Sequences and series 3I

1 **a** Initial amount = £4000
(start of month 1)

Start of month 2 = £(4000 + 200)

Start of month 3 = £(4000 + 200 + 200)
= £(4000 + 2 × 200)

Start of month 10 = £(4000 + 9 ×200)
= £(4000 + 1800)
= £5800

b Start of mth month
= £(4000 + (m − 1) × 200)
= £(4000 + 200m − 200)
= £(3800 + 200m)

2

$$\underset{\uparrow}{20\,000} + \underset{\uparrow}{20\,500} + \underset{\uparrow}{21\,000} + \underset{\uparrow}{21\,500} + \ldots$$

$$\text{Year 1} \xrightarrow[\text{1st increment}]{} \text{Year 2} \xrightarrow[\text{2nd increment}]{} \text{Year 3} \xrightarrow[\text{3rd increment}]{} \text{Year 4}$$

Carol will reach her maximum salary after

$$\frac{25\,000 - 20\,000}{500} = 10 \text{ increments}$$

This will be after 11 years.

a Total amount after 10 years

$$= \underbrace{20\,000 + 20\,500 + 21\,000 + \ldots}$$

This is an arithmetic series with a = 20 000, d = 500 and n = 10. Use $S_n = \frac{n}{2}\left(2a + (n-1)d\right)$.

$$= \frac{10}{2}\left(40\,000 + 9 \times 500\right)$$
$$= 5 \times 44\,500$$
$$= £222\,500$$

b From year 11 to year 15 she will continue to earn £25 000.

Total in this time = 5 × 25 000
= £125 000.

Total amount in the first 15 years is

£222 500 + £125 000 = £347 500

c It is unlikely her salary will rise by the same amount each year.

3 Amount saved by James

$$= \underbrace{1 + 2 + 3 + \ldots + 42}$$

This is an arithmetic series with a = 1, d = 1, n = 42 and L = 42.

a Use $S_n = \frac{n}{2}(a + L)$

$$= \frac{42}{2}(1 + 42)$$
$$= 21 \times 43$$
$$= 903\text{p}$$
$$= £9.03$$

b To save £100 we need

$$\underbrace{1 + 2 + 3 + \ldots}_{\text{Sum to } n \text{ terms}} = 10\,000$$

$$\frac{n}{2}\left(2 \times 1 + (n-1) \times 1\right) = 10\,000$$
$$\frac{n}{2}(n+1) = 10\,000$$
$$n(n+1) = 20\,000$$
$$n^2 + n - 20\,000 = 0$$

$$n = \frac{-1 \pm \sqrt{(1)^2 - 4 \times 1 \times (-20\,000)}}{2}$$
$$n = 140.9 \text{ or } -141.9$$

It takes James 141 days to save £100.

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4 A growth of 10% a year gives a multiplication factor of 1.1.

a After 1 year number is $200 \times 1.1 = 220$

b After 2 years number is $200 \times 1.1^2 = 242$

c After 3 years number is

$200 \times 1.1^3 = 266.2 = 266$
(to nearest whole number)

d After 10 years number is

$200 \times 1.1^{10} = 518.748\ldots = 519$
(to nearest whole number)

5 Let maximum speed in bottom gear be a km h^{-1}

This gives maximum speeds in each successive gear **of** ar, ar^2, ar^3, where r is the common ratio.

We are given

$a = 40$ **(1)**

$ar^3 = 120$ **(2)**

Substitute **(1)** into **(2)**:

$$40r^3 = 120 \quad (\div 40)$$
$$r^3 = 3$$
$$r = \sqrt[3]{3}$$
$$r = 1.442\ldots \ (3 \text{ d.p.})$$

Maximum speed in 2nd gear is
$ar = 40 \times 1.442\ldots = 57.7$ km h^{-1}

Maximum speed in 3rd gear is
$ar^2 = 40 \times (1.442\ldots)^2 = 83.2$ km h^{-1}

6 a $r = 0.85$
$a \times 0.85^3 = 11\,054.25$
$a = £18\,000$

b $18\,000 \times 0.85^n > 5000$

$$0.85^n > \frac{5}{18}$$
$$n > \frac{\log\left(\frac{5}{18}\right)}{\log(0.85)}$$
$$n > 7.88$$

7 a Total commission

$$= \underbrace{10 + 20 + 30 + \ldots + 520}$$

Arithmetic series with $a = 10$, $d = 10$, $n = 52$.

$$= \frac{52}{2}\left(2 \times 10 + (52-1) \times 10\right) \text{ using}$$
$$S_n = \frac{n}{2}\left(2a + (n-1)d\right)$$
$$= 26\left(20 + 51 \times 10\right)$$
$$= 26\left(20 + 510\right)$$
$$= 26 \times 530$$
$$= £13\,780$$

b Commission = policies for year 1 + policies for 2nd week of year 2
= 520 + 22 = £542

c Total commission for year 2

= Commission for year 1 policies + Commission for year 2 policies

= 520 × 52 + (11 + 22 + 33 + ... 52 × 11)

Use $S_n = \frac{n}{2} = \left(2a + (n-1)d\right)$

with $n = 52,\ a = 11,\ d = 11$

$$= 27\,040 + \frac{52}{2}\left(2 \times 11 + (52-1) \times 11\right)$$
$$= 27\,040 + 26 \times \left(22 + 51 \times 11\right)$$
$$= 27\,040 + £15158$$
$$= £42198$$

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8 a Cost of drilling to 500 m

$$= \underset{\substack{\uparrow \\ 1\text{st} \\ 50\,\text{m}}}{500} + \underset{\substack{\uparrow \\ 2\text{nd} \\ 50\,\text{m}}}{640} + \underset{\substack{\uparrow \\ 3\text{rd} \\ 50\,\text{m}}}{780} + \ldots$$

There would be 10 terms because there are 10 lots of 50 m in 500 m.

Arithmetic series with $a = 500$, $d = 140$ and $n = 10$.

$$\text{Using } S_n = \frac{n}{2}\left(2a + (n-1)d\right)$$
$$= \frac{10}{2}\left(2\times 500 + (10-1)\times 140\right)$$
$$= 5\left(1000 + 9\times 140\right)$$
$$= 5\times 2260$$
$$= £11300$$

b This time we are given $S = 76\,000$. The first term will still be 500 and d remains 140.

Use $S = \frac{n}{2}\left(2a + (n-1)d\right)$ with $S = 76\,000$, $a = 500$, $d = 140$, and solve for n.

$$76\,000 = \frac{n}{2}\left(2\times 500 + (n-1)\times 140\right)$$
$$76\,000 = \frac{n}{2}\left(1000 + 140(n-1)\right)$$
$$76\,000 = n\left(500 + 70(n-1)\right)$$
$$76\,000 = n\left(500 + 70n - 70\right)$$
$$76\,000 = n\left(70n + 430n\right) \quad (\text{multiply out})$$
$$76\,000 = 70n^2 + 430n \quad (\div 10)$$
$$7600 = 7n^2 + 43n$$

$$0 = 7n^2 + 43n - 7600$$
$$n = \frac{-43 \pm \sqrt{(43)^2 - 4\times 7\times(-7600)}}{2\times 7}$$
$$n = 30.02,\ (-36.16)$$

Only accept the positive answer, so there are 30 terms (to the nearest term).

So the greatest depth that can be drilled is 30 × 50 = 1500 m (to the nearest 50 m).

9 a 1st year = 500
2nd year = 550 = 500 + 1 × 50
3rd year = 600 = 500 + 2 × 50

⋮

40th year = 500 + 39 × 50 = £2450

b Total amount paid in

$$= \underbrace{£500 + £550 + £600 + \ldots + £2450}$$

This is an arithmetic series with $a = 500$, $d = 50$, $L = 2450$ and $n = 40$.

$$S_n = \frac{n}{2}(a + L)$$
$$S_{40} = \frac{40}{2}(500 + 2450)$$
$$= 20\times 2950$$
$$= £59\,000$$

c Brian's amount

$$= \underbrace{890 + (890 + d) + (890 + 2d) + \ldots}_{40\ \text{years}}$$

Use $S_n = \frac{n}{2}\left(2a + (n-1)d\right)$ with $n = 40$, $a = 890$ and d.

$$S_{40} = \frac{40}{2}\left(2\times 890 + (40-1)d\right)$$
$$= 20\left(1780 + 39d\right)$$

Use the fact that

Brian's saving = Anne's savings

$$20\left(1780 + 39d\right) = 59\,000 \quad (\div 20)$$
$$1780 + 39d = 2950 \quad (-1780)$$
$$39d = 1170 \quad (\div 39)$$
$$d = 30$$

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10 If the number of people infected increases by 4% the multiplication factor is 1.04.

After n days $100 \times (1.04)^n$ people will be infected.

If 1000 people are infected

$$100 \times (1.04)^n = 1000$$
$$(1.04)^n = 10$$
$$\log(1.04)^n = \log 10$$
$$n \log(1.04) = 1$$
$$n = \frac{1}{\log(1.04)}$$
$$n = 58.708\ldots$$

It would take 59 days.

11 If the increase is 3.5% per annum the multiplication factor is 1.035.

Therefore after n years I will have £$A \times (1.035)^n$.

If the money is doubled it will equal 2A, therefore

$$A \times (1.035)^n = 2A$$
$$(1.035)^n = 2$$
$$\log(1.035)^n = \log 2$$
$$n \log(1.035) = \log 2$$
$$n = \frac{\log 2}{\log(1.035)} = 20.14879\ldots$$

My money will double after 20.15 years.

12 The reduction is 6% which gives a multiplication factor of 0.94.

Let the number of fish now be F.

After n years there will be $F \times (0.94)^n$.

When their number is halved the number will be $\frac{1}{2}F$.

Set these equal to each other:

$$F \times (0.94)^n = \frac{1}{2}F$$
$$(0.94)^n = \frac{1}{2}$$
$$\log(0.94)^n = \log\left(\frac{1}{2}\right)$$
$$n \log(0.94) = \log\left(\frac{1}{2}\right)$$
$$n = \frac{\log\left(\frac{1}{2}\right)}{\log(0.94)}$$
$$n = 11.2$$

The fish stocks will halve in 11.2 years.

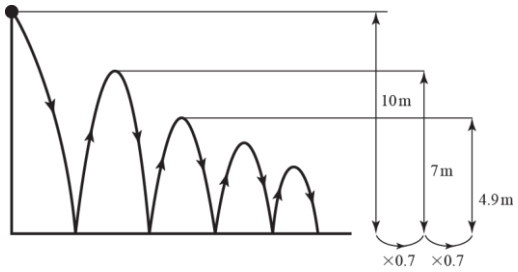
13 No. grains $= \underbrace{1 + 2 + 4 + 8 + \ldots}_{64 \text{ terms}}$

This is a geometric series with $a = 1$, $r = 2$ and $n = 64$.

As $|r| > 1$ use $S_n = \frac{a(r^n - 1)}{r - 1}$.

$$\text{Number of grains} = \frac{1(2^{64} - 1)}{2 - 1} = 2^{64} - 1$$

14 a



After the 1st bounce it bounces to 7 cm

After the 2nd bounce it bounces to 4.9 cm (× 0.7)

After the 3rd bounce it bounces to 3.43 cm (× 0.7)

After the 4th bounce it bounces to 2.401 cm (× 0.7)

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14 b Total distance travelled

$$= \underset{\text{1st bounce}}{10} + 7 + \underset{\text{2nd bounce}}{7} + 4.9 + \underset{\text{3rd bounce}}{4.9} + \ldots$$

$$= 2 \times \underbrace{(10 + 7 + 4.9 + \ldots)}_{\substack{6 \text{ terms} \\ a=10,\ r=0.7,\ n=6}} - 10$$

$$= 2 \times \frac{10\left(1 - 0.7^6\right)}{1 - 0.7} - 10$$

$$= 48.8234\,\text{m}$$

15 a $a = 10$, $r = 1.1$

$$S_n = \frac{10(1.1^n - 1)}{1.1 - 1} = 1000$$

$$1.1^n - 1 = 10$$

$$1.1^n = 11$$

$$n = \frac{\log 11}{\log 1.1}$$

$= 25.16$

So 26 days

b On the 25th day:

$ar^{24} = 10 \times 1.1^{24} = 98.5$ miles

16 Jan. 1st, year 1 = £500

Dec. 31st, year 1 = 500×1.035

Jan. 1st, year 2 = $500 \times 1.035 + 500$

Dec. 31st, year 2

$= (500 \times 1.035 + 500) \times 1.035$

$= 500 \times 1.035^2 + 500 \times 1.035$

$\vdots$

Dec. 31st, year n

$= 500 \times 1.035^n + \ldots + 500 \times 1.035^2 + 500 \times 1.035$

$$= 500 \times \underbrace{\left(1.035^n + \ldots + 1.035^2 + 1.035\right)}$$

A geometric series with $a = 1.035$, $r = 1.035$ and n.

Use $S_n = \dfrac{a\left(r^n - 1\right)}{r - 1}$.

Dec. 31st year $n = 500 \times \dfrac{1.035\left(1.035^n - 1\right)}{1.035 - 1}$

Set this equal to £20 000.

$$20\,000 = 500 \times \frac{1.035\left(1.035^n - 1\right)}{1.035 - 1}$$

$$\left(1.035^n - 1\right) = \frac{20\,000 \times (1.035 - 1)}{500 \times 1.035}$$

$$1.035^n - 1 = 1.3526570\ldots$$

$$1.035^n = 2.3526570\ldots$$

$$\log\left(1.035^n\right) = \log 2.3526570\ldots$$

$$n \log(1.035) = \log 2.3526570\ldots$$

$$n = \frac{\log 2.3526570\ldots}{\log 1.035}$$

$n = 24.9$ years (3 s.f.)

It takes Alan 25 years to save £20 000.

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