Integration 11H

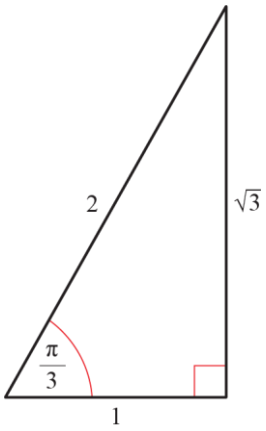
1 a $\text{Area} = \int_0^1 \frac{2}{1+x}\,dx = \left[2\ln|1+x|\right]_0^1$

$= (2\ln 2) - (2\ln 1)$

$\therefore$ Area = $2\ln 2$

b $\text{Area} = \int_0^{\frac{\pi}{3}} \sec x\,dx$

$= \left[\ln|\sec x + \tan x|\right]_0^{\frac{\pi}{3}}$



$= [\ln(2+\sqrt{3})] - [\ln(1)]$

$\therefore \text{Area} = \ln(2+\sqrt{3})$

c $\text{Area} = \int_1^2 \ln x\,dx$

$u = \ln x \Rightarrow \frac{du}{dx} = \frac{1}{x}$

$\frac{dv}{dx} = 1 \Rightarrow v = x$

$\therefore \text{Area} = \left[x\ln x\right]_1^2 - \int_1^2 x \times \frac{1}{x}\,dx$

$= (2\ln 2) - (0) - \left[x\right]_1^2$

$= 2\ln 2 - 1$

d $\text{Area} = \int_0^{\frac{\pi}{4}} \sec x\tan x\,dx$

$= \left[\sec x\right]_0^{\frac{\pi}{4}}$

$= (\sqrt{2}) - (1)$

$\therefore \text{Area} = \sqrt{2} - 1$

e $\text{Area} = \int_0^2 x\sqrt{4-x^2}\,dx$

Let $y = (4-x^2)^{\frac{3}{2}}$

$\Rightarrow \frac{dy}{dx} = \frac{3}{2}(4-x^2)^{\frac{1}{2}} \times (-2x) = -3x(4-x^2)^{\frac{1}{2}}$

$\therefore \text{Area} = \left[-\frac{1}{3}(4-x^2)^{\frac{3}{2}}\right]_0^2$

$= (0) - \left(-\frac{1}{3} \times 2^3\right) = \frac{8}{3}$

2 a $f(x) = \frac{4x-1}{(x+2)(2x+1)}$

$\frac{4x-1}{(x+2)(2x+1)} = \frac{A}{x+2} + \frac{B}{2x+1}$

$4x-1 = A(2x+1) + B(x+2)$

$x = -2 \Rightarrow -9 = -3A \Rightarrow A = 3$

$x = -\frac{1}{2} \Rightarrow -3 = \frac{3}{2}B \Rightarrow B = -2$

$f(x) = \frac{3}{x+2} - \frac{2}{2x+1}$

$\int_0^2 \left(\frac{3}{x+2} - \frac{2}{2x+1}\right) dx$

$= \left[3\ln(x+2) - \ln(2x+1)\right]_0^2$

$= 3\ln 4 - \ln 5 - 3\ln 2 + \ln 1$

$= \ln 64 - \ln 5 - \ln 8$

$= \ln\frac{8}{5}$

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2 b $\dfrac{x}{(x+1)^2} \equiv \dfrac{A}{(x+1)^2} + \dfrac{B}{x+1}$

$\Rightarrow x \equiv A + B(x+1)$

Compare coefficient of x: $1 = B \Rightarrow B = 1$

Compare constants: $0 = A + B \Rightarrow A = -1$

$\therefore$ area $= \displaystyle\int_0^2 \frac{x}{(x+1)^2}\,\mathrm{d}x$

$= \displaystyle\int_0^2 \left(\frac{1}{x+1} - \frac{1}{(x+1)^2}\right)\mathrm{d}x$

$= \left[\ln|x+1| + \dfrac{1}{x+1}\right]_0^2$

$= \left(\ln 3 + \dfrac{1}{3}\right) - (\ln 1 + 1)$

$= \ln 3 - \dfrac{2}{3}$

c Area $= \displaystyle\int_0^{\frac{\pi}{2}} x \sin x\,\mathrm{d}x$

$u = x \Rightarrow \dfrac{\mathrm{d}u}{\mathrm{d}x} = 1$

$\dfrac{\mathrm{d}v}{\mathrm{d}x} = \sin x \Rightarrow v = -\cos x$

$\therefore$ area $= \left[-x\cos x\right]_0^{\frac{\pi}{2}} - \displaystyle\int_0^{\frac{\pi}{2}} (-\cos x)\,\mathrm{d}x$

$= \left(-\dfrac{\pi}{2}\cos\dfrac{\pi}{2}\right) - (0) + \displaystyle\int_0^{\frac{\pi}{2}} \cos x\,\mathrm{d}x$

$= 0 + \left[\sin x\right]_0^{\frac{\pi}{2}}$

$= \left(\sin\dfrac{\pi}{2} - 0\right)$

$= 1$

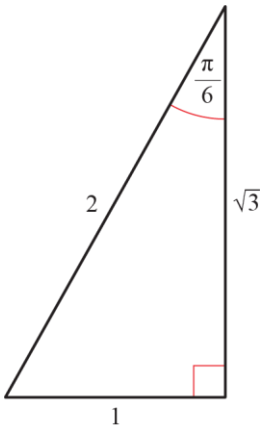
d Area $= \displaystyle\int_0^{\frac{\pi}{6}} \cos x\sqrt{2\sin x + 1}\,\mathrm{d}x$

Let $y = (2\sin x + 1)^{\frac{3}{2}}$

$\Rightarrow \dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{3}{2}(2\sin x + 1)^{\frac{1}{2}} \times 2\cos x$

$= 3\cos x(2\sin x + 1)^{\frac{1}{2}}$

$\therefore$ area $= \left[\dfrac{1}{3}(2\sin x + 1)^{\frac{3}{2}}\right]_0^{\frac{\pi}{6}}$



$= \left(\dfrac{1}{3}2^{\frac{3}{2}}\right) - \left(\dfrac{1}{3}1^{\frac{3}{2}}\right)$

$= \dfrac{2\sqrt{2}}{3} - \dfrac{1}{3}$

$= \dfrac{2\sqrt{2-1}}{3}$

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2 e $\text{Area} = \int_0^{1\ln 2} xe^{-x}dx$

$u = x \Rightarrow \frac{du}{dx} = 1$

$\frac{dv}{dx} = e^{-x} \Rightarrow v = e^{-x}$

$\therefore \text{area} = \left[-xe^{-x}\right]_0^{1\ln 2} - \int_0^{1\ln 2}(-e^{-x})dx$

$= (-1\ln 2 \times e^{-1\ln 2}) - (0) + \int_0^{1\ln 2} e^{-x}dx$

$= -1\ln 2 \times \frac{1}{2} + \left[-e^{-x}\right]_0^{1\ln 2}$

$= -\frac{1}{2}1\ln 2 + (-e^{-1\ln 2}) - (-e^{-0})$

$= -\frac{1}{2}1\ln 2 - \frac{1}{2} + 1$

$= \frac{1}{2}(1 - 1\ln 2)$

3 $\text{Area} = \int_1^2 \frac{4x+3}{(x+2)(2x-1)}\,dx$

$\frac{4x+3}{(x+2)(2x-1)} = \frac{A}{x+2} + \frac{B}{2x-1}$

$4x+3 = A(2x-1) + B(x+2)$

Let $x = -2 : -5 = -5A \Rightarrow A = 1$

Let $x = \frac{1}{2} : 5 = \frac{5}{2}B \Rightarrow B = 2$

$\text{Area} = \int_1^2 \left(\frac{1}{x+2} + \frac{2}{2x-1}\right)dx$

$= \left[\ln|x+2| + \ln|2x-1|\right]_1^2$

$= (\ln 4 + \ln 3) - (\ln 3 + \ln 1)$

$= \ln 4$

4 $\text{Area} = \int_2^4 \left(e^{0.5x} + \frac{1}{x}\right)dx$

$= \left[2e^{0.5x} + \ln|x|\right]_2^4$

$= (2e^2 + \ln 4) - (2e + \ln 2)$

$= 2e^2 - 2e + \ln\frac{4}{2}$

$= 2e^2 - 2e + \ln 2$

5 a $g(x) = 0 \Rightarrow A(0,0), B(\pi,0), C(2\pi,0)$

b $I = \int x\sin x\,dx$

Let $u = x \Rightarrow \frac{du}{dx} = 1$

$\frac{dv}{dx} = \sin x \Rightarrow v = -\cos x$

$I = -x\cos x + \int \cos x\,dx$

$= -x\cos x + \sin x + c$

Area between A and B:

$\left[-x\cos x + \sin x\right]_0^{\pi} = \pi$

Area between B and C:

$\left[-x\cos x + \sin x\right]_{\pi}^{2\pi} = -2\pi - \pi = -3\pi$

Total area = $\pi + 3\pi = 4\pi$

6 a $I = \int x^2 \ln x\,dx$

Let $u = \ln x \Rightarrow \frac{du}{dx} = \frac{1}{x}$

$\frac{dv}{dx} = x^2 \Rightarrow v = \frac{x^3}{3}$

$I = \frac{x^3}{3}\ln x - \int \frac{x^3}{3} \times \frac{1}{x}\,dx$

$= \frac{x^3}{3}\ln x - \frac{x^3}{9} + c$

b $x^2 \ln x = 0 \Rightarrow x = 0 \text{ or } 1$

Area between $x = 0$ and $x = 1$:

$\left[\frac{x^3}{3}\ln x - \frac{x^3}{9}\right]_0^1 = -\frac{1}{9}$

Area between $x = 1$ and $x = 2$:

$\left[\frac{x^3}{3}\ln x - \frac{x^3}{9}\right]_1^2 = \left(\frac{8}{3}\ln 2 - \frac{8}{9}\right) + \frac{1}{9}$

$= \frac{8}{3}\ln 2 - \frac{7}{9}$

Total area $= \frac{8}{3}\ln 2 - \frac{7}{9} + \frac{1}{9} = \frac{8}{3}\ln 2 - \frac{2}{3}$

$= \frac{2}{3}(4\ln 2 - 1)$

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7 a $y = 3\cos x\sqrt{\sin x + 1}$

Curve crosses the x axis when $y = 0$.

$\cos x = 0$ or $\sin x = -1$

$x = -\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}$

$A\left(-\frac{\pi}{2}, 0\right), B\left(\frac{\pi}{2}, 0\right), C\left(\frac{3\pi}{2}, 0\right)$

Curve crosses the y axis when $x = 0$.

So $D(0, 3)$.

b $I = \int 3\cos x\sqrt{\sin x + 1}\ \mathrm{d}x$

Let $u = \sin x + 1 \Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x} = \cos x$

$I = \int 3\sqrt{u}\ \mathrm{d}u = 2u^{\frac{3}{2}} + c$

$= 2(\sin x + 1)^{\frac{3}{2}} + c$

c $R_1 = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 3\cos x\sqrt{\sin x + 1}\ \mathrm{d}x$

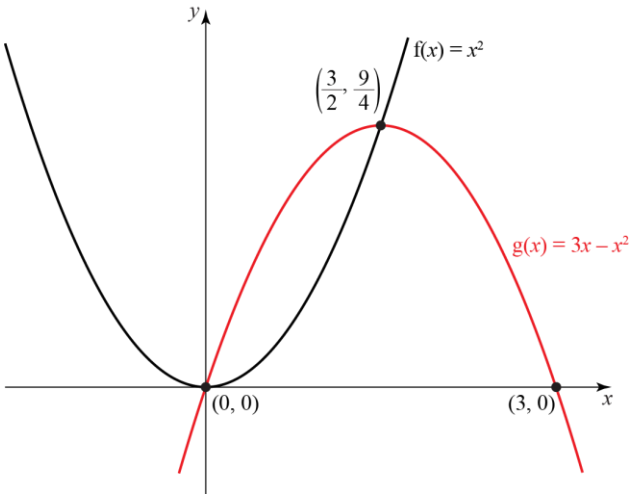
$\left[2(\sin x + 1)^{\frac{3}{2}}\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = 4\sqrt{2} = \sqrt{32}$

$R_2 = \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} 3\cos x\sqrt{\sin x + 1}\ \mathrm{d}x$

$\left[2(\sin x + 1)^{\frac{3}{2}}\right]_{\frac{\pi}{2}}^{\frac{3\pi}{2}} = -4\sqrt{2} = -\sqrt{32}$

$R_1 = R_2 = \sqrt{32}$, so $a = 32$.

8 a



$f(x) = g(x) \Rightarrow x^2 = 3x - x^2$

$2x^2 - 3x = 0$

$x(2x - 3) = 0$

$x = 0$ or $\frac{3}{2}$

$x = 0 \Rightarrow y = 0$

$x = \frac{3}{2} \Rightarrow y = \frac{9}{4}$

Points of intersection are

$(0, 0)$ and $\left(\frac{3}{2}, \frac{9}{4}\right)$

b Area under $f(x)$ between 0 and $\frac{3}{2}$:

$\int_0^{\frac{3}{2}} x^2\ \mathrm{d}x = \left[\frac{x^3}{3}\right]_0^{\frac{3}{2}} = \frac{27}{24} = \frac{9}{8}$

Area under $g(x)$ between 0 and $\frac{3}{2}$:

$\int_0^{\frac{3}{2}} 3x - x^2\ \mathrm{d}x = \left[\frac{3x^2}{2} - \frac{x^3}{3}\right]_0^{\frac{3}{2}}$

$= \frac{27}{8} - \frac{27}{24} = \frac{9}{4}$

Area between the two curves

$= \frac{9}{4} - \frac{9}{8} = \frac{9}{8}$

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9 a Points of intersection are when:

$2\cos x + 2 = -2\cos x + 4$

$4\cos x = 2$

$\cos x = \frac{1}{2} \Rightarrow x = -\frac{\pi}{3}, \frac{\pi}{3}$ or $\frac{5\pi}{3}$

$x = -\frac{\pi}{3}, \frac{\pi}{3}$ or $\frac{5\pi}{3} \Rightarrow y = 3$

$A\left(-\frac{\pi}{3}, 3\right), B\left(\frac{\pi}{3}, 3\right), C\left(\frac{5\pi}{3}, 3\right)$

b $\int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} (2\cos x + 2)\,\mathrm{d}x$

$= \left[2\sin x + 2x\right]_{-\frac{\pi}{3}}^{\frac{\pi}{3}}$

$= \left(\sqrt{3} + \frac{2\pi}{3}\right) - \left(-\sqrt{3} - \frac{2\pi}{3}\right) = 2\sqrt{3} + \frac{4\pi}{3}$

$= 2\sqrt{3} + \frac{4\pi}{3}$

$\int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} (-2\cos x + 4)\,\mathrm{d}x$

$= \left[-2\sin x + 4x\right]_{-\frac{\pi}{3}}^{\frac{\pi}{3}}$

$= \left(-\sqrt{3} + \frac{4\pi}{3}\right) - \left(\sqrt{3} - \frac{4\pi}{3}\right)$

$= -2\sqrt{3} + \frac{8\pi}{3}$

$R_1 = 2\sqrt{3} + \frac{4\pi}{3} - \left(-2\sqrt{3} + \frac{8\pi}{3}\right)$

$= 4\sqrt{3} - \frac{4\pi}{3}$

$a = 4, b = -4, c = 3$ (or $a = 4, b = 4, c = -3$)

c $\int_{\frac{\pi}{3}}^{\frac{5\pi}{3}} (2\cos x + 2)\,\mathrm{d}x$

$= \left[2\sin x + 2x\right]_{\frac{\pi}{3}}^{\frac{5\pi}{3}}$

$= \left(-\sqrt{3} + \frac{10\pi}{3}\right) - \left(\sqrt{3} + \frac{2\pi}{3}\right)$

$= -2\sqrt{3} + \frac{8\pi}{3}$

$\int_{\frac{\pi}{3}}^{\frac{5\pi}{3}} (-2\cos x + 4)\,\mathrm{d}x$

$= \left[-2\sin x + 4x\right]_{\frac{\pi}{3}}^{\frac{5\pi}{3}}$

$= \left(\sqrt{3} + \frac{20\pi}{3}\right) - \left(-\sqrt{3} + \frac{4\pi}{3}\right)$

$= 2\sqrt{3} + \frac{16\pi}{3}$

$R_2 = \left(2\sqrt{3} + \frac{16\pi}{3}\right) - \left(-2\sqrt{3} + \frac{8\pi}{3}\right)$

$= 4\sqrt{3} + \frac{8\pi}{3}$

$R_2 : R_1 = 4\sqrt{3} + \frac{8\pi}{3} : 4\sqrt{3} - \frac{4\pi}{3}$

$= \left(3\sqrt{3} + 2\pi\right) : \left(3\sqrt{3} - \pi\right)$

10 $y = \sin\theta$

Area under curve $= 2\int_0^{\pi} \sin\theta\,\mathrm{d}\theta$ because of the symmetry of the curve.

$= 2\left[-\cos\theta\right]_0^{\pi} = 2 + 2 = 4$

$y = \sin 2\theta$

Area under curve $= 4\int_0^{\frac{\pi}{2}} \sin 2\theta\,\mathrm{d}\theta$ because of the symmetry of the curve.

$= 4\left[-\frac{1}{2}\cos 2\theta\right]_0^{\frac{\pi}{2}} = 2 + 2 = 4$

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11 a At A, $\cos x = \sin x$

$\tan x = 1 \Rightarrow x = \frac{\pi}{4} \Rightarrow y = \frac{1}{\sqrt{2}}$

$A\left(\frac{\pi}{4}, \frac{1}{\sqrt{2}}\right)$

b i $R_1 =$ Area under $y = \cos x$ − Area under $y = \sin x$ between $x = 0$ and $x = \frac{\pi}{4}$

$R_1 = \int_0^{\frac{\pi}{4}} \cos x \, dx - \int_0^{\frac{\pi}{4}} \sin x \, dx$

$= \left[\sin x + \cos x\right]_0^{\frac{\pi}{4}}$

$= \frac{2}{\sqrt{2}} - 1 = \sqrt{2} - 1$

ii $R_2 = 2 \times$ Area under $y = \sin x$ between $x = 0$ and $x = \frac{\pi}{4}$

$R_2 = 2\int_0^{\frac{\pi}{4}} -\cos x \, dx$

$= 2\left(1 - \frac{1}{\sqrt{2}}\right) = 2 - \sqrt{2}$

iii $R_3 = \int_{\frac{\pi}{4}}^{\pi} \sin x \, dx - \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cos x \, dx$

$= \left[-\cos x\right]_{\frac{\pi}{4}}^{\pi} - \left[\sin x\right]_{\frac{\pi}{4}}^{\frac{\pi}{2}}$

$= 1 + \frac{1}{\sqrt{2}} - \left(1 - \frac{1}{\sqrt{2}}\right) = \sqrt{2}$

c $R_1 : R_2 = \left(\sqrt{2} - 1\right) : \left(2 - \sqrt{2}\right)$

$= \left(\sqrt{2} - 1\right)\left(2 + \sqrt{2}\right) : \left(2 - \sqrt{2}\right)\left(2 + \sqrt{2}\right)$

$= \sqrt{2} : 2$

Challenge

Curves intersect at $\sin 2x = \cos x$

$2\sin x \cos x = \cos x$

$\sin x = \frac{1}{2}, \cos x \neq 0$

$x = \frac{\pi}{6}$

Shaded area

$= \int_0^{\frac{\pi}{6}} \left(\cos x - \sin 2x\right) \, dx + \int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \left(\sin 2x - \cos x\right) \, dx$

$= \left[\sin x + \frac{1}{2}\cos 2x\right]_0^{\frac{\pi}{6}} + \left[-\frac{1}{2}\cos 2x - \sin x\right]_{\frac{\pi}{6}}^{\frac{\pi}{4}}$

$= \left(\frac{1}{2} + \frac{1}{4}\right) - \left(0 + \frac{1}{2}\right) + \left(0 - \frac{1}{\sqrt{2}}\right) - \left(-\frac{1}{4} - \frac{1}{2}\right)$

$= 1 - \frac{1}{\sqrt{2}} = \frac{\sqrt{2} - 1}{\sqrt{2}}$

$= \frac{2 - \sqrt{2}}{2}$

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