Integration 11J

1 a $\dfrac{dy}{dx} = (1+y)(1-2x)$

$\Rightarrow \int \dfrac{1}{1+y}\,dy = \int (1-2x)dx$

$\Rightarrow \ln|1+y| = x - x^2 + c$

$\Rightarrow 1+y = e^{(x-x^2+c)}$

$\Rightarrow 1+y = A\,e^{x-x^2}, \quad (A = e^c)$

$\Rightarrow y = A\,e^{x-x^2} - 1$

b $\dfrac{dy}{dx} = y\tan x$

$\Rightarrow \int \dfrac{1}{y}\,dy = \int \tan x\ dx$

$\Rightarrow \ln|y| = \ln|\sec x| + c$

$\Rightarrow \ln|y| = \ln|k\sec x|, \quad (c = \ln k)$

$\Rightarrow y = k\sec x$

c $\cos^2 x \dfrac{dy}{dx} = y^2 \sin^2 x$

$\Rightarrow \int \dfrac{1}{y^2}\,dy = \int \dfrac{\sin^2 x}{\cos^2 x}\,dx$

$\Rightarrow \int \dfrac{1}{y^2}\,dy = \int \tan^2 x\,dx = \int (\sec^2 x - 1)dx$

$\Rightarrow -\dfrac{1}{y} = \tan x - x + c$

$\Rightarrow y = \dfrac{-1}{\tan x - x + c}$

d $\dfrac{dy}{dx} = 2e^{x-y} = 2e^x e^{-y}$

$\Rightarrow \int \dfrac{1}{e^{-y}}\,dy = \int 2e^x dx$

i.e. $\Rightarrow \int e^y dy = \int 2e^x dx$

$\Rightarrow e^y = 2e^x + c$

$\Rightarrow y = \ln(2e^x + c)$

2 a $\dfrac{dy}{dx} = \sin x \cos^2 x$

$\Rightarrow \int dy = \int \sin x \cos^2 x\,dx$

$\Rightarrow y = -\dfrac{\cos^3 x}{3} + c$

$y = 0, x = \dfrac{\pi}{3} \Rightarrow 0 = -\dfrac{\left(\frac{1}{8}\right)}{3} + c \Rightarrow c = \dfrac{1}{24}$

$\therefore\ y = \dfrac{1}{24} - \dfrac{1}{3}\cos^3 x$

b $\dfrac{dy}{dx} = \sec^2 x \sec^2 y$

$\Rightarrow \int \dfrac{1}{\sec^2 y}\,dy = \int \sec^2 x\,dx$

$\Rightarrow \int \cos^2 y\,dy = \int \sec^2 x\,dx$

$\Rightarrow \int \left(\dfrac{1}{2} + \dfrac{1}{2}\cos 2y\right)dy = \int \sec^2 x\,dx$

$\Rightarrow \dfrac{1}{2}y + \dfrac{1}{4}\sin 2y = \tan x + c$

or $\sin 2y + 2y = 4\tan x + k$

$y = 0, x = \dfrac{\pi}{4} \Rightarrow 0 = 4 + k \Rightarrow k = -4$

$\therefore \sin 2y + 2y = 4\tan x - 4$

c $\dfrac{dy}{dx} = 2\cos^2 y \cos^2 x$

$\Rightarrow \int \dfrac{1}{\cos^2 y}\,dy = \int 2\cos^2 x\,dx$

$\Rightarrow \int \sec^2 y\,dy = \int (1 + \cos 2x)dx$

$\Rightarrow \tan y = x + \dfrac{1}{2}\sin 2x + c$

$x = 0, \quad y = \dfrac{\pi}{4} \Rightarrow 1 = 0 + c$

$\therefore\ \tan y = x + \dfrac{1}{2}\sin 2x + 1$

© Pearson Education Ltd 2017. Copying permitted for purchasing institution only. This material is not copyright free.

2 d $\sin y \cos x \dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{\cos y}{\cos x}$

$\tan y \dfrac{\mathrm{d}y}{\mathrm{d}x} = \sec^2 x$

$\int \tan y \, \mathrm{d}y = \int \sec^2 x \, \mathrm{d}x$

$-\ln|\cos y| = \tan x + c$

$x = 0, y = 0 \Rightarrow 0 = 0 + c \Rightarrow c = 0$

$-\ln|\cos y| = \tan x$

$\cos y = \mathrm{e}^{-\tan x}$

$y = \arccos\left(\mathrm{e}^{-\tan x}\right)$

3 a $x^2 \dfrac{\mathrm{d}y}{\mathrm{d}x} = y + xy$

$x^2 \dfrac{\mathrm{d}y}{\mathrm{d}x} = y(1+x)$

$\dfrac{1}{y}\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{1+x}{x^2} = \dfrac{1}{x^2} + \dfrac{1}{x}$

$\int \dfrac{1}{y} \, \mathrm{d}y = \int \left(\dfrac{1}{x^2} + \dfrac{1}{x}\right) \mathrm{d}x$

$\ln y = -\dfrac{1}{x} + \ln x + \ln A$

$y = \mathrm{e}^{-\frac{1}{x} + \ln x + \ln A} = \mathrm{e}^{-\frac{1}{x}} \times \mathrm{e}^{\ln x} \times \mathrm{e}^{\ln A}$

$y = Ax\mathrm{e}^{-\frac{1}{x}}$

b $y = \mathrm{e}^4, x = -1 \Rightarrow \mathrm{e}^4 = -A\mathrm{e}$

$A = -\mathrm{e}^3$

$y = -\mathrm{e}^3 x \mathrm{e}^{-\frac{1}{x}} = -x\mathrm{e}^{\left(\frac{3x-1}{x}\right)}$

4 $(2y + 2yx)\dfrac{\mathrm{d}y}{\mathrm{d}x} = 1 - y^2$

$2y(1+x)\dfrac{\mathrm{d}y}{\mathrm{d}x} = 1 - y^2$

$\int \dfrac{2y}{1-y^2} \, \mathrm{d}y = \int \dfrac{1}{1+x} \, \mathrm{d}x$

$\ln k - \ln|1-y^2| = \ln|1+x|$

$\ln\left|\dfrac{k}{1-y^2}\right| = \ln|1+x|$

$\dfrac{k}{1-y^2} = 1 + x$

$x = 0, y = 0 \Rightarrow k = 1$

$\dfrac{1}{1-y^2} = 1 + x$

$1 - y^2 = \dfrac{1}{x+1}$

$y^2 = 1 - \dfrac{1}{x+1} = \dfrac{x}{x+1}$

$y = \sqrt{\dfrac{x}{x+1}}$

5 $\mathrm{e}^{x+y}\dfrac{\mathrm{d}y}{\mathrm{d}x} = 2x + x\mathrm{e}^y$

$\mathrm{e}^x\mathrm{e}^y\dfrac{\mathrm{d}y}{\mathrm{d}x} = x(2 + \mathrm{e}^y)$

$\dfrac{\mathrm{e}^y}{2+\mathrm{e}^y}\dfrac{\mathrm{d}y}{\mathrm{d}x} = x\mathrm{e}^{-x}$

$\int \dfrac{\mathrm{e}^y}{2+\mathrm{e}^y} \, \mathrm{d}y = \int x\mathrm{e}^{-x} \, \mathrm{d}x$

$\ln|2+\mathrm{e}^y| = \int x\mathrm{e}^{-x} \, \mathrm{d}x$

Let $u = x \Rightarrow \dfrac{\mathrm{d}u}{\mathrm{d}x} = 1$

$\dfrac{\mathrm{d}v}{\mathrm{d}x} = \mathrm{e}^{-x} \Rightarrow v = -\mathrm{e}^{-x}$

So $\ln|2+\mathrm{e}^y| = -x\mathrm{e}^{-x} + \int \mathrm{e}^{-x} \, \mathrm{d}x$

$\ln|2+\mathrm{e}^y| = -x\mathrm{e}^{-x} - \mathrm{e}^{-x} + c$

© Pearson Education Ltd 2017. Copying permitted for purchasing institution only. This material is not copyright free.

6 $(1-x^2)\dfrac{dy}{dx} = xy + y$

$(1-x^2)\dfrac{dy}{dx} = y(x+1)$

$\dfrac{1}{y}\dfrac{dy}{dx} = \dfrac{x+1}{(1-x^2)} = \dfrac{x+1}{(1-x)(1+x)} = \dfrac{1}{1-x}$

$\int \dfrac{1}{y}\, dy = \int \dfrac{1}{1-x} dx$

$\ln y = \ln k - \ln|1-x| = \ln\left|\dfrac{k}{1-x}\right|$

$y = \dfrac{k}{1-x}$

$y = 6, x = 0.5 \Rightarrow 6 = 2k \Rightarrow k = 3$

$y = \dfrac{3}{1-x}$

7 $(1+x^2)\dfrac{dy}{dx} = x - xy^2$

$(1+x^2)\dfrac{dy}{dx} = x(1-y^2)$

$\int \dfrac{1}{1-y^2}\, dy = \int \dfrac{x}{1+x^2}\, dx$

$\dfrac{1}{2}\int\left(\dfrac{1}{1+y} + \dfrac{1}{1-y}\right) dy = \int \dfrac{x}{1+x^2}\, dx$

$\dfrac{1}{2}\left(\ln|1+y| - \ln|1-y|\right) = \dfrac{1}{2}\ln|1+x^2| + c$

$x = 0, y = 2 \Rightarrow c = \dfrac{1}{2}\ln 3$

$\left(\ln|1+y| - \ln|1-y|\right) = \ln|3(1+x^2)|$

$\dfrac{1+y}{y-1} = 3 + 3x^2$

$1 + y = 3y + 3x^2y - 3 - 3x^2$

$1 + 3 + 3x^2 = 3y + 3x^2y - y$

$3(x^2+1) + 1 = y(3(x^2+1) - 1)$

$y = \dfrac{3(1+x^2)+1}{3(1+x^2)-1}$

8 $\dfrac{dy}{dx} = xe^{-y}$

$\int e^y\, dy = \int x\, dx$

$e^y = \dfrac{x^2}{2} + c$

$x = 4, y = \ln 2 \Rightarrow 2 = 8 + c \Rightarrow c = -6$

$e^y = \dfrac{x^2 - 12}{2}$

$y = \ln\left|\dfrac{x^2-12}{2}\right|$

9 $\dfrac{dy}{dx} = \cos^2 y + \cos 2x \cos^2 y$

$\dfrac{dy}{dx} = \cos^2 y(1 + \cos 2x)$

$\int \sec^2 y\, dy = \int (1 + \cos 2x)\, dx$

$\tan y = x + \dfrac{1}{2}\sin 2x + c$

$x = \dfrac{\pi}{4}, y = \dfrac{\pi}{4} \Rightarrow 1 = \dfrac{\pi}{4} + \dfrac{1}{2}\sin\dfrac{\pi}{2} + c$

$c = \dfrac{1}{2} - \dfrac{\pi}{4} = \dfrac{2-\pi}{4}$

$\tan y = x + \dfrac{1}{2}\sin 2x + \dfrac{2-\pi}{4}$

10 $\dfrac{dy}{dx} = xy\sin x$

$\int \dfrac{1}{y}\, dy = \int x\sin x\, dx$

$\ln|y| = -x\cos x + \int \cos x\, dx$

$\ln|y| = -x\cos x + \sin x + c$

$x = \dfrac{\pi}{2}, y = 1 \Rightarrow c = -1$

$\ln|y| = -x\cos x + \sin x - 1$

11 a $I = \int \dfrac{3x+4}{x}\, dx$

$I = \int 3 + \dfrac{4}{x}\, dx = 3x + 4\ln x + c$

© Pearson Education Ltd 2017. Copying permitted for purchasing institution only. This material is not copyright free.

11 b $\dfrac{dy}{dx} = \dfrac{3x\sqrt{y} + 4\sqrt{y}}{x} = \sqrt{y}\,\dfrac{3x+4}{x}$

$\displaystyle\int \frac{1}{\sqrt{y}}\,dy = 3x + 4\ln x + c$ (from **a**)

$2\sqrt{y} = 3x + 4\ln x + c$

$x = 1, y = 16 \Rightarrow 8 = 3 + c \Rightarrow c = 5$

$\sqrt{y} = \dfrac{3}{2}x + 2\ln x + \dfrac{5}{2}$

$y = \left(\dfrac{3}{2}x + 2\ln x + \dfrac{5}{2}\right)^2$

12 a $\dfrac{8x-18}{(3x-8)(x-2)} = \dfrac{A}{3x-8} + \dfrac{B}{x-2}$

$8x - 18 = A(x-2) + B(3x-8)$

$x = 2 : -2 = -2B \Rightarrow B = 1$

$x = \dfrac{8}{3} : \dfrac{64}{3} - 18 = \dfrac{2}{3}A \Rightarrow A = 5$

$\dfrac{8x-18}{(3x-8)(x-2)} = \dfrac{5}{3x-8} + \dfrac{1}{x-2}$

b $(x-2)(3x-8)\dfrac{dy}{dx} = (8x-18)y$

$\displaystyle\int \frac{1}{y}\,dy = \int \frac{8x-18}{(3x-8)(x-2)}\,dx$

$\displaystyle\ln|y| = \int\left(\frac{5}{3x-8} + \frac{1}{x-2}\right)dx$

$\ln|y| = \dfrac{5}{3}\ln|3x-8| + \ln|x-2| + c$

c $x = 3, y = 8 \Rightarrow c = \ln 8$

$\ln|y| = \ln|3x-8|^{\frac{5}{3}} + \ln|8x-2|$

$\ln|y| = \ln\left|(3x-8)^{\frac{5}{3}}(8x-2)\right|$

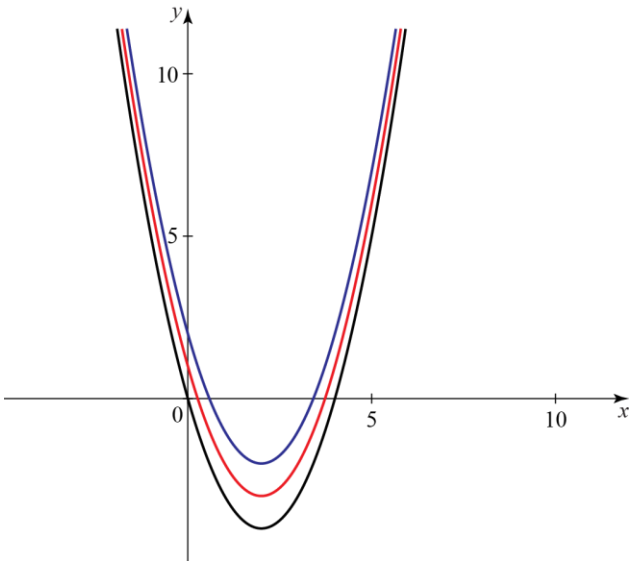
$y = (3x-8)^{\frac{5}{3}}(8x-2)$

$y = 8(x-2)(3x-8)^{\frac{5}{3}}$

13 a $\dfrac{dy}{dx} = 2x - 4$

$y = x^2 - 4x + c$

b



$y = x^2 - 4x, \quad y = x^2 - 4x + 1, \quad y = x^2 - 4x + 2$

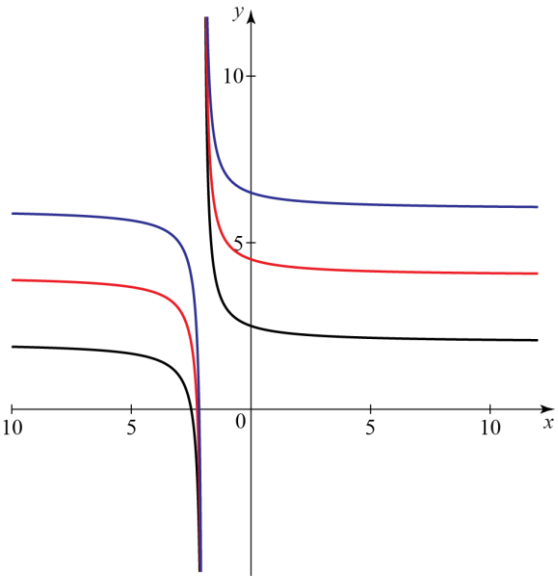
Any graphs of the form $y = x^2 - 4x + c$, where c is any real number.

14 a $\dfrac{dy}{dx} = -\dfrac{1}{(x+2)^2}$

$\displaystyle y = -\int \frac{1}{(x+2)^2}\,dx$

$y = \dfrac{1}{x+2} + c$

b



$y = \dfrac{1}{x+2} + 2, \quad y = \dfrac{1}{x+2} + 4, \quad y = \dfrac{1}{x+2} + 6$

Any graphs of the form $y = \dfrac{1}{x+2} + c$, where c is any real number.

© Pearson Education Ltd 2017. Copying permitted for purchasing institution only. This material is not copyright free.

14 c $3.1 = \frac{1}{10} + c \Rightarrow c = 3$

$y = \frac{1}{x+2} + 3$

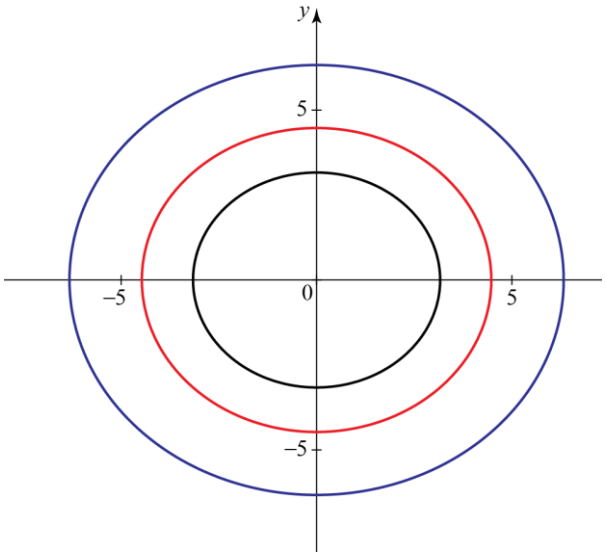
15 a $\frac{dy}{dx} = -\frac{x}{y}$

$\int y \, dy = \int x \, dx$

$\frac{y^2}{2} = -\frac{x^2}{2} + k$

$x^2 + y^2 = c$

15 b



$x^2 + y^2 = 10, \quad x^2 + y^2 = 20, \quad x^2 + y^2 = 40$

Circles with centre (0, 0) and radius $\sqrt{c}$ where c is any positive real number.

c $x^2 + y^2 = 49$

© Pearson Education Ltd 2017. Copying permitted for purchasing institution only. This material is not copyright free.