x$

$= 3x^2 \cos 3x + 2x \sin 3x$

**2 a** $2y = x - \sin x \cos x$

$\therefore y = \frac{x}{2} - \frac{1}{2}\sin x \cos x$

Using the product rule,

$\frac{dy}{dx} = \frac{1}{2} - \frac{1}{2}(\sin x(-\sin x) + \cos x \cos x)$

$= \frac{1}{2} + \frac{1}{2}\sin^2 x - \frac{1}{2}\cos^2 x$

$= \frac{1}{2}(1 - \cos^2 x) + \frac{1}{2}\sin^2 x$

$= \frac{1}{2}\sin^2 x + \frac{1}{2}\sin^2 x$

$= \sin^2 x$

**b** $y = \frac{x}{2} - \frac{1}{2}\sin x \cos x$

$\frac{dy}{dx} = \sin^2 x$

$\frac{d^2y}{dx^2} = 2\sin x \cos x = \sin 2x$

At points of inflection $\frac{d^2y}{dx^2} = 0$

i.e. $\sin 2x = 0$

$2x = \pi, 2\pi$ or $3\pi$

$x = \frac{\pi}{2}, \pi$ or $\frac{3\pi}{2}$

When $x = \frac{\pi}{2}$, $y = \frac{\pi}{4}$

At $x = \frac{\pi}{3}$, $\frac{d^2y}{dx^2} > 0$; at $x = \frac{3\pi}{4}$, $\frac{d^2y}{dx^2} < 0$

So $\frac{d^2y}{dx^2}$ changes sign either side of $x = \frac{\pi}{2}$

When $x = \pi$, $y = \frac{\pi}{2}$

At $x = \frac{3\pi}{4}$, $\frac{d^2y}{dx^2} < 0$; at $x = \frac{5\pi}{4}$, $\frac{d^2y}{dx^2} > 0$

So $\frac{d^2y}{dx^2}$ changes sign either side of $x = \pi$

When $x = \frac{3\pi}{2}$, $y = \frac{3\pi}{4}$

At $x = \frac{5\pi}{4}$, $\frac{d^2y}{dx^2} > 0$; at $x = \frac{7\pi}{4}$, $\frac{d^2y}{dx^2} < 0$

So $\frac{d^2y}{dx^2}$ changes sign either side of $x = \frac{3\pi}{2}$

Hence the points of inflection are

$\left(\frac{\pi}{2}, \frac{\pi}{4}\right)$, $\left(\pi, \frac{\pi}{2}\right)$ and $\left(\frac{3\pi}{2}, \frac{3\pi}{4}\right)$

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**3 a** $y = \dfrac{\sin x}{x}$

Using the quotient rule:

$$\frac{dy}{dx} = \frac{x\cos x - \sin x \times 1}{x^2}$$
$$= \frac{x\cos x - \sin x}{x^2}$$

**b** $y = \ln\dfrac{1}{x^2+9} = \ln 1 - \ln(x^2+9)$
$$= -\ln(x^2+9)$$

(by the laws of logarithms)

Using the chain rule:

$$\frac{dy}{dx} = -\frac{1}{x^2+9} \times 2x = -\frac{2x}{x^2+9}$$

**4 a** $f(x) = \dfrac{x}{x^2+2}$

$$f'(x) = \frac{(x^2+2)\times 1 - x \times 2x}{(x^2+2)^2} = \frac{2-x^2}{(x^2+2)^2}$$

The function is increasing when $f'(x) \geqslant 0$

i.e. $\dfrac{2-x^2}{(x^2+2)^2} \geqslant 0$

$$x^2 \leqslant 2$$
$$-\sqrt{2} \leqslant x \leqslant \sqrt{2}$$

Hence $f(x)$ is increasing on the interval $[-k, k]$ where $k = \sqrt{2}$.

**b** $f''(x) = \dfrac{-2x(x^2+2)^2 - 4x(x^2+2)(2-x^2)}{(x^2+2)^4}$

$$= \frac{2x(x^2+2)\left(-(x^2+2) - 2(2-x^2)\right)}{(x^2+2)^4}$$
$$= \frac{2x(x^2+2)(x^2-6)}{(x^2+2)^4}$$

$f''(x)$ changes sign when the numerator $2x(x^2+2)(x^2-6)$ is zero

i.e. at $x = 0$ and $x = \pm\sqrt{6}$

where $y = 0$ and $y = \dfrac{\pm\sqrt{6}}{6+2}$

Points of inflection are

(0, 0) and $\left(\pm\sqrt{6}, \pm\dfrac{\sqrt{6}}{8}\right)$

**5 a** $f(x) = 12\ln x + x^{\frac{3}{2}}, \quad x > 0$

$$f'(x) = \frac{12}{x} + \frac{3}{2}x^{\frac{1}{2}} = \frac{12}{x} + \frac{3}{2}\sqrt{x}$$

f($x$) is an increasing function when $f'(x) \geq 0$

As $x > 0$, $\dfrac{12}{x} + \dfrac{3}{2}\sqrt{x}$ is always positive.

$\therefore$ f($x$) is increasing for all $x > 0$.

**b** $f''(x) = -\dfrac{12}{x^2} + \dfrac{3}{4}x^{-\frac{1}{2}} = -\dfrac{12}{x^2} + \dfrac{3}{4\sqrt{x}}$

At a point of inflection $f''(x) = 0$

$$-\frac{12}{x^2} + \frac{3}{4\sqrt{x}} = 0$$
$$\frac{12}{x^2} = \frac{3}{4\sqrt{x}}$$
$$x^2 = 16\sqrt{x}$$
$$x^{\frac{3}{2}} = 16$$
$$x = \sqrt[3]{256}$$
$$f\left(\sqrt[3]{256}\right) = 12\ln(256)^{\frac{1}{3}} + 256^{\frac{1}{2}}$$
$$= 4\ln 256 + 16$$
$$= 4\ln 2^8 + 16 = 32\ln 2 + 16$$

Coordinates of the point of inflection are $\left(\sqrt[3]{256}, 32\ln 2 + 16\right)$

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**6** $y = \cos^2 x + \sin x$

$\dfrac{dy}{dx} = -2\cos x \sin x + \cos x$

$= \cos x(1 - 2\sin x)$

At stationary points $\dfrac{dy}{dx} = 0$

$\cos x(1 - 2\sin x) = 0$

$\cos x = 0$ or $\sin x = \dfrac{1}{2}$

Solutions in the interval $(0, 2\pi]$ are

$x = \dfrac{\pi}{6}, \dfrac{\pi}{2}, \dfrac{5\pi}{6}$ and $\dfrac{3\pi}{2}$

$x = \dfrac{\pi}{6} \Rightarrow y = \dfrac{3}{4} + \dfrac{1}{2} = \dfrac{5}{4}$

$x = \dfrac{\pi}{2} \Rightarrow y = 1$

$x = \dfrac{5\pi}{6} \Rightarrow y = \dfrac{3}{4} + \dfrac{1}{2} = \dfrac{5}{4}$

$x = \dfrac{3\pi}{2} \Rightarrow y = -1$

So the stationary points are

$\left(\dfrac{\pi}{6}, \dfrac{5}{4}\right), \left(\dfrac{\pi}{2}, 1\right), \left(\dfrac{5\pi}{6}, \dfrac{5}{4}\right)$ and $\left(\dfrac{3\pi}{2}, -1\right)$

**7** $y = x\sqrt{\sin x} = x(\sin x)^{\frac{1}{2}}$

$\dfrac{dy}{dx} = x \times \dfrac{1}{2}(\sin x)^{-\frac{1}{2}} \cos x + (\sin x)^{\frac{1}{2}} \times 1$

$= \dfrac{1}{2}(\sin x)^{-\frac{1}{2}}\left(x\cos x + 2\sin x\right)$

At the maximum point $\dfrac{dy}{dx} = 0$

$\dfrac{1}{2}(\sin x)^{-\frac{1}{2}}\left(x\cos x + 2\sin x\right) = 0$

$\therefore\ x\cos x + 2\sin x = 0$

(as $(\sin x)^{-\frac{1}{2}} = \dfrac{1}{\sqrt{\sin x}} \neq 0$)

Dividing through by $\cos x$ gives

$x + 2\tan x = 0$

So the $x$-coordinate of the maximum point satisfies $2\tan x + x = 0$.

**8 a** $f(x) = e^{0.5x} - x^2$

$f'(x) = 0.5e^{0.5x} - 2x$

**b** $f'(6) = -1.957... < 0$

$f'(7) = 2.557... > 0$

As the sign changes between $x = 6$ and $x = 7$ and $f'(x)$ is continuous, $f'(x) = 0$ has a root $p$ between 6 and 7.

Therefore $y = f(x)$ has a stationary point at $x = p$ where $6 < p < 7$.

**9 a** $f(x) = e^{2x}\sin 2x$

$f'(x) = e^{2x}(2\cos 2x) + \sin 2x(2e^{2x})$

$= 2e^{2x}(\cos 2x + \sin 2x)$

At turning points $f'(x) = 0$

$2e^{2x}(\cos 2x + \sin 2x) = 0$

$\cos 2x + \sin 2x = 0$

$\sin 2x = -\cos 2x$

Divide both sides by $\cos 2x$:

$\tan 2x = -1$

$2x = \dfrac{3\pi}{4}$ or $\dfrac{7\pi}{4}$

$\therefore x = \dfrac{3\pi}{8}$ or $\dfrac{7\pi}{8}$ (in the interval $0 < x < \pi$)

When $x = \dfrac{3\pi}{8}$, $y = \dfrac{1}{\sqrt{2}}e^{\frac{3\pi}{4}}$

When $x = \dfrac{7\pi}{8}$, $y = -\dfrac{1}{\sqrt{2}}e^{\frac{7\pi}{4}}$

So the coordinates of the turning points are $\left(\dfrac{3\pi}{8}, \dfrac{1}{\sqrt{2}}e^{\frac{3\pi}{4}}\right)$ and $\left(\dfrac{7\pi}{8}, -\dfrac{1}{\sqrt{2}}e^{\frac{7\pi}{4}}\right)$.

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**9 b** $f'(x) = 2e^{2x}(\cos 2x + \sin 2x)$

$$f''(x) = 2e^{2x}(-2\sin 2x + 2\cos 2x) + 4e^{2x}(\cos 2x + \sin 2x)$$
$$= e^{2x}(-4\sin 2x + 4\cos 2x + 4\cos 2x + 4\sin 2x)$$
$$= 8e^{2x}\cos 2x$$

**c** $f''\left(\frac{3\pi}{8}\right) = 8e^{\frac{3\pi}{4}}\cos\frac{3\pi}{4}$

$$= 8e^{\frac{3\pi}{4}}\left(-\frac{\sqrt{2}}{2}\right) = -4\sqrt{2}\,e^{\frac{3\pi}{4}} < 0$$

$\therefore \left(\frac{3\pi}{8}, \frac{1}{\sqrt{2}}e^{\frac{3\pi}{4}}\right)$ is a maximum.

$f''\left(\frac{7\pi}{8}\right) = 8e^{\frac{7\pi}{4}}\cos\frac{7\pi}{4}$

$$= 8e^{\frac{7\pi}{4}}\left(\frac{\sqrt{2}}{2}\right) = 4\sqrt{2}\,e^{\frac{7\pi}{4}} > 0$$

$\therefore \left(\frac{7\pi}{8}, -\frac{1}{\sqrt{2}}e^{\frac{7\pi}{4}}\right)$ is a minimum.

**d** At points of inflection $f''(x) = 0$

$8e^{2x}\cos 2x = 0$

$\cos 2x = 0$

$2x = \frac{\pi}{2}$ or $\frac{3\pi}{2}$

$\therefore x = \frac{\pi}{4}$ or $\frac{3\pi}{4}$

When $x = \frac{\pi}{4}$, $y = e^{\frac{\pi}{2}}\sin\frac{\pi}{2} = e^{\frac{\pi}{2}}$

When $x = \frac{3\pi}{4}$, $y = e^{\frac{3\pi}{2}}\sin\frac{3\pi}{2} = -e^{\frac{3\pi}{2}}$

Points of inflection are $\left(\frac{\pi}{4}, e^{\frac{\pi}{2}}\right)$ and $\left(\frac{3\pi}{4}, -e^{\frac{3\pi}{2}}\right)$.

**10** $y = 2e^x + 3x^2 + 2$

$\frac{dy}{dx} = 2e^x + 6x$

When $x = 0$, $y = 4$ and $\frac{dy}{dx} = 2$

Equation of normal at (0, 4) is

$y - 4 = -\frac{1}{2}(x - 0)$

$2y - 8 = -x$

or $x + 2y - 8 = 0$

**11 a** $f(x) = 3\ln x + \frac{1}{x}$

$f'(x) = \frac{3}{x} - \frac{1}{x^2}$

At a stationary point $\frac{dy}{dx} = 0$

$\frac{3}{x} - \frac{1}{x^2} = 0$

$3x - 1 = 0$

$x = \frac{1}{3}$

So the $x$-coordinate of the stationary point $P$ is $\frac{1}{3}$

**b** At the point $Q$, $x = 1$ so $y = f(1) = 1$

The gradient of the curve at point $Q$ is $f'(1) = 3 - 1 = 2$

So the gradient of the normal to the curve at $Q$ is $-\frac{1}{2}$

Equation of the normal at $Q$ is

$y - 1 = -\frac{1}{2}(x - 1)$

i.e. $y = -\frac{1}{2}x + \frac{3}{2}$

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**12 a** Let $f(x) = e^{2x}\cos x$

Then $f'(x) = e^{2x}(-\sin x) + \cos x(2e^{2x})$
$= e^{2x}(2\cos x - \sin x)$

Turning points occur when $f'(x) = 0$

$e^{2x}(2\cos x - \sin x) = 0$
$\sin x = 2\cos x$

Dividing both sides by $\cos x$ gives

$\tan x = 2$

**b** When $x = 0, y = f(0) = e^0 \cos 0 = 1$

The gradient of the curve at (0, 1) is
$f'(0) = e^0(2 - 0) = 2$

This is also the gradient of the tangent at (0, 1).

So the equation of the tangent at (0, 1) is

$y - 1 = 2(x - 0)$

$y = 2x + 1$

**13 a** $x = y^2 \ln y$

Using the product rule:

$$\frac{dx}{dy} = y^2\left(\frac{1}{y}\right) + \ln y \times 2y = y + 2y\ln y$$

**b** $$\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}} = \frac{1}{y + 2y\ln y}$$

When $y$ = e,

$$\frac{dy}{dx} = \frac{1}{e + 2e\ln e} = \frac{1}{3e}$$

**14 a** $f(x) = (x^3 - 2x)e^{-x}$
$f'(x) = (x^3 - 2x)(-e^{-x}) + (3x^2 - 2)e^{-x}$
$= e^{-x}(-x^3 + 3x^2 + 2x - 2)$

**b** When $x = 0$, $f'(x) = -2$

Gradient of normal is $\frac{1}{2}$

$\therefore$ equation of normal to the curve at the origin is

$y = \frac{1}{2}x$

This line will intersect the curve again when

$\frac{1}{2}x = (x^3 - 2x)e^{-x}$

$1 = 2(x^2 - 2)e^{-x}$

$e^x = 2x^2 - 4$

$2x^2 = e^x + 4$

**15 a** $f(x) = x(1 + x)\ln x = (x + x^2)\ln x$

$f'(x) = (x + x^2) \times \frac{1}{x} + \ln x \times (1 + 2x)$
$= 1 + x + (1 + 2x)\ln x$

**b** At minimum point $A$, $f'(x) = 0$

$1 + x + (1 + 2x)\ln x = 0$

$(1 + 2x)\ln x = -(1 + x)$

$$\ln x = -\frac{1 + x}{1 + 2x}$$

So $x$-coordinate of $A$ is the solution to the equation $x = e^{-\frac{1+x}{1+2x}}$

**16 a** $x = 4t - 3$, $y = \frac{8}{t^2} = 8t^{-2}$

$\frac{dx}{dt} = 4$, $\frac{dy}{dt} = -16t^{-3}$

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{-16t^{-3}}{4} = \frac{-4}{t^3}$$

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**16 b** When $t = 2$, the curve has gradient

$$\frac{dy}{dx} = \frac{-4}{2^3} = -\frac{1}{2}$$

$\therefore$ the normal has gradient 2.

Also, when $t = 2$, $x = 5$ and $y = 2$, so the point $A$ has coordinates (5, 2).

$\therefore$ the equation of the normal at $A$ is

$$y - 2 = 2(x - 5)$$

i.e. $y = 2x - 8$

**17** $x = 2t,\ y = t^2$

$$\frac{dx}{dt} = 2,\ \frac{dy}{dt} = 2t$$

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{2t}{2} = t$$

At the point $P$ where $t = 3$, the gradient of the curve is $\frac{dy}{dx} = 3$

$\therefore$ gradient of the normal is $-\frac{1}{3}$

Also, when $t = 3$, the coordinates are (6, 9).

$\therefore$ the equation of the normal at $P$ is

$$y - 9 = -\frac{1}{3}(x - 6)$$

i.e. $3y + x = 33$

**18** $x = t^3,\ y = t^2$

$$\frac{dx}{dt} = 3t^2,\ \frac{dy}{dt} = 2t$$

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{2t}{3t^2} = \frac{2}{3t}$$

At the point (1, 1) the value of $t$ is 1.

$\therefore$ the gradient of the curve is $\frac{2}{3}$, which is also the gradient of the tangent.

$\therefore$ the equation of the tangent is

$$y - 1 = \frac{2}{3}(x - 1)$$

i.e. $y = \frac{2}{3}x + \frac{1}{3}$

**19 a** $x = 2\cos t + \sin 2t,\ y = \cos t - 2\sin 2t$

$$\frac{dx}{dt} = -2\sin t + 2\cos 2t$$

$$\frac{dy}{dt} = -\sin t - 4\cos 2t$$

**b**

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{-\sin t - 4\cos 2t}{-2\sin t + 2\cos 2t}$$

When $t = \frac{\pi}{4}$, $\frac{dy}{dx} = \frac{-\frac{1}{\sqrt{2}} - 0}{-\frac{2}{\sqrt{2}} + 0} = \frac{1}{2}$

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**19 c** The gradient of the normal at the point $P$ where $t = \frac{\pi}{4}$ is –2.

The coordinates of $P$ are found by substituting $t = \frac{\pi}{4}$ into the parametric equations:

$$x = \frac{2}{\sqrt{2}} + 1,\ y = \frac{1}{\sqrt{2}} - 2$$

$\therefore$ the equation of the normal at $P$ is

$$y - \left(\frac{1}{\sqrt{2}} - 2\right) = -2\left(x - \left(\frac{2}{\sqrt{2}} + 1\right)\right)$$

$$y - \frac{1}{\sqrt{2}} + 2 = -2x + 2\sqrt{2} + 2$$

i.e. $y + 2x = \frac{5\sqrt{2}}{2}$

**20 a** $x = 2t + 3,\ y = t^3 - 4t$

At point $A$, where $t = -1$, $x = 1$ and $y = 3$.

$\therefore$ the coordinates of $A$ are (1, 3).

$$\frac{dx}{dt} = 2,\ \frac{dy}{dt} = 3t^2 - 4$$

$$\therefore \frac{dy}{dx} = \frac{3t^2 - 4}{2}$$

At the point $A$, $\frac{dy}{dx} = -\frac{1}{2}$

$\therefore$ gradient of the tangent at $A$ is $-\frac{1}{2}$

Equation of the tangent at $A$ is

$$y - 3 = -\frac{1}{2}(x - 1)$$

$$2y - 6 = -x + 1$$

i.e. $2y + x = 7$

**b** The tangent line $l$ meets the curve $C$ at points $A$ and $B$.

Substitute $x = 2t + 3$ and $y = t^3 - 4t$ into the equation of $l$:

$$2(t^3 - 4t) + (2t + 3) = 7$$

$$2t^3 - 6t = 4$$

$$t^3 - 3t - 2 = 0$$

At point $A$, $t = -1$, so $t = -1$ is a root of this equation, and hence $(t + 1)$ is a factor of the left-hand side expression.

$$\begin{aligned} t^3 - 3t - 2 &= (t + 1)(t^2 - t - 2) \\ &= (t + 1)(t + 1)(t - 2) \\ &= 2(t + 1)^2(t - 2) \end{aligned}$$

So line $l$ meets the curve $C$ at $t = -1$ (repeated root because the line is tangent to the curve there) and at $t = 2$.

Therefore, at point $B$, $t = 2$.

**21** The rate of change of $V$ is $\frac{dV}{dt}$

$$\therefore \frac{dV}{dt} \propto V$$

i.e. $\frac{dV}{dt} = -kV$

where $k$ is a positive constant.
(The negative sign is needed as the value of the car is *decreasing*.)

**22** The rate of change of mass is $\frac{dM}{dt}$

$$\therefore \frac{dM}{dt} \propto M$$

i.e. $\frac{dM}{dt} = -kM$

where $k$ is a positive constant.
(The negative sign represents *loss* of mass.)

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**23** The rate of change of pondweed is $\frac{dP}{dt}$

The growth rate is proportional to $P$:

$\text{growth rate} \propto P$

i.e. $\text{growth rate} = kP$
where $k$ is a positive constant.

But pondweed is also being removed at a constant rate $Q$.

$\therefore \frac{dP}{dt} = \text{growth rate} - \text{removal rate}$

$\frac{dP}{dt} = kP - Q$

**24** The rate of increase of the radius is $\frac{dr}{dt}$

$\therefore \frac{dr}{dt} \propto \frac{1}{r}$, as the rate is *inversely* proportional to the radius.

Hence $\frac{dr}{dt} = \frac{k}{r}$
where $k$ is the constant of proportion.

**25** The rate of change of temperature is $\frac{d\theta}{dt}$

$\therefore \frac{d\theta}{dt} \propto (\theta - \theta_0)$

i.e. $\frac{d\theta}{dt} = -k(\theta - \theta_0)$,
where $k$ is a positive constant.
(The negative sign indicates that the temperature is decreasing, i.e. *loss* of temperature.)

**26 a** $x = 4\cos 2t,\ y = 3\sin t$

The point $A\left(2, \frac{3}{2}\right)$ is on the curve, so

$4\cos 2t = 2$ and $3\sin t = \frac{3}{2}$

$\cos 2t = \frac{1}{2}$ and $\sin t = \frac{1}{2}$

The only value of $t$ in the interval $-\frac{\pi}{2} < t < \frac{\pi}{2}$ that satisfies both equations is $\frac{\pi}{6}$. Therefore $t = \frac{\pi}{6}$ at the point $A$.

**b** $\frac{dx}{dt} = -8\sin 2t,\ \frac{dy}{dt} = 3\cos t$

$\therefore \frac{dy}{dx} = \frac{3\cos t}{-8\sin 2t}$

$= -\frac{3\cos t}{16\sin t\cos t}$

(using a double angle formula)

$= -\frac{3}{16\sin t}$

$= -\frac{3}{16}\operatorname{cosec} t$

**c** At point $A$, where $t = \frac{\pi}{6}$, $\frac{dy}{dx} = -\frac{3}{8}$

$\therefore$ gradient of the normal at $A$ is $\frac{8}{3}$

Equation of the normal is

$y - \frac{3}{2} = \frac{8}{3}(x - 2)$

Multiply through by 6 and rearrange to give:

$6y - 9 = 16x - 32$

$6y - 16x + 23 = 0$

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**26 d** To find where the normal cuts the curve, substitute $x = 4\cos 2t$ and $y = 3\sin t$ into the equation of the normal:

$6(3\sin t) - 16(4\cos 2t) + 23 = 0$

$18\sin t - 64\cos 2t + 23 = 0$

$18\sin t - 64(1 - 2\sin^2 t) + 23 = 0$

(using a double angle formula)

$128\sin^2 t + 18\sin t - 41 = 0$

But $\sin t = \frac{1}{2}$ is one solution of this equation, as point *A* lies on the line and on the curve. So

$128\sin^2 t + 18\sin t - 41$

$= (2\sin t - 1)(64\sin t + 41)$

$\therefore (2\sin t - 1)(64\sin t + 41) = 0$

Therefore, at point *B*, $\sin t = -\frac{41}{64}$

$\therefore$ the *y*-coordinate of point *B* is

$3 \times \left(-\frac{41}{64}\right) = -\frac{123}{64}$

**27 a** $x = a\sin^2 t,\ y = a\cos t$

$\frac{dx}{dt} = 2a\sin t\cos t,\ \frac{dy}{dt} = -a\sin t$

$\therefore \frac{dy}{dx} = \frac{-a\sin t}{2a\sin t\cos t} = \frac{-1}{2\cos t} = -\frac{1}{2}\sec t$

**b** As $P\left(\frac{3}{4}a, \frac{1}{2}a\right)$ lies on the curve,

$a\sin^2 t = \frac{3}{4}a$ and $a\cos t = \frac{1}{2}a$

$\sin t = \pm\frac{\sqrt{3}}{2}$ and $\cos t = \frac{1}{2}$

The only value of *t* in the interval

$0 \leqslant t \leqslant \frac{\pi}{2}$ that satisfies both equations is $\frac{\pi}{3}$. Therefore $t = \frac{\pi}{3}$ at the point *P*.

Gradient of the curve at point *P* is $-\frac{1}{2}\sec\frac{\pi}{3} = -1$.

$\therefore$ equation of the tangent at *P* is

$y - \frac{1}{2}a = -1\left(x - \frac{3}{4}a\right)$

$y - \frac{1}{2}a = -x + \frac{3}{4}a$

Multiply equation by 4 and rearrange to give

$4y + 4x = 5a$

**c** Equation of the tangent at *C* is $4y + 4x = 5a$

At *A*, $x = 0 \Rightarrow y = \frac{5a}{4}$

At *B*, $y = 0 \Rightarrow x = \frac{5a}{4}$

Area of $AOB = \frac{1}{2}\left(\frac{5a}{4}\right)^2 = \frac{25}{32}a^2$,

which is of the form $ka^2$ with $k = \frac{25}{32}$

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**28** $x=(t+1)^2,\ y=\frac{1}{2}t^3+3$

$\frac{dx}{dt}=2(t+1),\ \frac{dy}{dt}=\frac{3}{2}t^2$

$\therefore\ \frac{dy}{dx}=\frac{\frac{3}{2}t^2}{2(t+1)}=\frac{3t^2}{4(t+1)}$

When $t=2,\ \frac{dy}{dx}=\frac{3\times4}{4\times3}=1$

$\therefore$ gradient of the normal at the point $P$, where $t = 2$, is $-1$.

The coordinates of $P$ are (9, 7).

$\therefore$ equation of the normal is

$y-7=-1(x-9)$

$y-7=-x+9$

i.e. $y+x=16$

**29** $5x^2+5y^2-6xy=13$

Differentiate with respect to $x$:

$10x+10y\frac{dy}{dx}-6\left(x\frac{dy}{dx}+y\right)=0$

$(10y-6x)\frac{dy}{dx}=6y-10x$

$\therefore\ \frac{dy}{dx}=\frac{6y-10x}{10y-6x}$

At the point (1, 2)

$\frac{dy}{dx}=\frac{12-10}{20-6}=\frac{2}{14}=\frac{1}{7}$

So the gradient of the curve at (1, 2) is $\frac{1}{7}$

**30** $e^{2x}+e^{2y}=xy$

Differentiate with respect to $x$:

$2e^{2x}+2e^{2y}\frac{dy}{dx}=x\frac{dy}{dx}+y\times1$

$2e^{2y}\frac{dy}{dx}-x\frac{dy}{dx}=y-2e^{2x}$

$(2e^{2y}-x)\frac{dy}{dx}=y-2e^{2x}$

$\therefore\ \frac{dy}{dx}=\frac{y-2e^{2x}}{2e^{2y}-x}$

**31** $y^3+3xy^2-x^3=3$

Differentiate with respect to $x$:

$3y^2\frac{dy}{dx}+\left(3x\times2y\frac{dy}{dx}+y^2\times3\right)-3x^2=0$

$(3y^2+6xy)\frac{dy}{dx}=3x^2-3y^2$

$\therefore\ \frac{dy}{dx}=\frac{3(x^2-y^2)}{3y(y+2x)}=\frac{x^2-y^2}{y(y+2x)}$

Turning points occur when $\frac{dy}{dx}=0$

$\frac{x^2-y^2}{y(y+2x)}=0$

$x^2=y^2$

$x=\pm y$

When $x=y,\ y^3+3y^3-y^3=3$

so $3y^3=3$

$y=1$ and hence $x=1$

When $x=-y,\ y^3-3y^3+y^3=3$

so $-y^3=3$

$y=\sqrt[3]{-3}$ and hence $x=-\sqrt[3]{-3}$

$\therefore$ the coordinates of the turning points are (1, 1) and $(-\sqrt[3]{-3},\ \sqrt[3]{-3})$.

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**32 a** $(1+x)(2+y) = x^2 + y^2$

Differentiate with respect to $x$:

$$(1+x)\left(\frac{dy}{dx}\right) + (2+y)(1) = 2x + 2y\frac{dy}{dx}$$

$$(1+x-2y)\frac{dy}{dx} = 2x - y - 2$$

$$\therefore \frac{dy}{dx} = \frac{2x-y-2}{1+x-2y}$$

**b** When the curve meets the $y$-axis, $x = 0$.

Substitute $x = 0$ into the equation of the curve:

$2 + y = y^2$

i.e. $y^2 - y - 2 = 0$

$(y-2)(y+1) = 0$

$y = 2$ or $y = -1$

At (0, 2), $\frac{dy}{dx} = \frac{0-2-2}{1+0-4} = \frac{4}{3}$

At (0, −1), $\frac{dy}{dx} = \frac{0+1-2}{1+0+2} = -\frac{1}{3}$

**c** A tangent that is parallel to the $y$-axis has infinite gradient.

For $\frac{dy}{dx} = \frac{2x-y-2}{1+x-2y}$ to be infinite,

the denominator $1 + x - 2y = 0$,

i.e. $x = 2y - 1$

Substitute $x = 2y - 1$ into the equation of the curve:

$(1 + 2y - 1)(2 + y) = (2y-1)^2 + y^2$

$2y^2 + 4y = 4y^2 - 4y + 1 + y^2$

$3y^2 - 8y + 1 = 0$

$y = \frac{8 \pm \sqrt{64-12}}{6} = \frac{4 \pm \sqrt{13}}{3}$

When $y = \frac{4+\sqrt{13}}{3}$, $x = \frac{5+2\sqrt{13}}{3}$

When $y = \frac{4-\sqrt{13}}{3}$, $x = \frac{5-2\sqrt{13}}{3}$

$\therefore$ there are two points at which the tangents are parallel to the $y$-axis.

They are $\left(\frac{5+2\sqrt{13}}{3}, \frac{4+\sqrt{13}}{3}\right)$ and $\left(\frac{5-2\sqrt{13}}{3}, \frac{4-\sqrt{13}}{3}\right)$.

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**33** $7x^2 + 48xy - 7y^2 + 75 = 0$

Implicit differentiation with respect to $x$ gives

$$14x + 48\left(x\frac{dy}{dx} + y\right) - 14y\frac{dy}{dx} = 0$$

$$(48x - 14y)\frac{dy}{dx} = -14x - 48y$$

$$\therefore \frac{dy}{dx} = \frac{-14x - 48y}{48x - 14y} = \frac{7x + 24y}{7y - 24x}$$

When $\frac{dy}{dx} = \frac{2}{11}$,

$$\frac{7x + 24y}{7y - 24x} = \frac{2}{11}$$

$$14y - 48x = 77x + 264y$$

$$125x + 250y = 0$$

$$\therefore x + 2y = 0$$

So the coordinates of the points at which the gradient is $\frac{2}{11}$ satisfy $x + 2y = 0$, which means that the points lie on the line $x + 2y = 0$.

**34** $y = x^x$

Take natural logs of both sides:

$$\ln y = \ln x^x$$

$$\ln y = x \ln x \text{ (using properties of logarithms)}$$

Differentiate with respect to $x$:

$$\frac{1}{y}\frac{dy}{dx} = x \times \frac{1}{x} + \ln x \times 1$$

$$= 1 + \ln x$$

$$\therefore \frac{dy}{dx} = y(1 + 1\text{n } x)$$

But $y = x^x$

$$\therefore \frac{dy}{dx} = x^x(1 + 1\text{n } x)$$

**35 a** $a^x = e^{kx}$

Take natural 1ogs of both sides:

$$\ln a^x = \ln e^{kx}$$

$$x \ln a = kx$$

As this is true for all values of $x$, $k = \ln a$.

**b** Taking $a = 2$,

$$y = 2^x = e^{kx} \text{ where } k = \ln 2$$

$$\frac{dy}{dx} = ke^{kx} = (\ln 2)e^{(\ln 2)x} = 2^x \ln 2$$

**c** At the point (2, 4), $x = 2$.

$\therefore$ gradient of the curve at (2, 4) is

$$\frac{dy}{dx} = 2^2 \ln 2 = 4 \ln 2 = \ln 2^4 = \ln 16$$

**36 a** $P = P_0(1.09)^t$

Take natural logs of both sides:

$$\ln P = \ln\left(P_0(1.09)^t\right)$$

$$= \ln P_0 + \ln(1.09)^t$$

$$= \ln P_0 + t \ln 1.09$$

$$\therefore t \ln 1.09 = \ln P - \ln P_0$$

$$t = \frac{\ln P - \ln P_0}{\ln 1.09} \text{ or } \frac{\ln\left(\frac{P}{P_0}\right)}{\ln 1.09}$$

**b** When $t = T$, $P = 2P_0$.

Substituting these into the expression in part **a** gives

$$T = \frac{\ln 2}{\ln 1.09} = 8.04 \text{ (3 s.f.)}$$

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**36 c** $\dfrac{dP}{dt} = P_0(1.09)^t(\ln 1.09)$

When $t = T$, $P = 2P_0$ so $(1.09)^T = 2$

Hence $\dfrac{dP}{dt} = P_0(1.09)^T(\ln 1.09)$

$= P_0 \times 2 \times \ln 1.09$

$= 0.172P_0$ (3 s.f.)

**37** $y = (\arcsin x)^2$

$$\frac{dy}{dx} = 2\arcsin x \times \frac{1}{\sqrt{1-x^2}} = \frac{2\arcsin x}{\sqrt{1-x^2}}$$

Using the quotient rule and the fact that $\dfrac{d\sqrt{1-x^2}}{dx} = \dfrac{1}{2}(1-x^2)^{-\frac{1}{2}}(-2x) = \dfrac{-x}{\sqrt{1-x^2}}$,

$$\frac{d^2y}{dx^2} = \frac{\sqrt{1-x^2}\left(\dfrac{2}{\sqrt{1-x^2}}\right) - 2\arcsin x\left(\dfrac{-x}{\sqrt{1-x^2}}\right)}{(1-x^2)}$$

$$= \frac{2 + x\left(\dfrac{2\arcsin x}{\sqrt{1-x^2}}\right)}{(1-x^2)}$$

$$(1-x^2)\frac{d^2y}{dx^2} = 2 + x\left(\frac{2\arcsin x}{\sqrt{1-x^2}}\right) = 2 + x\frac{dy}{dx}$$

$$\therefore (1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} - 2 = 0$$

**38** $y = x - \arctan x$

$$\frac{dy}{dx} = 1 - \frac{1}{1+x^2} = 1 - (1+x^2)^{-1}$$

$$\frac{d^2y}{dx^2} = 0 + 2x(1+x^2)^{-2} = 2x\left(\frac{1}{(1+x^2)^2}\right)$$

$$= 2x\left(1 - \left(1 - \frac{1}{1+x^2}\right)\right)^2$$

so $\dfrac{d^2y}{dx^2} = 2x\left(1 - \dfrac{dy}{dx}\right)^2$

**39** $y = \arcsin\dfrac{x}{\sqrt{1+x^2}}$

Let $t = \dfrac{x}{\sqrt{1+x^2}}$; then $y = \arcsin t$

$$\frac{dt}{dx} = \frac{(1+x^2)^{\frac{1}{2}} \times 1 - x \times \frac{1}{2}(1+x^2)^{-\frac{1}{2}}(2x)}{(1+x^2)}$$

$$= \frac{(1+x^2)^{-\frac{1}{2}}(1+x^2-x^2)}{(1+x^2)} = \frac{1}{(1+x^2)^{\frac{3}{2}}}$$

$$\frac{dy}{dt} = \frac{1}{\sqrt{1-t^2}}$$

$$\therefore \frac{dy}{dx} = \frac{\dfrac{1}{(1+x^2)^{\frac{3}{2}}}}{\sqrt{1-t^2}} = \frac{\dfrac{1}{(1+x^2)^{\frac{3}{2}}}}{\sqrt{1-\dfrac{x^2}{1+x^2}}}$$

$$= \frac{1}{(1+x^2)^{\frac{3}{2}}\sqrt{\dfrac{1}{1+x^2}}}$$

$$= \frac{1}{(1+x^2)^{\frac{3}{2}-\frac{1}{2}}} = \frac{1}{1+x^2}$$

**40 a** $y = \ln(\sin x)$

$$\frac{dy}{dx} = \cos x \times \frac{1}{\sin x} = \cot x$$

At a stationary point $\dfrac{dy}{dx} = 0$

$\cot x = 0 \Rightarrow x = \dfrac{\pi}{2}$

(in the interval $0 < x < \pi$)

When $x = \dfrac{\pi}{2}$, $y = \ln\left(\sin\dfrac{\pi}{2}\right) = \ln 1 = 0$

$\therefore$ stationary point is at $\left(\dfrac{\pi}{2}, 0\right)$.

**b** $\dfrac{d^2y}{dx^2} = -\text{cosec}^2 x$

$\text{cosec}^2 x = \dfrac{1}{\sin^2 x} > 0$ for all $0 < x < \pi$

$\therefore \dfrac{d^2y}{dx^2} < 0$ for all $0 < x < \pi$

Hence the curve $C$ is concave for all values of $x$ in its domain.

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**41 a** $m = 40\,e^{-0.244t}$

After 9 months, $t = 0.75$, so

$m = 40\,e^{-0.244\times0.75} = 40\,e^{-0.183} = 33.31...$

**b** $\dfrac{dm}{dt} = -0.244\times40\,e^{-0.244t} = -9.76\,e^{-0.244t}$

**c** The negative sign indicates that the mass is decreasing.

**42 a** $f(x) = \dfrac{\cos 2x}{e^x}$

$f'(x) = \dfrac{-2e^x\sin 2x - e^x\cos 2x}{e^{2x}}$

$= -\dfrac{2\sin 2x + \cos 2x}{e^x}$

At $A$ and $B$, $f'(x) = 0$

$2\sin 2x + \cos 2x = 0$

$2\tan 2x + 1 = 0$

$\tan 2x = -0.5$

$2x = 2.678$ or $5.820$

$x = 1.339$ or $2.910$

(**in the interval** $0 \leqslant x \leqslant \pi$)

$x = 1.339 \Rightarrow y = f(x) = -0.2344$

$x = 2.910 \Rightarrow y = f(x) = 0.04874$

Therefore, to 3 significant figures:
coordinates of $A$ are (1.34, −0.234);
coordinates of $B$ are (2.91, 0.0487).

**b** The curve of $y = 2 + 4f(x-4)$ is a transformation of $f(x)$, obtained via a translation of 4 units to the right, a stretch by a factor of 4 in the $y$-direction, and then a translation of 2 units upwards.

Turning points are:
minimum $(1.34+4, -0.234\times4+2)$ and
maximum $(2.91+4, 0.0487\times4+2)$,
i.e. minimum $(5.34, 1.06)$
and maximum $(6.91, 2.19)$.

**c** $f''(x)$

$= -\dfrac{e^x(4\cos 2x - 2\sin 2x) - e^x(2\sin 2x + \cos 2x)}{e^{2x}}$

$= \dfrac{4\sin 2x - 3\cos 2x}{e^x}$

$f(x)$ is concave when $f''(x) \leqslant 0$

$f''(x) = 0$ when

$4\sin 2x - 3\cos 2x = 0$

$\tan 2x = \dfrac{3}{4}$

$2x = 0.644$ or $3.785$

$x = 0.322$ or $1.893$

The curve has a minimum point and hence is convex between these values, so it is concave for
$0 \leqslant x \leqslant 0.322$ and $1.892 \leqslant x \leqslant \pi$.

**Challenge**

**a** $y = 2\sin 2t,\ x = 5\cos\left(t + \dfrac{\pi}{12}\right)$

$\dfrac{dy}{dt} = 4\cos 2t,\ \dfrac{dx}{dt} = -5\sin\left(t + \dfrac{\pi}{12}\right)$

$\therefore \dfrac{dy}{dx} = -\dfrac{4\cos 2t}{5\sin\left(t + \dfrac{\pi}{12}\right)}$

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**b** $\frac{dy}{dx} = 0$ when $4\cos 2t = 0$

$2t = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}$ or $\frac{7\pi}{2}$

$t = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}$ or $\frac{7\pi}{4}$

(in the interval $0 \leqslant x \leqslant 2\pi$)

$t = \frac{\pi}{4} \Rightarrow x = 5\cos\left(\frac{\pi}{3}\right) = \frac{5}{2}$

and $y = 2\sin\frac{\pi}{2} = 2$, i.e. point $\left(\frac{5}{2}, 2\right)$

$t = \frac{3\pi}{4} \Rightarrow x = 5\cos\left(\frac{5\pi}{6}\right) = -\frac{5\sqrt{3}}{2}$

and $y = 2\sin\frac{3\pi}{2} = -2$, i.e. point $\left(-\frac{5\sqrt{3}}{2}, -2\right)$

$t = \frac{5\pi}{4} \Rightarrow x = 5\cos\left(\frac{4\pi}{3}\right) = -\frac{5}{2}$

and $y = 2\sin\frac{5\pi}{2} = 2$, i.e. point $\left(-\frac{5}{2}, 2\right)$

$t = \frac{7\pi}{4} \Rightarrow x = 5\cos\left(\frac{11\pi}{6}\right) = \frac{5\sqrt{3}}{2}$

and $y = 2\sin\frac{7\pi}{2} = -2$, i.e. point $\left(\frac{5\sqrt{3}}{2}, -2\right)$

**c** The curve cuts the $x$-axis when $y = 0$,

i.e. when $2\sin 2t = 0$

$2t = 0, \pi, 2\pi, 3\pi, 4\pi$

$t = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi$

$t = 0 \Rightarrow x = 5\cos\frac{\pi}{12} = 4.83$, i.e. (4.83, 0)

with gradient $\frac{dy}{dx} = \frac{-4}{5\sin\frac{\pi}{12}} = -3.09$

$t = \frac{\pi}{2} \Rightarrow x = 5\cos\frac{7\pi}{12} = -1.29$, i.e. (−1.29, 0)

with gradient $\frac{dy}{dx} = \frac{4}{5\sin\frac{7\pi}{12}} = 0.828$

$t = \pi \Rightarrow x = 5\cos\frac{13\pi}{12} = -4.83$, i.e. (−4.83, 0)

with gradient $\frac{dy}{dx} = \frac{-4}{5\sin\frac{13\pi}{12}} = 3.09$

$t = \frac{3\pi}{2} \Rightarrow x = 5\cos\frac{19\pi}{12} = 1.29$, i.e. (1.29, 0)

with gradient $\frac{dy}{dx} = \frac{4}{5\sin\frac{19\pi}{12}} = -0.828$

The curve cuts the $y$-axis when $x = 0$.

i.e. when $5\cos\left(t + \frac{\pi}{12}\right) = 0$

$t + \frac{\pi}{12} = \frac{\pi}{2}, \frac{3\pi}{2}$

$t = \frac{5\pi}{12}, \frac{17\pi}{12}$

$t = \frac{5\pi}{12} \Rightarrow y = 2\sin\frac{5\pi}{6} = 1$, i.e. (0, 1)

with gradient $\frac{dy}{dx} = \frac{-4\cos\frac{5\pi}{6}}{5\sin\frac{\pi}{2}} = 0.693$

$t = \frac{17\pi}{12} \Rightarrow y = 2\sin\frac{17\pi}{6} = 1$, i.e. (0, 1)

with gradient $\frac{dy}{dx} = \frac{-4\cos\frac{17\pi}{6}}{5\sin\frac{3\pi}{2}} = -0.693$

So the curve cuts the $y$-axis twice at (0, 1) with gradients 0.693 and −0.693.

**d** $\frac{dx}{dy} = -\frac{5\sin\left(t + \frac{\pi}{12}\right)}{4\cos 2t}$

$\frac{dx}{dy} = 0$ when $\sin\left(t + \frac{\pi}{12}\right) = 0$

$t + \frac{\pi}{12} = \pi, 2\pi$

$t = \frac{11\pi}{12}, \frac{23\pi}{12}$

$t = \frac{11\pi}{12} \Rightarrow y = 2\sin\frac{11\pi}{6} = -1$

and $x = 5\cos\left(\frac{11\pi}{12} + \frac{\pi}{12}\right) = -5$

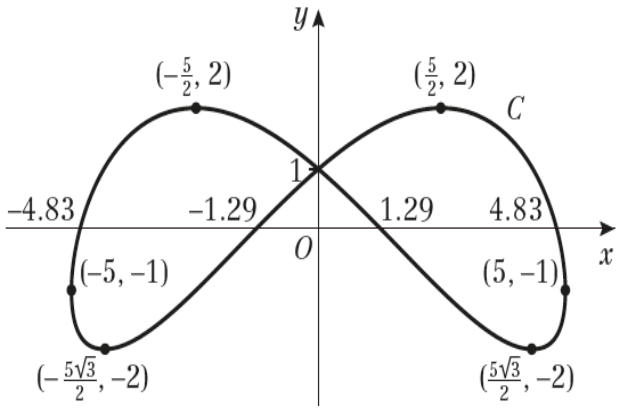
$t = \frac{23\pi}{12} \Rightarrow y = 2\sin\frac{23\pi}{6} = -1$

and $x = 5\cos\left(\frac{23\pi}{12} + \frac{\pi}{12}\right) = 5$

So points where curve is vertical are (−5, −1) and (5, −1).

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**e**




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