Trigonometric Functions 6E

1 a arccos(0) is the angle a in $0 \leq a \leq \pi$ for which $\cos a = 0$

Refer to graph of $y = \cos\theta \Rightarrow a = \frac{\pi}{2}$

So $\arccos(0) = \frac{\pi}{2}$

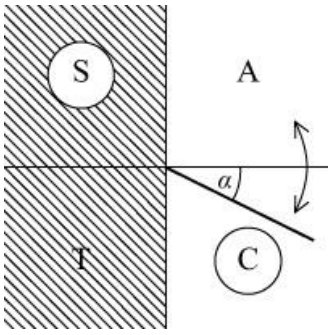
b arcsin(1) is the angle a in $-\frac{\pi}{2} \leq a \leq \frac{\pi}{2}$ for which $\sin a = 1$

Refer to graph of $y = \sin\theta \Rightarrow a = \frac{\pi}{2}$

So $\arcsin(1) = \frac{\pi}{2}$

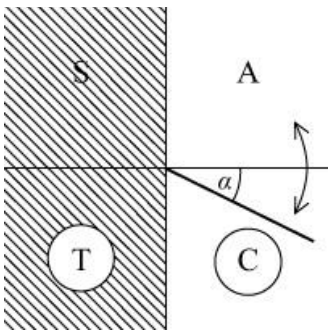
c arctan(-1) is the angle a in $-\frac{\pi}{2} < a < \frac{\pi}{2}$ for which $\tan a = -1$

So $\arctan(-1) = -\frac{\pi}{4}$



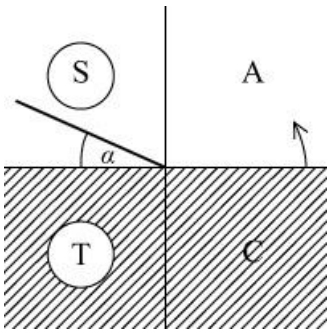
d $\arcsin\left(-\frac{1}{2}\right)$ is the angle a in $-\frac{\pi}{2} \leq a \leq \frac{\pi}{2}$ for which $\sin a = -\frac{1}{2}$

So $\arcsin\left(-\frac{1}{2}\right) = -\frac{\pi}{6}$



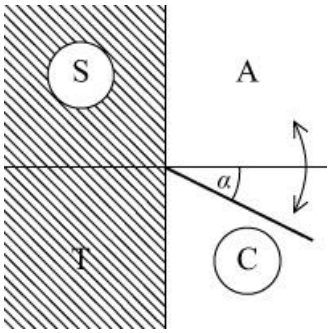
e $\arccos\left(-\frac{1}{\sqrt{2}}\right)$ is the angle a in $0 \leq a \leq \pi$ for which $\cos a = -\frac{1}{\sqrt{2}}$

So $\arccos\left(-\frac{1}{\sqrt{2}}\right) = \frac{3\pi}{4}$



f $\arctan\left(-\frac{1}{\sqrt{3}}\right)$ is the angle a in $-\frac{\pi}{2} < a < \frac{\pi}{2}$ for which $\tan a = -\frac{1}{\sqrt{3}}$

So $\arctan\left(-\frac{1}{\sqrt{3}}\right) = -\frac{\pi}{6}$



g $\arcsin\left(\sin\frac{\pi}{3}\right)$ is the angle a in $-\frac{\pi}{2} \leq a \leq \frac{\pi}{2}$ for which $\sin a = \sin\frac{\pi}{3}$

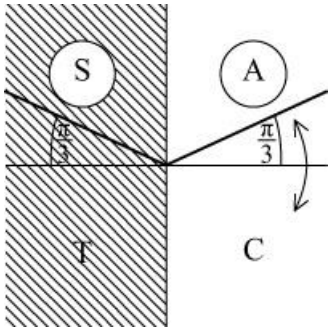
So $\arcsin\left(\sin\frac{\pi}{3}\right) = \frac{\pi}{3}$

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1 h $\arcsin\left(\sin\frac{2\pi}{3}\right)$ is the angle α in

$-\frac{\pi}{2} \le \alpha \le \frac{\pi}{2}$ for which $\sin\alpha = \sin\frac{2\pi}{3}$

So $\arcsin\left(\sin\frac{2\pi}{3}\right) = \frac{\pi}{3}$



2 a $\arcsin\left(\frac{1}{2}\right) + \arcsin\left(-\frac{1}{2}\right) = \frac{\pi}{6} + \left(-\frac{\pi}{6}\right) = 0$

b $\arccos\left(\frac{1}{2}\right) - \arccos\left(-\frac{1}{2}\right) = \frac{\pi}{3} - \frac{2\pi}{3} = -\frac{\pi}{3}$

c $\arctan(1) - \arctan(-1) = \frac{\pi}{4} - \left(-\frac{\pi}{4}\right) = \frac{\pi}{2}$

3 a $\sin\left(\arcsin\frac{1}{2}\right)$

$\arcsin\frac{1}{2} = \alpha$ where $\sin\alpha = \frac{1}{2}$,

and $-\frac{\pi}{2} \le \alpha \le \frac{\pi}{2}$

So $\arcsin\frac{1}{2} = \frac{\pi}{6}$

$\Rightarrow \sin\left(\arcsin\frac{1}{2}\right) = \sin\frac{\pi}{6} = \frac{1}{2}$

b $\sin\left(\arcsin\left(-\frac{1}{2}\right)\right)$

$\arcsin\left(-\frac{1}{2}\right) = \alpha$

where $\sin\alpha = -\frac{1}{2}$, $-\frac{\pi}{2} \le \alpha \le \frac{\pi}{2}$

So $\arcsin\left(-\frac{1}{2}\right) = -\frac{\pi}{6}$

$\Rightarrow \sin\left(\arcsin\left(-\frac{1}{2}\right)\right) = \sin\left(-\frac{\pi}{6}\right) = -\frac{1}{2}$

c $\tan\left(\arctan(-1)\right)$

$\arctan(-1) = \alpha$

where $\tan\alpha = -1$, $-\frac{\pi}{2} < \alpha < \frac{\pi}{2}$

So $\arctan(-1) = -\frac{\pi}{4}$

$\Rightarrow \tan\left(\arctan(-1)\right) = \tan\left(-\frac{\pi}{4}\right) = -1$

d $\cos(\arccos 0)$

$\arccos 0 = \alpha$ where $\cos\alpha = 0$, $0 \le \alpha \le \pi$

So $\arccos 0 = \frac{\pi}{2}$

$\therefore \cos(\arccos 0) = \cos\frac{\pi}{2} = 0$

4 a $\sin\left(\arccos\frac{1}{2}\right)$

$\arccos\frac{1}{2} = \frac{\pi}{3}$

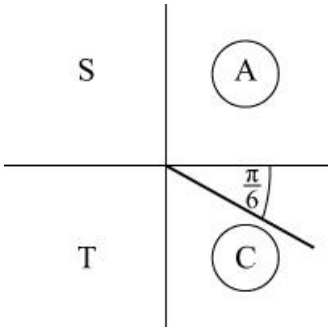
$\sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$

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4 b $\cos\left(\arcsin\left(-\frac{1}{2}\right)\right)$

$\arcsin\left(-\frac{1}{2}\right) = -\frac{\pi}{6}$

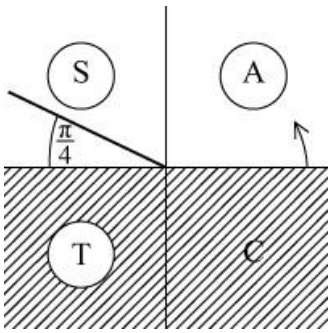
$\cos\left(-\frac{\pi}{6}\right) = +\frac{\sqrt{3}}{2}$



c $\tan\left(\arccos\left(-\frac{\sqrt{2}}{2}\right)\right)$

$\arccos\left(-\frac{\sqrt{2}}{2}\right) = a$

where $\cos a = -\frac{\sqrt{2}}{2}$, $0 \leq a \leq \pi$



So $\arccos\left(-\frac{\sqrt{2}}{2}\right) = \frac{3\pi}{4}$

$\tan\frac{3\pi}{4} = -1$

d $\sec(\arctan\sqrt{3})$

$\arctan\sqrt{3} = \frac{\pi}{3}$

(the angle whose tan is $\sqrt{3}$)

$\sec\frac{\pi}{3} = \frac{1}{\cos\frac{\pi}{3}} = \frac{1}{\frac{1}{2}} = 2$

e $\operatorname{cosec}\left(\arcsin(-1)\right)$

$\arcsin(-1) = a$

where $\sin a = -1$, $-\frac{\pi}{2} \leq a \leq \frac{\pi}{2}$

So $\arcsin(-1) = -\frac{\pi}{2}$

$\Rightarrow \operatorname{cosec}\left(\arcsin(-1)\right) = \frac{1}{\sin\left(-\frac{\pi}{2}\right)}$

$= \frac{1}{-1} = -1$

f $\sin\left(2\arcsin\left(\frac{\sqrt{2}}{2}\right)\right)$

$\arcsin\frac{\sqrt{2}}{2} = \frac{\pi}{4}$

So $\sin\left(2\arcsin\left(\frac{\sqrt{2}}{2}\right)\right) = \sin\frac{\pi}{2} = 1$

5 As k is positive, the first two positive solutions of $\sin x = k$ are $\arcsin k$ and $\pi - \arcsin k$ i.e. a and $\pi - a$

(Try a few examples, taking specific values for k).

6 a $\arcsin x$ is the angle a in $-\frac{\pi}{2} \leq a \leq \frac{\pi}{2}$ such that $\sin a = x$

In this case $x = \sin k$ where $0 < k < \frac{\pi}{2}$

As sin is an increasing function

$\sin 0 < x < \sin\frac{\pi}{2}$

$\Rightarrow 0 < x < 1$

b i $\cos k = \pm\sqrt{1 - \sin^2 k} = \pm\sqrt{1 - x^2}$

k is in the 1st quadrant $\Rightarrow \cos k > 0$

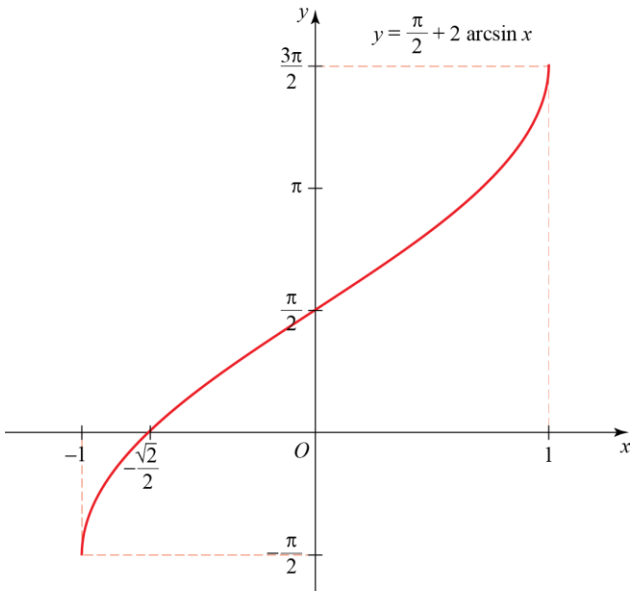
So $\cos k = \sqrt{1 - x^2}$

ii $\tan k = \frac{\sin k}{\cos k} = \frac{x}{\sqrt{1 - x^2}}$

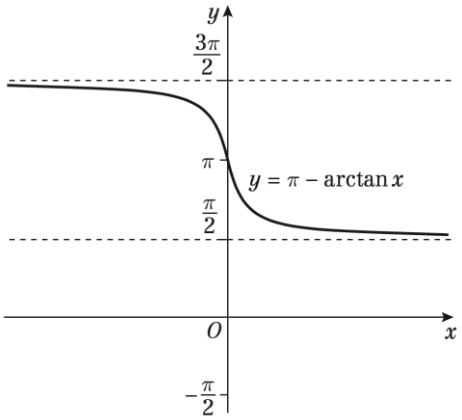
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6 c k is now in the 4th quadrant, where $\cos k$ is positive. So the value of $\cos k$ remains the same and there is no change to $\tan k$.

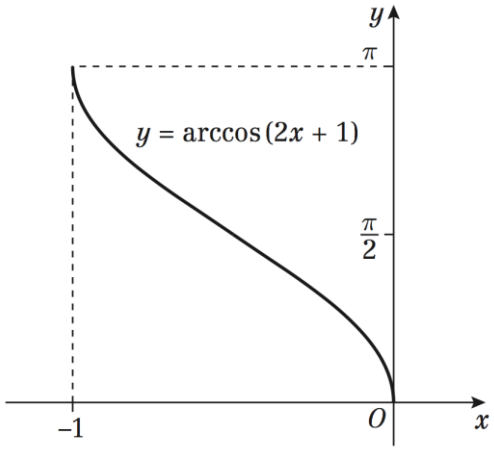
7 a The graph of $y = \frac{\pi}{2} + 2\arcsin x$ is $y = \arcsin x$ stretched by a scale factor 2 in the y direction and then translated by the vector $\begin{pmatrix} 0 \\ \frac{\pi}{2} \end{pmatrix}$



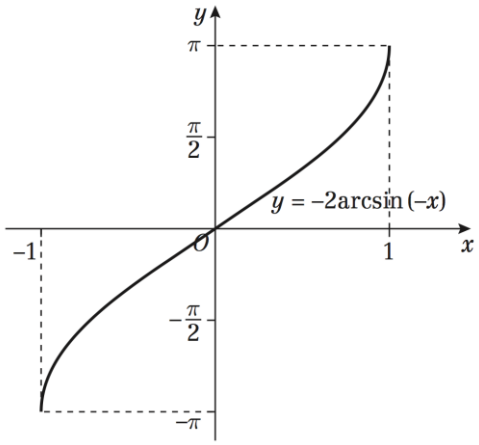
b The graph of $y = \pi - \arctan x$ is $y = \arctan x$ reflected in the x-axis and then translated by the vector $\begin{pmatrix} 0 \\ \pi \end{pmatrix}$



7 c The graph of $y = \arccos(2x + 1)$ is $y = \arccos x$ translated by the vector $\begin{pmatrix} -1 \\ 0 \end{pmatrix}$ and then stretched by scale factor $\frac{1}{2}$ in the x direction

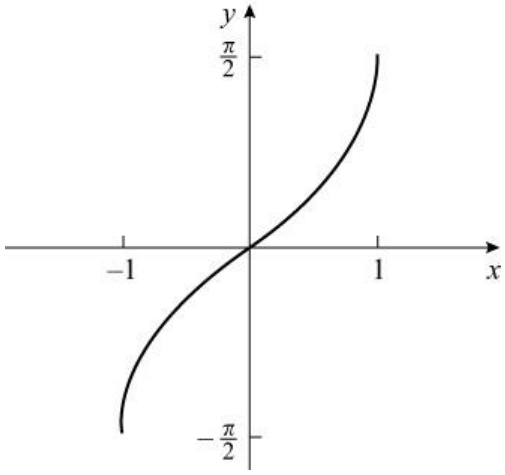


d The graph of $y = -2\arcsin(-x)$ is $y = \arcsin x$ reflected in the y-axis, then reflected in the x-axis and then stretched by a scale factor 2 in the y direction



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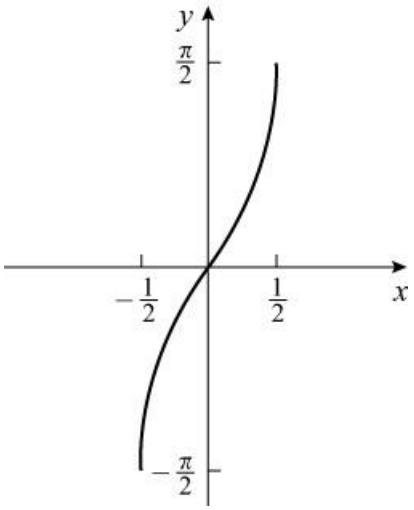
8 a $y = \arcsin x$



Range is $-\frac{\pi}{2} \leq f(x) \leq \frac{\pi}{2}$

b The graph of $y = f(2x)$ is the graph of $y = f(x)$ stretched in the x direction by scale factor $\frac{1}{2}$

$y = g(x)$



c $g : x \mapsto \arcsin 2x$

The domain is $-\frac{1}{2} \leq x \leq \frac{1}{2}$

d Let $y = \arcsin 2x$

$\Rightarrow 2x = \sin y$

$\Rightarrow x = \frac{1}{2}\sin y$

So $g^{-1} : x \mapsto \frac{1}{2}\sin x, \ -\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$

9 a Let $y = \arccos x$

As $0 \leq x \leq 1 \Rightarrow 0 \leq y \leq \frac{\pi}{2}$

$\cos y = x$, and using $\cos^2 y + \sin^2 y \equiv 1$

$\Rightarrow \sin^2 y = 1 - \cos^2 y$

$\Rightarrow \sin y = \sqrt{1 - \cos^2 y} = \sqrt{1 - x^2}$

Note, $\sin y \geqslant 0$ since $0 \leq y \leq \frac{\pi}{2}$

so $\sin y \geqslant \sqrt{1 - x^2}$

$\sin y = \sqrt{1 - x^2}$

$\Rightarrow y = \arcsin\sqrt{1 - x^2}$

Therefore, $\arccos x = \arcsin\sqrt{1 - x^2}$ for $0 \leq x \leq 1$

b For $-1 \leq x \leq 0$, $\frac{\pi}{2} \leq \arccos x \leq \pi$

But arcsin has a range of $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

So $\arccos x \neq \arcsin\sqrt{1 - x^2}$, for $-1 \leq x \leq 0$

An alternative approach is to provide a counterexample.

Let $x = -\frac{1}{\sqrt{2}}$

$\arccos x = \frac{3\pi}{4}$

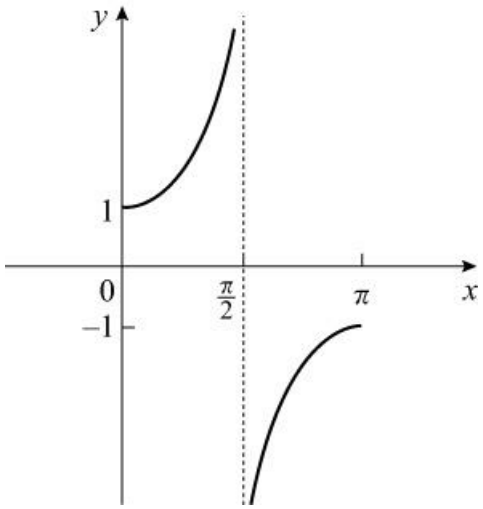
$\arcsin\sqrt{1 - x^2} = \arcsin\frac{1}{\sqrt{2}} = \frac{\pi}{4}$

So $\arccos x \neq \arcsin\sqrt{1 - x^2}$

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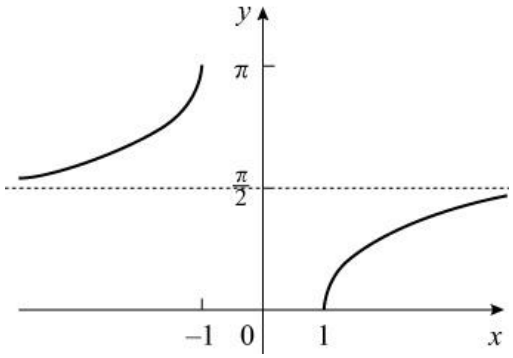
Challenge

a $y = \sec x$



b Reflect the graph drawn for part (a) in the line $y = x$

$y = \text{arcsec}\, x, x \leq -1, x \geq 1$



Range is $0 \leq \text{arcsec}\, x \leq \pi$, for $\text{arcsec}\, x \neq \frac{\pi}{2}$

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