Differentiation 9A

1 a $f(x) = \cos x$

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$
$$= \lim_{h\to 0}\frac{\cos(x+h)-\cos x}{h}$$
$$= \lim_{h\to 0}\frac{\cos x\cos h-\sin x\sin h-\cos x}{h}$$
$$= \lim_{h\to 0}\left(\left(\frac{\cos h-1}{h}\right)\cos x-\frac{\sin h}{h}\sin x\right)$$

b Since $\frac{\cos h-1}{h}\to 0$ and $\frac{\sin h}{h}\to 1$
the expression inside the limit in part **a** tends to $0\times\cos x-1\times\sin x=-\sin x$
So $f'(x)=-\sin x$

2 a $y = 2\cos x$

$$\frac{dy}{dx} = 2\times(-\sin x) = -2\sin x$$

b $y = 2\sin\frac{1}{2}x$

$$\frac{dy}{dx} = 2\times\frac{1}{2}\cos\frac{1}{2}x = \cos\frac{1}{2}x$$

c $y = \sin 8x$

$$\frac{dy}{dx} = 8\cos 8x$$

d $y = 6\sin\frac{2}{3}x$

$$\frac{dy}{dx} = 6\times\frac{2}{3}\cos\frac{2}{3}x = 4\cos\frac{2}{3}x$$

3 a $f(x) = 2\cos x$

$$f'(x) = 2\times(-\sin x) = -2\sin x$$

b $f(x) = 6\cos\frac{5}{6}x$

$$f'(x) = 6\times\left(-\frac{5}{6}\sin\frac{5}{6}x\right) = -5\sin\frac{5}{6}x$$

c $f(x) = 4\cos\frac{1}{2}x$

$$f'(x) = 4\times\left(-\frac{1}{2}\sin\frac{1}{2}x\right) = -2\sin\frac{1}{2}x$$

d $f(x) = 3\cos 2x$

$$f'(x) = 3(-2\sin 2x) = -6\sin 2x$$

4 a $y = \sin 2x + \cos 3x$

$$\frac{dy}{dx} = 2\cos 2x + (-3\sin 3x)$$
$$= 2\cos 2x - 3\sin 3x$$

b $y = 2\cos 4x - 4\cos x + 2\cos 7x$

$$\frac{dy}{dx} = 2\times(-4\sin 4x) - 4\times(-\sin x)$$
$$+ 2\times(-7\sin 7x)$$
$$= -8\sin 4x + 4\sin x - 14\sin 7x$$

c $y = x^2 + 4\cos 3x$

$$\frac{dy}{dx} = 2x + 4(-3\sin 3x) = 2x - 12\sin 3x$$

d $y = \frac{1+2x\sin 5x}{x} = \frac{1}{x} + 2\sin 5x$

$$\frac{dy}{dx} = -\frac{1}{x^2} + 2\times(5\cos 5x)$$
$$= -\frac{1}{x^2} + 10\cos 5x$$

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5 $y = x - \sin 3x$

$\dfrac{dy}{dx} = 1 - 3\cos 3x$

At stationary points $\dfrac{dy}{dx} = 0$

$1 - 3\cos 3x = 0$

$\cos 3x = \dfrac{1}{3}$

$3x = 1.23..., 5.05...$ or $7.51...$

$x = 0.410$, 1.68 or 2.50 (3 s.f.)

$x = 0.410 \Rightarrow y = 0.41 - \sin 1.23 = -0.532$

$x = 1.68 \Rightarrow y = 1.68 - \sin 5.04 = 2.63$

$x = 2.50 \Rightarrow y = 2.50 - \sin 7.50 = 1.56$

Stationary points in the interval $0 \leq x \leq \pi$ are (0.410, −0.532), (1.68, 2.63) and (2.50, 1.56).

6 $y = 2\sin 4x - 4\cos 2x$

$\dfrac{dy}{dx} = 2 \times 4\cos 4x - 4 \times (-2\sin 2x)$

$= 8\cos 4x + 8\sin 2x$

When $x = \dfrac{\pi}{2}$:

$\dfrac{dy}{dx} = 8\cos 2\pi + 8\sin \pi$

$= 8 \times 1 + 8 \times 0 = 8$

So the gradient of the curve at the point where $x = \dfrac{\pi}{2}$ is 8.

7 $y = 2\sin 2x + \cos 2x$

$\dfrac{dy}{dx} = 2 \times 2\cos 2x + (-2\sin 2x)$

$= 4\cos 2x - 2\sin 2x$

At stationary points $\dfrac{dy}{dx} = 0$

$4\cos 2x - 2\sin 2x = 0$

$4 - 2\tan 2x = 0$

$\tan 2x = 2$

$2x = 1.107...$ or $4.248...$

$x = 0.554$ or 2.12 (3 s.f.)

When $x = 0.554$:

$y = 2\sin(2 \times 0.554) + \cos(2 \times 0.554) = 2.24$

When $x = 2.12$:

$y = 2\sin(2 \times 2.12) + \cos(2 \times 2.12) = -2.24$

Stationary points **in the interval $0 \leqslant x \leqslant \pi$ are** (0.554, 2.24) and (2.12, −2.24).

8 $y = \sin 5x + \cos 3x$

$\dfrac{dy}{dx} = 5\cos 5x - 3\sin 3x$

At $(\pi, -1)$, $\dfrac{dy}{dx} = 5\cos 5\pi - 3\sin 3\pi$

$= 5 \times (-1) - 3 \times 0 = -5$

Equation of tangent is $y - (-1) = -5(x - \pi)$

or $y = -5x + 5\pi - 1$

9 $y = 2x^2 - \sin x$

$\dfrac{dy}{dx} = 4x - \cos x$

When $x = \pi$, $y = 2\pi^2$ and

$\dfrac{dy}{dx} = 4\pi - \cos \pi = 4\pi + 1$

Gradient of normal is $-\dfrac{1}{4\pi + 1}$

Equation of normal is

$y - 2\pi^2 = -\dfrac{1}{4\pi + 1}(x - \pi)$

Multiplying through by $(4\pi + 1)$ and rearranging gives

$x + (4\pi + 1)y - \pi(8\pi^2 + 2\pi + 1) = 0$

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10 Let $f(x) = \sin x$

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \frac{\sin(x+h)-\sin x}{h}$$

$$= \lim_{h\to 0}\left(\frac{\sin x\cos h+\cos x\sin h-\sin x}{h}\right)$$

$$= \lim_{h\to 0}\left(\left(\frac{\cos h-1}{h}\right)\sin x+\left(\frac{\sin h}{h}\right)\cos x\right)$$

Since $\frac{\cos h-1}{h}\to 0$ and $\frac{\sin h}{h}\to 1$,

the expression inside the limit tends to

$(0\times\sin x+1\times\cos x)$

So $\lim_{h\to 0}\frac{\sin(x+h)-\sin x}{h} = \cos x$

Hence the derivative of $\sin x$ is $\cos x$.

Challenge

Let $f(x) = \sin(kx)$

$$f'(x) = \lim_{h\to 0}\left(\frac{f(x+h)-f(x)}{h}\right)$$

$$= \lim_{h\to 0}\left(\frac{\sin(kx+kh)-\sin(kx)}{h}\right)$$

$$= \lim_{h\to 0}\left(\frac{\sin kx\cos kh+\cos kx\sin kh-\sin kx}{h}\right)$$

$$= \lim_{h\to 0}\left(\left(\frac{\cos kh-1}{h}\right)\sin kx+\left(\frac{\sin kh}{h}\right)\cos kx\right)$$

As $h\to 0$, $\left(\frac{\sin kh}{h}\right)\to k$ and $\left(\frac{\cos kh-1}{h}\right)\to 0$,

so the expression inside the limit tends to

$0\times\sin kx+k\times\cos kx = k\cos kx$

Hence the derivative of $\sin(kx)$ is $k\cos(kx)$.

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