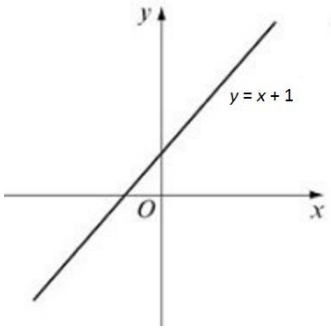
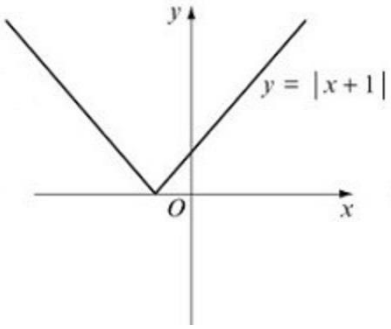
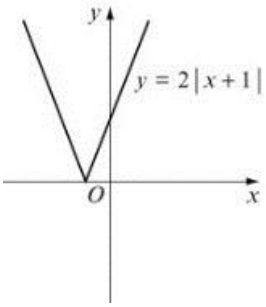
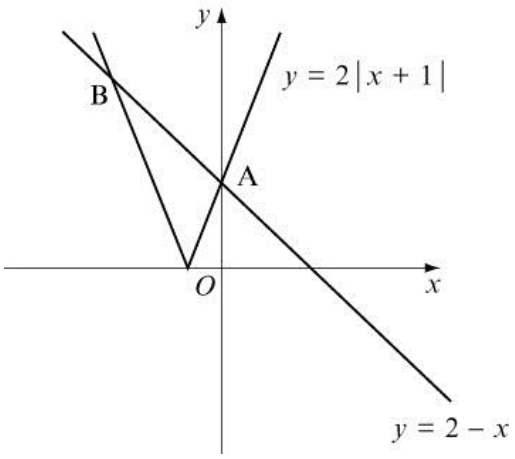
Functions and graphs Mixed exercise 2

1 a









b Intersection point A:

$2(x+1) = 2-x$

$2x+2 = 2-x$

$3x = 0$

$x = 0$

Intersection point B is on the reflected part of the modulus graph.

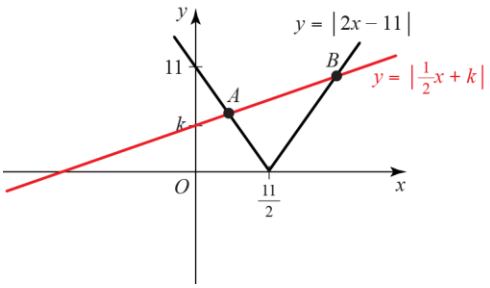
$-2(x+1) = 2-x$

$-2x-2 = 2-x$

$-x = 4$

$x = -4$

2



Minimum value of $y = |2x-11|$ is $y = 0$ at $x = \frac{11}{2}$

For two distinct solutions to $|2x-11| = \frac{1}{2}x - k$, we must have

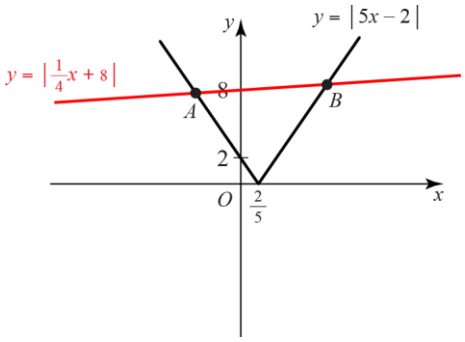
$\frac{1}{2}x - k > 0$ at $x = \frac{11}{2}$

$\frac{1}{2} \times \frac{11}{2} + k > 0$

$k > -\frac{11}{4}$

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3



At A:

$$-(5x-2)=-\frac{1}{4}x+8$$
$$-20x+8=-x+32$$
$$-19x=24$$
$$x=-\frac{24}{19}$$

At B:

$$5x-2=-\frac{1}{4}x+8$$
$$20x-8=-x+32$$
$$21x=40$$
$$x=\frac{40}{21}$$

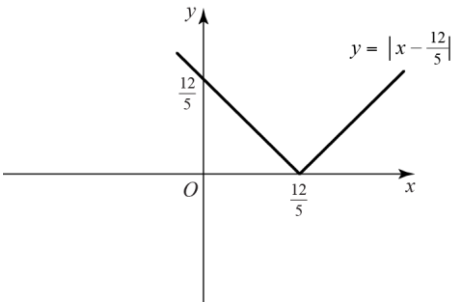
So the solution are

$x=-\frac{24}{19}$ and $x=\frac{40}{21}$

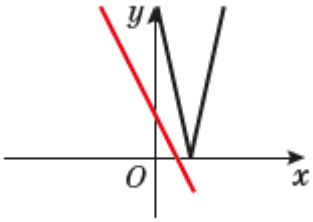
4 a $y=\left|12-5x\right|=5\left|-\left(x-\frac{12}{5}\right)\right|$

Start with $y=\left|x\right|$

$y=\left|x-\frac{12}{5}\right|$ is a horizontal translation of $+\frac{12}{5}$



$y=5\left|x-\frac{12}{5}\right|$ is a vertical stretch, scale factor 5

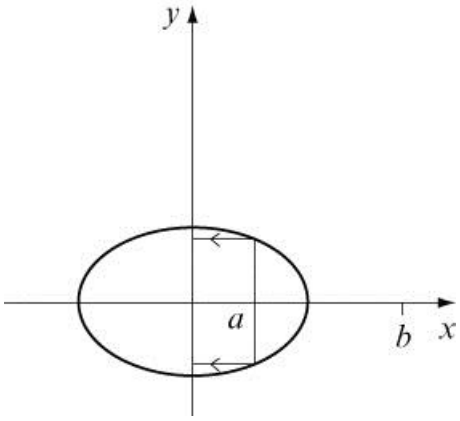


b The graphs do not intersect, so there are no solutions.

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5 a i One-to-many.

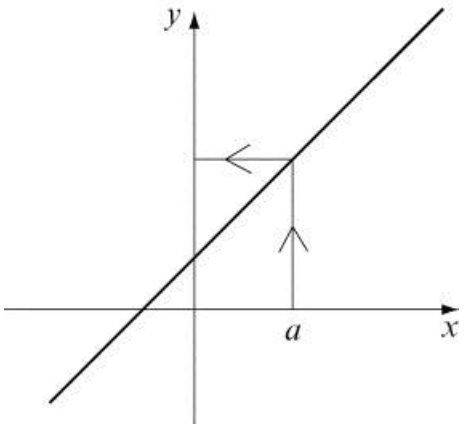
ii Not a function.



x value a gets mapped to two values of y.
x value b gets mapped to no values.

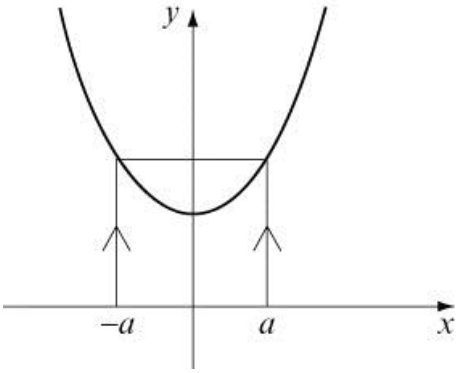
b i One-to-one.

ii Is a function.



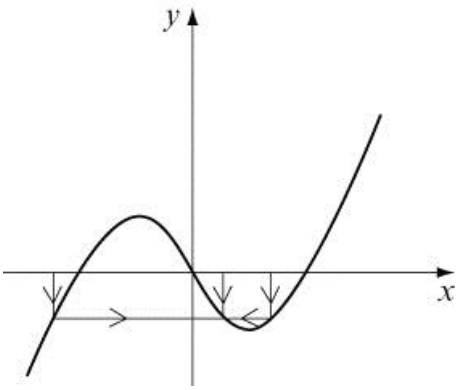
c i Many-to-one.

ii Is a function.



d i Many-to-one.

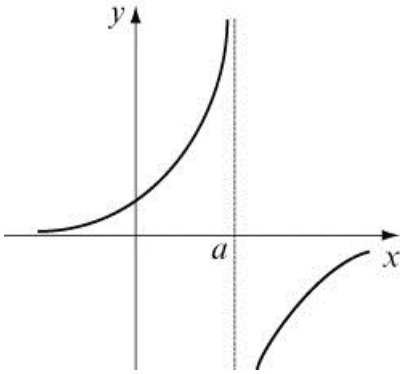
ii Is a function.



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5 e i One-to-one.

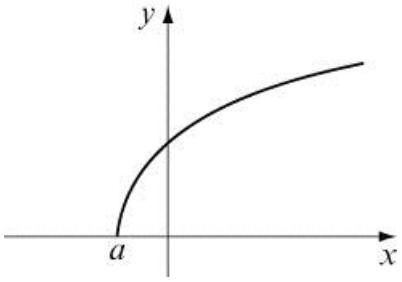
ii Not a function.



x value a doesn't get mapped to any value of y. It could be redefined as a function if the domain is said to exclude point a.

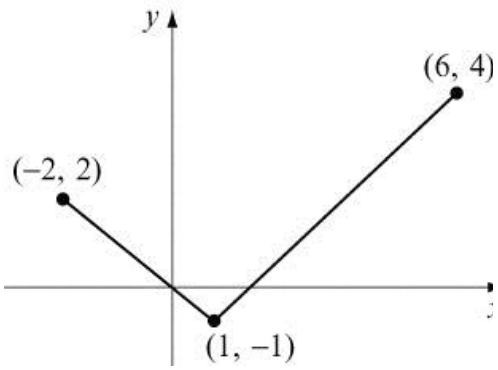
f i One-to-one.

ii Not a function for this domain.



x values less than a don't get mapped anywhere. Again, we could define the domain to be $x \leqslant a$ and then it would be a function.

6 a



For $x \leqslant 1$, $f(x) = -x$

This is a straight line of gradient -1.

At point $x = 1$, its y-coordinate is -1.

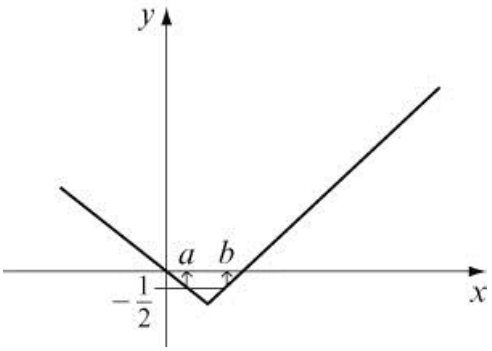
For $x > 1$, $f(x) = x - 2$

This is a straight line of gradient $+1$.

At point $x = 1$, its y-coordinate is also -1.

Hence, the graph is said to be continuous.

b There are two values x in the range $-2 \leqslant x \leqslant 6$ for which $f(x) = -\frac{1}{2}$



Point a is where

$$-x = -\frac{1}{2} \Rightarrow x = \frac{1}{2}$$

Point b is where

$$x - 2 = -\frac{1}{2} \Rightarrow x = 1\frac{1}{2}$$

Hence, the values of x for which $f(x) = -\frac{1}{2}$ are $x = \frac{1}{2}$ and $x = 1\frac{1}{2}$

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7 a $pq(x) = p(2x + 1)$
$= (2x + 1)^2 + 3(2x + 1) - 4$
$= 4x^2 + 4x + 1 + 6x + 3 - 4$
$= 4x^2 + 10x$

b $qq(x) = q(2x + 1)$
$= 2(2x + 1) + 1$
$= 4x + 3$

$pq(x) = qq(x)$ gives
$4x^2 + 10x = 4x + 3$
$4x^2 + 6x - 3 = 0$

Using the formula:

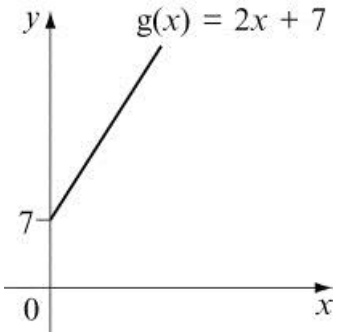
$$x = \frac{-6 \pm \sqrt{6^2 - 4 \times 4 \times (-3)}}{2 \times 4}$$

$$x = \frac{-6 \pm \sqrt{84}}{8}$$

$$x = \frac{-6 \pm 2\sqrt{21}}{8}$$

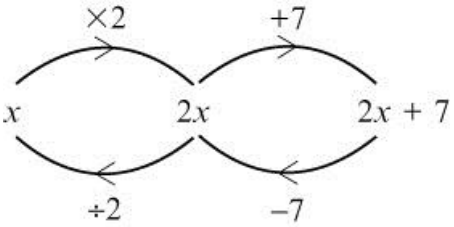
$$x = \frac{-3 \pm \sqrt{21}}{4}$$

8 a $y = 2x + 7$ is a straight line with gradient 2 and y-intercept 7



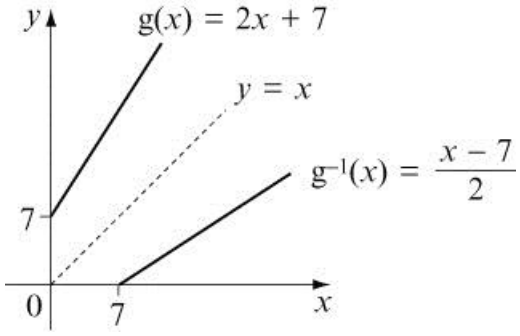
For $x \geqslant 0$, the range is $g(x) \geqslant 7$

b The range is $g^{-1}(x) \geqslant 0$.
To find the equation of the inverse function, you can use a flow chart.



$g^{-1}(x) = \frac{x-7}{2}$ and has domain $x \geqslant 7$

c



$g^{-1}(x)$ is the reflection of $g(x)$ in the line $y = x$.

9 a To find $f^{-1}(x)$, you can change the subject of the formula.

$$\text{Let } y = \frac{2x+3}{x-1}$$

$$y(x-1) = 2x+3$$

$$yx - y = 2x+3$$

$$yx - 2x = y+3$$

$$x(y-2) = y+3$$

$$x = \frac{y+3}{y-2}$$

Therefore $f^{-1}(x) = \frac{x+3}{x-2}$

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9 b i Domain $f(x) =$ Range $f^{-1}(x)$

$\therefore$ Range $f^{-1}(x) = \{y \in \mathbb{R}, y > 1\}$

ii Range $f(x) =$ Domain $f^{-1}(x)$

Now range of $f(x)$ is

$\{f(x) \in \mathbb{R}, f(x) > 2\}$

$\therefore$ Domain $f^{-1}(x) = \{x \in \mathbb{R}, x > 2\}$

10 a $f(x) = \dfrac{x}{x^2-1} - \dfrac{1}{x+1}$

$= \dfrac{x}{(x+1)(x-1)} - \dfrac{1}{(x+1)}$

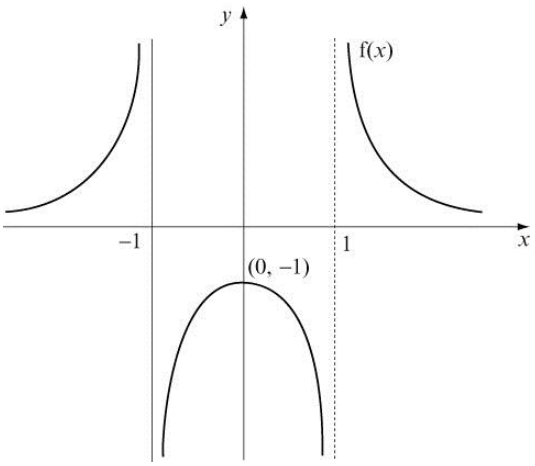
$= \dfrac{x}{(x+1)(x-1)} - \dfrac{x-1}{(x+1)(x-1)}$

$= \dfrac{x-(x-1)}{(x+1)(x-1)}$

$= \dfrac{1}{(x+1)(x-1)}$

b Consider the graph of

$y = \dfrac{1}{(x-1)(x+1)}$ for $x \in \mathbb{R}$:



For $x > 1, f(x) > 0$

c $gf(x) = g\left(\dfrac{1}{(x-1)(x+1)}\right)$

$= \dfrac{2}{\left(\dfrac{1}{(x-1)(x+1)}\right)}$

$= 2 \times \dfrac{(x-1)(x+1)}{1}$

$= 2(x-1)(x+1)$

$gf(x) = 70 \Rightarrow 2(x-1)(x+1) = 70$

$(x-1)(x+1) = 35$

$x^2 - 1 = 35$

$x^2 = 36$

$x = \pm 6$

11 a $f(7) = 4(7-2)$

$= 4 \times 5$

$= 20$

$g(3) = 3^3 + 1$

$= 27 + 1$

$= 28$

$h(-2) = 3^{-2}$

$= \dfrac{1}{3^2}$

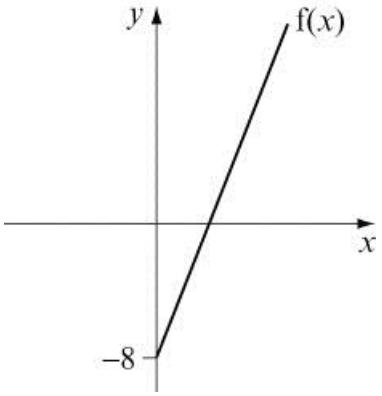
$= \dfrac{1}{9}$

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11 b $f(x) = 4(x-2) = 4x-8$

This is a straight line with gradient 4 and intercept −8.

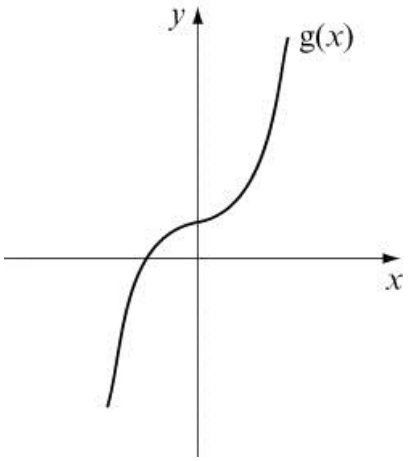
The domain tells us that $x \geqslant 0$, so the graph of $y = f(x)$ is:



The range of $f(x)$ is

$f(x) \in \mathbb{R}, f(x) \geqslant -8$

$g(x) = x^3 + 1$



The range of $g(x)$ is $g(x) \in \mathbb{R}$

c Let $y = x^3 + 1$

(change the subject of the formula)

$y - 1 = x^3$

$\sqrt[3]{y-1} = x$

Hence $g^{-1}(x) = \sqrt[3]{x-1}$ $\{x \in \mathbb{R}\}$

d $fg(x) = f(x^3 + 1)$

$= 4(x^3 + 1 - 2)$

$= 4(x^3 - 1)$

e First find $gh(x)$:

$gh(x) = g(3^x)$

$= (3^x)^3 + 1$

$= 3^{3x} + 1$

$gh(a) = 244$

$3^{3a} + 1 = 244$

$3^{3a} = 243$

$3^{3a} = 3^5$

$3a = 5$

$a = \frac{5}{3}$

12 a f^{-1} exists when f is one-to-one.

Now $f(x) = x^2 + 6x - 4$

Completing the square:

$f(x) = (x + 3)^2 - 13$

The minimum value is

$f(x) = -13$ when $x + 3 = 0$

$\Rightarrow x = -3$

Hence, f is one-to-one when $x > -3$

So least value of a is $a = -3$

b Let $y = f(x)$

$y = x^2 + 6x - 4$

$y = (x + 3)^2 - 13$

$y + 13 = (x + 3)^2$

$x + 3 = \sqrt{y+13}$

$x = \sqrt{y+13} - 3$

So $f^{-1} : x \mapsto \sqrt{x+13} - 3$

For $a = 0$, Range $f(x)$ is $y > -4$

So Domain $f^{-1}(x)$ is $x > -4$

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13 a $\mathrm{f}: x \mapsto 4x-1$

Let $y = 4x-1$ and change the subject of the formula.

$\Rightarrow\ y+1 = 4x$

$\Rightarrow\quad x = \dfrac{y+1}{4}$

Hence $\mathrm{f}^{-1}: x \mapsto \dfrac{x+1}{4},\ \ x \in \mathbb{R}$

b $\mathrm{gf}(x) = \mathrm{g}(4x-1)$

$$= \frac{3}{2(4x-1)-1}$$

$$= \frac{3}{8x-3}$$

Hence $\mathrm{gf}: x \mapsto \dfrac{3}{8x-3}$

$\mathrm{gf}(x)$ is undefined when $8x-3=0$

That is, at $x = \dfrac{3}{8}$

$\therefore$ Domain $\mathrm{gf}(x) = \left\{ x \in \mathbb{R}, x \neq \dfrac{3}{8} \right\}$

c If $2\mathrm{f}(x) = \mathrm{g}(x)$

$$2 \times (4x-1) = \frac{3}{2x-1}$$

$$8x-2 = \frac{3}{2x-1}$$

$$(8x-2)(2x-1) = 3$$

$$16x^2 - 12x + 2 = 3$$

$$16x^2 - 12x - 1 = 0$$

Use $x = \dfrac{-b \pm \sqrt{b^2-4ac}}{2a}$

with $a = 16,\ b = -12$ and $c = -1$.

Then $x = \dfrac{12 \pm \sqrt{144+64}}{32}$

$$= \frac{12 \pm \sqrt{208}}{32}$$

$$= 0.826, -0.076$$

Values of x are -0.076 and 0.826

14 a Let $y = \dfrac{x}{x-2}$

$$y(x-2) = x$$

$$yx - 2y = x \quad \text{(rearrange)}$$

$$yx - x = 2y$$

$$x(y-1) = 2y$$

$$x = \frac{2y}{y-1}$$

$$\mathrm{f}^{-1}(x) = \frac{2x}{x-1},\ \ x \neq 1$$

b The range of $\mathrm{f}^{-1}(x)$ is the domain of $\mathrm{f}(x)$:

$\{\mathrm{f}^{-1}(x) \in \mathbb{R},\ \mathrm{f}^{-1}(x) \neq 2\}$

c $\mathrm{gf}(1.5) = \mathrm{g}\left(\dfrac{1.5}{1.5-2}\right)$

$$= \mathrm{g}\left(\frac{1.5}{-0.5}\right)$$

$$= \mathrm{g}(-3)$$

$$= \frac{3}{-3}$$

$$= -1$$

d If $\mathrm{g}(x) = \mathrm{f}(x) + 4$

$$\frac{3}{x} = \frac{x}{x-2} + 4$$

$$3(x-2) = x^2 + 4x(x-2)$$

$$3x - 6 = x^2 + 4x^2 - 8x$$

$$0 = 5x^2 - 11x + 6$$

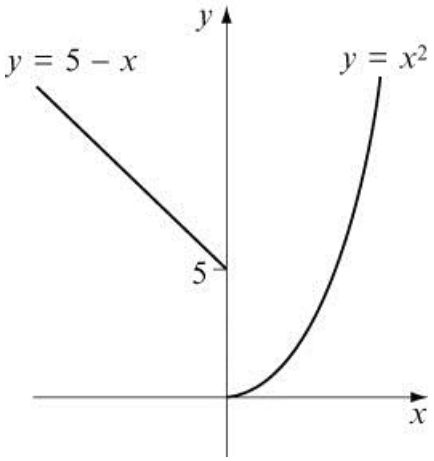
$$0 = (5x-6)(x-1)$$

$$\Rightarrow\ x = \frac{6}{5}, 1$$

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15 $y = 5 - x$ is a straight line with gradient -1 passing through 5 on the y axis.

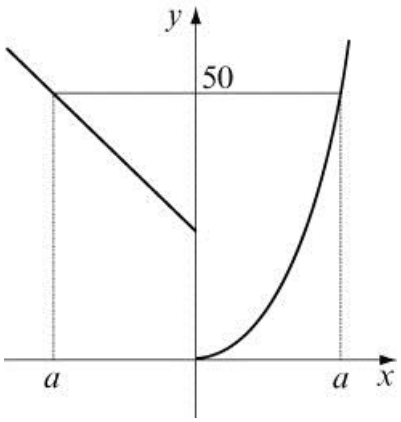
$y = x^2$ is a $\cup$-shaped quadratic passing through $(0, 0)$



a $\text{n}(-3) = 5 - (-3)$
$= 5 + 3$
$= 8$

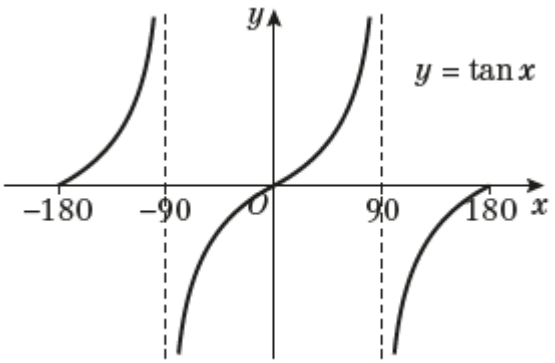
$\text{n}(3) = 32$
$= 9$

b From the diagram, you can see there are two values of x for which $\text{n}(x) = 50$

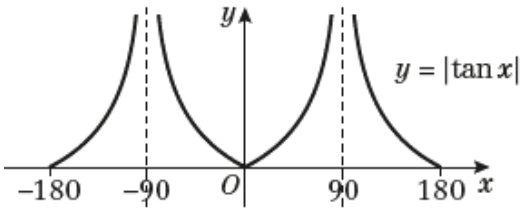


The negative value of x is where
$5 - x = 50$
$x = 5 - 50$
$x = -45$
The positive value of x is where
$x^2 = 50$
$x = \sqrt{50}$
$x = 5\sqrt{2}$
The values of x such that $\text{n}(x) = 50$ are -45 and $+5\sqrt{2}$

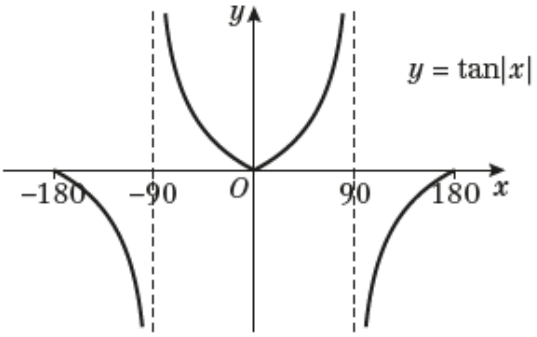
16 a



b $y = |\tan(x)|$ reflects the negative parts of $\tan x$ in the x axis.

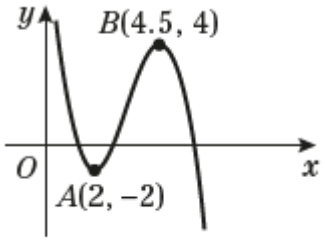


c $y = \tan(|x|)$ reflects $\tan x$ in the y-axis.

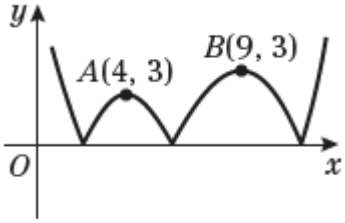


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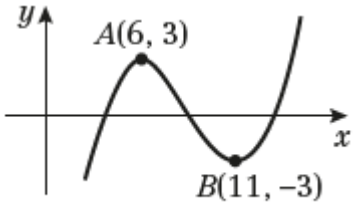
17 a



b



c

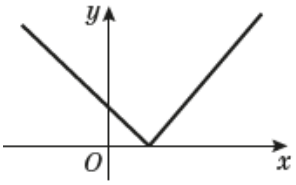


18 a $g(x) \geqslant 0$

b $gf(x) = g(4 - x)$
$= 3(4 - x)^2$
$= 3x^2 - 24x + 48$

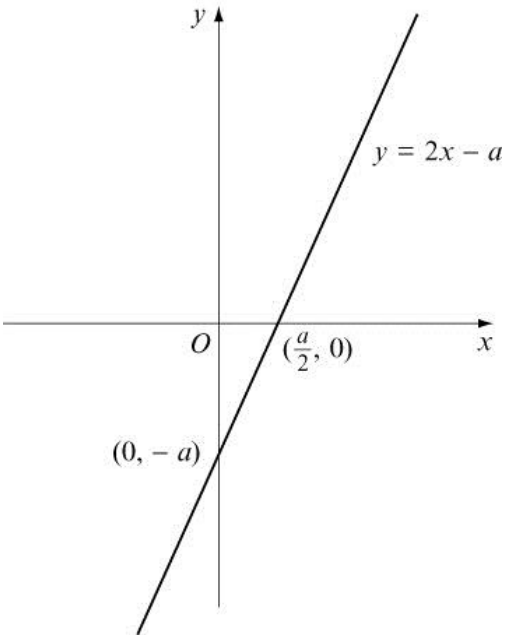
$gf(x) = 48$
$3x^2 - 24x + 48 = 48$
$3x^2 - 24x = 0$
$3x(x - 8) = 0$
$x = 0$ or $x = 8$

c

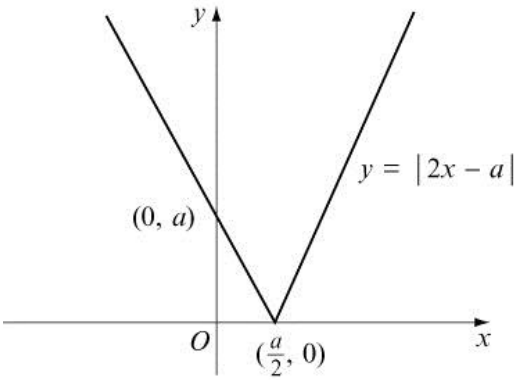


$|f(x)| = 2$ when $|4 - x| = 2$, so
$4 - x = 2 \Rightarrow x = 2$
or $-(4 - x) = 2 \Rightarrow x = 6$

19 a



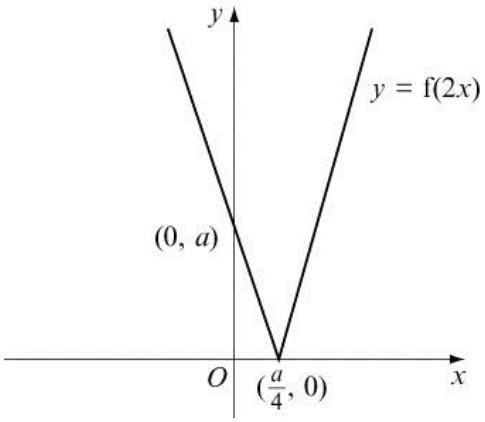
For $y = |2x - a|$:
When $x = 0$, $y = |-a| = a$ $(0, a)$
When $y = 0$, $2x - a = 0$
$\Rightarrow x = \frac{a}{2}$ $\left(\frac{a}{2}, 0\right)$



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19 b $y = \mathrm{f}(2x)$

Horizontal stretch, scale factor $\frac{1}{2}$



c $|2x - a| = \frac{1}{2}x$

Either $(2x - a) = \frac{1}{2}x$

$$\Rightarrow a = \frac{3}{2}x$$

Given that $x = 4$,

$$a = \frac{3 \times 4}{2} = 6$$

Or

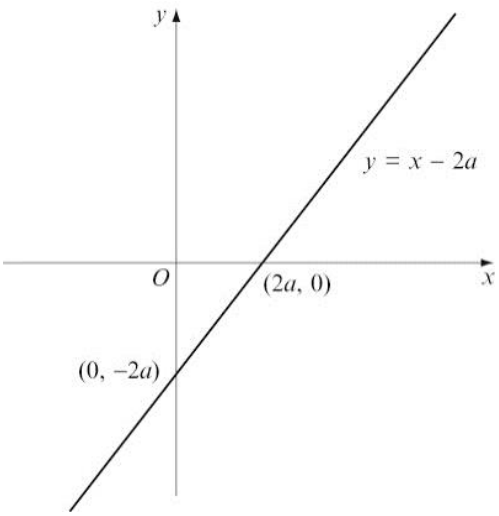
$$-(2x - a) = \frac{1}{2}x$$

$$\Rightarrow a = \frac{5}{2}x$$

Given that $x = 4$,

$$a = \frac{5 \times 4}{2} = 10$$

20 a

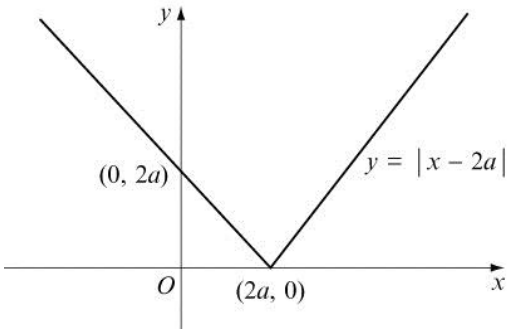


For $y = |x - 2a|$:

When $x = 0$, $y = |-2a| = 2a$ $\quad (0, 2a)$

When $y = 0$, $x - 2a = 0$

$\Rightarrow x = 2a$ $\quad (2a, 0)$



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20 b $|x-2a| = \frac{1}{3}x$

Either $(x-2a) = \frac{1}{3}x$

$\Rightarrow x - \frac{1}{3}x = 2a$

$\Rightarrow \frac{2}{3}x = 2a$

$\Rightarrow x = 3a$

or $-(x-2a) = \frac{1}{3}x$

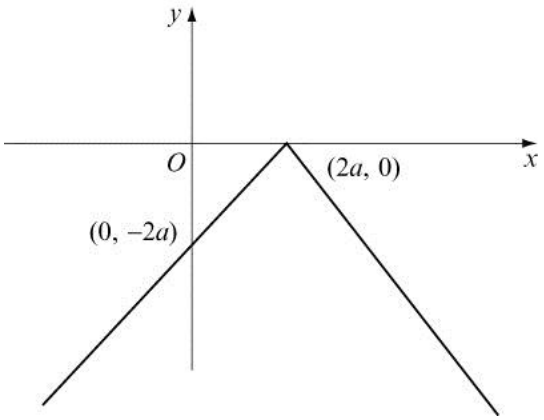
$\Rightarrow -x + 2a = \frac{1}{3}x$

$\Rightarrow \frac{4}{3}x = 2a$

$\Rightarrow x = \frac{3}{2}a$

c $y = -|x-2a|$

Reflect $y = |x-2a|$ in the x-axis



$y = a - |x-2a|$ Vertical translation by $+a$

For $y = a - |x-2a|$:

When $x = 0$,

$y = a - |-2a|$

$= a - 2a$

$= -a$ $(0, -a)$

When $y = 0$,

$a - |x-2a| = 0$

$|x-2a| = a$

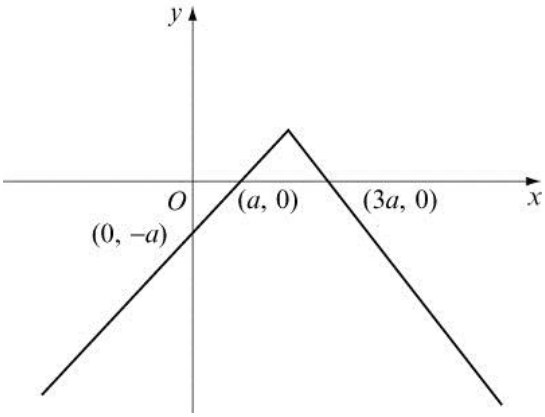
Either $x - 2a = a$

$\Rightarrow x = 3a$ $(3a, 0)$

or $-(x-2a) = a$

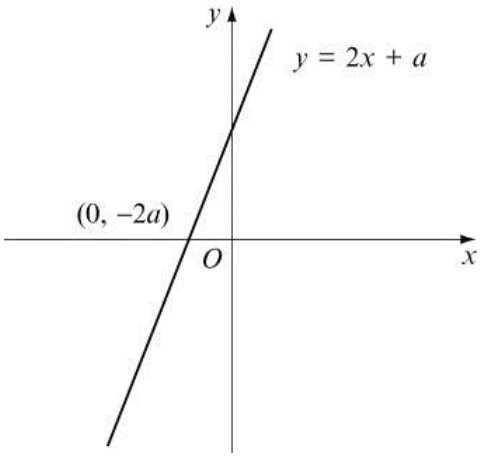
$\Rightarrow -x + 2a = a$

$\Rightarrow x = a$ $(a, 0)$



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21 a & b

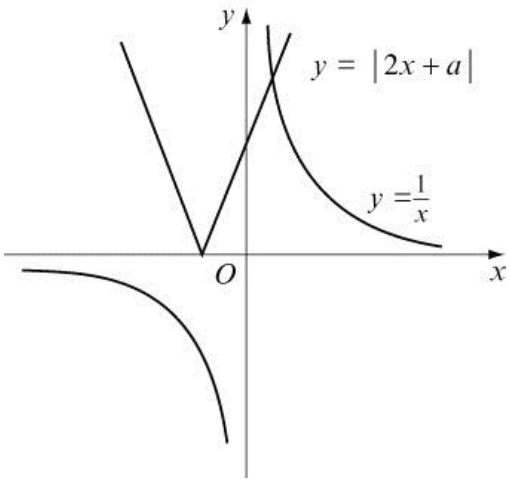


For $y = |2x + a|$:

When $x = 0,\ y = |a| = a \qquad (0,\ a)$

When $y = 0,\ 2x + a = 0$

$$\Rightarrow x = -\frac{a}{2} \qquad \left(-\frac{a}{2},\ 0\right)$$



c Intersection of graphs in **b** gives solutions to the equation:

$$|2x + a| = \frac{1}{x}$$

$$x|x + a| = 1$$

$$x|2x + a| - 1 = 0$$

The graphs intersect once only, so $x|2x + a| - 1 = 0$ has only one solution.

d The intersection point is on the non-reflected part of the modulus graph, so here $|2x - a| = 2x - a$

$$x(2x + a) - 1 = 0$$

$$2x^2 + ax - 1 = 0$$

$$x = \frac{-a \pm \sqrt{a^2 + 8}}{4}$$

As shown on the graph, x is positive at intersection,

so $x = \dfrac{-a + \sqrt{a^2 + 8}}{4}$

22 a $f(x) = x^2 - 7x + 5\ln x + 8$

$$f'(x) = 2x - 7 + \frac{5}{x}$$

At stationary points, $f'(x) = 0$:

$$2x - 7 + \frac{5}{x} = 0$$

$$2x^2 - 7x + 5 = 0$$

$$(2x - 5)(x - 1) = 0$$

$$x = \frac{5}{2},\ x = 1$$

Point A: $x = 1$,

$$f(x) = 1 - 7 + 5\ln 1 + 8$$

$$= 2$$

A is $(1,\ 2)$

Point B: $x = \dfrac{5}{2}$,

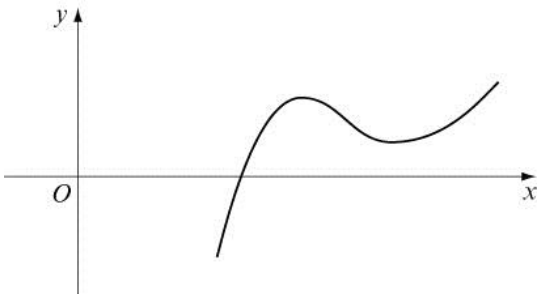
$$f(x) = \frac{25}{4} - \frac{35}{2} + 5\ln\frac{5}{2} + 8$$

$$= 5\ln\frac{5}{2} - \frac{13}{4}$$

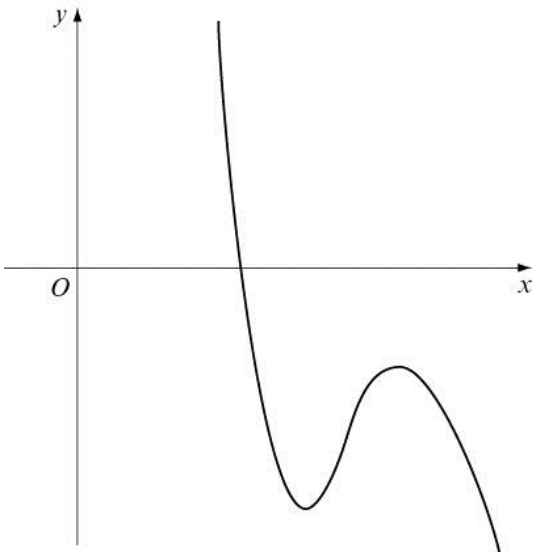
B is $\left(\dfrac{5}{2}, 5\ln\dfrac{5}{2} - \dfrac{13}{4}\right)$

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22 b $y = \mathrm{f}(x-2)$
Horizontal translation of $+2$.
Graph looks like:



$y = -3f(x-2)$
Reflection in the x-axis, and
vertical stretch, scale factor 3.
Graph looks like:



c Using the transformations,
point (X, Y)
becomes $(X+2, -3Y)$

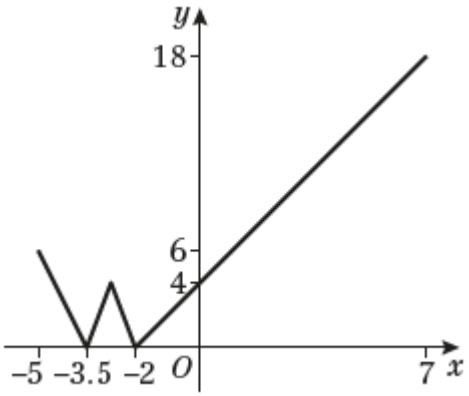
$(1, 2) \rightarrow (3, -6)$
Minimum

$\left(\frac{5}{2}, 5\ln\frac{5}{2} - \frac{13}{4}\right) \rightarrow$
$\left(\frac{9}{2}, \frac{39}{4} - 15\ln\frac{5}{2}\right)$
Maximum

23 a The range of f(x) is $-2 \leqslant \mathrm{f}(x) \leqslant 18$

b ff(−3) = f(−2)
Using f(x) = 2x + 4
f(−2) = 2 × (−2) + 4 = 0

c



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23 d Look at each section of $f(x)$ separately.

$-5 \leqslant x \leqslant -3$:

Gradient $= \dfrac{-2-6}{-3-(-5)} = -4$

$\therefore f(x)-(-2) = -4(x-(-3)) \Rightarrow f(x) = -4x-14$

So in this region, $f(x) = 2$ when $x = -4$

$\therefore fg(x) = 2$ has a corresponding solution if

$g(x) = -4 \Rightarrow g(x)+4 = x^2-7x+14 = 0$

Discriminant $(-7)^2 - 4(1)(14) = -7 < 0$

So no solution

$-3 \leqslant x \leqslant 7$: Gradient $= \dfrac{18-(-2)}{7-(-3)} = 2$

$\therefore f(x)-(-2) = 2(x-(-3)) \Rightarrow f(x) = 2x+4$

So in this region, $f(x) = 2$ when $x = -1$

$\therefore fg(x) = 2$ has a corresponding solution if

$g(x) = -1 \Rightarrow g(x)+1 = x^2-7x+11 = 0$

$$x = \frac{-(-7) \pm \sqrt{(-7)^2 - 4(1)(11)}}{2(1)} = \frac{7 \pm \sqrt{5}}{2}$$

$$\therefore x = \frac{7+\sqrt{5}}{2} \text{ or } x = \frac{7-\sqrt{5}}{2}$$

24 a The range of $p(x)$ is $p(x) \leqslant 10$

b $p(x)$ is many-to-one, therefore the inverse is one-to-many, which is not a function.

c At first point of intersection:

$2(x + 4) + 10 = -4$

$2x + 18 = -4$

$x = -11$

At the other point of intersection:

$-2(x + 4) + 10 = -4$

$-2x + 2 = -4$

$x = 3$

$-11 < x < 3$

d For no solutions, $p(x) > 10$ at $x = -4$

So $-\dfrac{1}{2}x + k > 10$ at $x = -4$

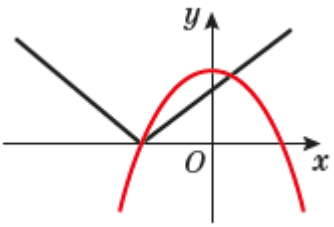
$-\dfrac{1}{2}(-4) + k > 10$

$2 + k > 10$

$k > 8$

Challenge

a



b $y = (a + x)(a - x)$

When $y = 0$, $x = -a$ or $x = a$

When $x = 0$, $y = a^2$

$(-a, 0)$, $(a, 0)$, $(0, a^2)$

c When $x = 4$, $y = a^2 - x^2$

$= a^2 - 16$

and $y = x + a$

$= 4 + a$

$a^2 - 16 = 4 + a$

$a^2 - a - 20 = 0$

$(a - 5)(a + 4) = 0$

As $a > 1$, $a = 5$

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