Integration 13G

1 a A, B are given by $6 = x^2 + 2$

$$x^2 = 4$$
$$x = \pm 2$$

So A is $(-2, 6)$ and B is $(2, 6)$.

b $\text{Area} = \int_{-2}^{2} \left(6 - \left(x^2 + 2\right)\right) \mathrm{d}x$

$$= \int_{-2}^{2} \left(4 - x^2\right) \mathrm{d}x$$
$$= \left(4x - \frac{x^3}{3}\right)_{-2}^{2}$$
$$= \left(8 - \tfrac{8}{3}\right) - \left(-8 + \tfrac{8}{3}\right)$$
$$= 16 - 2 \times \tfrac{16}{3}$$
$$= 5\tfrac{1}{3}$$

2 a A, B are given by $3 = 4x - x^2$

$$x^2 - 4x + 3 = 0$$
$$(x - 3)(x - 1) = 0$$
$$x = 1, 3$$

So A is $(1, 3)$ and B is $(3, 3)$.

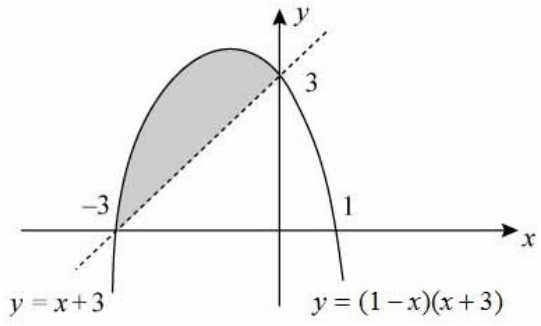
b $\text{Area} = \int_{1}^{3} \left(\left(4x - x^2\right) - 3\right) \mathrm{d}x$

$$= \int_{1}^{3} \left(4x - x^2 - 3\right) \mathrm{d}x$$
$$= \left(2x^2 - \frac{x^3}{3} - 3x\right)_{1}^{3}$$
$$= (18 - 9 - 9) - \left(2 - \tfrac{1}{3} - 3\right)$$
$$= 1\tfrac{1}{3}$$

3 $\text{Area} = \int_{-1}^{1} (\text{curve} - \text{line}) \mathrm{d}x$

$$= \int_{-1}^{1} \left(9 - 3x - 5x^2 - x^3 - (4 - 4x)\right) \mathrm{d}x$$
$$= \int_{-1}^{1} \left(5 + x - 5x^2 - x^3\right) \mathrm{d}x$$
$$= \left(5x + \frac{x^2}{2} - \frac{5}{3}x^3 - \frac{x^4}{4}\right)_{-1}^{1}$$
$$= \left(5 + \tfrac{1}{2} - \tfrac{5}{3} - \tfrac{1}{4}\right) - \left(-5 + \tfrac{1}{2} + \tfrac{5}{3} - \tfrac{1}{4}\right)$$
$$= 10 - \tfrac{10}{3}$$
$$= \tfrac{20}{3} \text{ or } 6\tfrac{2}{3}$$

4 $y = (1 - x)(x + 3)$ is $\cap$ shaped and crosses the x-axis at (1, 0) and (−3, 0).
$y = x + 3$ is a straight line passing through (−3, 0) and (0, 3)



Intersections occur when
$x + 3 = (1 - x)(x + 3)$

$$0 = (x + 3)(1 - x - 1)$$
$$0 = -x(x + 3)$$
$$x = -3 \text{ or } x = 0$$

$\text{Area} = \int_{-3}^{0} \left((1 - x)(x + 3) - (x + 3)\right) \mathrm{d}x$

$$= \int_{-3}^{0} \left(-x^2 - 3x\right) \mathrm{d}x$$
$$= \left(-\frac{x^3}{3} - \frac{3x^2}{2}\right)_{-3}^{0}$$
$$= (0) - \left(\tfrac{27}{3} - \tfrac{27}{2}\right)$$
$$= \tfrac{27}{6}$$
$$= \tfrac{9}{2} \text{ or } 4\tfrac{1}{2}$$

5 a A is given by $x(4 + x) = 12$

$$x^2 + 4x - 12 = 0$$
$$(x + 6)(x - 2) = 0$$
$$x = 2 \text{ or } x = -6$$

So A is (2, 12)

b R is found by $\int_{0}^{2} x(4 + x) \mathrm{d}x$ away from a rectangle of area 12 × 2 = 24.

So area of $R = 24 - \int_{0}^{2} \left(x^2 + 4x\right) \mathrm{d}x$

$$= 24 - \left(\frac{x^3}{3} + 2x^2\right)_{0}^{2}$$

© Pearson Education Ltd 2017. Copying permitted for purchasing institution only. This material is not copyright free.

5 b Area of $R = 24 - \left\{\left(\frac{8}{3}+8\right) - (0)\right\}$

$= 24 - \frac{32}{3}$

$= \frac{40}{3}$ or $13\frac{1}{3}$

6 a Intersections occur when $7 - x = x^2 + 1$

$0 = x^2 + x - 6$

$0 = (x+3)(x-2)$

$x = 2$ or -3

Area of R_1, is given by

$\int_{-3}^{2}\left(7 - x - \left(x^2+1\right)\right)\,\mathrm{d}x$

$= \int_{-3}^{2}\left(6 - x - x^2\right)\,\mathrm{d}x$

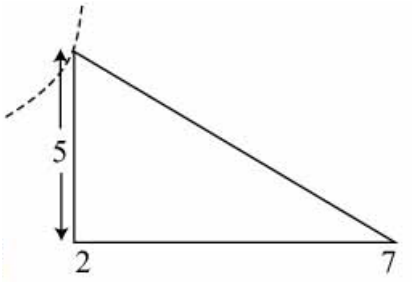
$= \left(6x - \frac{x^2}{2} - \frac{x^3}{3}\right)_{-3}^{2}$

$= \left(12 - \frac{4}{2} - \frac{8}{3}\right) - \left(-18 - \frac{9}{2} + \frac{27}{3}\right)$

$= 20\frac{5}{6}$

b Area of R_2, is given by

$\int_{0}^{2}\left(x^2+1\right)\,\mathrm{d}x$ + area of the triangle.



Area of $R_2 = \left(\frac{x^3}{3} + x\right)_0^2 + \frac{1}{2}\times 5\times 5$

$= \left(\frac{8}{3}+2\right) - (0) + \frac{25}{2}$

$= 17\frac{1}{6}$

7 a When $x = 1$, $y = 1 - \frac{2}{1} + 1$

$= 0$

So $(1, 0)$ lies on C.

b When $x = 8$, $y = 8^{\frac{2}{3}} - \frac{2}{8^{\frac{1}{3}}} + 1$

$= 2^2 - \frac{2}{2} + 1$

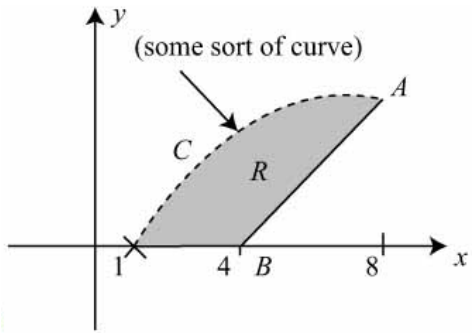
$= 4$

So $(8, 4)$ lies on C.

7 c A is the point (8, 4) and B is the point (4, 0).

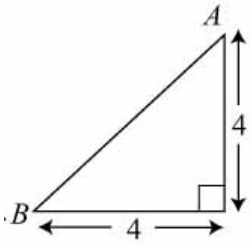
Gradient of line through AB is $\frac{4-0}{8-4} = 1$.

So the equation is $y - 0 = x - 4$ or $y = x - 4$



d Area of R is given by

$\int_{1}^{8}(\text{curve})\,\mathrm{d}x$ – area of the triangle.



Area $R = \int_1^8 \left(x^{\frac{2}{3}} - \frac{2}{x^{\frac{1}{3}}} + 1\right)\,\mathrm{d}x - \frac{1}{2}\times 4\times 4$

$= \left(\frac{3}{5}x^{\frac{5}{3}} - \frac{2x^{\frac{2}{3}}}{\frac{2}{3}} + x\right)_1^8 - 8$

$= \left(\frac{3}{5}\times 32 - 3\times 4 + 8\right)$

$\quad -\left(\frac{3}{5} - 3 + 1\right) - 8$

$= \frac{96}{5} - 4 - \frac{7}{5} - 8$

$= \frac{29}{5}$

$= 5\frac{4}{5}$

8 Area $= \int_{\frac{1}{2}}^{2}\left(\text{line } AB - \left(\frac{2}{x^2} + x\right)\right)\,\mathrm{d}x$

Substitute $\frac{1}{2}$ and 2 for x into the equation to find

A is $\left(\frac{1}{2}, 8\frac{1}{2}\right)$ and B is $\left(2, 2\frac{1}{2}\right)$.

The gradient of $AB = \frac{6}{-1\frac{1}{2}} = -4$

© Pearson Education Ltd 2017. Copying permitted for purchasing institution only. This material is not copyright free.

8 So the equation is $y-2\frac{1}{2}=-4(x-2)$

$$y=10\tfrac{1}{2}-4x$$

$$\text{Area}=\int_{\frac{1}{2}}^{2}\left(10\tfrac{1}{2}-5x-2x^{-2}\right)\mathrm{d}x$$

$$=\left(\frac{21}{2}x-\frac{5}{2}x^2-\frac{2x^{-1}}{-1}\right)_{\frac{1}{2}}^{2}$$

$$=\left(\frac{21}{2}x-\frac{5}{2}x^2+\frac{2}{x}\right)_{\frac{1}{2}}^{2}$$

$$=(21-10+1)-\left(\tfrac{21}{4}-\tfrac{5}{8}+4\right)$$

$$=12-8\tfrac{5}{8}$$

$$=3\tfrac{3}{8}\text{ or }3.375$$

$$=3.38\,(3\text{ s.f.})$$

9 a On the line, when $x=4$, $y=4-\frac{1}{2}\times 4$

$$=2$$

On the curve, when $x=4$,

$$y=3\times\sqrt{4}-\sqrt{64}+4$$

$$=6-8+4$$

$$=2$$

So the point (4, 2) lies on the line and the curve.

b $$\text{Area}=\int_0^4\left(3x^{\frac{1}{2}}-x^{\frac{3}{2}}+4-\left(4-\tfrac{1}{2}x\right)\right)\mathrm{d}x$$

$$=\int_0^4\left(3x^{\frac{1}{2}}-x^{\frac{3}{2}}+\tfrac{1}{2}x\right)\mathrm{d}x$$

$$=\left(\frac{3x^{\frac{3}{2}}}{\frac{3}{2}}-\frac{x^{\frac{5}{2}}}{\frac{5}{2}}+\frac{x^2}{4}\right)_0^4$$

$$=\left(2x^{\frac{3}{2}}-\tfrac{2}{5}x^{\frac{5}{2}}+\tfrac{1}{4}x^2\right)_0^4$$

$$=\left(2\times 8-\tfrac{2}{5}\times 32+4\right)-(0)$$

$$=20-\tfrac{64}{5}$$

$$=\tfrac{36}{5}\text{ or }7.2$$

10 a $y=x^2(x+4)$

$y=0\Rightarrow x=0\,(\text{twice}),-4$

10 a Area of R_1 is

$$\int_{-4}^{0}\left(x^3+4x^2\right)\mathrm{d}x=\left(\frac{x^4}{4}+\frac{4}{3}x^3\right)_{-4}^{0}$$

$$=(0)-\left(\frac{4^4}{4}-\frac{4^4}{3}\right)$$

$$=\frac{4^4}{12}$$

$$=\frac{4^3}{3}$$

$$=\tfrac{64}{3}\text{ or }21\tfrac{1}{3}$$

b Area of R_2 is $\int_0^2\left(x^3+4x^2\right)\mathrm{d}x$ + area of the triangle.

$$\text{Area of }R_2=\left(\frac{x^4}{4}+\frac{4}{3}x^3\right)_0^2+12(b-2)$$

$$=\left(\tfrac{16}{4}+\tfrac{32}{3}\right)-(0)+12(b-2)$$

$$=14\tfrac{2}{3}+12b-24$$

$$=-9\tfrac{1}{3}+12b$$

Area of R_2 = area of R_1

$\Rightarrow -9\frac{1}{3}+12b=21\frac{1}{3}$

So $12b=30\frac{2}{3}$

$\Rightarrow b=2\frac{5}{9}$ or $2.56\,(3\text{ s.f.})$

11 a The intersections occur when

$$10-x=2x^2-5x+4$$

$$0=2x^2-4x-6$$

$$0=2(x+1)(x-3)$$

$$x=-1\text{ or }x=3$$

When $x=-1$, $y=11$, A is $(-1, 11)$.
When $x=3$, $y=7$, B is $(3, 7)$.

b $$\text{Area}=\int_{-1}^{3}\left[(10-x)-\left(2x^2-5x+4\right)\right]\mathrm{d}x$$

$$=\int_{-1}^{3}(10-x-2x^2+5x-4)\,\mathrm{d}x$$

$$=\int_{-1}^{3}(6+4x-2x^2)\,\mathrm{d}x$$

$$=\left[6x+2x^2-\tfrac{2}{3}x^3\right]_{-1}^{3}$$

$$=(18+18-18)-\left(-6+2+\tfrac{2}{3}\right)$$

$$=18+3\tfrac{1}{3}$$

$$=21\tfrac{1}{3}$$

© Pearson Education Ltd 2017. Copying permitted for purchasing institution only. This material is not copyright free.