Integration, Mixed exercise 11

1 a $I = \int (2x-3)^7 \, dx$

Consider $y = (2x-3)^8$

$\frac{dy}{dx} = 16(2x-3)^7$

$I = \frac{(2x-3)^8}{16} + c$

b $I = \int x\sqrt{4x-1} \, dx$

Let $u = 4x-1 \Rightarrow \frac{du}{dx} = 4$

$I = \int \frac{u+1}{16}\sqrt{u} \, du$

$= \frac{2u^{\frac{5}{2}}}{80} + \frac{2u^{\frac{3}{2}}}{48} + c$

$= \frac{(4x-1)^{\frac{5}{2}}}{40} + \frac{(4x-1)^{\frac{3}{2}}}{24} + c$

c $I = \int \sin^2 x \cos x \, dx$

Consider $y = \sin^3 x \Rightarrow \frac{dy}{dx} = 3\sin^2 x \cos x$

$I = \frac{1}{3}\sin^3 x + c$

d $I = \int x \ln x \, dx$

Let $u = \ln x \Rightarrow \frac{du}{dx} = \frac{1}{x}$

$\frac{dv}{dx} = x \Rightarrow v = \frac{x^2}{2}$

$I = \frac{x^2}{2}\ln x - \int \frac{x^2}{2} \times \frac{1}{x} \, dx$

$= \frac{x^2}{2}\ln x - \frac{x^2}{4} + c$

e $I = \int \frac{4\sin x \cos x}{4 - 8\sin^2 x} \, dx$

$I = \int \frac{2\sin 2x}{4(1-2\sin^2 x)} \, dx$

$I = \int \frac{2\sin 2x}{4\cos 2x} \, dx$

$= -\frac{1}{4}\ln|\cos 2x| + c$

f $I = \int \frac{1}{3-4x} \, dx$

$= -\frac{1}{4}\ln|3-4x| + c$

2 a $I = \int_{-3}^{0} x(x^2+3)^5 \, dx$

Consider $y = (x^2+3)^6$

$\frac{dy}{dx} = 12x(x^2+3)^5$

$I = \left[\frac{1}{12}(x^2+3)^6\right]_{-3}^{0}$

$= \frac{1}{12}(729 - 2985984)$

$= -\frac{995085}{4}$

b $I = \int_0^{\frac{\pi}{4}} x\sec^2 x \, dx$

Let $u = x \Rightarrow \frac{du}{dx} = 1$

$\frac{dv}{dx} = \sec^2 x \Rightarrow v = \tan x$

$I = [x\tan x]_0^{\frac{\pi}{4}} - \int_0^{\frac{\pi}{4}} \tan x \, dx$

$= \frac{\pi}{4} + \left[\ln|\cos x|\right]_0^{\frac{\pi}{4}} = \frac{\pi}{4} + \ln\frac{1}{\sqrt{2}}$

$= \frac{\pi}{4} - \frac{1}{2}\ln 2$

c $I = \int_1^4 \left(16x^{\frac{3}{2}} - \frac{2}{x}\right) dx$

$= \left[\frac{32}{5}x^{\frac{5}{2}} - 2\ln|x|\right]_1^4$

$= \frac{1024}{5} - 2\ln 4 - \frac{32}{5}$

$= \frac{992}{5} - 2\ln 4$

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2 d $I = \int_{\frac{\pi}{12}}^{\frac{\pi}{3}} (\cos x + \sin x)(\cos x - \sin x)\ \mathrm{d}x$

$= \int_{\frac{\pi}{12}}^{\frac{\pi}{3}} (\cos^2 x - \sin^2 x)\ \mathrm{d}x$

$= \int_{\frac{\pi}{12}}^{\frac{\pi}{3}} \cos 2x\ \mathrm{d}x$

$= \left[\frac{1}{2}\sin 2x\right]_{\frac{\pi}{12}}^{\frac{\pi}{3}}$

$= \frac{\sqrt{3}}{4} - \frac{1}{4}$

$= \frac{\sqrt{3}-1}{4}$

e $I = \int_1^4 \frac{4}{16x^2 + 8x - 3}\ \mathrm{d}x$

$\frac{4}{16x^2+8x-3} = \frac{4}{(4x+3)(4x-1)}$

$\frac{4}{(4x+3)(4x-1)} = \frac{A}{4x+3} + \frac{B}{4x-1}$

$4 = A(4x-1) + B(4x+3)$

$x = \frac{1}{4} \Rightarrow 4 = 4B \Rightarrow B = 1$

$x = -\frac{3}{4} \Rightarrow 4 = -4A \Rightarrow A = -1$

$I = \int_1^4 \frac{1}{4x-1} - \frac{1}{4x+3}\ \mathrm{d}x$

$= \frac{1}{4}\left[\ln|4x-1| - \ln|4x+3|\right]_1^4$

$= \frac{1}{4}(\ln 15 - \ln 19 - \ln 3 + \ln 7)$

$= \frac{1}{4}\ln\frac{105}{57}$

$= \frac{1}{4}\ln\frac{35}{19}$

f $I = \int_0^{\ln 2} \frac{1}{1+\mathrm{e}^x}\ \mathrm{d}x$

Let $u = 1 + \mathrm{e}^x \Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x} = \mathrm{e}^x = u - 1$

$I = \int_2^3 \frac{1}{(u-1)u}\ \mathrm{d}u$

$I = \int_2^3 \left(\frac{1}{(u-1)} - \frac{1}{u}\right)\mathrm{d}u$

$= \left[\ln|u-1| - \ln|u|\right]_2^3$

$= \ln 2 - \ln 3 - \ln 1 + \ln 2$

$= \ln\frac{4}{3}$

3 a $I = \int_1^{\mathrm{e}} \frac{1}{x^2}\ln x\,\mathrm{d}x$

$u = \ln x \Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x} = \frac{1}{x}$

$\frac{\mathrm{d}v}{\mathrm{d}x} = \frac{1}{x^2} \Rightarrow v = -\frac{1}{x}$

$\therefore I = \left[-\frac{1}{x}\ln x\right]_1^{\mathrm{e}} - \int_1^{\mathrm{e}} \left(-\frac{1}{x^2}\right)\mathrm{d}x$

$= \left(-\frac{1}{\mathrm{e}}\right) - (0) + \left[-\frac{1}{x}\right]_1^{\mathrm{e}}$

$= -\frac{1}{\mathrm{e}} + \left(-\frac{1}{\mathrm{e}}\right) - (-1)$

$= 1 - \frac{2}{\mathrm{e}}$

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3 b $\dfrac{1}{(x+1)(2x-1)} \equiv \dfrac{A}{x+1} + \dfrac{B}{2x-1}$

$\Rightarrow 1 \equiv A(2x-1) + B(x+1)$

$x - \dfrac{1}{2} \Rightarrow 1 = \dfrac{3}{2}B \Rightarrow B = \dfrac{2}{3}$

$x = -1 \Rightarrow 1 = -3A \Rightarrow A = -\dfrac{1}{3}$

$\therefore \int_1^p \dfrac{1}{(x+1)(2x-1)}\,\mathrm{d}x = \int_1^p \left(\dfrac{\frac{2}{3}}{2x-1} + \dfrac{-\frac{1}{3}}{x+1} \right) \mathrm{d}x$

$= \left[\dfrac{1}{3}\ln|2x-1| - \dfrac{1}{3}\ln|x+1| \right]_1^p$

$= \left[\dfrac{1}{3}\ln\left|\dfrac{2x-1}{x+1}\right| \right]_1^p$

$= \dfrac{1}{3}\ln\left(\dfrac{2p-1}{p+1}\right) - \left(\dfrac{1}{3}\ln\dfrac{1}{2}\right)$

$= \dfrac{1}{3}\ln\left(\dfrac{4p-2}{p+1}\right)$

4 $\int_{\frac{1}{2}}^b \left(\dfrac{2}{x^3} - \dfrac{1}{x^2} \right) \mathrm{d}x = \dfrac{9}{4}$

$\left[-\dfrac{1}{x^2} + \dfrac{1}{x} \right]_{\frac{1}{2}}^b = \dfrac{9}{4}$

$-\dfrac{1}{b^2} + \dfrac{1}{b} + 4 - 2 = \dfrac{9}{4}$

$\dfrac{b-1}{b^2} = \dfrac{1}{4}$

$b^2 - 4b + 4 = 0$

$(b-2)^2 = 0$

$b = 2$

5 $I = \int_0^\theta \cos x \sin^3 x\,\mathrm{d}x = \dfrac{9}{64}$

$\left[\dfrac{\sin^4 x}{4} \right]_0^\theta = \dfrac{9}{64}$

$\sin^4\theta = \dfrac{9}{16}$

$\sin\theta = \pm\dfrac{\sqrt{3}}{2}$

$\theta = \dfrac{\pi}{3}$

6 a $t^2 = x+1 \Rightarrow 2t\,\mathrm{d}t = \mathrm{d}x$

$\therefore I = \int \dfrac{x}{\sqrt{x+1}}\,\mathrm{d}x$

$= \int \dfrac{t^2-1}{t} \times 2t\,\mathrm{d}t$

$= \int (2t^2 - 2)\mathrm{d}t$

$= \dfrac{2}{3}t^3 - 2t + c$

$= \dfrac{2}{3}(x+1)^{\frac{3}{2}} - 2\sqrt{x+1} + c$

$= \dfrac{2}{3}\sqrt{x+1}(x-2) + c$

b $\int_0^3 \dfrac{x}{\sqrt{x+1}}\,\mathrm{d}x = \left[\dfrac{2}{3}(x-2)\sqrt{x+1} \right]_0^3$

$= \left(\dfrac{2}{3} \times 2 \right) - \left(-\dfrac{4}{3} \right) = \dfrac{8}{3}$

7 a $I = \int x\sin 8x\,\mathrm{d}x$

Let $u = x \Rightarrow \dfrac{\mathrm{d}u}{\mathrm{d}x} = 1$

$\dfrac{\mathrm{d}v}{\mathrm{d}x} = \sin 8x \Rightarrow v = -\dfrac{1}{8}\cos 8x$

$I = -\dfrac{1}{8}x\cos 8x + \dfrac{1}{8}\int \cos 8x\,\mathrm{d}x$

$= -\dfrac{1}{8}x\cos 8x + \dfrac{1}{64}\sin 8x + c$

b $I = \int x^2 \cos 8x\,\mathrm{d}x$

Let $u = x^2 \Rightarrow \dfrac{\mathrm{d}u}{\mathrm{d}x} = 2x$

$\dfrac{\mathrm{d}v}{\mathrm{d}x} = \cos 8x \Rightarrow v = \dfrac{1}{8}\sin 8x$

$I = \dfrac{1}{8}x^2\sin 8x - \dfrac{2}{8}\int x\sin 8x\,\mathrm{d}x$

$= \dfrac{1}{8}x^2\sin 8x - \dfrac{2}{8}\left(-\dfrac{1}{8}x\cos 8x + \dfrac{1}{64}\sin 8x \right) + c$

$= \dfrac{1}{8}x^2\sin 8x + \dfrac{1}{32}x\cos 8x - \dfrac{1}{256}\sin 8x + c$

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8 a $f(x) \equiv \dfrac{5x^2-8x+1}{2x(x-1)^2} \equiv \dfrac{A}{x}+\dfrac{B}{x-1}+\dfrac{C}{(x-1)^2}$

$\Rightarrow 5x^2-8x+1 \equiv 2A(x-1)^2 + 2Bx(x-1)+2Cx$

$x=0 \Rightarrow 1=2A \Rightarrow A=\dfrac{1}{2}$

$x=1 \Rightarrow -2=2C \Rightarrow C=-1$

Coefficients of x^2: $5=2A+2B \Rightarrow B=2$

b $\int f(x)\,dx = \int\left(\dfrac{\frac{1}{2}}{x}+\dfrac{2}{x-1}-\dfrac{1}{(x-1)^2}\right)dx$

$=\dfrac{1}{2}\ln|x|+2\ln|x-1|+\dfrac{1}{x-1}+c$

c $\int_4^9 f(x)\,dx = \left[\dfrac{1}{2}\ln|x|+2\ln|x-1|+\dfrac{1}{x-1}\right]_4^9$

$=\left[\ln\left|\sqrt{x(x-1)^2}\right|+\dfrac{1}{x-1}\right]_4^9$

$=\left[\ln(3\times 64)+\dfrac{1}{8}\right]-\left[\ln(2\times 9)+\dfrac{1}{3}\right]$

$=\ln\left(\dfrac{3\times 64}{2\times 9}\right)+\dfrac{1}{8}-\dfrac{1}{3}$

$=\ln\dfrac{32}{3}-\dfrac{5}{24}$

9 a $y=x^{\frac{3}{2}}+48x^{-1} \Rightarrow \dfrac{dy}{dx}=\dfrac{3}{2}x^{\frac{1}{2}}-48x^{-2}$

$\dfrac{dy}{dx}=0 \Rightarrow \dfrac{3}{2}x^{\frac{1}{2}}=\dfrac{48}{x^2}$

$\Rightarrow x^{\frac{5}{2}}=\dfrac{2}{3}\times 48=32$

$\Rightarrow x=4,\ y=2^3+12=20$

$\Rightarrow x=4, y=20$

b $\dfrac{d^2y}{dx^2}=\dfrac{3}{4}x^{-\frac{1}{2}}+96x^{-3} >$ for all $x>0$

$\therefore$ 20 is a minimum value of y

c Area $=\int_1^4\left(x^{\frac{3}{2}}+\dfrac{48}{x}\right)dx$

$=\left[\dfrac{2}{5}x^{\frac{5}{2}}+48\ln|x|\right]_1^4$

$=\left(\dfrac{2}{5}\times 32+48\ln 4\right)-\left(\dfrac{2}{5}+0\right)$

$=\dfrac{62}{5}+48\ln 4$

10 a $I=\int x^2\ln 2x\ dx$

Let $u=\ln 2x \Rightarrow \dfrac{du}{dx}=\dfrac{1}{x}$

$\dfrac{dv}{dx}=x^2 \Rightarrow v=\dfrac{x^3}{3}$

$I=\dfrac{1}{3}x^3\ln 2x-\int\dfrac{x^2}{3}\ dx$

$=\dfrac{1}{3}x^3\ln 2x-\dfrac{1}{9}x^3+c$

b $\int_{\frac{1}{2}}^3 x^2\ln 2x\ dx=\left[\dfrac{1}{3}x^3\ln 2x-\dfrac{1}{9}x^3\right]_{\frac{1}{2}}^3$

$=9\ln 6-3-0+\dfrac{1}{72}$

$=9\ln 6-\dfrac{215}{9}$

11 a $y=(1+\sin 2x)^2$

$=1+2\sin 2x+\sin^2 2x$

$=1+2\sin 2x+\dfrac{1}{2}-\dfrac{1}{2}\cos 4x$

$=\dfrac{1}{2}(3+4\sin 2x-\cos 4x)$

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11 b Area of $R = \int_0^{\frac{3\pi}{4}} (1+\sin 2x)^2 \, dx$

$= \frac{1}{2}\int_0^{\frac{3\pi}{4}} (3+4\sin 2x-\cos 4x) \, dx$

$= \frac{1}{2}\left[3x-2\cos 2x-\frac{1}{4}\sin 4x\right]_0^{\frac{3\pi}{4}}$

$= \left(\frac{9\pi}{8}-0-0\right)-(0-1-0)$

$= \frac{9\pi}{8}+1$

c $\frac{dy}{dx} = 4\cos 2x + 2\sin 4x$

$\frac{dy}{dx} = 0 \Rightarrow 4\cos 2x + 2\sin 4x = 0$

$4\cos 2x + 4\sin 2x\cos 2x = 0$

$4\cos 2x(1+\sin 2x) = 0$

$2x = \frac{\pi}{2}$ for $x < \frac{3\pi}{4}$

$x = \frac{\pi}{4}, y = \left(1+\sin\frac{\pi}{2}\right)^2 = 4$

Coordinates of $A\left(\frac{\pi}{4}, 4\right)$

12 a $I = \int xe^{-x}dx$

$u = x \Rightarrow \frac{du}{dx} = 1$

$\frac{dv}{dx} = e^{-x} \Rightarrow v = -e^{-x}$

$\therefore I = -xe^{-x} - \int(-e^{-x})dx$

i.e. $I = -xe^{-x} - e^{-x} + c$

b $e^x \frac{dy}{dx} = \frac{x}{\sin 2y}$

$\Rightarrow \int \sin 2y \, dy = \int xe^{-x}dx$

$\Rightarrow -\frac{1}{2}\cos 2y = -xe^{-x} - e^{-x} + c$

$x = 0, y = \frac{\pi}{4} \Rightarrow 0 = 0 - 1 + c \Rightarrow c = 1$

$\therefore \frac{1}{2}\cos 2y = xe^{-x} + e^{-x} - 1$

or $\cos 2y = 2(xe^{-x} + e^{-x} - 1)$

13 a $I = \int x\sin 2x \, dx$

$u = x \Rightarrow \frac{du}{dx} = 1$

$\frac{dv}{dx} = \sin 2x \Rightarrow v = \frac{-1}{2}\cos 2x$

$\therefore I = -\frac{1}{2}x\cos 2x - \int \frac{-1}{2}\cos 2x \, dx$

$= -\frac{1}{2}x\cos 2x + \frac{1}{4}\sin 2x + c$

b $\frac{dy}{dx} = x\sin 2x\cos^2 y$

$\Rightarrow \int \sec^2 y \, dy = \int x\sin 2x \, dx$

$\Rightarrow \tan y = -\frac{1}{2}x\cos 2x + \frac{1}{4}\sin 2x + c$

$y = 0, \ x = \frac{\pi}{4} \Rightarrow 0 = 0 + \frac{1}{4} + c \Rightarrow c = -\frac{1}{4}$

$\therefore \tan y = -\frac{1}{2}x\cos 2x + \frac{1}{4}\sin 2x - \frac{1}{4}$

14 a $\frac{dy}{dx} = xy^2$

$\Rightarrow \int \frac{1}{y^2} dy = \int x \, dx$

$\Rightarrow -\frac{1}{y} = \frac{x^2}{2} + c$

or $y = \frac{-2}{x^2 + k}$ $\quad (k = 2c)$

b $y = 1, x = 1 \Rightarrow 1 = \frac{-2}{1+k} \Rightarrow k = -3$

$\therefore y = \frac{2}{3-x^2}$

for $x^2 \neq 3$ and $y > 0$, i.e. $-\sqrt{3} < x < \sqrt{3}$

c When $x = 1, y = 1$, $\frac{dy}{dx}$ is 1

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14 d Equation of tangent is:

$y-1=1(x-1)$

$y=x$

This meets the curve again when:

$x=\dfrac{2}{3-x^2}$

$3x-x^3=2$

$x^3-3x+2=0$

$(x-1)(x-1)(x+2)=0$

Other point is when $x=-2, y=-2$

i.e. $(-2, -2)$

15 a $I=\displaystyle\int\frac{4x}{(1+2x)^2}\,\mathrm{d}x$

$u=1+2x$

$\Rightarrow\dfrac{\mathrm{d}u}{2}=\mathrm{d}x$ and $4x=2(u-1)$

$\therefore I=\displaystyle\int\frac{2(u-1)}{u^2}\times\frac{\mathrm{d}u}{2}$

$=\displaystyle\int\left(\frac{1}{u}-u^{-2}\right)\mathrm{d}u$

$=1\ln|u|+\dfrac{1}{u}+c$

$=1\ln|1+2x|+\dfrac{1}{1+2x}+c$

b $(1+2x)^2\dfrac{\mathrm{d}y}{\mathrm{d}x}=\dfrac{x}{\sin^2 y}$

$\Rightarrow\displaystyle\int\sin^2 y\,\mathrm{d}y=\int\frac{x}{(1+2x)}\mathrm{d}x$

$\Rightarrow\displaystyle\int 4\sin^2 y\,\mathrm{d}y=\int\frac{4x}{(1+2x)^2}\mathrm{d}x$

$\Rightarrow\displaystyle\int(2-2\cos 2y)\,\mathrm{d}y=I$

$\Rightarrow 2y-\sin 2y=\ln|1+2x|+\dfrac{1}{1+2x}+c$

$x=0, y=\dfrac{\pi}{4}\Rightarrow\dfrac{\pi}{2}-1=\ln 1+1+c$

$\Rightarrow c=\dfrac{\pi}{2}-2$

$\therefore 2y-\sin 2y=\ln|1+2x|+\dfrac{1}{1+2x}+\dfrac{\pi}{2}-2$

16 a $\displaystyle\int xe^{2x}\mathrm{d}x$

$u=x\Rightarrow\dfrac{\mathrm{d}u}{\mathrm{d}x}=1$

$\dfrac{\mathrm{d}v}{\mathrm{d}x}=e^{2x}\Rightarrow v=\dfrac{1}{2}e^{2x}$

$\therefore\displaystyle\int xe^{2x}\mathrm{d}x=\frac{1}{2}xe^{2x}$

$\displaystyle-\int\frac{1}{2}e^{2x}\mathrm{d}x=\frac{1}{2}xe^{2x}-\frac{1}{4}e^{2x}+c$

$A_1=-\left[\dfrac{1}{2}xe^{2x}-\dfrac{1}{4}e^{2x}\right]_{-\frac{1}{2}}^{0}$

$=-\left(\left(0-\dfrac{1}{4}\right)-\left(-\dfrac{1}{4}e^{-1}-\dfrac{1}{4}e^{-1}\right)\right)$

$=\dfrac{1}{4}(1-2e^{-1})$

$A_2=\left[\dfrac{1}{2}xe^{2x}-\dfrac{1}{4}e^{2x}\right]_{0}^{\frac{1}{2}}$

$=\left(\dfrac{1}{4}e^{1}-\dfrac{1}{4}e^{1}\right)-\left(0-\dfrac{1}{4}\right)$

$=\dfrac{1}{4}$

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16 b $\dfrac{A_1}{A_2} = \dfrac{\frac{1}{4}\left(1-2\mathrm{e}^{-1}\right)}{\frac{1}{4}} = 1-2\mathrm{e}^{-1} = \dfrac{\mathrm{e}-2}{\mathrm{e}}$

$\therefore A_1 : A_2 = (\mathrm{e}-2):\mathrm{e}$

17 a $I = \int x^2\mathrm{e}^{-x}\ \mathrm{d}x$

Let $u = x^2 \Rightarrow \dfrac{\mathrm{d}u}{\mathrm{d}x} = 2x$

$\dfrac{\mathrm{d}v}{\mathrm{d}x} = \mathrm{e}^{-x} \Rightarrow v = -\mathrm{e}^{-x}$

$I = -x^2\mathrm{e}^{-x} + 2\int x\mathrm{e}^{-x}\ \mathrm{d}x$

Again, let $u = x \Rightarrow \dfrac{\mathrm{d}u}{\mathrm{d}x} = 1$

$\dfrac{\mathrm{d}v}{\mathrm{d}x} = \mathrm{e}^{-x} \Rightarrow v = -\mathrm{e}^{-x}$

$I = -x^2\mathrm{e}^{-x} - 2x\mathrm{e}^{-x} + 2\int \mathrm{e}^{-x}\ \mathrm{d}x$

$= -x^2\mathrm{e}^{-x} - 2x\mathrm{e}^{-x} - 2\mathrm{e}^{-x} + c$

$= -\mathrm{e}^{-x}\left(x^2+2x+2\right) + c$

b $\dfrac{\mathrm{d}y}{\mathrm{d}x} = x^2\mathrm{e}^{3y-x}$

$\dfrac{\mathrm{d}y}{\mathrm{d}x} = x^2\mathrm{e}^{3y}\mathrm{e}^{-x}$

$\int \mathrm{e}^{-3y}\ \mathrm{d}x = \int x^2\mathrm{e}^{-x}\ \mathrm{d}x$

$-\dfrac{1}{3}\mathrm{e}^{-3y} = -\mathrm{e}^{-x}\left(x^2+2x+2\right) + c$

$x = 0, y = 0 \Rightarrow -\dfrac{1}{3} = -2 + c \Rightarrow c = \dfrac{5}{3}$

$\mathrm{e}^{-3y} = 3\mathrm{e}^{-x}\left(x^2+2x+2\right) - 5$

$3y = -\ln\left(3\mathrm{e}^{-x}\left(x^2+2x+2\right) - 5\right)$

$y = -\dfrac{1}{3}\ln\left(3\mathrm{e}^{-x}\left(x^2+2x+2\right) - 5\right)$

18 a $y = \mathrm{e}^{3x} + 1$

$y = 8 \Rightarrow \mathrm{e}^{3h} = 7$

$3h = \ln 7$

$h = \dfrac{1}{3}\ln 7$

b Area of R is given by

$\int_0^{\frac{1}{3}\ln 7}\left(\mathrm{e}^{3x}+1\right)\mathrm{d}x = \left[\dfrac{1}{3}\mathrm{e}^{3x} + x\right]_0^{\frac{1}{3}\ln 7}$

$= \left(\dfrac{1}{3}\mathrm{e}^{\ln 7} + \dfrac{1}{3}\ln 7\right) - \left(\dfrac{1}{3} + 0\right)$

$= 2 + \dfrac{1}{3}\ln 7$

19 a $\dfrac{x^2}{x^2-1} \equiv A + \dfrac{B}{x-1} + \dfrac{C}{x+1}$

$\Rightarrow x^2 \equiv A(x-1)(x+1) + B(x+1) + C(x-1)$

$x = 1 \Rightarrow 1 = 2B \Rightarrow B = \dfrac{1}{2}$

$x = -1 \Rightarrow 1 = -2C \Rightarrow C = -\dfrac{1}{2}.$

Coefficients of x^2: $1 = A \Rightarrow A = 1$

b $\dfrac{\mathrm{d}x}{\mathrm{d}t} = 2\dfrac{(x^2-1)}{x^2}$

$\Rightarrow \int \dfrac{x^2}{x^2-1}\mathrm{d}x = \int 2\,\mathrm{d}t$

$\Rightarrow \int\left(1 + \dfrac{\left(\frac{1}{2}\right)}{x-1} - \dfrac{\left(\frac{1}{2}\right)}{x+1}\right)\mathrm{d}x = 2t$

$\Rightarrow x + \dfrac{1}{2}\ln\left|\dfrac{x-1}{x+1}\right| = 2t + c$

$x = 2, t = 1 \Rightarrow 2 + \dfrac{1}{2}\ln\dfrac{1}{3} = 2 + c \Rightarrow c = \dfrac{1}{2}\ln\dfrac{1}{3}$

$\therefore x + \dfrac{1}{2}\ln\left|\dfrac{x-1}{x+1}\right| = 2t + \dfrac{1}{2}\ln\dfrac{1}{3}$

20 a $y = \mathrm{e}^{2x} - \mathrm{e}^{-x}$

x	0	0.25	0.5	0.75	1
y	0	0.86992	2.11175	4.00932	7.02118

b Area $\approx \dfrac{1}{2}h\left(y_0 + 2(y_1 + \ldots) + y_n\right)$

$= \dfrac{1}{8}\left(7.02118 + 2\times 6.99099\right)$

$= 2.6254$

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20 c The curve is convex, so it is an overestimate.

d $\int_0^1 \left(e^{2x} - e^{-x}\right) dx = \left[\frac{1}{2}e^{2x} + e^{-x}\right]_0^1$

$= \frac{1}{2}e^2 + e^{-1} - \frac{1}{2} - 1$

$= \frac{1}{2}e^2 + \frac{1}{e} - \frac{3}{2}$

$= \frac{e^3 - 3e + 2}{2e}$

$P = -3,\ Q = 2$

e $\frac{e^3 - 3e + 2}{2e} = 2.5624$

Percentage error

$= \frac{2.5624 - 2.6254}{2.5624} \times 100\% \approx 2.5\%$

21 a $\frac{dv}{dt} = -kV$

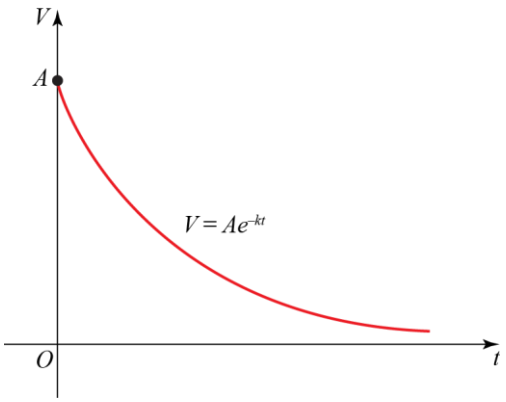
$\Rightarrow \int \frac{1}{V} dV = \int -k\, dt$

$\Rightarrow 1n|V| = -kt + C$

$\Rightarrow V = A_1 e^{-kt}$

$t = 0, V = A \Rightarrow V = Ae^{-kt} \qquad (A_1 = A)$

b



c $t = T, V = \frac{1}{2}A \Rightarrow \frac{1}{2}A = Ae^{-kT}$

$\Rightarrow -1n\,2 = -kT$

$\Rightarrow kT = 1n\,2$

22 a $\frac{dy}{dx} = \frac{x}{k - y}$

$\int (k - y)\, dy = \int x\, dx$

$-\frac{(k-y)^2}{2} + c = \frac{x^2}{2}$

$x^2 + (y - k)^2 = c$

b Concentric circles with centre (0, 2).

23 a

x	1	1.5	2	2.5
y	1	0.6825	0.5545	0.6454

x	3	3.5	4
y	0.9775	1.5693	2.4361

b Area $= \frac{1}{4}(3.4361 + 2 \times 4.4262)$

$= 3.074$

c If smaller intervals are used and consequently more values, the lines would follow the curve more closely.

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23 d $\text{Area} = \int_1^4 \frac{1}{5}x^2 \ln x - x + 2$

Now let $I = \int x^2 \ln x \, dx$

Let $u = \ln x \Rightarrow \frac{du}{dx} = \frac{1}{x}$

$\frac{dv}{dx} = x^2 \Rightarrow v = \frac{x^3}{3}$

$I = \frac{1}{3}x^3 \ln x - \int \frac{x^2}{3} \, dx$

$= \frac{1}{3}x^3 \ln x - \frac{1}{9}x^3 + c$

$\text{Area} = \left[\frac{1}{15}x^3 \ln x - \frac{1}{45}x^3 - \frac{x^2}{2} + 2x\right]_1^4$

$= \frac{64}{15}\ln 4 - \frac{64}{15} - 8 + 8 - \left(0 - \frac{1}{45} - \frac{1}{2} + 2\right)$

$= -\frac{29}{10} + \frac{64}{15}\ln 4$

e $-\frac{29}{10} + \frac{64}{15}\ln 4 = 3.015$

Percentage error

$= \frac{3.015 - 3.074}{3.015} \times 100\% \approx 2.0\%$

24 a $u = 1 + 2x^2 \Rightarrow du = 4x\,dx \Rightarrow x\,dx = \frac{du}{4}$

So $\int x(1+2x^2)^5 dx = \int \frac{u^5}{4} du = \frac{u^6}{24}$

$+ c_1 = \frac{(1+2x^2)^6}{24} + c_1$

b $\frac{dy}{dx} = x(1+2x^2)^5 \cos^2 2y$

$\Rightarrow \int \sec^2 2y \, dy = \int x(1+2x^2)^5 dx$

$\Rightarrow \frac{1}{2}\tan 2y = \frac{(1+2x^2)^6}{24} + c_2$

$y = \frac{\pi}{8}, x = 0 \Rightarrow \frac{1}{2} = \frac{1}{24} + c_2 \Rightarrow c_2 = \frac{11}{24}$

$\therefore \tan 2y = \frac{(1+2x^2)^6}{12} + \frac{11}{12}$

25 $I = \int \frac{1}{1+x^2} \, dx$

Let $x = \tan u \Rightarrow \frac{dx}{du} = \sec^2 u$

$I = \int \frac{1}{1+\tan^2 u} \sec^2 u \, du$

But $1 + \tan^2 u = \sec^2 u$

So $I = \int du = u + c$

$= \arctan x + c$

26 $x(x+2)\frac{dy}{dx} = y$

$\Rightarrow \int \frac{1}{y} dy = \int \frac{1}{x(x+2)} dx$

$\frac{1}{x(x+2)} \equiv \frac{A}{x} + \frac{B}{x+2}$

$\Rightarrow 1 \equiv A(x+2) + Bx$

$x = 0 \Rightarrow 1 = 2A \Rightarrow A = \frac{1}{2}$

$x = -2 \Rightarrow 1 = -2B \Rightarrow B = -\frac{1}{2}$

So $\ln y = \int \left(\frac{\left(\frac{1}{2}\right)}{x} - \frac{\left(\frac{1}{2}\right)}{x+2}\right) dx$

$= \frac{1}{2}\ln|x| - \frac{1}{2}\ln|x+2| + c$

$\therefore y = \sqrt{\frac{kx}{x+2}} \quad \left(c = \frac{1}{2}\ln k\right)$

$x = 2, y = 2 \Rightarrow 2 = \sqrt{\frac{2k}{4}} \Rightarrow 4 \times 2 = k$

$\therefore y = \sqrt{\frac{8x}{x+2}} \quad \text{or} \quad y^2 = \frac{8x}{x+2}$

27 a $A = \pi r^2 \Rightarrow \frac{dA}{dr} = 2\pi r$

$\frac{dr}{dt} = \frac{dr}{dA} \times \frac{dA}{dt} = \frac{1}{2\pi r} \times k \sin\left(\frac{t}{3\pi}\right)$

$\frac{dr}{dt} = \frac{k}{2\pi r}\sin\left(\frac{t}{3\pi}\right)$

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27 b $\int 2\pi r \, dr = \int k \sin\left(\frac{t}{3\pi}\right) dt$

$\pi r^2 = -3\pi k \cos\left(\frac{t}{3\pi}\right) + c$

$r^2 = -3k \cos\left(\frac{t}{3\pi}\right) + c$

$r = 1, t = 0 \Rightarrow 1 = -3k + c \Rightarrow c = 3k + 1$

$r = 2, t = \pi^2 \Rightarrow 4 = -\frac{3k}{2} + 3k + 1$

So $r^2 = -6\cos\left(\frac{t}{3\pi}\right) + 6 + 1$

$r^2 = -6\cos\left(\frac{t}{3\pi}\right) + 7$

c $r = 1.5 \Rightarrow 2.25 = -6\cos\left(\frac{t}{3\pi}\right) + 7$

$6\cos\left(\frac{t}{3\pi}\right) = 4.75$

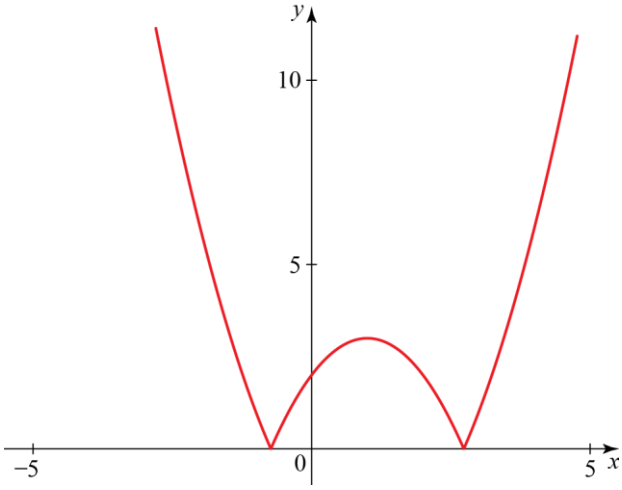
$\cos\left(\frac{t}{3\pi}\right) \approx 0.7917$

$\frac{t}{3\pi} = 0.6527$

$t = 6.19$ days

Challenge

a



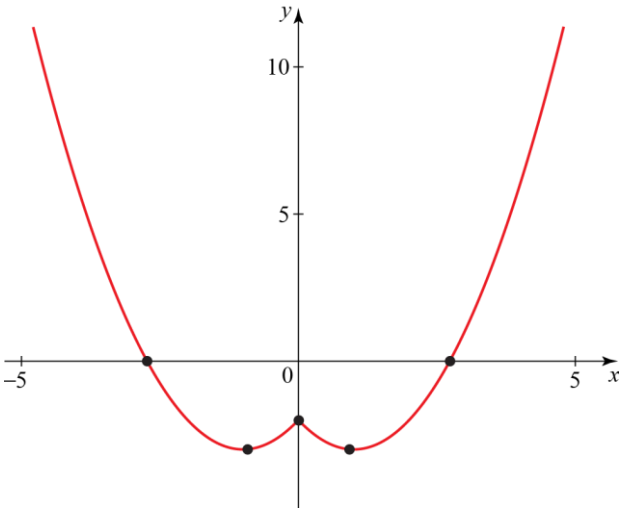
$\int_{-3}^{3} |f(x)| \, dx = \int_{-3}^{3} f(x) \, dx + 2 \times \left| \int_{-1}^{2} f(x) \, dx \right|$

$\int_{-3}^{3} f(x) \, dx = \left[\frac{x^3}{3} - \frac{x^2}{2} - 2x \right]_{-3}^{3} = 6$

$\int_{-1}^{2} f(x) \, dx = \left[\frac{x^3}{3} - \frac{x^2}{2} - 2x \right]_{-1}^{2} = \frac{9}{2}$

$\int_{-3}^{3} |f(x)| \, dx = 6 + 2 \times \frac{9}{2} = 15$

b



$\int_{-3}^{3} f(|x|) \, dx = 2 \times \int_{0}^{3} f(x) \, dx$

$\int_{0}^{3} f(x) \, dx = \left[\frac{x^3}{3} - \frac{x^2}{2} - 2x \right]_{0}^{3} = -\frac{3}{2}$

$\int_{-3}^{3} f(|x|) \, dx = 2 \times \left(-\frac{3}{2} \right) = -3$

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