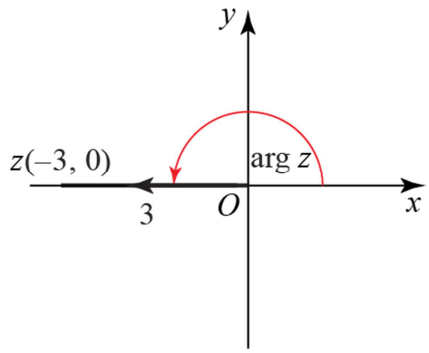
Complex numbers 1A

1 a -3

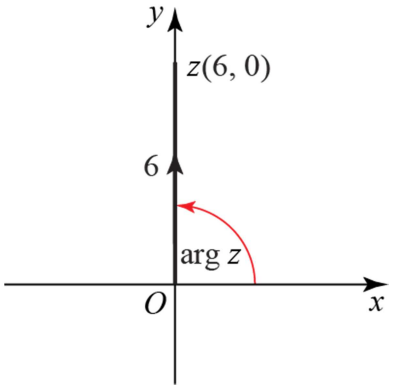


$r = 3$

$\theta = \arg z = \pi$

$\therefore -3 = 3e^{\pi i}$

b $6i$



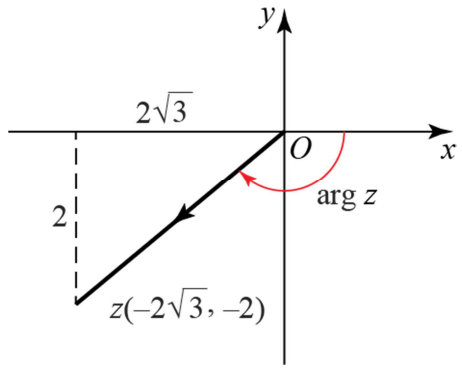
$r = 6$

$\theta = \arg z = \frac{\pi}{2}$

$\therefore \ 6i = 6e^{\frac{\pi i}{2}}$

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1 c $-2\sqrt{3}-2\mathrm{i}$

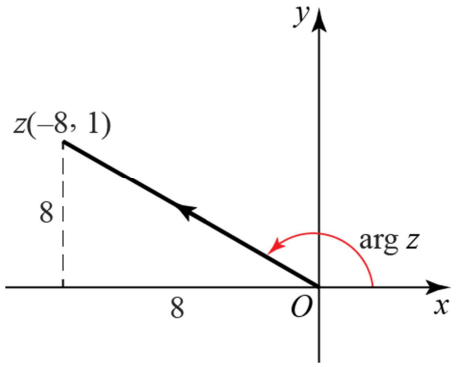


$r=\sqrt{(-2\sqrt{3})^2+(-2)^2}=\sqrt{12+4}=\sqrt{16}=4$

$\theta=\arg z=-\pi+\tan^{-1}\left(\frac{2}{2\sqrt{3}}\right)=-\pi+\frac{\pi}{6}=-\frac{5\pi}{6}$

$\therefore\ -2\sqrt{3}-2\mathrm{i}=4\mathrm{e}^{\frac{-5\pi\mathrm{i}}{6}}$

d $-8+\mathrm{i}$



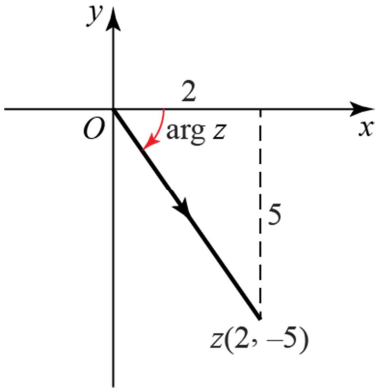
$r=\sqrt{(-8)^2+1^2}=\sqrt{65}$

$\theta=\pi-\tan^{-1}\left(\frac{1}{8}\right)=3.02^{\mathrm{c}}\,(2\,\mathrm{d.p.})$

$\therefore -8+\mathrm{i}=\sqrt{65}\,\mathrm{e}^{3.02\mathrm{i}}$

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1 e $2-5\mathrm{i}$

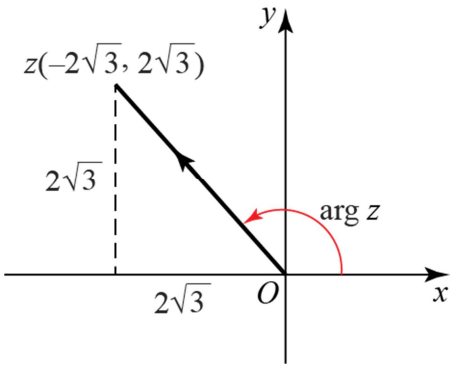


$r=\sqrt{2^2+(-5)^2}=\sqrt{29}$

$\theta=-\tan^{-1}\left(\frac{5}{2}\right)=-1.19^{\mathrm{c}}\ (2\,\mathrm{d.p.})$

$\therefore 2-5\mathrm{i}=\sqrt{29}\,\mathrm{e}^{-1.19\mathrm{i}}$

1 f $-2\sqrt{3}+2\sqrt{3}\mathrm{i}$



$r=\sqrt{(-2\sqrt{3})^2+(2\sqrt{3})^2}=\sqrt{12+12}=\sqrt{24}$

$=\sqrt{4}\sqrt{6}=2\sqrt{6}$

$\theta=\pi-\tan^{-1}\left(\frac{2\sqrt{3}}{2\sqrt{3}}\right)=\pi-\frac{\pi}{4}=\frac{3\pi}{4}$

$\therefore -2\sqrt{3}+2\sqrt{3}\,\mathrm{i}=2\sqrt{6}\,\mathrm{e}^{\frac{3\pi\mathrm{i}}{4}}$

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1 g $\sqrt{8}\left(\cos\frac{\pi}{4}+\mathrm{i}\sin\frac{\pi}{4}\right)$

$=2\sqrt{2}\,\mathrm{e}^{\frac{\pi\mathrm{i}}{4}}$

$r=\sqrt{8}=\sqrt{4}\sqrt{2}=2\sqrt{2}$

$\theta=\frac{\pi}{4}$

h $8\left(\cos\frac{\pi}{6}-\mathrm{i}\sin\frac{\pi}{6}\right)$

$=8\left(\cos\left(-\frac{\pi}{6}\right)+\mathrm{i}\sin\left(-\frac{\pi}{6}\right)\right)$

$=8\mathrm{e}^{-\frac{\pi\mathrm{i}}{6}}$

$r=8,\ \theta=-\frac{\pi}{6}$

i $2\left(\cos\frac{\pi}{5}-\mathrm{i}\sin\frac{\pi}{5}\right)$

$=2\left(\cos\left(-\frac{\pi}{5}\right)+\mathrm{i}\sin\left(-\frac{\pi}{5}\right)\right)$

$=2\mathrm{e}^{\frac{-\pi\mathrm{i}}{5}}$

$r=2,\ \theta=-\frac{\pi}{5}$

2 a $\mathrm{e}^{\frac{\pi\mathrm{i}}{3}}=\cos\frac{\pi}{3}+\mathrm{i}\sin\frac{\pi}{3}$

$=\frac{1}{2}+\frac{\sqrt{3}}{2}\mathrm{i}$

b $4\mathrm{e}^{\pi\mathrm{i}}=4(\cos\pi+\mathrm{i}\sin\pi)$

$=4(-1+\mathrm{i}(0))$

$=-4$

c $3\sqrt{2}\,\mathrm{e}^{\frac{\pi\mathrm{i}}{4}}=3\sqrt{2}\left(\cos\frac{\pi}{4}+\mathrm{i}\sin\frac{\pi}{4}\right)$

$=3\sqrt{2}\left(\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}\mathrm{i}\right)$

$=3+3\mathrm{i}$

2 d $8\mathrm{e}^{\frac{\pi\mathrm{i}}{6}}=8\left(\cos\frac{\pi}{6}+\mathrm{i}\sin\frac{\pi}{6}\right)$

$=8\left(\frac{\sqrt{3}}{2}+\frac{1}{2}\mathrm{i}\right)$

$=4\sqrt{3}+4\mathrm{i}$

e $3\mathrm{e}^{-\frac{\pi\mathrm{i}}{2}}=3\left(\cos\left(-\frac{\pi}{2}\right)+\mathrm{i}\sin\left(-\frac{\pi}{2}\right)\right)$

$=3(0-\mathrm{i})$

$=-3\mathrm{i}$

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2 f $e^{\frac{5\pi i}{6}} = \cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}$

$= -\frac{\sqrt{3}}{2} + \frac{1}{2}i$

g $e^{-\pi i} = \cos(-\pi) + i\sin(-\pi)$

$= -1 + i(0)$

$= -1$

h $3\sqrt{2}e^{-\frac{3\pi}{4}i} = 3\sqrt{2}\left(\cos\left(-\frac{3\pi}{4}\right) + i\sin\left(-\frac{3\pi}{4}\right)\right)$

$= 3\sqrt{2}\left(-\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i\right)$

$= -3 - 3i$

i $8e^{-\frac{4\pi i}{3}} = 8\left(\cos\left(-\frac{4\pi}{3}\right) + i\sin\left(-\frac{4\pi}{3}\right)\right)$

$= 8\left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)$

$= -4 + 4\sqrt{3}i$

3 a $e^{\frac{16\pi i}{13}} = \cos\left(\frac{16\pi}{13}\right) + i\sin\left(\frac{16\pi}{13}\right)$

$= \cos\left(-\frac{10\pi}{13}\right) + i\sin\left(-\frac{10\pi}{13}\right)$

-2π from the argument.

b $4e^{\frac{17\pi i}{5}} = 4\left(\cos\left(\frac{17\pi}{5}\right) + i\sin\left(\frac{17\pi}{5}\right)\right)$

$= 4\left(\cos\left(\frac{7\pi}{5}\right) + i\sin\left(\frac{7\pi}{5}\right)\right)$

$= 4\left(\cos\left(-\frac{3\pi}{5}\right) + i\sin\left(-\frac{3\pi}{5}\right)\right)$

c $5e^{-\frac{9\pi i}{8}} = 5\left(\cos\left(-\frac{9\pi}{8}\right) + i\sin\left(-\frac{9\pi}{8}\right)\right)$

$= 5\left(\cos\left(\frac{7\pi}{8}\right) + i\sin\left(\frac{7\pi}{8}\right)\right)$

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4 $e^{i\theta} = \cos\theta + i\sin\theta$ **(1)**

$e^{-i\theta} = \cos(-\theta) + i\sin(-\theta) = \cos\theta - i\sin\theta$ **(2)**

(1) – (2) $\Rightarrow e^{i\theta} - e^{-i\theta} = 2i\sin\theta$

$\frac{1}{2i}(e^{i\theta} - e^{-i\theta}) = \sin\theta$

$\therefore \sin\theta = \frac{1}{2i}(e^{i\theta} - e^{-i\theta})$ (as required)

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Complex numbers 1B

1 a $e^{\frac{\pi i}{3}} \times e^{\frac{\pi i}{4}} = e^{\pi i\left(\frac{1}{3}+\frac{1}{4}\right)} = e^{\frac{7\pi i}{12}}$

$= \cos\left(\frac{7\pi}{12}\right) + i\sin\left(\frac{7\pi}{12}\right)$

b

$\sqrt{5}e^{i\theta} \times 3e^{3i\theta} = 3\sqrt{5}e^{4i\theta}$

$= 3\sqrt{5}\left(\cos 4\theta + i\sin 4\theta\right)$

c

$\sqrt{2}e^{\frac{2\pi i}{3}} \times e^{-\frac{7\pi i}{3}} \times 3e^{\frac{\pi i}{6}}$

$= 3\sqrt{2}e^{\pi i\left(\frac{2}{3}-\frac{7}{3}+\frac{1}{6}\right)} = 3\sqrt{2}e^{-\frac{9\pi i}{6}}$

$= 3\sqrt{2}\left(\cos\frac{-3\pi}{2} + i\sin\frac{-3\pi}{2}\right) = 3\sqrt{2}i$

2 a

$\frac{2e^{\frac{7\pi i}{2}}}{8e^{\frac{9\pi i}{2}}} = \frac{2}{8}e^{\frac{7\pi i}{2}-\frac{9\pi i}{2}} = \frac{1}{4}e^{-\pi i}$

$= \frac{1}{4}\left(\cos(-\pi) + i\sin(-\pi)\right) = -\frac{1}{4}$

b

$\frac{\sqrt{3}e^{\frac{3\pi i}{7}}}{4e^{-\frac{2\pi i}{7}}} = \frac{\sqrt{3}}{4}e^{\frac{3\pi i}{7}+\frac{2\pi i}{7}} = \frac{\sqrt{3}}{4}e^{\frac{5\pi i}{7}}$

$= \frac{\sqrt{3}}{4}\left(\cos\frac{5\pi}{7} + i\sin\frac{5\pi}{7}\right)$

c

$\frac{\sqrt{2}e^{-\frac{15\pi i}{6}}}{2e^{\frac{\pi i}{3}}} \times \sqrt{2}e^{\frac{19\pi i}{3}} = \frac{\sqrt{2}\times\sqrt{2}}{2}e^{-\frac{15\pi i}{6}-\frac{\pi i}{3}+\frac{19\pi i}{3}}$

$= e^{\frac{21\pi i}{6}} = e^{\frac{7\pi i}{2}} = e^{\frac{3\pi i}{2}} = -i$

3 a

$\left(\cos 2\theta + i\sin 2\theta\right)(\cos 3\theta + i\sin 3\theta)$

$= e^{2i\theta} \times e^{3i\theta} = e^{5i\theta}$

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3 b

$$\left(\cos\frac{3\pi}{11}+i\sin\frac{3\pi}{11}\right)\left(\cos\frac{8\pi}{11}+i\sin\frac{8\pi}{11}\right)$$
$$=\mathrm{e}^{\frac{3\pi i}{11}}\times\mathrm{e}^{\frac{8\pi i}{11}}=\mathrm{e}^{\pi i}$$

c

$$3\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right)\times 2\left(\cos\frac{\pi}{12}+i\sin\frac{\pi}{12}\right)$$
$$=3\mathrm{e}^{\frac{\pi i}{4}}\times 2\mathrm{e}^{\frac{\pi i}{12}}=6\mathrm{e}^{\frac{3\pi i}{12}+\frac{\pi i}{12}}=6\mathrm{e}^{\frac{\pi i}{3}}$$

d

$$\sqrt{6}\left(\cos\frac{-\pi}{12}+i\sin\frac{-\pi}{12}\right)\times\sqrt{3}\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right)$$
$$=\sqrt{6}\mathrm{e}^{-\frac{\pi i}{12}}\times\sqrt{3}\mathrm{e}^{\frac{\pi i}{3}}=\sqrt{18}\mathrm{e}^{-\frac{\pi i}{12}+\frac{4\pi i}{12}}=3\sqrt{2}\mathrm{e}^{\frac{\pi i}{4}}$$

4 a

$$\frac{\cos 5\theta+i\sin 5\theta}{\cos 2\theta+i\sin 2\theta}=\frac{\mathrm{e}^{5i\theta}}{\mathrm{e}^{2i\theta}}=\mathrm{e}^{5i\theta-2i\theta}=\mathrm{e}^{3i\theta}$$

b

$$\frac{\sqrt{2}\left(\cos\frac{\pi}{2}+i\sin\frac{\pi}{2}\right)}{\frac{1}{2}\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right)}=2\sqrt{2}\,\frac{\cos\frac{\pi}{2}+i\sin\frac{\pi}{2}}{\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}}=2\sqrt{2}\,\frac{\mathrm{e}^{\frac{\pi i}{2}}}{\mathrm{e}^{\frac{\pi i}{4}}}$$
$$=2\sqrt{2}\mathrm{e}^{\frac{\pi i}{2}-\frac{\pi i}{4}}=2\sqrt{2}\mathrm{e}^{\frac{\pi i}{4}}$$

c

$$\frac{3\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right)}{4\left(\cos\frac{5\pi}{6}+i\sin\frac{5\pi}{6}\right)}=\frac{3}{4}\times\frac{\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}}{\cos\frac{5\pi}{6}+i\sin\frac{5\pi}{6}}=\frac{3}{4}\times\frac{\mathrm{e}^{\frac{\pi i}{3}}}{\mathrm{e}^{\frac{5\pi i}{6}}}$$
$$=\frac{3}{4}\mathrm{e}^{\frac{\pi i}{3}-\frac{5\pi i}{6}}=\frac{3}{4}\mathrm{e}^{-\frac{\pi i}{2}}$$

5 a We have $z=-9+3\sqrt{3}i$ so that $|z|=\sqrt{81+27}=\sqrt{108}=6\sqrt{3}$

if $\arg z=\theta$ then we have that

$$\tan\theta=-\frac{3\sqrt{3}}{9}=-\frac{\sqrt{3}}{3}$$

Hence

$$\theta=\frac{5\pi}{6}$$

So $z=6\sqrt{3}\mathrm{e}^{\frac{5\pi i}{6}}$

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5 b We have $|w| = \sqrt{3}$ and $\arg w = \dfrac{7\pi}{12}$ so by definition we have

$w = \sqrt{3}\mathrm{e}^{\frac{7\pi i}{12}}$

c $zw = 6\sqrt{3}\mathrm{e}^{\frac{5\pi i}{6}} \times \sqrt{3}\mathrm{e}^{\frac{7\pi i}{12}} = 18\mathrm{e}^{\frac{17\pi i}{12}} = 18\mathrm{e}^{-\frac{7\pi i}{12}}$

d $\dfrac{z}{w} = \dfrac{6\sqrt{3}\mathrm{e}^{\frac{5\pi i}{6}}}{\sqrt{3}\mathrm{e}^{\frac{7\pi i}{12}}} = 6\mathrm{e}^{\frac{5\pi i}{6} - \frac{7\pi i}{12}} = 6\mathrm{e}^{\frac{\pi i}{4}}$

6 We have

$$\frac{(\cos 9\theta + i\sin 9\theta)(\cos 4\theta + i\sin 4\theta)}{\cos 7\theta + i\sin 7\theta} = \frac{\mathrm{e}^{9i\theta} \times \mathrm{e}^{4i\theta}}{\mathrm{e}^{7i\theta}}$$

$$= \mathrm{e}^{9i\theta + 4i\theta - 7i\theta} = \mathrm{e}^{6i\theta} = \cos 6\theta + i\sin 6\theta$$

7 $z = 1 + i\sqrt{3}$ so $|z| = \sqrt{1+3} = 2$ and if $\arg z = \theta$ then we have

$\tan\theta = \sqrt{3}$ so that $\theta = \dfrac{\pi}{3}$

Now the equation $\left|\dfrac{z^2}{w}\right| = |z|$ implies that $|w| = |z|$ hence it only remains to find the possible values of $\arg w = \varphi$ we have that

$\mathrm{Re}\left(\dfrac{z^2}{w}\right) = 0$

Which means that $z^2 w^{-1}$ is purely imaginary i.e. that $\arg\left(z^2 w^{-1}\right) = \pm\dfrac{\pi}{2}$

So there are two cases to consider, we first consider the case $\arg\left(z^2 w^{-1}\right) = \dfrac{\pi}{2}$

Then if $w = 2\mathrm{e}^{i\varphi}$ we have $\frac{2\pi}{3} - \varphi = \frac{\pi}{2}$ hence $\varphi = \frac{\pi}{6}$ so we have

$w = 2\mathrm{e}^{\frac{\pi i}{6}}$

In the second case we have $\frac{2\pi}{3} - \varphi = -\frac{\pi}{2}$ hence $\varphi = \frac{7\pi}{6}$ so we have

$w = 2\mathrm{e}^{\frac{7\pi i}{6}} = 2\mathrm{e}^{-\frac{5\pi i}{6}}$

8 a Note that $|1+i| = \sqrt{2}$ and $\arg(1+i) = \frac{\pi}{4}$ so we can write it in exponential form as

$1 + i = \sqrt{2}\mathrm{e}^{\frac{i\pi}{4}}$ hence we have $(1+i)^2 = \sqrt{2}\mathrm{e}^{\frac{i\pi}{4}} \times \sqrt{2}\mathrm{e}^{\frac{i\pi}{4}} = 2e^{\frac{i\pi}{2}}$

b We wish to prove by induction that

$(1+i)^n = 2^{\frac{n}{2}} e^{\frac{in\pi}{4}}$

Note that the base case is already true for $n = 1$ by the first part of the question so assume the statement is true up to $n = k$ then

$(1+i)^{k+1} = (1+i) \times 2^{\frac{k}{2}} e^{\frac{ik\pi}{4}} = \sqrt{2}\mathrm{e}^{\frac{i\pi}{4}} \times 2^{\frac{k}{2}} e^{\frac{ik\pi}{4}} = 2^{\frac{k+1}{2}} e^{\frac{i(k+1)\pi}{4}}$

Proving the statement is true for $n = k+1$ hence the claim is true by induction

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8 c $(1+i)^{16} = 2^8 e^{\frac{16\pi i}{2}} = 256$

9 We have

$e^{i\theta} = \cos\theta + i\sin\theta$

$e^{-i\theta} = \cos -\theta + i\sin -\theta = \cos\theta - i\sin\theta$

Multiplying the two equations gives

$1 = e^{i\theta} \times e^{-i\theta} = (\cos\theta + i\sin\theta)(\cos\theta - i\sin\theta)$

$= \cos^2\theta + \sin^2\theta$

Challenge

a We want to prove by induction that

$(re^{i\theta})^n = r^n e^{in\theta}$

Clearly the statement is true when $n = 1$, suppose now the statement is true for $n = k$ then we have

$(re^{i\theta})^{k+1} = re^{i\theta} \times (re^{i\theta})^k = re^{i\theta} \times r^k e^{ik\theta} = r^{k+1} e^{i\theta + ik\theta} = r^{k+1} e^{i(k+1)\theta}$

Proving the statement is true for $n = k+1$ hence the claim is true by induction.

b Now we want to show that

$(re^{i\theta})^{-n} = r^{-n} e^{-in\theta}$

Again by definition this is true when $n = 1$ so suppose it is true for $n = k$, then we have

$(re^{i\theta})^{-(k+1)} = \frac{1}{(re^{i\theta})^{k+1}} = \frac{1}{re^{i\theta} \times (re^{i\theta})^k} = \frac{r^{-k} e^{-ik\theta}}{re^{i\theta}} = r^{-(k+1)} e^{-ik\theta - i\theta}$

$= r^{-(k+1)} e^{-i(k+1)\theta}$

Proving the statement is true for $n = k+1$ hence the claim is true by induction.

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Complex numbers 1C

1 a $(\cos\theta + i\sin\theta)^6 = \cos 6\theta + i\sin 6\theta$

b $(\cos 3\theta + i\sin 3\theta)^4 = \cos 12\theta + i\sin 12\theta$

c

$$\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)^5 = \cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}$$
$$= -\frac{\sqrt{3}}{2} + \frac{i}{2}$$

d

$$\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)^8 = \cos\frac{8\pi}{3} + i\sin\frac{8\pi}{3} = \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}$$
$$= -\frac{1}{2} + \frac{i\sqrt{3}}{2}$$

e

$$\left(\cos\frac{2\pi}{5} + i\sin\frac{2\pi}{5}\right)^5 = \cos\frac{10\pi}{5} + i\sin\frac{10\pi}{5} = \cos 2\pi + i\sin 2\pi$$
$$= 1$$

f

$$\left(\cos\frac{\pi}{10} - i\sin\frac{\pi}{10}\right)^{15} = \left(\cos\frac{-\pi}{10} + i\sin\frac{-\pi}{10}\right)^{15} = \cos(-\frac{3\pi}{2}) + i\sin(-\frac{3\pi}{2})$$
$$= i$$

2 a $$\frac{\cos 5\theta + i\sin 5\theta}{(\cos 2\theta + i\sin 2\theta)^2} = \frac{(\cos\theta + i\sin\theta)^5}{(\cos\theta + i\sin\theta)^4} = \cos\theta + i\sin\theta = \mathrm{e}^{i\theta}$$

b $$\frac{(\cos 2\theta + i\sin 2\theta)^7}{(\cos 4\theta + i\sin 4\theta)^3} = \frac{(\cos\theta + i\sin\theta)^{14}}{(\cos\theta + i\sin\theta)^{12}} = (\cos\theta + i\sin\theta)^2 = \mathrm{e}^{2i\theta}$$

c $$\frac{1}{(\cos 2\theta + i\sin 2\theta)^3} = \frac{1}{(\cos\theta + i\sin\theta)^6} = \frac{1}{\mathrm{e}^{6i\theta}} = \mathrm{e}^{-6i\theta}$$

d $$\frac{(\cos 2\theta + i\sin 2\theta)^4}{(\cos 3\theta + i\sin 3\theta)^3} = \frac{(\cos\theta + i\sin\theta)^8}{(\cos\theta + i\sin\theta)^9} = \frac{1}{\cos\theta + i\sin\theta} = \frac{1}{\mathrm{e}^{i\theta}} = \mathrm{e}^{-i\theta}$$

e $$\frac{\cos 5\theta + i\sin 5\theta}{(\cos 3\theta - i\sin 3\theta)^2} = \frac{(\cos\theta + i\sin\theta)^5}{(\cos(-3\theta) + i\sin(-3\theta))^2} = \frac{(\cos\theta + i\sin\theta)^5}{(\cos-\theta + i\sin-\theta)^6} = \frac{\mathrm{e}^{5i\theta}}{\mathrm{e}^{-6i\theta}} = \mathrm{e}^{11i\theta}$$

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2 f $\dfrac{\cos\theta - i\sin\theta}{(\cos 2\theta - i\sin 2\theta)^3} = \dfrac{\cos(-\theta) + i\sin(-\theta)}{(\cos(-2\theta) + i\sin(-2\theta))^3} = \dfrac{e^{-i\theta}}{(\cos -\theta + i\sin -\theta)^6} = \dfrac{e^{-i\theta}}{e^{-6i\theta}} = e^{5i\theta}$

3 a

$$\frac{\left(\cos\frac{7\pi}{13} + i\sin\frac{7\pi}{13}\right)^4}{\left(\cos\frac{4\pi}{13} - i\sin\frac{4\pi}{13}\right)^6} = \frac{\cos\frac{28\pi}{13} + i\sin\frac{28\pi}{13}}{\cos\frac{-24\pi}{13} + i\sin\frac{-24\pi}{13}} = \frac{e^{\frac{28\pi i}{13}}}{e^{\frac{-24\pi i}{13}}} = e^{\frac{52\pi i}{13}}$$
$$= e^{4\pi i} = 1$$

b Recall the identity $\sin(\pi + x) = -\sin(\pi - x)$ then

$$\frac{\left(\cos\frac{3\pi}{7} - i\sin\frac{11\pi}{7}\right)^3}{\left(\cos\frac{15\pi}{7} + i\sin\frac{\pi}{7}\right)^2} = \frac{\left(\cos\frac{3\pi}{7} + i\sin\frac{3\pi}{7}\right)^3}{\left(\cos\frac{\pi}{7} + i\sin\frac{\pi}{7}\right)^2} = \frac{\cos\frac{9\pi}{7} + i\sin\frac{9\pi}{7}}{\cos\frac{2\pi}{7} + i\sin\frac{2\pi}{7}} = \frac{e^{\frac{9\pi i}{7}}}{e^{\frac{2\pi i}{7}}} = e^{\frac{7\pi i}{7}}$$
$$= e^{\pi i} = -1$$

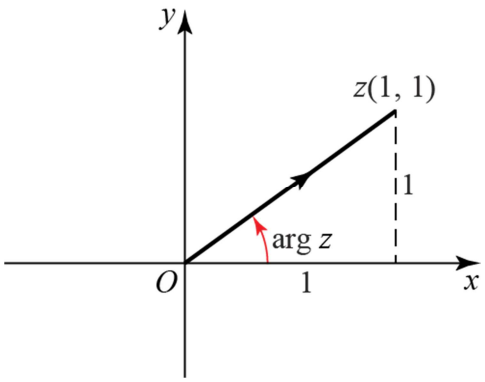
c

$$\frac{\left(\cos\frac{4\pi}{3} - i\sin\frac{2\pi}{3}\right)^7}{\left(\cos\frac{10\pi}{3} + i\sin\frac{4\pi}{3}\right)^4} = \frac{\left(\cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3}\right)^7}{\left(\cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3}\right)^4} = \frac{\cos\frac{28\pi}{3} + i\sin\frac{28\pi}{3}}{\cos\frac{16\pi}{3} + i\sin\frac{16\pi}{3}} = \frac{e^{\frac{28\pi i}{3}}}{e^{\frac{16\pi i}{3}}} = e^{\frac{12\pi i}{3}}$$
$$= e^{4\pi i} = 1$$

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4 a $(1+\mathrm{i})^5$

If $z = 1+\mathrm{i}$, then



$r = \sqrt{1^2+1^2} = \sqrt{2}$

$\theta = \arg z = \tan^{-1}\left(\frac{1}{1}\right) = \frac{\pi}{4}$

So, $1+\mathrm{i} = \sqrt{2}\left(\cos\frac{\pi}{4}+\mathrm{i}\sin\frac{\pi}{4}\right)$

$\therefore (1+\mathrm{i})^5 = \left[\sqrt{2}\left(\cos\frac{\pi}{4}+\mathrm{i}\sin\frac{\pi}{4}\right)\right]^5$

$= (\sqrt{2})^5\left(\cos\frac{5\pi}{4}+\mathrm{i}\sin\frac{5\pi}{4}\right)$

$= 4\sqrt{2}\left(\cos\frac{5\pi}{4}+\mathrm{i}\sin\frac{5\pi}{4}\right)$

$= 4\sqrt{2}\left(-\frac{1}{\sqrt{2}}-\frac{1}{\sqrt{2}}\mathrm{i}\right)$

$= -4-4\mathrm{i}$

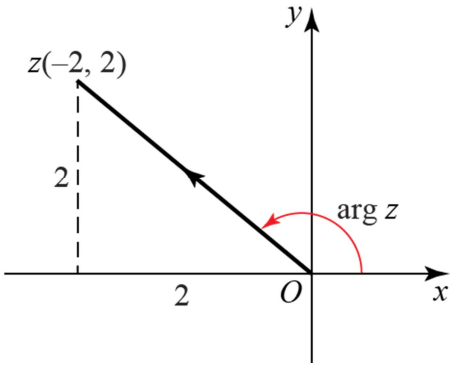
Therefore, $(1+\mathrm{i})^5 = -4-4\mathrm{i}$

$(\sqrt{2}) = \sqrt{2}\sqrt{2}\sqrt{2}\sqrt{2}\sqrt{2}$
$= 4\sqrt{2}$

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4 b $(-2+2\mathrm{i})^8$

If $z=-2+\mathrm{i}$, then



$r=\sqrt{(-2)^2+2^2}=\sqrt{4+4}=\sqrt{8}=\sqrt{4}\sqrt{2}=2\sqrt{2}$

$\theta=\arg z=\pi-\tan^{-1}\left(\frac{2}{2}\right)=\pi-\frac{\pi}{4}=\frac{3\pi}{4}$

So, $-2+2\mathrm{i}=2\sqrt{2}\left(\cos\frac{3\pi}{4}+\mathrm{i}\sin\frac{3\pi}{4}\right)$

$$\therefore (-2+2\mathrm{i})^8=\left[2\sqrt{2}\left(\cos\left(\frac{3\pi}{4}\right)+\mathrm{i}\sin\left(\frac{3\pi}{4}\right)\right)\right]^8$$

$$=(2\sqrt{2})^8\left(\cos\left(\frac{24\pi}{4}\right)+\mathrm{i}\sin\left(\frac{24\pi}{4}\right)\right)$$

$$=(256)(16)\ (\cos 6\pi+\mathrm{i}\sin 6\pi)$$

$$=4096(1+\mathrm{i}(0))$$

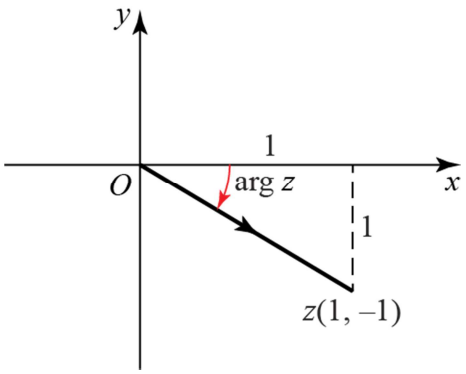
$$=4096$$

Therefore, $(-2+2\mathrm{i})^8=4096$

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4 c $(1-\mathrm{i})^6$

If $z = 1-\mathrm{i}$, then



$r = \sqrt{1^2+1^2} = \sqrt{2}$

$\theta = \arg z = -\tan^{-1}\left(\frac{1}{1}\right) = -\frac{\pi}{4}$

So, $1-\mathrm{i} = \sqrt{2}\left(\cos\left(-\frac{\pi}{4}\right)+\mathrm{i}\sin\left(-\frac{\pi}{4}\right)\right)$

$\therefore\ (1-\mathrm{i})^6 = \left[\sqrt{2}\left(\cos\left(-\frac{\pi}{4}\right)+\mathrm{i}\sin\left(-\frac{\pi}{4}\right)\right)\right]^6$

$= (\sqrt{2})^6\left(\cos\left(-\frac{6\pi}{4}\right)+\mathrm{i}\sin\left(-\frac{6\pi}{4}\right)\right)$

$= 8\left(\cos\left(-\frac{3\pi}{2}\right)+\mathrm{i}\sin\left(-\frac{3\pi}{2}\right)\right)$

$= 8(0+\mathrm{i})$

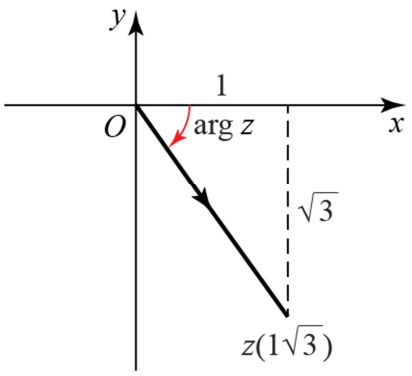
$= 8\mathrm{i}$

Therefore, $(1-\mathrm{i})^6 = 8\mathrm{i}$

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4 d $(1-\sqrt{3}\,\mathrm{i})^6$

If $z = 1-\sqrt{3}\,\mathrm{i}$, then



$$r = \sqrt{1^2 + (-\sqrt{3})^2} = \sqrt{1+3} = \sqrt{4} = 2$$

$$\theta = \arg z = -\tan^{-1}\left(\frac{\sqrt{3}}{1}\right) = -\frac{\pi}{3}$$

$$\text{So, } 1-\sqrt{3}\,\mathrm{i} = 2\left(\cos\left(-\frac{\pi}{3}\right) + \mathrm{i}\sin\left(-\frac{\pi}{3}\right)\right)$$

$$\therefore (1-\sqrt{3}\,\mathrm{i})^6 = \left[2\left(\cos\left(-\frac{\pi}{3}\right) + \mathrm{i}\sin\left(-\frac{\pi}{3}\right)\right)\right]^6$$

$$= (2)^6\left(\cos\left(-\frac{6\pi}{3}\right) + \mathrm{i}\sin\left(-\frac{6\pi}{3}\right)\right)$$

$$= 64\left(\cos(-2\pi) + \mathrm{i}\sin(-2\pi)\right)$$

$$= 64(1 + \mathrm{i}(0))$$

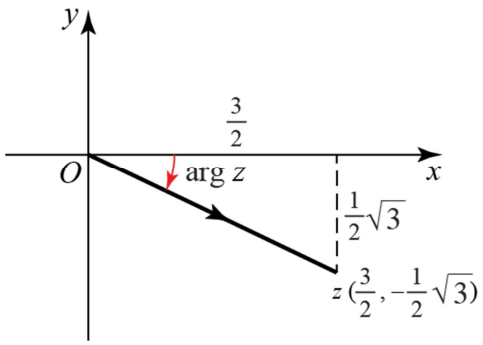
$$= 64$$

Therefore, $(1-\sqrt{3}\,\mathrm{i})^6 = 64$

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4 e $\left(\frac{3}{2}-\frac{1}{2}\sqrt{3}\,\mathrm{i}\right)^9$

If $z=\frac{3}{2}-\frac{1}{2}\sqrt{3}\,\mathrm{i}$, then



$$r=\sqrt{\left(\frac{3}{2}\right)^2+\left(-\frac{1}{2}\sqrt{3}\right)^2}=\sqrt{\frac{9}{4}+\frac{3}{4}}=\sqrt{\frac{12}{4}}=\sqrt{3}$$

$$\theta=\arg z=-\tan^{-1}\left(\frac{\frac{1}{2}\sqrt{3}}{\frac{3}{2}}\right)=-\tan^{-1}\frac{\sqrt{3}}{3}=-\frac{\pi}{6}$$

So, $\frac{3}{2}-\frac{1}{2}\sqrt{3}\,\mathrm{i}=\sqrt{3}\left(\cos\left(-\frac{\pi}{6}\right)+\mathrm{i}\sin\left(-\frac{\pi}{6}\right)\right)$

$$\therefore\left(\frac{3}{2}-\frac{1}{2}\sqrt{3}\,\mathrm{i}\right)^9=\left[\sqrt{3}\left(\cos\left(-\frac{\pi}{6}\right)+\mathrm{i}\sin\left(-\frac{\pi}{6}\right)\right)\right]^9$$

$$=(\sqrt{3})^9\left(\cos\left(-\frac{9\pi}{6}\right)+\mathrm{i}\sin\left(-\frac{9\pi}{6}\right)\right)$$

$$=81\sqrt{3}\left(\cos\left(-\frac{3\pi}{2}\right)+\mathrm{i}\sin\left(-\frac{3\pi}{2}\right)\right)$$

$$=81\sqrt{3}(0+\mathrm{i})$$

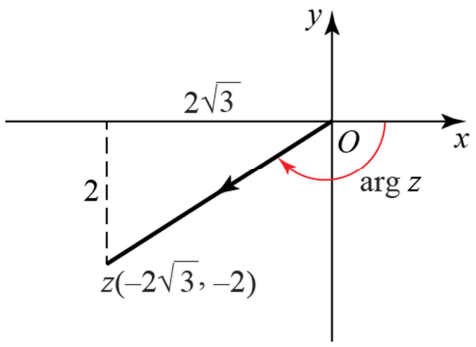
$$=81\sqrt{3}\,\mathrm{i}$$

Therefore, $\left(\frac{3}{2}-\frac{1}{2}\sqrt{3}\,\mathrm{i}\right)^9=81\sqrt{3}\,\mathrm{i}$

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4 f $(-2\sqrt{3}-2\mathrm{i})^5$

If $z=-2\sqrt{3}-2\mathrm{i}$, then



$r=\sqrt{(-2\sqrt{3})^2+(-2)^2}=\sqrt{12+4}=\sqrt{16}=4$

$\theta=\arg z=-\pi+\tan^{-1}\left(\frac{2}{2\sqrt{3}}\right)=-\pi+\frac{\pi}{6}=-\frac{5\pi}{6}$

So, $-2\sqrt{3}-2\mathrm{i}=4\left(\cos\left(-\frac{5\pi}{6}\right)+\mathrm{i}\sin\left(-\frac{5\pi}{6}\right)\right)$

$\therefore\ (-2\sqrt{3}-2\mathrm{i})^5=\left[4\left(\cos\left(-\frac{5\pi}{6}\right)+\mathrm{i}\sin\left(-\frac{5\pi}{6}\right)\right)\right]^5$

$=4^5\left(\cos\left(-\frac{25\pi}{6}\right)+\mathrm{i}\sin\left(-\frac{25\pi}{6}\right)\right)$

$=1024\left(\cos\left(-\frac{\pi}{6}\right)+\mathrm{i}\sin\left(-\frac{\pi}{6}\right)\right)$

$=1024\left(\frac{\sqrt{3}}{2}-\frac{1}{2}\mathrm{i}\right)$

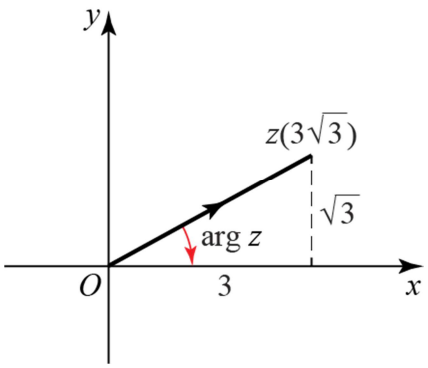
$=512\sqrt{3}-512\mathrm{i}$

Therefore, $(-2\sqrt{3}-2\mathrm{i})^5=512\sqrt{3}-512\mathrm{i}$

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5 $(3+\sqrt{3}\mathrm{i})^5$

If $z = 3+\sqrt{3}\,\mathrm{i}$, then



$r = \sqrt{3^2+(\sqrt{3})^2} = \sqrt{9+3} = \sqrt{12} = \sqrt{4}\sqrt{3} = 2\sqrt{3}$

$\theta = \arg z = \tan^{-1}\left(\dfrac{\sqrt{3}}{3}\right) = \dfrac{\pi}{6}$

So, $3+\sqrt{3}\,\mathrm{i} = 2\sqrt{3}\left(\cos\dfrac{\pi}{6}+\mathrm{i}\sin\dfrac{\pi}{6}\right)$

$\therefore\ (3+\sqrt{3}\,\mathrm{i})^5 = \left[2\sqrt{3}\left(\cos\dfrac{\pi}{6}+\mathrm{i}\sin\dfrac{\pi}{6}\right)\right]^5$

$= (2\sqrt{3})^5\left(\cos\dfrac{5\pi}{6}+\mathrm{i}\sin\dfrac{5\pi}{6}\right)$

$= 32(9\sqrt{3})\left(-\dfrac{\sqrt{3}}{2}+\dfrac{1}{2}\mathrm{i}\right)$

$= 288\sqrt{3}\left(-\dfrac{\sqrt{3}}{2}+\dfrac{1}{2}\mathrm{i}\right)$

$= -144\sqrt{3}\sqrt{3}+144\sqrt{3}\,\mathrm{i}$

$= -432+144\sqrt{3}\,\mathrm{i}$

Therefore, $(3+\sqrt{3}\,\mathrm{i})^5 = -432+144\sqrt{3}\,\mathrm{i}$

6

$w = 2\left(\cos\dfrac{\pi}{6}+i\sin\dfrac{\pi}{6}\right)$

$w^4 = 2^4\left(\cos\dfrac{\pi}{6}+i\sin\dfrac{\pi}{6}\right)^4 = 16\left(\cos\dfrac{2\pi}{3}+i\sin\dfrac{2\pi}{3}\right) = 16\left(-\dfrac{1}{2}+\dfrac{i\sqrt{3}}{2}\right)$

$= -8+8i\sqrt{3}$

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7

$$z=\sqrt{3}\left(\cos\frac{3\pi}{4}-i\sin\frac{3\pi}{4}\right)$$

$$z^6=\left(\sqrt{3}\right)^6\left(\cos\frac{-3\pi}{4}+i\sin\frac{-3\pi}{4}\right)^6=27\left(\cos\frac{-18\pi}{4}+i\sin\frac{-18\pi}{4}\right)$$

$$=27\left(\cos\frac{-\pi}{2}+i\sin\frac{-\pi}{2}\right)=-27i$$

8 a Let $z=1+i\sqrt{3}$ and $w=1-i\sqrt{3}$ first of all note that $|z|=|w|=\sqrt{1+3}=2$ so it follows that we can write $z=2\mathrm{e}^{i\theta}$ and $w=2\mathrm{e}^{i\varphi}$ with $\tan\theta=\sqrt{3}$ and $\tan\varphi=-\sqrt{3}$ so that $\theta=\frac{\pi}{3}$ and $\varphi=-\frac{\pi}{3}$ then it follows that

$$\frac{1+i\sqrt{3}}{1-i\sqrt{3}}=\frac{z}{w}=\frac{2\mathrm{e}^{\frac{i\pi}{3}}}{2\mathrm{e}^{-\frac{i\pi}{3}}}=\mathrm{e}^{\frac{i2\pi}{3}}$$

b We have

$$\left(\frac{1+i\sqrt{3}}{1-i\sqrt{3}}\right)^n=\mathrm{e}^{\frac{i2n\pi}{3}}=\cos\frac{2n\pi}{3}+i\sin\frac{2n\pi}{3}$$

This is real precisely when $\frac{2n\pi}{3}=k\pi$ for some integer k so the smallest value of n for which this is real is $n=\frac{3}{2}$ and we have $\cos\frac{2\pi}{3}\frac{3}{2}=-1$ so this is not admissible, the next smallest value for which this is real is $n=3$ in which case $\left(\frac{1+i\sqrt{3}}{1-i\sqrt{3}}\right)^3=\cos 2\pi=1$ which is positive hence this is the smallest value.

9 We start by noting that we can write

$a+bi=r\left(\cos\theta+i\sin\theta\right)=r\mathrm{e}^{i\theta}$

$a-bi=r(\cos\theta-i\sin\theta)=r\mathrm{e}^{-i\theta}$

Then it follows that

$$\left(a+bi\right)^n+\left(a-bi\right)^n=r^n\mathrm{e}^{in\theta}+r^n\mathrm{e}^{-in\theta}=r^n\left(\mathrm{e}^{in\theta}+\mathrm{e}^{-in\theta}\right)$$

$$=r^n\left(\cos n\theta+i\sin n\theta+\cos n\theta-i\sin n\theta\right)=2r^n\cos n\theta$$

which is real.

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Challenge

Given $n \in \mathbb{Z}^+$, we have:

$$(r(\cos\theta + \mathrm{i}\sin))^{-n} = \frac{1}{\left(r\left(\cos\theta + \mathrm{i}\sin\right)\right)^n} = \frac{1}{r^n\left(\cos n\theta + \mathrm{i}\sin n\theta\right)}$$

by de Moivre's theorem for positive integer exponents.

$$= \frac{1}{r^n\left(\cos n\theta + \mathrm{i}\sin n\theta\right)} \times \frac{\cos n\theta - \mathrm{i}\sin n\theta}{\cos n\theta - \mathrm{i}\sin n\theta}$$

$$= \frac{\cos n\theta - \mathrm{i}\sin n\theta}{r^n\left(\cos^2 n\theta - \mathrm{i}^2\sin^2 n\theta\right)} = \frac{\cos n\theta - \mathrm{i}\sin n\theta}{r^n\left(\cos^2 n\theta + \sin^2 n\theta\right)}$$

$$= r^{-n}\left(\cos n\theta - \mathrm{i}\sin n\theta\right) = r^{-n}\left(\cos\left(-n\theta\right) + \mathrm{i}\sin\left(-n\theta\right)\right)$$

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Complex numbers 1D

1 a $(\cos\theta + \mathrm{i}\sin\theta)^3 = \cos^3\theta + \mathrm{i}\sin^3\theta$

de Moivre's Theorem.

$= \cos^3\theta + {}^3\mathrm{C}_1\cos^2\theta(\mathrm{i}\sin\theta)$

$+ {}^3\mathrm{C}_2\cos\theta(\mathrm{i}\sin\theta)^2 + (\mathrm{i}\sin\theta)^3$

Binomial expansion.

$= \cos^3\theta + 3\mathrm{i}\cos^2\theta\sin\theta + 3\mathrm{i}^2\cos\theta\sin^2\theta + \mathrm{i}^3\sin^3\theta$

$= \cos^3\theta + 3\mathrm{i}\cos^2\theta\sin\theta - 3\cos\theta\sin^2\theta - \mathrm{i}\sin^3\theta$

Hence,

$\cos 3\theta + \mathrm{i}\sin 3\theta = \cos^3\theta + 3\mathrm{i}\cos^2\theta\sin\theta - 3\cos\theta\sin^2\theta - \mathrm{i}\sin^3\theta$

Equating the imaginary parts gives,

$\sin 3\theta = 3\cos^2\theta\sin\theta - \sin^3\theta$

$= 3(1-\sin^2\theta)\sin\theta - \sin^3\theta$

Applying $\cos^2\theta = 1 - \sin^2\theta$.

$= 3\sin\theta(1-\sin^2\theta) - \sin^3\theta$

$= 3\sin\theta - 3\sin^3\theta - \sin^3\theta$

$= 3\sin\theta - 4\sin^3\theta$

Hence, $\sin 3\theta = 3\sin\theta - 4\sin^3\theta$ (as required)

b $(\cos\theta + \mathrm{i}\sin\theta)^5 = \cos 5\theta + \mathrm{i}\sin 5\theta$

$= \cos^5\theta + {}^5\mathrm{C}_1\cos^4\theta(\mathrm{i}\sin\theta) + {}^5\mathrm{C}_2\cos^3\theta(\mathrm{i}\sin\theta)^2$

de Moivre's Theorem.

$+ {}^5\mathrm{C}_3\cos^2\theta(\mathrm{i}\sin\theta)^3 + {}^5\mathrm{C}_4\cos\theta(\mathrm{i}\sin\theta)^4 + (\mathrm{i}\sin\theta)^5$

$= \cos^5\theta + 5\mathrm{i}\cos^4\theta\sin\theta + 10\mathrm{i}^2\cos^3\theta\sin^2\theta + 10\mathrm{i}^3\cos^2\theta\sin^3\theta$

Binomial expansion.

$+ 5\mathrm{i}^4\cos\theta\sin^4\theta + \mathrm{i}^5\sin^5\theta$

Hence,

$\cos 5\theta + \mathrm{i}\sin 5\theta = \cos^5\theta + 5\mathrm{i}\cos^4\theta\sin\theta - 10\cos^3\theta\sin^2\theta - 10\mathrm{i}\cos^2\theta\sin^3\theta$

$+ 5\cos\theta\sin^4\theta + \mathrm{i}\sin^5\theta$

Equating the imaginary parts gives,

$\sin 5\theta = 5\cos^4\theta\sin\theta - 10\cos^2\theta\sin^3\theta + \sin^5\theta$

$= 5(\cos^2\theta)^2\sin\theta - 10\cos^2\theta\sin^3\theta + \sin^5\theta$

Applying $\cos^2\theta = 1 - \sin^2\theta$.

$= 5(1-\sin^2\theta)^2\sin\theta - 10(1-\sin^2\theta)\sin^3\theta + \sin^5\theta$

$= 5\sin\theta(1 - 2\sin^2\theta + \sin^4\theta) - 10\sin^3\theta(1-\sin^2\theta) + \sin^5\theta$

$= 5\sin\theta - 10\sin^3\theta + 5\sin^5\theta - 10\sin^3\theta + 10\sin^5\theta + \sin^5\theta$

$= 16\sin^5\theta - 20\sin^3\theta + 5\sin\theta$

Hence, $\sin 5\theta = 16\sin^5\theta - 20\sin^3\theta + 5\sin\theta$ (as required)

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1 c $(\cos\theta + \mathrm{i}\sin\theta)^7 = \cos 7\theta + \mathrm{i}\sin 7\theta$

$= \cos^7\theta + {}^7\mathrm{C}_1\cos^6\theta(\mathrm{i}\sin\theta) + {}^7\mathrm{C}_2\cos^5\theta(\mathrm{i}\sin\theta)^2$

$+ {}^7\mathrm{C}_3\cos^4\theta(\mathrm{i}\sin\theta)^3 + {}^7\mathrm{C}_4\cos^3\theta(\mathrm{i}\sin\theta)^4 + {}^7\mathrm{C}_5\cos^2\theta(\mathrm{i}\sin\theta)^5$

$+ {}^7\mathrm{C}_6\cos\theta(\mathrm{i}\sin\theta)^6 + (\mathrm{i}\sin\theta)^7$

$= \cos^7\theta + 7\mathrm{i}\cos^6\theta\sin\theta + 21\mathrm{i}^2\cos^5\theta\sin^2\theta$

$+ 35\mathrm{i}^3\cos^4\theta\sin^3\theta + 35\mathrm{i}^4\cos^3\theta\sin^4\theta + 21\mathrm{i}^5\cos^2\theta\sin^5\theta$

$+ 7\mathrm{i}^6\cos\theta\sin^6\theta + \mathrm{i}^7\sin^7\theta$

de Moivre's Theorem.

Binomial expansion.

Hence,

$\cos 7\theta + \mathrm{i}\sin 7\theta = \cos^7\theta + 7\mathrm{i}\cos^6\theta\sin\theta - 21\cos^5\theta\sin^2\theta$

$- 35\mathrm{i}^3\cos^4\theta\sin^3\theta + 35\mathrm{i}^4\cos^3\theta\sin^4\theta + 21\mathrm{i}^5\cos^2\theta\sin^5\theta$

$- 7\cos\theta\sin^6\theta - \mathrm{i}\sin^7\theta$

Equating the imaginary parts gives,

$\cos 7\theta = \cos^7\theta - 21\cos^5\theta\sin^2\theta + 35\cos^3\theta\sin^4\theta - 7\cos\theta\sin^6\theta$

$= \cos^7\theta - 21\cos^5\theta(1-\cos^2\theta) + 35\cos^3\theta(1-\cos^2\theta)^2$

$- 7\cos\theta(1-\cos^2\theta)^3$

$= \cos^7\theta - 21\cos^5\theta + 21\cos^7\theta + 35\cos^3\theta(1 - 2\cos^2\theta + \cos^4\theta)$

$- 7\cos\theta(1 - 3\cos^2\theta + 3\cos^4\theta - \cos^6\theta)$

$= \cos^7\theta - 21\cos^5\theta + 21\cos^7\theta + 35\cos^3\theta - 70\cos^5\theta + 35\cos^7\theta$

$- 7\cos\theta + 21\cos^3\theta - 21\cos^5\theta + 7\cos^7\theta$

$= 64\cos^7\theta - 112\cos^5\theta + 56\cos^3\theta - 7\cos\theta$

Applying $\cos^2\theta = 1 - \sin^2\theta$.

Hence, $\cos 7\theta = 64\cos^7\theta - 112\cos^5\theta + 56\cos^3\theta - 7\cos^5\theta$ (as required)

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1 d Let $z = \cos\theta + \mathrm{i}\sin\theta$

$$\left(z + \frac{1}{z}\right)^4 = (2\cos\theta)^4 = 16\cos^4\theta \quad \longleftarrow \quad z + \frac{1}{z} = 2\cos\theta$$

$$= z^4 + {}^4\mathrm{C}_1 z^3\left(\frac{1}{z}\right) + {}^4\mathrm{C}_2 z^2\left(\frac{1}{z}\right)^2 + {}^4\mathrm{C}_3 z\left(\frac{1}{z}\right)^3 + \left(\frac{1}{z}\right)^4$$

$$= z^4 + 4z^3\left(\frac{1}{z}\right) + 6z^2\left(\frac{1}{z^2}\right) + 4z^2\left(\frac{1}{z^3}\right) + \frac{1}{z^4}$$

$$= z^4 + 4z^2 + 6 + \frac{4}{z^2} + \frac{1}{z^4}$$

$$= \left(z^4 + \frac{1}{z^4}\right) + 4\left(z^2 + \frac{1}{z^2}\right) + 6$$

$$= 2\cos 4\theta + 4(2\cos 2\theta) + 6 \quad \longleftarrow \quad z^n + \frac{1}{z^n} = 2\cos n\theta$$

So, $16\cos^4\theta = 2\cos 4\theta + 8\cos 2\theta + 6$

$$16\cos^4\theta = 2(\cos 4\theta + 4\cos 2\theta + 3)$$

$$\cos^4\theta = \frac{2}{16}(\cos 4\theta + 4\cos 2\theta + 3)$$

Therefore, $\cos^4\theta = \frac{1}{8}(\cos 4\theta + 4\cos 2\theta + 3)$ (as required)

e Let $z = \cos\theta + \mathrm{i}\sin\theta$

$$\left(z - \frac{1}{z}\right)^5 = (2\mathrm{i}\sin\theta)^5 = 32\mathrm{i}^5\sin^5\theta = 32\mathrm{i}\sin^5\theta \quad \longleftarrow \quad z - \frac{1}{z} = 2\mathrm{i}\sin\theta$$

$$= z^5 + {}^5\mathrm{C}_1 z^4\left(-\frac{1}{z}\right) + {}^5\mathrm{C}_2 z^3\left(-\frac{1}{z}\right)^2 + {}^5\mathrm{C}_3 z^2\left(-\frac{1}{z}\right)^3 + {}^5\mathrm{C}_4 z\left(-\frac{1}{z}\right)^4 + \left(-\frac{1}{z}\right)^5$$

$$= z^5 + 5z^4\left(-\frac{1}{z}\right) + 10z^3\left(-\frac{1}{z}\right)^2 + 10z^2\left(-\frac{1}{z}\right)^3 + 5z\left(-\frac{1}{z}\right)^4 + \left(-\frac{1}{z}\right)^5$$

$$= z^5 - 5z^4\left(\frac{1}{z}\right) + 10z^3\left(\frac{1}{z^2}\right) - 10z^2\left(\frac{1}{z^3}\right) + 5z\left(\frac{1}{z^4}\right) - \frac{1}{z^5}$$

$$= z^5 - 5z^3 + 10z - \frac{10}{z} + \frac{5}{z^3} - \frac{1}{z^5}$$

$$= \left(z^5 - \frac{1}{z^5}\right) - 5\left(z^3 - \frac{1}{z^3}\right) + 10\left(z - \frac{1}{z}\right)$$

$$= 2\mathrm{i}\sin 5\theta - 5(2\mathrm{i}\sin 3\theta) + 10(2\mathrm{i}\sin\theta) \quad \longleftarrow \quad z^n + \frac{1}{z^n} = 2\mathrm{i}\sin n\theta$$

So, $32\mathrm{i}\sin^5\theta = 2\mathrm{i}\sin 5\theta - 10\mathrm{i}\sin 3\theta + 20\mathrm{i}\sin\theta \quad (\div 2\mathrm{i})$

$$16\sin^5\theta = \sin 5\theta - 5\sin 3\theta + 10\sin\theta$$

$$\sin^5\theta = \frac{1}{16}(\sin 5\theta - 5\sin 3\theta + 10\sin\theta)$$

Therefore, $\sin^5\theta = \frac{1}{16}(\sin 5\theta - 5\sin 3\theta + 10\sin\theta)$

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2 a Let $z = \cos\theta + i\sin\theta$ then we have $z^5 + z^{-5} = 2\cos 5\theta$ hence

$$\cos 5\theta = \frac{1}{2}\left(z^5 + z^{-5}\right)$$

We have

$$z^5 = \left(\cos\theta + i\sin\theta\right)^5$$
$$= \left(\cos^5\theta + 5i\cos^4\theta\sin\theta - 10\cos^3\theta\sin^2\theta - 10i\cos^2\theta\sin^3\theta + 5\cos\theta\sin^4\theta + i\sin^5\theta\right)$$

and

$$z^{-5} = \left(\cos\theta - i\sin\theta\right)^5$$
$$= \left(\cos^5\theta - 5i\cos^4\theta\sin\theta - 10\cos^3\theta\sin^2\theta + 10i\cos^2\theta\sin^3\theta + 5\cos\theta\sin^4\theta - i\sin^5\theta\right)$$

Hence

$$\cos 5\theta = \frac{1}{2}\left(2\cos^5\theta - 20\cos^3\theta\sin^2\theta + 10\cos\theta\sin^4\theta\right)$$
$$= \cos^5\theta - 10\cos^3\theta\sin^2\theta + 5\cos\theta\sin^4\theta$$
$$= \cos^5\theta - 10\cos^3\theta\left(1 - \cos^2\theta\right) + 5\cos\theta(1 - \cos^2\theta)^2$$
$$= \cos^5\theta - 10\cos^3\theta + 10\cos^5\theta + 5\cos\theta(1 - 2\cos^2\theta + \cos^4\theta)$$
$$= 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta$$

b We have that $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$ and we wish to solve

$\cos 5\theta + 5\cos 3\theta = 0$

Using the above identities the equation becomes

$$16\cos^5\theta - 20\cos^3\theta + 5\cos\theta + 5\left(4\cos^3\theta - 3\cos\theta\right) = 0$$

Which simplifies to

$$16\cos^5\theta - 10\cos\theta = 0$$

Hence we either have $\cos\theta = 0$ in which case $\theta = \frac{\pi}{2} = 1.571$ or we have

$$16\cos^4\theta = 10$$

Hence

$$\cos\theta = \pm\sqrt[4]{\frac{5}{8}}$$

Which implies we have $\theta = 0.475$ or $\theta = 2.666$

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3 Let $z = \cos\theta + i\sin\theta$

a $\left(z + \frac{1}{z}\right)^6 = (2\cos\theta)^6 = 64\cos^6\theta$ ← $z - \frac{1}{z} = 2\cos\theta$

$$= z^6 + {}^6C_1 z^5\left(\frac{1}{z}\right) + {}^6C_2 z^4\left(\frac{1}{z}\right)^2 + {}^6C_3 z^3\left(\frac{1}{z}\right)^3 + {}^6C_4 z^2\left(\frac{1}{z}\right)^4 + {}^6C_5\, z\left(\frac{1}{z}\right)^5 + \left(\frac{1}{z}\right)^6$$

$$= z^6 + 6z^5\left(\frac{1}{z}\right) + 15z^4\left(\frac{1}{z^2}\right) + 20z^3\left(\frac{1}{z^3}\right) + 15z^2\left(\frac{1}{z^4}\right) + 6z\left(\frac{1}{z^5}\right) + \left(\frac{1}{z^6}\right)$$

$$= z^6 - 6z^4 + 15z^2 + 20 + \frac{15}{z^2} + \frac{6}{z^4} + \frac{1}{z^6}$$

$$= \left(z^6 + \frac{1}{z^6}\right) + 6\left(z^4 + \frac{1}{z^4}\right) + 15\left(z^2 + \frac{1}{z^2}\right) + 20$$

$= 2\cos 6\theta + 6(2\cos 4\theta) + 15(2\sin 2\theta) + 20$ ← $z^n + \frac{1}{z^n} = 2\cos\theta$

So, $64\cos^6\theta = 2\cos 6\theta + 12\cos 4\theta + 30\cos 2\theta + 20$

$32\cos^6\theta = \cos 6\theta + 6\cos 4\theta + 15\cos 2\theta + 10$ (as required)

b $\int_0^{\frac{\pi}{6}} \cos^6\theta \,\mathrm{d}\theta = \frac{1}{32}\int_0^{\frac{\pi}{6}} (\cos 6\theta + 6\cos 4\theta + 15\cos 2\theta + 10)\,\mathrm{d}\theta$

$$= \frac{1}{32}\left[\frac{\sin 6\theta}{6} + \frac{6\sin 4\theta}{4} + \frac{15\sin 2\theta}{2} + 10\theta\right]_0^{\frac{\pi}{6}}$$

$$= \frac{1}{32}\left[\left(\frac{\sin(\pi)}{6} + \frac{6\sin\left(\frac{2\pi}{3}\right)}{4} + \frac{15\sin\left(\frac{\pi}{3}\right)}{2} + \frac{10\pi}{6}\right) - (0)\right]$$

$$= \frac{1}{32}\left[0 + \frac{3}{2}\frac{\sqrt{3}}{2} + \frac{15}{2}\frac{\sqrt{3}}{2} + \frac{5\pi}{3}\right]$$

$$= \frac{1}{32}\left[\frac{3}{4}\sqrt{3} + \frac{15}{4}\sqrt{3} + \frac{5\pi}{3}\right]$$

$$= \frac{1}{32}\left[\frac{9}{2}\sqrt{3} + \frac{5\pi}{3}\right]$$

$$= \frac{5\pi}{96} + \frac{9}{64}\sqrt{3}$$

$\therefore \int_0^{\frac{\pi}{6}} \cos^6\theta = \frac{5\pi}{96} + \frac{9}{64}\sqrt{3}$

$a = \frac{5}{96}$, $b = \frac{9}{64}$

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4 a We wish to show that

$32\cos^2\theta\sin^4\theta = \cos 6\theta - 2\cos 4\theta - \cos 2\theta + 2$

Let us start with the right hand side of the equation, letting $z = \cos\theta + i\sin\theta$ we have

$$\cos 6\theta = \frac{1}{2}\left(z^6 + z^{-6}\right)$$
$$= \frac{1}{2}\left(2\cos^6\theta - 30\cos^4\theta\sin^2\theta + 30\cos^2\theta\sin^4\theta - 2\sin^6\theta\right)$$
$$= \cos^6\theta - 15\cos^4\theta\sin^2\theta + 15\cos^2\theta\sin^4\theta - \sin^6\theta$$
$$\cos 4\theta = \frac{1}{2}\left(z^4 + z^{-4}\right)$$
$$= \frac{1}{2}\left(2\cos^4\theta - 12\cos^2\theta\sin^2\theta + 2\sin^4\theta\right)$$
$$= \cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta$$
$$\cos 2\theta = \cos^2\theta - \sin^2\theta$$

Hence the right hand side becomes

$$\cos 6\theta - 2\cos 4\theta - \cos 2\theta + 2$$
$$= \cos^6\theta - 15\cos^4\theta\sin^2\theta + 15\cos^2\theta\sin^4\theta - \sin^6\theta - 2\left(\cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta\right)$$
$$-\left(\cos^2\theta - \sin^2\theta\right) + 2$$
$$= \cos^2\theta\left(1 - \sin^2\theta\right)^2 - 15\left(1 - \sin^2\theta\right)\cos^2\theta\sin^2\theta + 15\cos^2\theta\sin^4\theta - \left(1 - \cos^2\theta\right)\sin^4\theta$$
$$-2\cos^4\theta + 12\cos^2\theta\sin^2\theta - 2\sin^4\theta - \cos^2\theta + \sin^2\theta + 2$$
$$= 32\cos^2\theta\sin^4\theta + \cos^2\theta - 2\cos^2\theta\sin^2\theta - 15\cos^2\theta\sin^2\theta - \sin^4\theta - 2\cos^4\theta + 12\cos^2\theta\sin^2\theta$$
$$-2\sin^4\theta - \cos^2\theta + \sin^2\theta + 2$$
$$= 32\cos^2\theta\sin^4\theta - 5\cos^2\theta\sin^2\theta - 3\sin^4\theta - 2\cos^4\theta + \sin^2\theta + 2$$
$$= 32\cos^2\theta\sin^4\theta - \cos^2\theta\left(5\sin^2\theta + 2\cos^2\theta\right) - 3\sin^4\theta + \sin^2\theta + 2$$
$$= 32\cos^2\theta\sin^4\theta - \left(1 - \sin^2\theta\right)\left(3\sin^2\theta + 2\right) - 3\sin^4\theta + \sin^2\theta + 2$$
$$= 32\cos^2\theta\sin^4\theta$$

b We have

$$\int_0^{\frac{\pi}{3}} \cos^2\theta\sin^4\theta\,\mathrm{d}\theta = \frac{1}{32}\int_0^{\frac{\pi}{3}} \cos 6\theta - 2\cos 4\theta - \cos 2\theta + 2\,\mathrm{d}\theta$$
$$= \frac{1}{32}\left[\frac{1}{6}\sin 6\theta - \frac{1}{2}\sin 4\theta - \frac{1}{2}\sin 2\theta + 2\theta\right]_0^{\frac{\pi}{3}}$$
$$= \frac{1}{32}\left(\frac{\sqrt{3}}{4} - \frac{\sqrt{3}}{4} + \frac{2\pi}{3}\right) = \frac{\pi}{48}$$

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5 a We wish to compute $\int_0^{\frac{\pi}{2}} \sin^6\theta \,\mathrm{d}\theta$, let $z = \cos\theta + i\sin\theta$ then we have

$\sin\theta = \left(\frac{1}{2i}\left(z - z^{-1}\right)\right)$ so that

$$\sin^6\theta = \left(\frac{1}{2i}\left(z - z^{-1}\right)\right)^6 = \frac{-1}{64}\left(z^6 - 6z^4 + 15z^2 - 20 + 15z^{-2} - 6z^{-4} + z^{-6}\right)$$

$$= \frac{-1}{64}\left(2\cos 6\theta - 12\cos 4\theta + 30\cos 2\theta - 20\right)$$

$$= \frac{1}{32}\left(-\cos 6\theta + 6\cos 4\theta - 15\cos 2\theta + 10\right)$$

Hence the integral becomes

$$\frac{1}{32}\int_0^{\frac{\pi}{2}}\left(-\cos 6\theta + 6\cos 4\theta - 15\cos 2\theta + 10\right)\mathrm{d}\theta = \frac{1}{32}\left[\frac{-1}{6}\sin 6\theta + \frac{3}{2}\sin 4\theta - \frac{15}{2}\sin 2\theta + 10\theta\right]_0^{\frac{\pi}{2}}$$

$$= \frac{5\pi}{32}$$

b We wish to compute $\int_0^{\frac{\pi}{4}} \sin^2\theta\cos^4\theta \,\mathrm{d}\theta$ we have

$$\sin^2\theta\cos^4\theta = \sin^2\theta\left(1 - \sin^2\theta\right)^2 = \sin^6\theta - 2\sin^4\theta + \sin^2\theta$$

From the previous part we know that

$$\sin^6\theta = \frac{1}{32}\left(-\cos 6\theta + 6\cos 4\theta - 15\cos 2\theta + 10\right)$$

and

$$\sin^4\theta = \left(\frac{1}{2i}\left(z - z^{-1}\right)\right)^4 = \frac{1}{16}\left(z^4 - 4z^2 + 6 - 4z^{-2} + z^{-4}\right)$$

$$= \frac{1}{16}\left(2\cos 4\theta - 8\cos 2\theta + 6\right) = \frac{1}{32}\left(4\cos 4\theta - 16\cos 2\theta + 12\right)$$

$$\sin^2\theta = \frac{1}{2}\left(1 - \cos 2\theta\right) = \frac{1}{32}\left(16 - 16\cos 2\theta\right)$$

Hence we have

$$\sin^2\theta\cos^4\theta = \sin^6\theta - 2\sin^4\theta + \sin^2\theta$$

$$= \frac{1}{32}\left(-\cos 6\theta - 2\cos 4\theta + \cos 2\theta + 2\right)$$

So the integral becomes

$$\frac{1}{32}\int_0^{\frac{\pi}{4}}\left(-\cos 6\theta - 2\cos 4\theta + \cos 2\theta + 2\right)\mathrm{d}\theta = \frac{1}{32}\left[-\frac{1}{6}\sin 6\theta - \frac{1}{2}\sin 4\theta + \frac{1}{2}\sin 2\theta + 2\theta\right]_0^{\frac{\pi}{4}}$$

$$= \frac{1}{32}\left(\frac{1}{6} + \frac{1}{2} + \frac{\pi}{2}\right) = \frac{1}{64}\left(\frac{4}{3} + \pi\right) = \frac{4 + 3\pi}{192} = \frac{\pi}{64} + \frac{1}{48}$$

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5 c We wish to compute

$\int_0^{\frac{\pi}{6}} \sin^3\theta\cos^5\theta \,\mathrm{d}\theta$

Let $z = \cos\theta + i\sin\theta$ then we have

$$\sin^3\theta\cos^5\theta = \left(\frac{1}{2i}\left(z - z^{-1}\right)\right)^3\left(\frac{1}{2}\left(z + z^{-1}\right)\right)^5$$

$$= \frac{i}{256}\left(z^3 - 3z + 3z^{-1} - z^{-3}\right)\left(z^5 - 5z^3 + 10z - 10z^{-1} + 5z^{-3} - z^{-5}\right)$$

$$= \frac{i}{256}\left(z^8 + 2z^6 - 2z^4 - 6z^2 + 6z^{-2} + 2z^{-4} - 2z^{-6} - z^{-8}\right)$$

$$= \frac{i}{256}\left(2i\left(\sin 8\theta + 2\sin 6\theta - 2\sin 4\theta - 6\sin 2\theta\right)\right)$$

$$= \frac{-1}{128}\left(\sin 8\theta + 2\sin 6\theta - 2\sin 4\theta - 6\sin 2\theta\right)$$

Hence the integral becomes

$$\frac{-1}{128}\int_0^{\frac{\pi}{6}}\left(\sin 8\theta + 2\sin 6\theta - 2\sin 4\theta - 6\sin 2\theta\right)\mathrm{d}\theta$$

$$= \frac{-1}{128}\left[\frac{-1}{8}\cos 8\theta - \frac{1}{3}\cos 6\theta + \frac{1}{2}\cos 4\theta + 3\cos 2\theta\right]_0^{\frac{\pi}{6}}$$

$$= \frac{-1}{128}\left(\left(\frac{1}{16} + \frac{1}{3} - \frac{1}{4} + \frac{3}{2}\right) - \left(\frac{-1}{8} - \frac{1}{3} + \frac{1}{2} + 3\right)\right) = \frac{-1}{128}\left(\frac{79}{48} - \frac{146}{48}\right) = \frac{67}{6144}$$

6 a Let $z = \cos\theta + i\sin\theta$ then we have

$$\cos 6\theta = \frac{1}{2}\left(z^6 + z^{-6}\right) = \frac{1}{2}\left(\left(\cos\theta + i\sin\theta\right)^6 + \left(\cos\theta - i\sin\theta\right)^6\right)$$

Noting that odd powers will cancel this simplifies to

$$= \frac{1}{2}\left(2\cos^6\theta - 30\cos^4\theta\sin^2\theta + 30\cos^2\theta\sin^4\theta - 2\sin^6\theta\right)$$

$$= \cos^6\theta - 15\cos^4\theta\sin^2\theta + 15\cos^2\theta\sin^4\theta - \sin^6\theta$$

$$= \cos^6\theta - 15\cos^4\theta\left(1 - \cos^2\theta\right) + 15\cos^2\theta\left(1 - \cos^2\theta\right)^2 - \left(1 - \cos^2\theta\right)^3$$

$$32\cos^6\theta - 48\cos^4\theta + 18\cos^2\theta - 1$$

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6 b We wish to solve

$32x^6 - 48x^4 + 18x - \frac{3}{2} = 0$

We use the substitution $x = \cos\theta$ so that the equation becomes

$$32\cos^6\theta - 48\cos^4\theta + 18\cos^2\theta - \frac{3}{2} = 0$$

i.e.

$$32\cos^6\theta - 48\cos^4\theta + 18\cos^2\theta - 1 = \frac{1}{2}$$

Hence

$$\cos 6\theta = \frac{1}{2}$$

The general solution to this is given by

$6\theta = \pm\frac{\pi}{3} + 2k\pi$ where k is any integer i.e.

$$\theta = \pm\frac{\pi}{18} + \frac{k\pi}{3}$$

Trying both choices of sign and varying k gives the following values of $x = \cos\theta$

$x = \pm 0.985$

$x = \pm 0.342$

$x = \pm 0.643$

But these are 6 distinct solutions and since the polynomial has order 6 there are at most 6 unique solutions hence these are all solutions.

7 a $(\cos\theta + \mathrm{i}\sin\theta)^4 = \cos 4\theta + \mathrm{i}\sin 4\theta$ de Moivre's Theorem.

$$= \cos^4\theta + {}^4\mathrm{C}_1\cos^3\theta(\mathrm{i}\sin\theta) + {}^4\mathrm{C}_2\cos^2\theta(\mathrm{i}\sin\theta)^2$$
$$+ {}^4\mathrm{C}_3\cos\theta(\mathrm{i}\sin\theta)^3 + (\mathrm{i}\sin\theta)^4$$ Binomial expansion.

$$= \cos^4\theta + 4\mathrm{i}\cos^3\theta\sin\theta + 6\mathrm{i}^2\cos^2\theta\sin^2\theta$$
$$+ 4\mathrm{i}^3\cos\theta\sin^3\theta + \mathrm{i}^4\sin^4\theta$$
$$= \cos^4\theta + 4\mathrm{i}\cos^3\theta\sin\theta - 6\cos^2\theta\sin^2\theta - 4\mathrm{i}\cos\theta\sin^3\theta + \sin^4\theta$$

Hence,

$\cos 4\theta + \mathrm{i}\sin 4\theta = \cos^4\theta + 4\mathrm{i}\cos^3\theta\sin\theta - 6\cos^2\theta\sin^2\theta - 4\mathrm{i}\cos\theta\sin^3\theta + \sin^4\theta$ **(1)**

Equating the imaginary parts of **(1)** gives:

$\sin 4\theta = 4\cos^3\theta\sin\theta - 4\cos\theta\sin^3\theta$ (as required)

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7 b Equating the real parts of **(1)** gives:

$$\cos 4\theta = \cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta$$

$$\tan 4\theta = \frac{\sin 4\theta}{\cos 4\theta} = \frac{4\cos^3\theta\sin\theta - 4\cos\theta\sin^3\theta}{\cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta} \qquad \frac{(\sin 4\theta \div \cos^4\theta)}{(\cos 4\theta \div \cos^4\theta)}$$

$$= \frac{\frac{4\cos^3\theta\sin\theta}{\cos^4\theta} - \frac{4\cos\theta\sin^3\theta}{\cos^4\theta}}{\frac{\cos^4\theta}{\cos^4\theta} - \frac{6\cos^2\theta\sin^2\theta}{\cos^4\theta} + \frac{\sin^4\theta}{\cos^4\theta}}$$

$$= \frac{\frac{4\cancel{\cos^3\theta}}{\cancel{\cos^3\theta}}\frac{\sin\theta}{\cos\theta} - \frac{4\cancel{\cos\theta}\sin^3\theta}{\cancel{\cos\theta}\cos^3\theta}}{\frac{\cancel{\cos^4\theta}}{\cancel{\cos^4\theta}} - \frac{6\cancel{\cos^2\theta}\sin^2\theta}{\cancel{\cos^2\theta}\cos^2\theta} + \frac{\sin^4\theta}{\cos^4\theta}}$$

$$= \frac{4\tan\theta - 4\tan^3\theta}{1 - 6\tan^2\theta + \tan^4\theta}$$

Therefore, $\tan 4\theta = \dfrac{4\tan\theta - 4\tan^3\theta}{1 - 6\tan^2\theta + \tan^4\theta}$ (as required)

c $x^4 + 4x^3 - 6x^2 - 4x + 1 = 0$

$x^4 - 6x^2 + 1 = 4x - 4x^3$

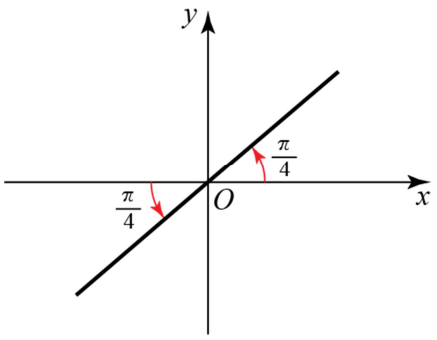
$1 = \dfrac{4x - 4x^3}{x^4 - 6x^2 + 1}$ **(2)**

Let $x = \tan\theta$; then

(2) $\Rightarrow \dfrac{4\tan\theta - 4\tan^3\theta}{\tan^4\theta - 6\tan^2\theta + 1} = 1$

$\tan 4\theta = 1$ ← From part **b**.

$\alpha = \dfrac{\pi}{4}$



$$4\theta = \left\{\frac{\pi}{4}, \frac{5\pi}{4}, \frac{9\pi}{4}, \frac{13\pi}{4}, \ldots\right\}$$

$$\theta = \left\{\frac{\pi}{16}, \frac{5\pi}{16}, \frac{9\pi}{16}, \frac{13\pi}{16}, \ldots\right\}$$

$\therefore x = \tan\theta = \tan\dfrac{\pi}{16}, \tan\dfrac{5\pi}{16}, \tan\dfrac{9\pi}{16}, \tan\dfrac{13\pi}{16}$

$x = 0.19891\ldots, 1.49660\ldots, -5.02733\ldots, -0.66817\ldots,$

$x = 0.20, 1,50, -5.03, -0.67$ (2 d.p.)

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Complex numbers 1E

1 a $z = \mathrm{e}^{\frac{\pi i}{n}}$ we have

$$1+z+z^2+\ldots+z^{2n-1} = \frac{1-z^{2n}}{1-z} = \frac{1-e^{2\pi i}}{1-\mathrm{e}^{\frac{\pi i}{n}}} = \frac{1-1}{1-\mathrm{e}^{\frac{\pi i}{n}}} = 0$$

b

$$1+z+z^2+\ldots+z^n = \frac{1-z^{n+1}}{1-z} = \frac{1-e^{\frac{(n+1)\pi i}{n}}}{1-\mathrm{e}^{\frac{\pi i}{n}}} = \frac{1+\mathrm{e}^{\frac{\pi i}{n}}}{\mathrm{e}^{\frac{\pi i}{2n}}\left(\mathrm{e}^{-\frac{\pi i}{2n}}-\mathrm{e}^{\frac{\pi i}{2n}}\right)} = \frac{\mathrm{e}^{\frac{\pi i}{2n}}\left(\mathrm{e}^{-\frac{\pi i}{2n}}+\mathrm{e}^{\frac{\pi i}{2n}}\right)}{\mathrm{e}^{\frac{\pi i}{2n}}\left(\mathrm{e}^{-\frac{\pi i}{2n}}-\mathrm{e}^{\frac{\pi i}{2n}}\right)}$$

$$= \frac{\mathrm{e}^{-\frac{\pi i}{2n}}+\mathrm{e}^{\frac{\pi i}{2n}}}{\mathrm{e}^{-\frac{\pi i}{2n}}-\mathrm{e}^{\frac{\pi i}{2n}}} = \frac{2\cos\frac{\pi}{2n}}{-2i\sin\frac{\pi}{2n}} = i\cot\frac{\pi}{2n}$$

2 $z = \mathrm{e}^{\frac{\pi i}{2}}$ then we have

$$\sum_{r=0}^{12} z^r = \frac{1-z^{13}}{1-z} = \frac{1-z}{1-z} = 1$$

since $z^4 = 1$

3 Let $z = (1+i) = \sqrt{2}\mathrm{e}^{\frac{\pi i}{4}}$ then

$$\sum_{r=0}^{7}(1+i)^r = \sum_{r=0}^{7}\left(\sqrt{2}\mathrm{e}^{\frac{\pi i}{4}}\right)^r = \frac{1-\left(\sqrt{2}\mathrm{e}^{\frac{\pi i}{4}}\right)^8}{1-\sqrt{2}\mathrm{e}^{\frac{\pi i}{4}}} = \frac{1-16}{1-\sqrt{2}\mathrm{e}^{\frac{\pi i}{4}}} = \frac{-15}{1-(1+i)} = \frac{15}{i} = -15i$$

4 a We have

$$C = 1+\frac{1}{3}\cos\theta+\frac{1}{9}\cos 2\theta+\frac{1}{27}\cos 3\theta+\ldots$$

$$S = \frac{1}{3}\sin\theta+\frac{1}{9}\sin 2\theta+\frac{1}{27}\sin 3\theta$$

So that

$$C+iS = 1+\frac{1}{3}(\cos\theta+i\sin\theta)+\frac{1}{9}(\cos 2\theta+i\sin 2\theta)+\frac{1}{27}(\cos 2\theta+i\sin 2\theta)+\ldots$$

$$= 1+\frac{1}{3}\mathrm{e}^{i\theta}+\frac{1}{9}\mathrm{e}^{2i\theta}+\frac{1}{27}\mathrm{e}^{3i\theta}+\ldots$$

$$= \sum_{r=0}^{\infty}\frac{1}{3^r}\mathrm{e}^{ir\theta} = \frac{1}{1-\frac{1}{3}\mathrm{e}^{i\theta}} = \frac{3}{3-\mathrm{e}^{i\theta}}$$

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4 b We have

$$\frac{3}{3-e^{i\theta}}=\frac{3\left(3-e^{-i\theta}\right)}{\left(3-e^{i\theta}\right)\left(3-e^{-i\theta}\right)}=\frac{9-3e^{-i\theta}}{10-3\left(e^{i\theta}+e^{-i\theta}\right)}=\frac{9-3(\cos\theta-i\sin\theta)}{10-3(2\cos\theta)}$$

$$=\frac{9-3\cos\theta+3i\sin\theta)}{10-3(2\cos\theta)}$$

Equating real and imaginary parts then gives

$$C=\frac{9-3\cos\theta}{10-6\cos\theta}$$

$$S=\frac{3\sin\theta}{10-6\cos\theta}$$

5 a We have

$$P=1+\cos\theta+\cos2\theta+\cos3\theta+...+\cos11\theta+\cos12\theta$$

$$Q=\sin\theta+\sin2\theta+\sin3\theta+...+\sin11\theta+\sin12\theta$$

So that

$$P+iQ=1+e^{i\theta}+e^{2i\theta}+e^{3i\theta}+...+e^{11i\theta}+e^{12i\theta}$$

$$=\frac{1-e^{13i\theta}}{1-e^{i\theta}}=\frac{e^{\frac{13i\theta}{2}}\left(e^{-\frac{13i\theta}{2}}-e^{\frac{13i\theta}{2}}\right)}{e^{\frac{i\theta}{2}}\left(e^{-\frac{i\theta}{2}}-e^{\frac{i\theta}{2}}\right)}=\frac{e^{6i\theta}\left(e^{-\frac{13i\theta}{2}}-e^{\frac{13i\theta}{2}}\right)}{e^{\frac{i\theta}{2}}-e^{-\frac{i\theta}{2}}}$$

b We have

$$P+iQ=\frac{e^{6i\theta}\left(e^{-\frac{13i\theta}{2}}-e^{\frac{13i\theta}{2}}\right)}{e^{\frac{i\theta}{2}}-e^{-\frac{i\theta}{2}}}=\frac{e^{6i\theta}\times2i\sin\frac{13\theta}{2}}{2i\sin\frac{\theta}{2}}=\sin\frac{13\theta}{2}\operatorname{cosec}\frac{\theta}{2}\left(\cos6\theta+i\sin6\theta\right)$$

Hence equating real and imaginary parts gives

$$P=\sin\frac{13\theta}{2}\operatorname{cosec}\frac{\theta}{2}\cos6\theta$$

$$Q=\sin\frac{13\theta}{2}\operatorname{cosec}\frac{\theta}{2}\sin6\theta$$

c $P+iQ$ is real when $Q=0$ i.e. when $\sin6\theta\operatorname{cosec}\frac{\theta}{2}\sin\frac{13\theta}{2}=0$

$\operatorname{cosec}\frac{\theta}{2}$ can never be zero so we either have $\sin6\theta=0$ in which case $\theta=\frac{k\pi}{6}$ for $k=1,2,3,4,5$

or we have $\sin\frac{13\theta}{2}=0$ in which case $\theta=\frac{2k\pi}{13}$ for $k=1,2,3,4,5,6$

6 a We have

$$C=1+\binom{n}{1}\cos\theta+\binom{n}{2}\cos2\theta+\binom{n}{3}\cos3\theta+...+\binom{n}{n}\cos n\theta$$

$$S=\binom{n}{1}\sin\theta+\binom{n}{2}\sin2\theta+\binom{n}{3}\sin3\theta+...+\binom{n}{n}\sin n\theta$$

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6 a So that

$$C+iS=1+\binom{n}{1}e^{i\theta}+\binom{n}{2}e^{2i\theta}+\binom{n}{3}e^{3i\theta}+\ldots+\binom{n}{n}e^{ni\theta}$$

$$=\left(1+e^{i\theta}\right)^n=\left(e^{\frac{i\theta}{2}}\left(e^{-\frac{i\theta}{2}}+e^{\frac{i\theta}{2}}\right)\right)^n=e^{\frac{in\theta}{2}}\left(e^{-\frac{i\theta}{2}}+e^{\frac{i\theta}{2}}\right)^n=e^{\frac{in\theta}{2}}\left(2\cos\frac{\theta}{2}\right)^n$$

Equating real and imaginary parts then gives

$$C=\cos\frac{n\theta}{2}\left(2\cos\frac{\theta}{2}\right)^n$$

$$S=\sin\frac{n\theta}{2}\left(2\cos\frac{\theta}{2}\right)^n$$

b We have

$$\frac{S}{C}=\frac{\sin\frac{n\theta}{2}}{\cos\frac{n\theta}{2}}=\tan\frac{n\theta}{2}$$

7 a We have

$$\left(2+e^{i\theta}\right)\left(2+e^{-i\theta}\right)=4+2e^{-i\theta}+2e^{i\theta}+1=5+4\cos\theta$$

b We have

$$C=1-\frac{1}{2}\cos\theta+\frac{1}{4}\cos2\theta-\frac{1}{8}\cos3\theta+\ldots$$

$$S=\frac{1}{2}\sin\theta-\frac{1}{4}\sin2\theta+\frac{1}{8}\sin3\theta+\ldots$$

So that

$$C-iS=1-\frac{1}{2}e^{i\theta}+\frac{1}{4}e^{2i\theta}-\frac{1}{8}e^{3i\theta}+\ldots$$

$$=\frac{1}{1-\left(-\frac{1}{2}e^{i\theta}\right)}=\frac{1}{1+\frac{1}{2}e^{i\theta}}=\frac{2}{2+e^{i\theta}}=\frac{2\left(2+e^{-i\theta}\right)}{\left(2+e^{i\theta}\right)\left(2+e^{-i\theta}\right)}=\frac{4+2e^{-i\theta}}{5+4\cos\theta}$$

Equating real and imaginary parts then gives

$$C=\frac{4+2\cos\theta}{5+4\cos\theta}$$

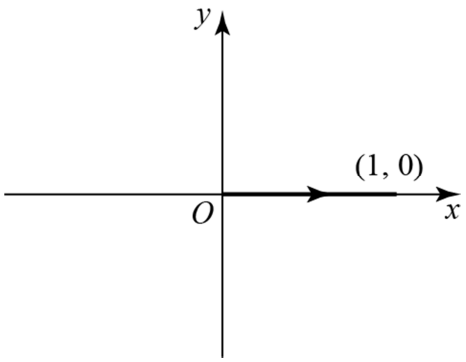
$$S=\frac{2\sin\theta}{5+4\cos\theta}$$

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Complex numbers 1F

1 a $z^4 - 1 = 0$

$z^4 = 1$



for 1, $r = 1$ and $\theta = 0$

So $z^4 = 1(\cos 0 + \mathrm{i}\sin 0)$

$z^4 = \cos(0 + 2k\pi) + \mathrm{i}\sin(0 + 2k\pi) \quad k \in \mathbb{Z}$

Hence, $z = [\cos(2k\pi) + \mathrm{i}\sin(2k\pi)]^{\frac{1}{4}}$

$$z = \cos\left(\frac{2k\pi}{4}\right) + \mathrm{i}\sin\left(\frac{2k\pi}{4}\right)$$

de Moivre's Theorem.

$$z = \cos\left(\frac{k\pi}{2}\right) + \mathrm{i}\sin\left(\frac{k\pi}{2}\right)$$

$k = 0,\ z = \cos 0 + \mathrm{i}\sin 0 = 1$

$k = 1,\ z = \cos\frac{\pi}{2} + \mathrm{i}\sin\frac{\pi}{2} = \mathrm{i}$

$k = 2,\ z = \cos\pi + \mathrm{i}\sin\pi = -1$

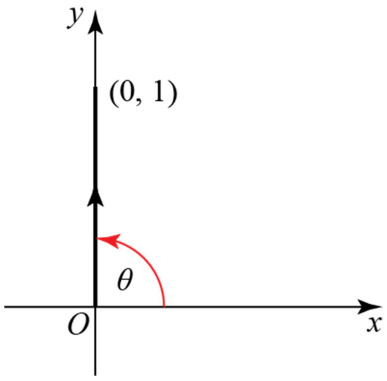
$k = -1,\ z = \cos\left(-\frac{\pi}{2}\right) + \mathrm{i}\sin\left(-\frac{\pi}{2}\right) = -\mathrm{i}$

Therefore, $z = 1,\ \mathrm{i},\ -1,\ -\mathrm{i}$

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1 b $z^3 - \mathrm{i} = 0$

$z^3 = \mathrm{i}$



for i, $r = 1$ and $\theta = \frac{\pi}{2}$

So $z^3 = 1\left(\cos\frac{\pi}{2} + \mathrm{i}\sin\frac{\pi}{2}\right)$

$z^3 = \cos\left(\frac{\pi}{2} + 2k\pi\right) + \mathrm{i}\sin\left(\frac{\pi}{2} + 2k\pi\right), k \in \mathbb{Z}$

Hence, $z = \left[\cos\left(\frac{\pi}{2} + 2k\pi\right) + \mathrm{i}\sin\left(\frac{\pi}{2} + 2k\pi\right)\right]^{\frac{1}{3}}$

$z = \cos\left(\frac{\frac{\pi}{2} + 2k\pi}{3}\right) + \mathrm{i}\sin\left(\frac{\frac{\pi}{2} + 2k\pi}{3}\right)$

de Moivre's Theorem.

$z = \cos\left(\frac{\pi}{6} + \frac{2k\pi}{3}\right) + \mathrm{i}\sin\left(\frac{\pi}{6} + \frac{2k\pi}{3}\right)$

$\therefore k = 0,\ z = \cos\frac{\pi}{6} + \mathrm{i}\sin\frac{\pi}{6} = \frac{\sqrt{3}}{2} + \frac{1}{2}\mathrm{i}$

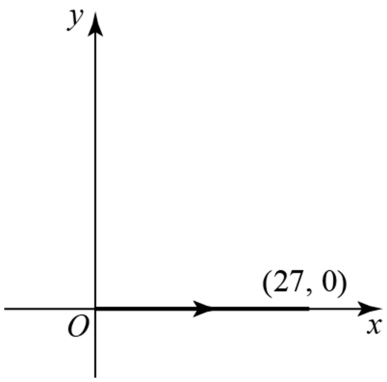
$k = 1,\ z = \cos\frac{5\pi}{6} + \mathrm{i}\sin\frac{5\pi}{6} = -\frac{\sqrt{3}}{2} + \frac{1}{2}\mathrm{i}$

$k = -1,\ z = \cos\left(-\frac{\pi}{2}\right) + \mathrm{i}\sin\left(-\frac{\pi}{2}\right) = 0 - \mathrm{i}$

Therefore, $z = \frac{\sqrt{3}}{2} + \frac{1}{2}\mathrm{i}, -\frac{\sqrt{3}}{2} + \frac{1}{2}\mathrm{i}, -\mathrm{i}$

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1 c $z^3 = 27$



for 27, $r = 27$ and $\theta = 0$

So $z^3 = 27(\cos 0 + \mathrm{i}\sin 0)$

$z^3 = 27[\cos(0 + 2k\pi) + \mathrm{i}\sin(0 + 2k\pi)] \quad k \in \mathbb{Z}$

Hence, $z = [27(\cos 2k\pi) + \mathrm{i}\sin(2k\pi)]^{\frac{1}{3}}$

de Moivre's Theorem.

$$z = 3\left[\cos\left(\frac{2k\pi}{3}\right) + \mathrm{i}\sin\left(\frac{2k\pi}{3}\right)\right]$$

$k = 0;\ z = 3(\cos 0 + \mathrm{i}\sin 0) = 3$

$$k = 1;\ z = 3\left(\cos\frac{2\pi}{3} + \mathrm{i}\sin\frac{2\pi}{3}\right) = 3\left(-\frac{1}{2} + \frac{\sqrt{3}}{2}\mathrm{i}\right) = -\frac{3}{2} + \frac{3\sqrt{3}}{2}\mathrm{i}$$

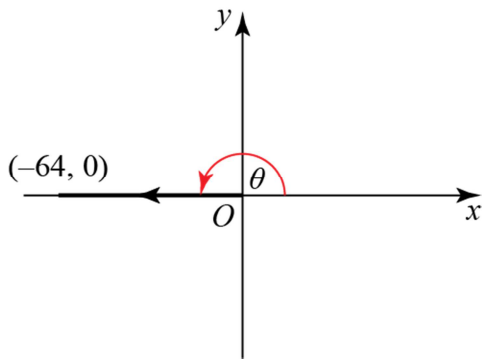
$$k = -1;\ z = 3\left(\cos\left(\frac{-2\pi}{3}\right) + \mathrm{i}\sin\left(\frac{-2\pi}{3}\right)\right) = 3\left(-\frac{1}{2} + \frac{\sqrt{3}}{2}\mathrm{i}\right) = -\frac{3}{2} - \frac{3\sqrt{3}}{2}\mathrm{i}$$

Therefore, $z = 3, -\frac{3}{2} + \frac{3\sqrt{3}}{2}\mathrm{i}, -\frac{3}{2} - \frac{3\sqrt{3}}{2}\mathrm{i}$

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1 d $z^4 + 64 = 0$

$z^4 = -64$



for -64, $r = 64$ and $\theta = \pi$

So $z^4 = 64(\cos\pi + i\sin\pi)$

$z^4 = 64(\cos(\pi + 2k\pi) + i\sin(\pi + 2k\pi)) \quad k \in \mathbb{Z}$

Hence, $z = [64(\pi + 2k\pi) + i\sin(\pi + 2k\pi))]^{\frac{1}{4}}$

$$z = 64^{\frac{1}{4}}\left(\cos\left(\frac{\pi + 2k\pi}{4}\right) + i\sin\left(\frac{\pi + 2k\pi}{4}\right)\right)$$

de Moivre's Theorem.

$$z = 2\sqrt{2}\left(\cos\left(\frac{\pi}{4} + \frac{k\pi}{2}\right) + i\sin\left(\frac{\pi}{4} + \frac{k\pi}{2}\right)\right)$$

$k = 0;\ z = 2\sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right) = 2\sqrt{2}\left(\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right) = 2 + 2i$

$k = 1;\ z = 2\sqrt{2}\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{2}\right) = 2\sqrt{2}\left(-\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i\right) = -2 + 2i$

$k = -1;\ z = 2\sqrt{2}\left(\cos\left(-\frac{\pi}{4}\right) + i\sin\left(-\frac{\pi}{4}\right)\right) = 2\sqrt{2}\left(\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i\right) = 2 - 2i$

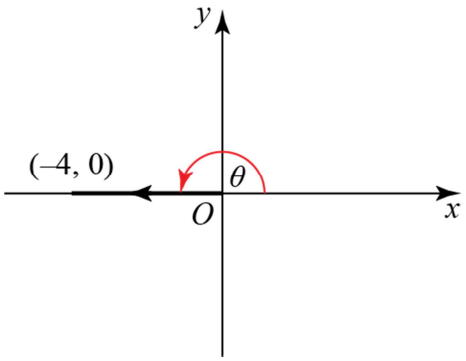
$k = -2;\ z = 2\sqrt{2}\left(\cos\left(-\frac{3\pi}{4}\right) + i\sin\left(-\frac{3\pi}{4}\right)\right) = 2\sqrt{2}\left(-\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i\right) = -2 - 2i$

Therefore, $z = 2 + 2i,\ -2 + 2i,\ 2 - 2i, -2 - 2i$

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1 e $z^4 + 4 = 0$

$z^4 = -4$



for -4, $r = 4$ and $\theta = \pi$

So $z^4 = 4(\cos\pi + i\sin\pi)$

$z^4 = 4(\cos(\pi + 2k\pi) + i\sin(\pi + 2k\pi)) \quad k \in \mathbb{Z}$

Hence, $z = [4(\cos(\pi + 2k\pi) + i\sin(\pi + 2k\pi))]^{\frac{1}{4}}$

$$z = 4^{\frac{1}{4}}\left(\cos\left(\frac{\pi + 2k\pi}{4}\right) + i\sin\left(\frac{\pi + 2k\pi}{4}\right)\right)$$

de Moivre's Theorem.

$$z = \sqrt{2}\left(\cos\left(\frac{\pi}{4} + \frac{k\pi}{2}\right) + i\sin\left(\frac{\pi}{4} + \frac{k\pi}{2}\right)\right)$$

$k = 0;\ z = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right) = \sqrt{2}\left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i\right) = 1 + i$

$k = 1;\ z = \sqrt{2}\left(\cos\left(\frac{3\pi}{4}\right) + i\sin\left(\frac{3\pi}{4}\right)\right) = \sqrt{2}\left(-\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i\right) = -1 + i$

$k = -1;\ z = \sqrt{2}\left(\cos\left(-\frac{\pi}{4}\right) + i\sin\left(-\frac{\pi}{4}\right)\right) = \sqrt{2}\left(\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i\right) = 1 - i$

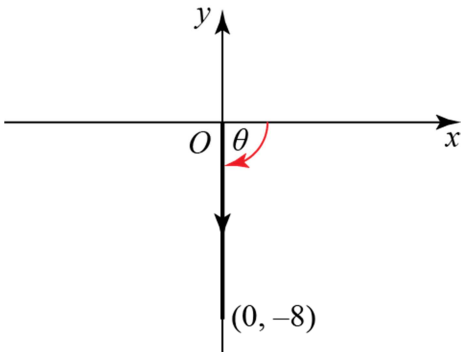
$k = -2;\ z = \sqrt{2}\left(\cos\left(-\frac{3\pi}{4}\right) + i\sin\left(-\frac{3\pi}{4}\right)\right) = \sqrt{2}\left(-\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i\right) = -1 - i$

Therefore, $z = 1 + i,\ -1 + i,\ 1 - i,\ -1 - i$

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1 f $z^3 + 8\mathrm{i} = 0$

$z^3 = -8\mathrm{i}$



for $-8\mathrm{i}$, $r = 8$, $\theta = -\frac{\pi}{2}$

So $z^3 = 8\left(\cos\left(-\frac{\pi}{2}\right) + \mathrm{i}\sin\left(-\frac{\pi}{2}\right)\right)$

$z^4 = 8\left(\cos\left(-\frac{\pi}{2} + 2k\pi\right) + \mathrm{i}\sin\left(-\frac{\pi}{2} + 2k\pi\right)\right) \quad k \in \mathbb{Z}$

Hence, $z = \left[8\left(\cos\left(-\frac{\pi}{2} + 2k\pi\right) + \mathrm{i}\sin\left(-\frac{\pi}{2} + 2k\pi\right)\right)\right]^{\frac{1}{3}}$

$z = 8^{\frac{1}{3}}\left(\cos\left(\frac{-\frac{\pi}{2} + 2k\pi}{3}\right) + \mathrm{i}\sin\left(\frac{-\frac{\pi}{2} + 2k\pi}{3}\right)\right)$ de Moivre's Theorem.

$z = 2\left(\cos\left(-\frac{\pi}{6} + \frac{2k\pi}{3}\right) + \mathrm{i}\sin\left(-\frac{\pi}{6} + \frac{2k\pi}{3}\right)\right)$

$k = 0;\ z = 2\left(\cos\left(-\frac{\pi}{6}\right) + \mathrm{i}\sin\left(-\frac{\pi}{6}\right)\right) = 2\left(\frac{\sqrt{3}}{2} - \frac{1}{2}\mathrm{i}\right) = \sqrt{3} - \mathrm{i}$

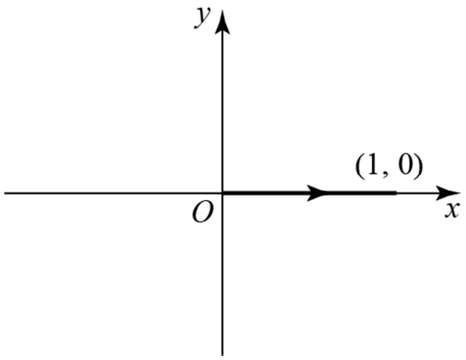
$k = 1;\ z = 2\left(\cos\frac{\pi}{2} + \mathrm{i}\sin\frac{\pi}{2}\right) = 2(0 + \mathrm{i}) = 2\mathrm{i}$

$k = -1;\ z = 2\left(\cos\left(-\frac{5\pi}{6}\right) + \mathrm{i}\sin\left(-\frac{5\pi}{6}\right)\right) = 2\left(-\frac{\sqrt{3}}{2} - \frac{1}{2}\mathrm{i}\right) = -\sqrt{3} - \mathrm{i}$

Therefore, $z = \sqrt{3} - \mathrm{i},\ 2\mathrm{i},\ -\sqrt{3} - \mathrm{i}$

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2 a $z^7 = 1$



for 1, $r = 1$ and $\theta = 0$

So $z^7 = 1(\cos 0 + i\sin 0)$

$z^7 = \cos(0 + 2k\pi) + i\sin(0 + 2k\pi) \quad k \in \mathbb{Z}$

Hence, $z = (\cos(2k\pi) + i\sin(2k\pi))^{\frac{1}{7}}$

$$z = \cos\left(\frac{2k\pi}{7}\right) + i\sin\left(\frac{2k\pi}{7}\right)$$

de Moivre's Theorem.

$k = 0,\ z = \cos 0 + i\sin 0$

$k = 1,\ z = \cos\left(\frac{2\pi}{7}\right) + i\sin\left(\frac{2\pi}{7}\right)$

$k = 2,\ z = \cos\left(\frac{4\pi}{7}\right) + i\sin\left(\frac{4\pi}{7}\right)$

$k = 3,\ z = \cos\left(\frac{6\pi}{7}\right) + i\sin\left(\frac{6\pi}{7}\right)$

$k = -1,\ z = \cos\left(-\frac{2\pi}{7}\right) + i\sin\left(-\frac{2\pi}{7}\right)$

$k = -2,\ z = \cos\left(-\frac{4\pi}{7}\right) + i\sin\left(-\frac{4\pi}{7}\right)$

$k = -3,\ z = \cos\left(-\frac{6\pi}{7}\right) + i\sin\left(-\frac{6\pi}{7}\right)$

Therefore, $z = \cos 0 + i\sin 0,\ \cos\frac{2\pi}{7} + i\sin\frac{2\pi}{7}$

$\cos\frac{4\pi}{7} + i\sin\frac{4\pi}{7},\ \cos\frac{6\pi}{7} + i\sin\frac{6\pi}{7}$

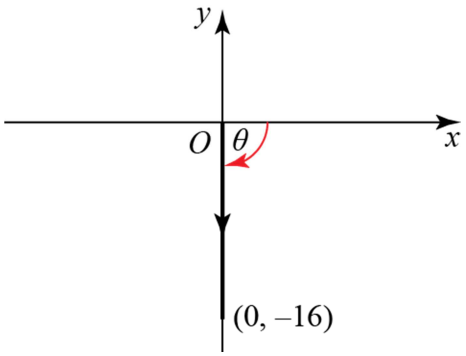
$\cos\left(-\frac{2\pi}{7}\right) + i\sin\left(-\frac{2\pi}{7}\right),\ \cos\left(-\frac{4\pi}{7}\right) + i\sin\left(-\frac{4\pi}{7}\right)$

$\cos\left(-\frac{6\pi}{7}\right) + i\sin\left(-\frac{6\pi}{7}\right)$

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2 b $z^4 + 16\text{i} = 0$

$z^4 = -16\text{i}$



for -16i, $r = 16$ and $\theta = -\frac{\pi}{2}$

So $z^4 = 16\left(\cos\left(-\frac{\pi}{2}\right) + \text{i}\sin\left(-\frac{\pi}{2}\right)\right)$

$z^4 = 16\left(\cos\left(-\frac{\pi}{2} + 2k\pi\right) + \text{i}\sin\left(-\frac{\pi}{2} + 2k\pi\right)\right) \quad k \in \mathbb{Z}$

Hence, $z = \left[16\left(\cos\left(-\frac{\pi}{2} + 2k\pi\right) + \text{i}\sin\left(-\frac{\pi}{2} + 2k\pi\right)\right)\right]^{\frac{1}{4}}$

$z = 16^{\frac{1}{4}}\left(\cos\left(\frac{-\frac{\pi}{2} + 2k\pi}{4}\right) + \text{i}\sin\left(\frac{-\frac{\pi}{2} + 2k\pi}{4}\right)\right)$ de Moivre's Theorem.

$z = \left(\cos\left(-\frac{\pi}{8} + \frac{k\pi}{2}\right) + \text{i}\sin\left(-\frac{\pi}{8} + \frac{k\pi}{2}\right)\right)$

$k = 0,\ z = 2\left(\cos\left(-\frac{\pi}{8}\right) + \text{i}\sin\left(-\frac{\pi}{8}\right)\right)$

$k = 1,\ z = 2\left(\cos\frac{3\pi}{8} + \text{i}\sin\frac{3\pi}{8}\right)$

$k = 2,\ z = 2\left(\cos\frac{7\pi}{8} + \text{i}\sin\frac{7\pi}{8}\right)$

$k = -1,\ z = 2\left(\cos\left(-\frac{5\pi}{8}\right) + \text{i}\sin\left(-\frac{5\pi}{8}\right)\right)$

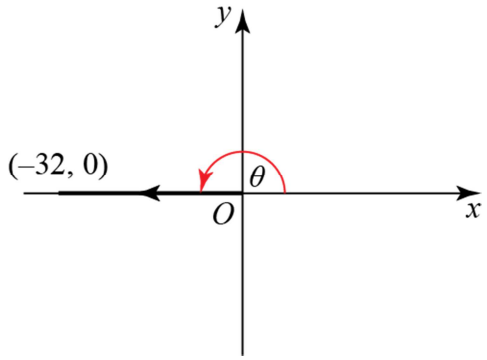
Therefore, $z = 2\left(\cos\left(-\frac{\pi}{8}\right) + \text{i}\sin\left(-\frac{\pi}{8}\right)\right), 2\left(\cos\left(\frac{3\pi}{8}\right) + \text{i}\sin\left(\frac{3\pi}{8}\right)\right)$

$2\left(\cos\left(\frac{7\pi}{8}\right) + \text{i}\sin\left(\frac{7\pi}{8}\right)\right), 2\left(\cos\left(-\frac{5\pi}{8}\right) + \text{i}\sin\left(-\frac{5\pi}{8}\right)\right)$

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2 c $z^5 + 32 = 0$

$z^5 = -32$



for -32, $r = 32$ and $\theta = \pi$

So $z^5 = 32(\cos\pi + i\sin\pi)$

$z^5 = 32(\cos(\pi + 2k\pi) + i\sin(\pi + 2k\pi)) \quad k \in \mathbb{Z}$

Hence, $z = [32(\cos(\pi + 2k\pi) + i\sin(\pi + 2k\pi))]^{\frac{1}{5}}$

$$z = 32^{\frac{1}{5}}\left(\cos\left(\frac{\pi + 2k\pi}{5}\right) + i\sin\left(\frac{\pi + 2k\pi}{5}\right)\right)$$

$$z = 2\left(\cos\left(\frac{\pi}{5} + \frac{2k\pi}{5}\right) + i\sin\left(\frac{\pi}{5} + \frac{2k\pi}{5}\right)\right)$$

de Moivre's Theorem.

$k = 0,\ z = 2\left(\cos\frac{\pi}{5} + i\sin\frac{\pi}{5}\right)$

$k = 1,\ z = 2\left(\cos\frac{3\pi}{5} + i\sin\frac{3\pi}{5}\right)$

$k = 1,\ z = 2(\cos\pi + i\sin\pi)$

$k = 2,\ z = 2\left(\cos\left(-\frac{\pi}{5}\right) + i\sin\left(-\frac{\pi}{5}\right)\right)$

$k = -1,\ z = 2\left(\cos\left(-\frac{5\pi}{8}\right) + i\sin\left(-\frac{5\pi}{8}\right)\right)$

$k = -2,\ z = 2\left(\cos\left(-\frac{3\pi}{5}\right) + i\sin\left(-\frac{3\pi}{5}\right)\right)$

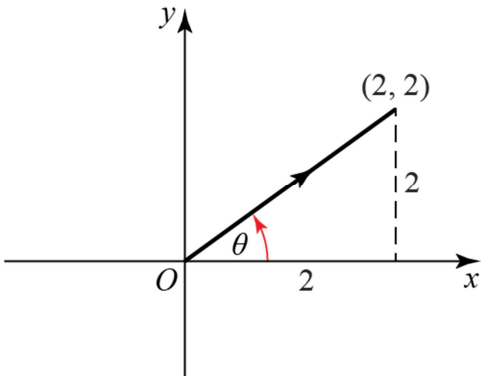
Therefore, $z = 2\left(\cos\frac{\pi}{5} + i\sin\frac{\pi}{5}\right), 2\left(\cos\frac{3\pi}{5} + i\sin\frac{3\pi}{5}\right),$

$2(\cos\pi + i\sin\pi), 2\left(\cos\left(-\frac{\pi}{5}\right) + i\sin\left(-\frac{\pi}{5}\right)\right),$

$2\left(\cos\left(-\frac{3\pi}{5}\right) + i\sin\left(-\frac{3\pi}{5}\right)\right)$

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2 d $z^3 = 2 + 2\text{i}$



$r = \sqrt{2^2 + 2^2} = \sqrt{4+4} = \sqrt{8} = 2\sqrt{2}$

$\theta = \tan^{-1}\left(\frac{2}{2}\right) = \frac{\pi}{4}$

So $z^3 = 2\sqrt{2}\left(\cos\frac{\pi}{4} + \text{i}\sin\frac{\pi}{4}\right)$

$$z^3 = 2\sqrt{2}\left(\cos\left(\frac{\pi}{4} + 2k\pi\right) + \text{i}\sin\left(\frac{\pi}{4} + 2k\pi\right)\right) \quad k \in \mathbb{Z}$$

Hence, $z = \left[2\sqrt{2}\left(\cos\left(\frac{\pi}{4} + 2k\pi\right) + \text{i}\sin\left(\frac{\pi}{4} + 2k\pi\right)\right)\right]^{\frac{1}{3}}$

$$z = (2\sqrt{2})^{\frac{1}{3}}\left(\cos\left(\frac{\frac{\pi}{4} + 2k\pi}{3}\right) + \text{i}\sin\left(\frac{\frac{\pi}{4} + 2k\pi}{3}\right)\right)$$

de Moivre's Theorem.

$$z = \sqrt{2}\left(\cos\left(\frac{\pi}{12} + \frac{2k\pi}{3}\right) + \text{i}\sin\left(\frac{\pi}{12} + \frac{2k\pi}{3}\right)\right)$$

$k = 0,\ z = \sqrt{2}\left(\cos\frac{\pi}{12} + \text{i}\sin\frac{\pi}{12}\right)$

$k = 1,\ z = \sqrt{2}\left(\cos\frac{3\pi}{4} + \text{i}\sin\frac{3\pi}{4}\right)$

$k = -1,\ z = \sqrt{2}\left(\cos\left(-\frac{7\pi}{12}\right) + \text{i}\sin\left(\frac{-7\pi}{12}\right)\right)$

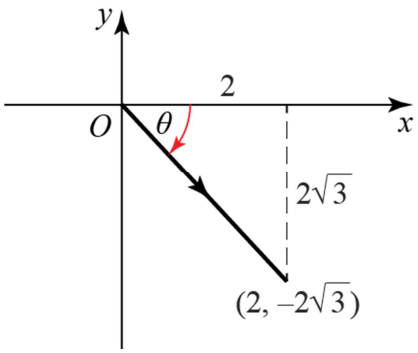
Therefore, $z = \sqrt{2}\left(\cos\frac{\pi}{12} + \text{i}\sin\frac{\pi}{12}\right), \sqrt{2}\left(\cos\frac{3\pi}{4} + \text{i}\sin\frac{3\pi}{4}\right),$

$$\sqrt{2}\left(\cos\left(-\frac{7\pi}{12}\right) + \text{i}\sin\left(\frac{-7\pi}{12}\right)\right)$$

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2 e $z^4 + 2\sqrt{3}\,\mathrm{i} = 2$

$z^4 = 2 - 2\sqrt{3}\,\mathrm{i}$



$r = \sqrt{2^2 + (-2\sqrt{3})^2} = \sqrt{4+12} = \sqrt{16} = 4$

$\theta = -\tan^{-1}\left(\frac{2\sqrt{3}}{2}\right) = -\frac{\pi}{3}$

So $z^4 = 4\left(\cos\left(-\frac{\pi}{3}\right) + \mathrm{i}\sin\left(-\frac{\pi}{3}\right)\right)$

$z^4 = 4\left(\cos\left(-\frac{\pi}{3} + 2k\pi\right) + \mathrm{i}\sin\left(-\frac{\pi}{3} + 2k\pi\right)\right) \quad k \in \mathbb{Z}$

Hence, $z = \left[4\left[\cos\left(-\frac{\pi}{3} + 2k\pi\right) + \mathrm{i}\sin\left(-\frac{\pi}{3} + 2k\pi\right)\right]\right]^{\frac{1}{4}}$

$z = 4^{\frac{1}{4}}\left(\cos\left(\frac{-\frac{\pi}{3} + 2k\pi}{4}\right) + \mathrm{i}\sin\left(\frac{-\frac{\pi}{3} + 2k\pi}{4}\right)\right)$ de Moivre's Theorem.

$z = \sqrt{2}\left(\cos\left(-\frac{\pi}{12} + \frac{k\pi}{2}\right) + \mathrm{i}\sin\left(-\frac{\pi}{12} + \frac{k\pi}{2}\right)\right)$

$k = 0,\ z = \sqrt{2}\left(\cos\left(-\frac{\pi}{12}\right) + \mathrm{i}\sin\left(-\frac{\pi}{12}\right)\right)$

$k = 1,\ z = \sqrt{2}\left(\cos\left(\frac{\pi}{12}\right) + \mathrm{i}\sin\left(\frac{5\pi}{12}\right)\right)$

$k = 1,\ z = \sqrt{2}\left(\cos\left(\frac{11\pi}{12}\right) + \mathrm{i}\sin\left(\frac{11\pi}{12}\right)\right)$

$k = -1, z = \sqrt{2}\left(\cos\left(-\frac{7\pi}{12}\right) + \mathrm{i}\sin\left(-\frac{7\pi}{12}\right)\right)$

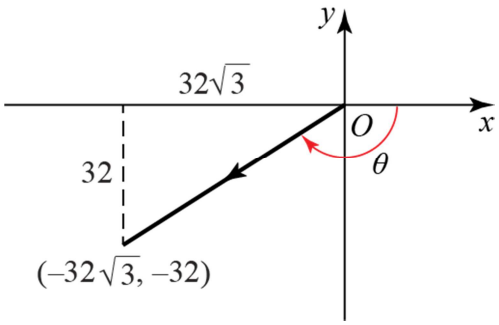
Therefore, $z = \sqrt{2}\left(\cos\left(-\frac{\pi}{12}\right) + \mathrm{i}\sin\left(-\frac{\pi}{12}\right)\right), \sqrt{2}\left(\cos\left(\frac{5\pi}{12}\right) + \mathrm{i}\sin\left(\frac{5\pi}{12}\right)\right),$

$\sqrt{2}\left(\cos\left(\frac{11\pi}{12}\right) + \mathrm{i}\sin\left(\frac{11\pi}{12}\right)\right), \sqrt{2}\left(\cos\left(-\frac{7\pi}{12}\right) + \mathrm{i}\sin\left(-\frac{7\pi}{12}\right)\right)$

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2 f $z^3 + 32\sqrt{3} + 32\mathrm{i} = 0$

$z^3 = -32\sqrt{3} - 32\mathrm{i}$



$r = \sqrt{(-32\sqrt{3})^2 + (-32)^2} = \sqrt{3072 + 1024} = \sqrt{4096} = 64$

$\theta = -\pi + \tan^{-1}\left(\frac{32}{32\sqrt{3}}\right) = -\pi + \frac{\pi}{6} = -\frac{5\pi}{6}$

So $z^4 = 64\left(\cos\left(-\frac{5\pi}{6}\right) + \mathrm{i}\sin\left(-\frac{5\pi}{6}\right)\right)$

$z^3 = 64\left(\cos\left(-\frac{5\pi}{6} + 2k\pi\right) + \mathrm{i}\sin\left(-\frac{5\pi}{6} + 2k\pi\right)\right) \quad k \in \mathbb{Z}$

Hence, $z = \left[64\left(\cos\left(-\frac{5\pi}{6} + 2k\pi\right) + \mathrm{i}\sin\left(-\frac{5\pi}{6} + 2k\pi\right)\right)\right]^{\frac{1}{3}}$

$z = 64^{\frac{1}{3}}\left(\cos\left(\frac{-\frac{5\pi}{6} + 2k\pi}{3}\right) + \mathrm{i}\sin\left(\frac{-\frac{5\pi}{6} + 2k\pi}{3}\right)\right)$ de Moivre's Theorem.

$z = 4\left(\cos\left(-\frac{5\pi}{18} + \frac{2k\pi}{3}\right) + \mathrm{i}\sin\left(-\frac{5\pi}{18} + \frac{2k\pi}{3}\right)\right)$

$k = 0,\ z = 4\left(\cos\left(-\frac{5\pi}{18}\right) + \mathrm{i}\sin\left(-\frac{5\pi}{18}\right)\right)$

$k = 1,\ z = 4\left(\cos\left(\frac{7\pi}{18}\right) + \mathrm{i}\sin\left(\frac{7\pi}{18}\right)\right)$

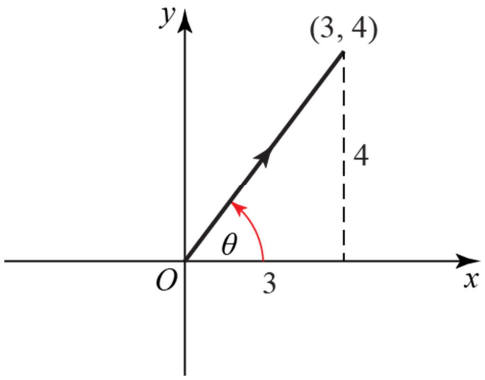
$k = -1,\ z = 4\left(\cos\left(-\frac{17\pi}{18}\right) + \mathrm{i}\sin\left(-\frac{17\pi}{18}\right)\right)$

Therefore, $z = 4\left(\cos\left(-\frac{5\pi}{18}\right) + \mathrm{i}\sin\left(-\frac{5\pi}{18}\right)\right), 4\left(\cos\left(\frac{7\pi}{18}\right) + \mathrm{i}\sin\left(\frac{7\pi}{18}\right)\right),$

$4\left(\cos\left(-\frac{17\pi}{18}\right) + \mathrm{i}\sin\left(-\frac{17\pi}{18}\right)\right)$

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3 a $z^4 = 3 + 4\mathrm{i}$



$r = \sqrt{3^2 + 4^2} = \sqrt{25} = 5$

$\theta = \tan^{-1}\left(\frac{4}{3}\right) = 0.927295\ldots$

So $z^4 = 5\mathrm{e}^{\mathrm{i}(0.927295\ldots)}$

$z^4 = 5\mathrm{e}^{\mathrm{i}(0.927295\ldots + 2k\pi)}, \quad k \in \mathbb{Z}$

Hence, $z = [5\mathrm{e}^{\mathrm{i}(0.927295\ldots + 2k\pi)}]^{\frac{1}{4}}$

$= 5^{\frac{1}{4}}\mathrm{e}^{\mathrm{i}\left(\frac{0.927295\ldots + 2k\pi}{4}\right)}$ de Moivre's Theorem.

$= 5^{\frac{1}{4}}\mathrm{e}^{\mathrm{i}\left(\frac{0.927295\ldots}{4} + \frac{k\pi}{2}\right)}$

$k = 0,\ z = 5^{\frac{1}{4}}\mathrm{e}^{\mathrm{i}(0.2318\ldots)}$

$k = 1,\ z = 5^{\frac{1}{4}}\mathrm{e}^{\mathrm{i}(1.8026\ldots)}$

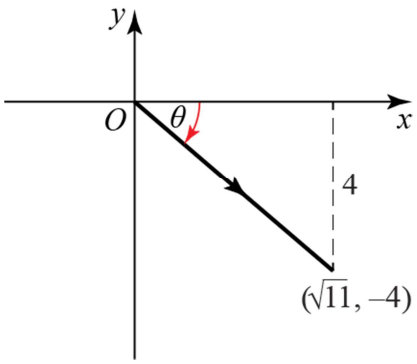
$k = -1,\ z = 5^{\frac{1}{4}}\mathrm{e}^{\mathrm{i}(-1.3389\ldots)}$

$k = -2,\ z = 5^{\frac{1}{4}}\mathrm{e}^{\mathrm{i}(-2.9097\ldots)}$

Therefore, $z = 5^{\frac{1}{4}}\mathrm{e}^{0.23\mathrm{i}}, 5^{\frac{1}{4}}\mathrm{e}^{1.80\mathrm{i}}, 5^{\frac{1}{4}}\mathrm{e}^{-1.34\mathrm{i}}, 5^{\frac{1}{4}}\mathrm{e}^{-2.91\mathrm{i}}$

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3 b $z^3 = \sqrt{11} + 4\mathrm{i}$



$r = \sqrt{(\sqrt{11})^2 + (-4)^2} = \sqrt{11+16} = \sqrt{27}$

$\theta = -\tan^{-1}\left(\dfrac{4}{\sqrt{11}}\right) = 0.878528\ldots$

So, $z^3 = \sqrt{27}\,\mathrm{e}^{\mathrm{i}(-0.878528\ldots)}$

$z^3 = \sqrt{27}\,\mathrm{e}^{\mathrm{i}(-0.878528\ldots+2k\pi)}, \quad k \in \mathbb{Z}$

Hence, $z = [\sqrt{270}\,\mathrm{e}^{\mathrm{i}(-0.878528\ldots+2k\pi)}]^{\frac{1}{3}}$

de Moivre's Theorem.

$= (\sqrt{27})^{\frac{1}{3}}\mathrm{e}^{\mathrm{i}\left(\frac{-0.878528\ldots+2k\pi}{3}\right)}$

$= \sqrt{3}\,\mathrm{e}^{\mathrm{i}\frac{-0.878528\ldots}{3}+\frac{2k\pi}{3}}$

$k = 0,\ z = \sqrt{3}\,\mathrm{e}^{\mathrm{i}(-0.2928\ldots)}$

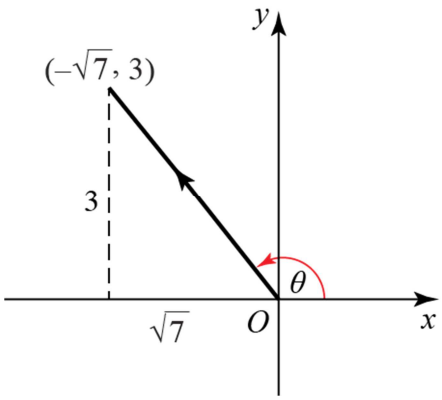
$k = 1,\ z = \sqrt{3}\,\mathrm{e}^{\mathrm{i}(1.8015\ldots)}$

$k = -1,\ z = \sqrt{3}\,\mathrm{e}^{\mathrm{i}(-2.3872\ldots)}$

Therefore, $z = \sqrt{3}\,\mathrm{e}^{-0.29\mathrm{i}}, \sqrt{3}\,\mathrm{e}^{1.80\mathrm{i}}, \sqrt{3}\,\mathrm{e}^{-2.39\mathrm{i}}$

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3 c $z^4 = -\sqrt{7} + 3\mathrm{i}$



$r = \sqrt{(-\sqrt{7})^2 + 3^2} = \sqrt{7+9} = \sqrt{16} = 4$

$\theta = \pi - \tan^{-1}\left(\frac{3}{\sqrt{7}}\right) = 2.293530\ldots$

So, $z^4 = 4\mathrm{e}^{\mathrm{i}(2.293530\ldots)}$

$z^4 = 4\mathrm{e}^{\mathrm{i}(2.293530\ldots + 2k\pi)}, \quad k \in \mathbb{Z}$

Hence, $z = [4\,\mathrm{e}^{\mathrm{i}(2.293530\ldots + 2k\pi)}]^{\frac{1}{4}}$

$= 4^{\frac{1}{4}}\,\mathrm{e}^{\mathrm{i}\left(\frac{2.293530\ldots + 2k\pi}{4}\right)}$ de Moivre's Theorem.

$= \sqrt{2}\,\mathrm{e}^{\mathrm{i}\left(\frac{2.293530\ldots}{4} + \frac{k\pi}{2}\right)}$

$k = 0,\ z = \sqrt{2}\,\mathrm{e}^{\mathrm{i}(0.5733\ldots)}$

$k = 1,\ z = \sqrt{2}\,\mathrm{e}^{\mathrm{i}(2.1441\ldots)}$

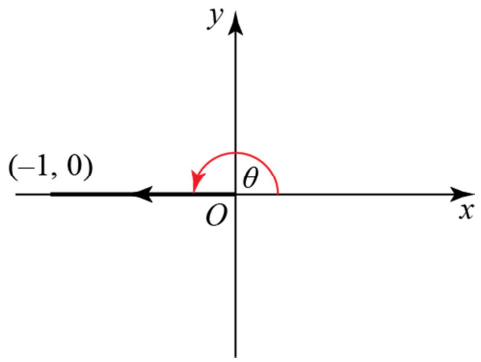
$k = -1,\ z = \sqrt{2}\,\mathrm{e}^{\mathrm{i}(-0.9974\ldots)}$

$k = -2,\ z = \sqrt{2}\,\mathrm{e}^{\mathrm{i}(-2.5682\ldots)}$

Therefore, $z = \sqrt{2}\,\mathrm{e}^{0.57\mathrm{i}}$, $z = \sqrt{2}\,\mathrm{e}^{2.14\mathrm{i}}$, $z = \sqrt{2}\,\mathrm{e}^{-1.00\mathrm{i}}$, $z = \sqrt{2}\,\mathrm{e}^{-2.57\mathrm{i}}$

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4 a $(z+1)^3 = -1$



For -1, $r = 1$ and $\theta = \pi$

So, $(z+1)^3 = 1(\cos\pi + i\sin\pi)$

$(z+1)^3 = \cos(\pi + 2k\pi) + i\sin(\pi + 2k\pi) \quad k \in \mathbb{Z}$

Hence, $z+1 = [\cos(\pi + 2k\pi) + i\sin(\pi + 2k\pi)]^{\frac{1}{3}}$

de Moivre's Theorem.

$$z+1 = \cos\left(\frac{\pi + 2k\pi}{3}\right) + i\sin\left(\frac{\pi + 2k\pi}{3}\right)$$

$$z+1 = \cos\left(\frac{\pi}{3} + \frac{2k\pi}{3}\right) + i\sin\left(\frac{\pi}{3} + \frac{2k\pi}{3}\right)$$

$k = 0$, $z+1 = \cos\frac{\pi}{3} + i\sin\frac{\pi}{3} = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$

$\Rightarrow z = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$

$k = 1$, $z+1 = \cos\pi + i\sin\pi = -1 + 0i$

$\Rightarrow z = -2$

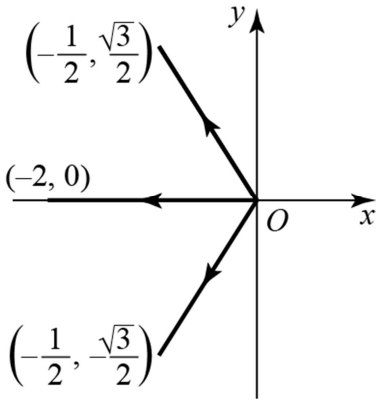
$k = -1$, $z+1 = \cos\left(-\frac{\pi}{3}\right) + i\sin\left(-\frac{\pi}{3}\right) = \frac{1}{2} - \frac{\sqrt{3}}{2}i$

$\Rightarrow z = -\frac{1}{2} - \frac{\sqrt{3}}{2}i$

Therefore, $z = -\frac{1}{2} + \frac{\sqrt{3}}{2}i, -2, -\frac{1}{2} - \frac{\sqrt{3}}{2}i$

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4 b

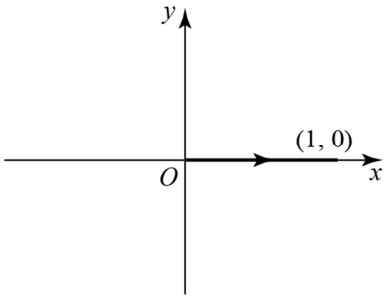


c The solution to $w^3 = -1$, lie on a circle centre (0, 0), radius 1.
As $w = z + 1$, then the three solutions for z are the three solutions for w translated by $\begin{pmatrix} -1 \\ 0 \end{pmatrix}$.

Hence the three points (the solutions for z), lie on a circle centre (–1, 0), radius 1.

5 a $z^5 - 1 = 0$

$z^5 = 1$



For 1, $r = 1$ and $\theta = 0$

So, $z^5 = 1(\cos 0 + i \sin 0)$

$z^5 = \cos(0 + 2k\pi) + i\sin(0 + 2k\pi) \quad k \in \mathbb{Z}$

Hence, $z = [\cos(2k\pi) + i\sin(2k\pi)]^{\frac{1}{5}}$

$$z = \cos\left(\frac{2k\pi}{5}\right) + i\sin\left(\frac{2k\pi}{5}\right)$$

de Moivre's Theorem.

$k = 0,\ z_1 = \cos 0 + i\sin 0 = 1 + i(0) = 1$

$k = 1,\ z_2 = \cos\left(\frac{2\pi}{5}\right) + i\sin\left(\frac{2\pi}{5}\right)$

$k = 2,\ z_3 = \cos\left(\frac{4\pi}{5}\right) + i\sin\left(\frac{4\pi}{5}\right)$

$k = -1,\ z_4 = \cos\left(-\frac{2\pi}{5}\right) + i\sin\left(-\frac{2\pi}{5}\right)$

$k = -2,\ z_5 = \cos\left(-\frac{4\pi}{5}\right) + i\sin\left(-\frac{4\pi}{5}\right)$

Therefore $z = 1, \cos\left(\frac{2\pi}{5}\right) + i\sin\left(\frac{2\pi}{5}\right), \cos\left(\frac{4\pi}{5}\right) + i\sin\left(\frac{4\pi}{5}\right),$

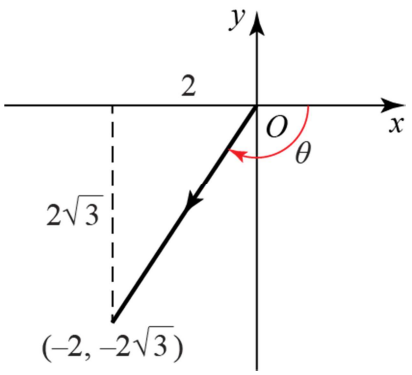
$$\cos\left(-\frac{2\pi}{5}\right) + i\sin\left(-\frac{2\pi}{5}\right), \cos\left(-\frac{4\pi}{5}\right) + i\sin\left(-\frac{4\pi}{5}\right)$$

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5 b So, $z_1+z_2+z_3+z_4+z_5=0$

$$1+\cos\left(\frac{2\pi}{5}\right)+\mathrm{i}\sin\left(\frac{2\pi}{5}\right)+\cos\left(\frac{4\pi}{5}\right)+\mathrm{i}\sin\left(\frac{4\pi}{5}\right)$$
$$+\cos\left(-\frac{2\pi}{5}\right)+\mathrm{i}\sin\left(-\frac{2\pi}{5}\right)+\cos\left(-\frac{4\pi}{5}\right)+\mathrm{i}\sin\left(-\frac{4\pi}{5}\right)=0$$
$$\Rightarrow 1+\cos\left(\frac{2\pi}{5}\right)+\mathrm{i}\sin\left(\frac{2\pi}{5}\right)+\cos\left(\frac{4\pi}{5}\right)+\mathrm{i}\sin\left(\frac{4\pi}{5}\right)$$
$$+\cos\left(\frac{2\pi}{5}\right)-\mathrm{i}\sin\left(\frac{2\pi}{5}\right)+\cos\left(\frac{4\pi}{5}\right)-\mathrm{i}\sin\left(\frac{4\pi}{5}\right)=0$$
$$1+2\cos\left(\frac{2\pi}{5}\right)+2\cos\left(\frac{4\pi}{5}\right)=0$$
$$2\cos\left(\frac{2\pi}{5}\right)+2\cos\left(\frac{4\pi}{5}\right)=-1$$
$$2\left(\cos\left(\frac{2\pi}{5}\right)+\cos\left(\frac{4\pi}{5}\right)\right)=-1$$
$$\cos\left(\frac{2\pi}{5}\right)+\cos\left(\frac{4\pi}{5}\right)=-\frac{1}{2}$$ (as required)

6 a $-2-2\sqrt{3}\,\mathrm{i}$.



$$\text{modulus} = r\sqrt{(-2)^2+(-2\sqrt{3})^2}=\sqrt{4+12}=\sqrt{16}=4$$
$$\text{argument} = \theta = 2\pi+\tan^{-1}\left(\frac{2\sqrt{3}}{2}\right)=-\pi+\frac{\pi}{3}=-\frac{2\pi}{3}$$

Therefore, $r=4$, $\theta=-\frac{2\pi}{3}$

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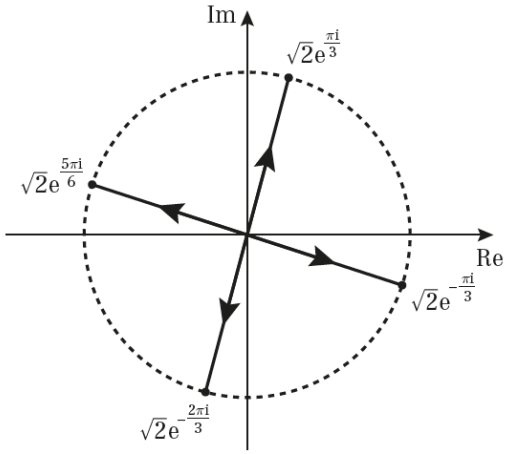
6 b We have $w = -2 - 2i\sqrt{3} = 4\mathrm{e}^{-\frac{2\pi i}{3}}$ we wish to solve

$z^4 + 2 + 2i\sqrt{3} = 0$ which is equivalent to solving $z^4 = -2 - 2i\sqrt{3} = 4\mathrm{e}^{-\frac{2\pi i}{3}}$

The solution has the form $z = \sqrt[4]{4}\mathrm{e}^{i\theta}$ where the argument satisfies

$4\theta = -\frac{2\pi}{3} + 2k\pi$ for $k \in \mathbb{Z}$ hence the 4 distinct solutions have arguments given by

$\theta = -\frac{\pi}{6} + \frac{k\pi}{2}$ for $k = -1, 0, 1, 2$, so $\theta = \sqrt{2}\mathrm{e}^{-\frac{2\pi i}{3}}, \sqrt{2}\mathrm{e}^{-\frac{\pi i}{6}}, \sqrt{2}\mathrm{e}^{\frac{\pi i}{3}}, \sqrt{2}\mathrm{e}^{\frac{5\pi i}{6}}$



7 We wish to solve $z^4 = 2\left(1 - i\sqrt{3}\right)$, to start off let $w = 2\left(1 - i\sqrt{3}\right)$ then if we write $w = r\mathrm{e}^{i\theta}$ then

$r = \sqrt{4 + 12} = 4$

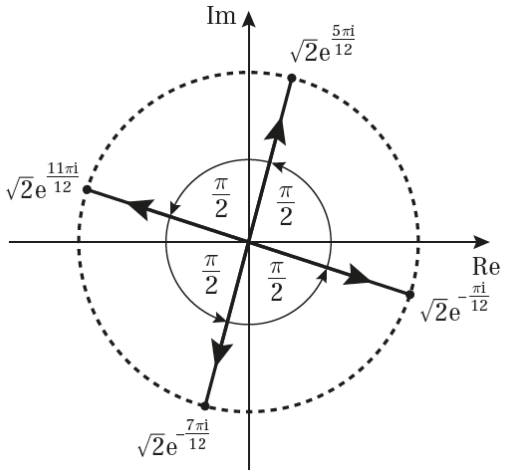
$\tan\theta = -\sqrt{3}$

So $\theta = -\frac{\pi}{3}$ and hence $w = 4\mathrm{e}^{-\frac{\pi i}{3}}$ now going back to the original equation we have

$z^4 = 4\mathrm{e}^{-\frac{\pi i}{3}}$ so that $z = \sqrt[4]{4}\mathrm{e}^{i\theta}$ where the argument must satisfy $4\theta = -\frac{\pi}{3} + 2k\pi$ for $k \in \mathbb{Z}$

So $\theta = -\frac{\pi}{12} + \frac{k\pi}{2}$ hence the 4 distinct roots are given by these values of θ for $k = 0, 1, 2, 3$

Hence $\theta = \sqrt{2}\mathrm{e}^{-\frac{7\pi i}{12}}, \sqrt{2}\mathrm{e}^{-\frac{\pi i}{12}}, \sqrt{2}\mathrm{e}^{\frac{5\pi i}{12}}, \sqrt{2}\mathrm{e}^{\frac{11\pi i}{12}}$



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8 a Let $z=\sqrt{6}+i\sqrt{2}$ then if we write $z=re^{i\theta}$ then we have $r=\sqrt{6+2}=2\sqrt{2}$ and $\tan\theta=\dfrac{\sqrt{2}}{\sqrt{6}}=\dfrac{1}{\sqrt{3}}$ so that $\theta=\dfrac{\pi}{6}$ so we have $z=2\sqrt{2}e^{\frac{\pi i}{6}}$

b We wish to solve

$$\omega^3=z^4=\left(2\sqrt{2}e^{\frac{\pi i}{6}}\right)^4=64e^{\frac{2\pi i}{3}}$$

If we write $\omega=re^{i\theta}$ then $r=\sqrt[3]{64}=4$ and $3\theta=\dfrac{2\pi}{3}+2k\pi$ for $k\in\mathbb{Z}$ so that the distinct solutions correspond to $\theta=\dfrac{2\pi}{9}+\dfrac{2k\pi}{3}$ for $k=0,1,2$, explicitly these are

$$\omega=4e^{\frac{2\pi i}{9}}=4\cos\frac{2\pi}{9}+4i\sin\frac{2\pi}{9}$$

$$\omega=4e^{\frac{8\pi i}{9}}=4\cos\frac{8\pi}{9}+4i\sin\frac{8\pi}{9}$$

$$\omega=4e^{\frac{14\pi i}{9}}=4e^{\frac{-4\pi i}{9}}=4\cos\frac{-4\pi}{9}+4i\sin\frac{-4\pi}{9}=4\cos\frac{4\pi}{9}-4i\sin\frac{4\pi}{9}$$

9 a We wish to solve $1+z+z^2+z^3+z^4+z^5+z^6+z^7=0$ note that the left hand side is a geometric series so if we apply the formula for its sum we get

$$\frac{1-z^8}{1-z}=0$$

Hence the equation is equivalent to $z^8=1$ whose solutions are the 8^{th} roots of unity $z=e^{\frac{2k\pi i}{8}}$ for $0\le k\le 7$ so $z=e^{\frac{\pi i}{4}},i,e^{\frac{3\pi i}{4}},-1,e^{\frac{5\pi i}{4}},-i,e^{\frac{7\pi i}{4}}$

b Now if we have $z^2+1=0$ then $z^2=-1$ so that $z^8=(-1)^4=1$ and hence z is a solution to z^8-1 hence z^2+1 is a factor of z^8-1 but we also have

$$z^8-1=(1-z)(1+z+z^2+z^3+z^4+z^5+z^6+z^7)$$

And since $(1-z)$ is not a factor of z^2+1 it must be the case that z^2+1 is a factor of $1+z+z^2+z^3+z^4+z^5+z^6+z^7$

In the second case we have $z^4+1=0$ and so $z^8=(-1)^2=1$ hence we have that z^4+1 is a factor of z^8-1 and since $(1-z)$ is not a factor of z^4+1 the same argument as above means z^4+1 must be a factor of z^8-1

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Challenge

a We wish to solve $z^6 = 1$, writing $z = e^{i\theta}$ implies we must have $\theta = \frac{2k\pi}{6}$ for $k \in \mathbb{Z}$ hence the 6 distinct solutions in the range $-\pi < \theta \leq \pi$ are given by

$z = e^{-\frac{2\pi i}{3}}, e^{-\frac{\pi i}{3}}, 1, e^{\frac{\pi i}{3}}, e^{\frac{2\pi i}{3}}, -1$

b We wish to solve $(z+1)^6 = z^6$ rearranging gives

$$\frac{z^6}{(z+1)^6} = 1$$

$$\left(\frac{z}{z+1}\right)^6 = 1$$

Hence $\frac{z}{z+1} = e^{\frac{k\pi i}{3}}$ for some $1 \leq k \leq 6$ further rearranging gives

$$z = (z+1)e^{\frac{k\pi i}{3}}$$

$$\left(1 - e^{\frac{k\pi i}{3}}\right)z = e^{\frac{k\pi i}{3}}$$

So

$$z = \frac{e^{\frac{k\pi i}{3}}}{1 - e^{\frac{k\pi i}{3}}} = \frac{\left(1 - e^{-\frac{k\pi i}{3}}\right)e^{\frac{k\pi i}{3}}}{\left(1 - e^{\frac{k\pi i}{3}}\right)\left(1 - e^{-\frac{k\pi i}{3}}\right)} = \frac{\left(1 - e^{-\frac{k\pi i}{3}}\right)e^{\frac{k\pi i}{3}}}{\left(1 - \cos\frac{k\pi}{3}\right)^2 + \sin^2\frac{k\pi}{3}} = \frac{e^{\frac{k\pi i}{3}} - 1}{2 - 2\cos\frac{k\pi}{3}}$$

$$= \frac{\cos\frac{k\pi}{3} - 1 + i\sin\frac{k\pi}{3}}{2 - 2\cos\frac{k\pi}{3}} = -\frac{1}{2} + i\frac{\sin\frac{k\pi}{3}}{2 - 2\cos\frac{k\pi}{3}}$$

Now by the double angle formula we have

$$1 - \cos\frac{k\pi}{3} = 1 - \cos^2\frac{k\pi}{6} + \sin^2\frac{k\pi}{6} = 2\sin^2\frac{k\pi}{6}$$

$$\sin\frac{k\pi}{3} = 2\cos\frac{k\pi}{6}\sin\frac{k\pi}{6}$$

Hence

$$z = -\frac{1}{2} + i\frac{\sin\frac{k\pi}{3}}{2 - 2\cos\frac{k\pi}{3}} = -\frac{1}{2} + i\frac{2\cos\frac{k\pi}{6}\sin\frac{k\pi}{6}}{42\sin^2\frac{k\pi}{6}} = -\frac{1}{2} + \frac{i}{2}\cot\frac{k\pi}{6}$$

and take $k = 1, 2, 3, 4, 5$

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Complex numbers 1G

1 a One vertex corresponds to the complex number $z = 4i = 4\mathrm{e}^{\frac{\pi i}{2}}$, so $z = (0, 4)$. Let $\omega = \mathrm{e}^{\frac{2\pi i}{3}}$ then the other two correspond to

$$z\omega = 4\mathrm{e}^{\frac{\pi i}{2}} \times \mathrm{e}^{\frac{2\pi i}{3}} = 4\mathrm{e}^{\frac{7\pi i}{6}} = 4\cos\frac{7\pi}{6} + 4i\sin\frac{7\pi}{6} = -2\sqrt{3} - 2i = \left(-2\sqrt{3}, -2\right)$$

and

$$z\omega^2 = 4\mathrm{e}^{\frac{\pi i}{2}} \times \mathrm{e}^{\frac{4\pi i}{3}} = 4\mathrm{e}^{\frac{11\pi i}{6}} = 4\cos\frac{11\pi}{6} + 4i\sin\frac{11\pi}{6} = 2\sqrt{3} - 2i = \left(2\sqrt{3}, -2\right)$$

b We are given one vertex corresponds to $z = (5, 0)$. Let $\omega = \mathrm{e}^{\frac{\pi i}{2}}$ be a primitive 4^{th} root of unity then the other three vertices are given by

$$z\omega = 5 \times \mathrm{e}^{\frac{\pi i}{2}} = 5i = (0, 5)$$

$$z\omega^2 = 5 \times \mathrm{e}^{i\pi} = -5 = (-5, 0)$$

$$z\omega^3 = 5 \times \mathrm{e}^{\frac{3\pi i}{2}} = -5i = (0, -5)$$

c We are given that one vertex corresponds to $z = -1 + i\sqrt{3} = 2\mathrm{e}^{\frac{2\pi i}{3}}$ which has the coordinate $(-1, \sqrt{3})$. Let $\omega = \mathrm{e}^{\frac{2\pi i}{5}}$ be a primitive 5^{th} root of unity then the other four vertices are given by

$$z\omega = 2\mathrm{e}^{\frac{2\pi i}{3}} \times \mathrm{e}^{\frac{2\pi i}{5}} = 2\mathrm{e}^{\frac{16\pi i}{15}} = 2\cos\frac{16\pi}{15} + 2i\sin\frac{16\pi}{15} = \left(2\cos\frac{16\pi}{15}, 2\sin\frac{16\pi}{15}\right)$$

$$z\omega^2 = 2\mathrm{e}^{\frac{2\pi i}{3}} \times \mathrm{e}^{\frac{4\pi i}{5}} = 2\mathrm{e}^{\frac{22\pi i}{15}} = 2\cos\frac{22\pi}{15} + 2i\sin\frac{22\pi}{15} = \left(2\cos\frac{22\pi}{15}, 2\sin\frac{22\pi}{15}\right)$$

$$z\omega^3 = 2\mathrm{e}^{\frac{2\pi i}{3}} \times \mathrm{e}^{\frac{6\pi i}{5}} = 2\mathrm{e}^{\frac{28\pi i}{15}} = 2\cos\frac{28\pi}{15} + 2i\sin\frac{28\pi}{15} = \left(2\cos\frac{28\pi}{15}, 2\sin\frac{28\pi}{15}\right)$$

$$z\omega^4 = 2\mathrm{e}^{\frac{2\pi i}{3}} \times \mathrm{e}^{\frac{8\pi i}{5}} = 2\mathrm{e}^{\frac{4\pi i}{15}} = 2\cos\frac{4\pi}{15} + 2i\sin\frac{4\pi}{15} = \left(2\cos\frac{4\pi}{15}, 2\sin\frac{4\pi}{15}\right)$$

d We are given that one vertex corresponds to $z = 2 + 2i = 2\sqrt{2}\mathrm{e}^{\frac{\pi i}{4}}$ which has the coordinate $(2, 2)$

Let $\omega = \mathrm{e}^{\frac{\pi i}{3}}$ be a primitive 6^{th} root of unity then the other five vertices of the hexagon are given by

$$z\omega = 2\sqrt{2}\mathrm{e}^{\frac{\pi i}{4}} \times \mathrm{e}^{\frac{\pi i}{3}} = 2\sqrt{2}\mathrm{e}^{\frac{7\pi i}{12}} = 2\sqrt{2}\cos\frac{7\pi}{12} + 2i\sqrt{2}\sin\frac{7\pi}{12} = \left(2\sqrt{2}\cos\frac{7\pi}{12}, 2\sqrt{2}\sin\frac{7\pi}{12}\right)$$

$$z\omega^2 = 2\sqrt{2}\mathrm{e}^{\frac{\pi i}{4}} \times \mathrm{e}^{\frac{2\pi i}{3}} = 2\sqrt{2}\mathrm{e}^{\frac{11\pi i}{12}} = 2\sqrt{2}\cos\frac{11\pi}{12} + 2i\sqrt{2}\sin\frac{11\pi}{12} = \left(2\sqrt{2}\cos\frac{11\pi}{12}, 2\sqrt{2}\sin\frac{11\pi}{12}\right)$$

$$z\omega^3 = 2\sqrt{2}\mathrm{e}^{\frac{\pi i}{4}} \times \mathrm{e}^{\frac{3\pi i}{3}} = 2\sqrt{2}\mathrm{e}^{\frac{15\pi i}{12}} = 2\sqrt{2}\cos\frac{15\pi}{12} + 2i\sqrt{2}\sin\frac{15\pi}{12} = \left(2\sqrt{2}\cos\frac{15\pi}{12}, 2\sqrt{2}\sin\frac{15\pi}{12}\right)$$

$$z\omega^4 = 2\sqrt{2}\mathrm{e}^{\frac{\pi i}{4}} \times \mathrm{e}^{\frac{4\pi i}{3}} = 2\sqrt{2}\mathrm{e}^{\frac{19\pi i}{12}} = 2\sqrt{2}\cos\frac{19\pi}{12} + 2i\sqrt{2}\sin\frac{19\pi}{12} = \left(2\sqrt{2}\cos\frac{19\pi}{12}, 2\sqrt{2}\sin\frac{19\pi}{12}\right)$$

$$z\omega^5 = 2\sqrt{2}\mathrm{e}^{\frac{\pi i}{4}} \times \mathrm{e}^{\frac{5\pi i}{3}} = 2\sqrt{2}\mathrm{e}^{\frac{23\pi i}{12}} = 2\sqrt{2}\cos\frac{23\pi}{12} + 2i\sqrt{2}\sin\frac{23\pi}{12} = \left(2\sqrt{2}\cos\frac{23\pi}{12}, 2\sqrt{2}\sin\frac{23\pi}{12}\right)$$

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2 First we translate so that the centre of the triangle is the origin, this maps the vertex at $(3,-2)=3-2i$ gets mapped to $z=3-2i-(2+3i)=1-5i$ since the centre of the translated triangle now lies at the origin we can find the other two vertices of the translated triangle by multiplying by a primitive 3rd root of unity $\omega=\mathrm{e}^{\frac{2\pi i}{3}}$, this gives

$$z\omega=(1-5i)\left(-\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)=\frac{5\sqrt{3}-1}{2}+\left(\frac{\sqrt{3}+5}{2}\right)i$$

$$z\omega^2=(1-5i)\left(-\frac{1}{2}-i\frac{\sqrt{3}}{2}\right)=\frac{-5\sqrt{3}-1}{2}+\left(\frac{-\sqrt{3}+5}{2}\right)i$$

Hence when we reverse the translation to take the centre of the triangle is at $(2,3)$ gives the other two vertices of the triangle at

$$z_2=\frac{5\sqrt{3}-1}{2}+\left(\frac{\sqrt{3}+5}{2}\right)i+2+3i=\frac{5\sqrt{3}+3}{2}+\left(\frac{\sqrt{3}+11}{2}\right)i$$

$$z_3=\frac{-5\sqrt{3}-1}{2}+\left(\frac{-\sqrt{3}+5}{2}\right)i+2+3i=\frac{-5\sqrt{3}+3}{2}+\left(\frac{-\sqrt{3}+11}{2}\right)i$$

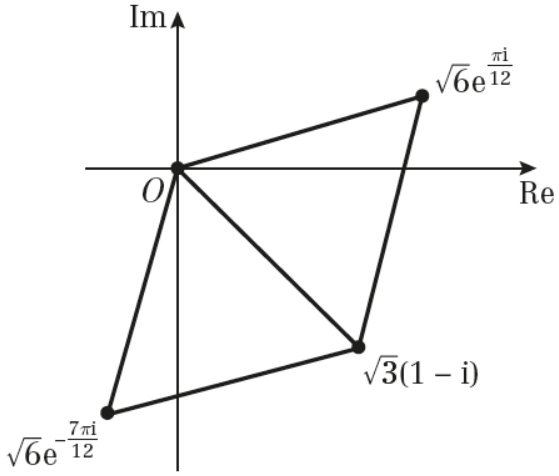
So the coordinates are $z_2=\left(\frac{5\sqrt{3}+3}{2},\frac{\sqrt{3}+11}{2}\right),z_3=\left(\frac{-5\sqrt{3}+3}{2},\frac{-\sqrt{3}+11}{2}\right)$

3 We have that A corresponds to the complex number $z=\sqrt{3}(1-i)=\sqrt{6}\mathrm{e}^{-\frac{\pi i}{4}}$, then B corresponds to rotating A by $\pm\frac{\pi}{3}$ hence the complex number b representing B is one of the two possible values

$$b=\sqrt{6}\mathrm{e}^{-\frac{\pi i}{4}}\times\mathrm{e}^{\frac{\pi i}{3}}=\sqrt{6}\mathrm{e}^{\frac{\pi i}{12}}$$

$$b=\sqrt{6}\mathrm{e}^{-\frac{\pi i}{4}}\times\mathrm{e}^{-\frac{\pi i}{3}}=\sqrt{6}\mathrm{e}^{-\frac{7\pi i}{12}}$$

$\sqrt{6}\,\mathrm{e}^{\frac{\pi i}{12}}, \sqrt{6}\,\mathrm{e}^{-\frac{7\pi i}{12}}$



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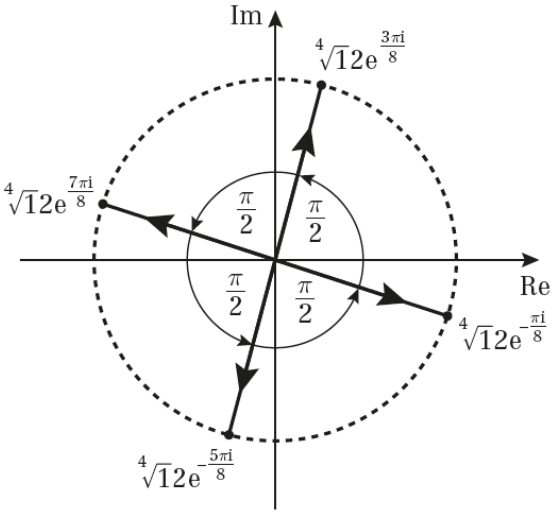
4 a We have $z = -12i = 12e^{-\frac{\pi i}{2}}$ so if $\omega = re^{i\theta}$ satisfies $\omega^4 = z$ then we have $r = \sqrt[4]{12}$ and $4\theta = -\frac{\pi}{2} + 2k\pi$ for some $k \in \mathbb{Z}$ hence 4 distinct roots are

$\omega_1 = \sqrt[4]{12}e^{-\frac{\pi i}{8}}$

$\omega_2 = \sqrt[4]{12}e^{\frac{3\pi i}{8}}$

$\omega_3 = \sqrt[4]{12}e^{\frac{7\pi i}{8}}$

$\omega_4 = \sqrt[4]{12}e^{-\frac{5\pi i}{8}}$



b Let the points representing these roots in order of increasing θ be A, B, C, D which form a square in the complex plane then by geometrical considerations the angle between adjacent midpoints of edges is $\frac{\pi}{2}$ and each midpoint has the same modulus. Hence all 4 midpoints are 4th roots of the same complex number w and to find w it suffices to compute the midpoint of one edge and take the fourth power, so without loss of generality we compute the midpoint of $\overrightarrow{AB}$ this is given by

$$\frac{1}{2}\left(\sqrt[4]{12}e^{-\frac{5\pi i}{8}} + \sqrt[4]{12}e^{-\frac{\pi i}{8}}\right)$$

Geometrically the argument is $\frac{1}{2}\left(-\frac{5\pi i}{8} + -\frac{\pi i}{8}\right) = -\frac{3\pi}{8}$ and by considering a triangle with vertices

At the origin, ω_4 and ω_1 the modulus is $\sqrt[4]{12}\sin\frac{\pi}{4} = \frac{\sqrt[4]{12}}{\sqrt{2}} = \sqrt[4]{3}$ hence the midpoint is given by

$\sqrt[4]{3}e^{-\frac{3\pi i}{8}}$ and so $w = \left(\sqrt[4]{3}e^{-\frac{3\pi i}{8}}\right)^4 = 3e^{-\frac{12\pi i}{8}} = 3e^{\frac{\pi i}{2}} = 3i$

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5 a Let $z = 8+8i = 8\sqrt{2}\mathrm{e}^{\frac{\pi i}{4}}$ and $\omega = \mathrm{e}^{\frac{\pi i}{3}}$ be a primitive 6th root of unity then the other 5 vertices are given by

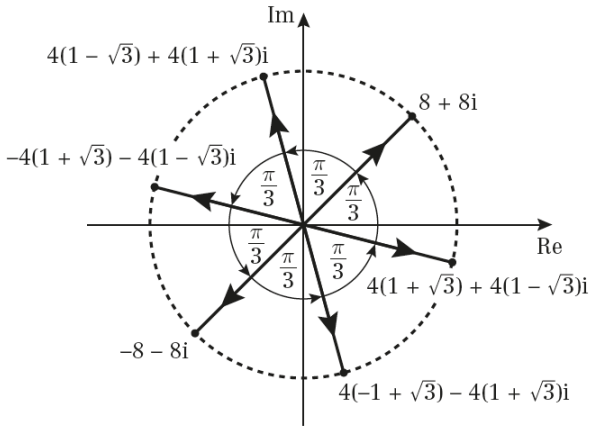
$$z\omega = 8\sqrt{2}\mathrm{e}^{\frac{\pi i}{4}} \times \mathrm{e}^{\frac{\pi i}{3}} = 8\sqrt{2}\mathrm{e}^{\frac{7\pi i}{12}} = 8\sqrt{2}\left(\cos\frac{7\pi}{12} + i\sin\frac{7\pi}{12}\right) = 4\left(1-\sqrt{3}\right) + 4i\left(1+\sqrt{3}\right)$$

$$z\omega^2 = 8\sqrt{2}\mathrm{e}^{\frac{\pi i}{4}} \times \mathrm{e}^{\frac{2\pi i}{3}} = 8\sqrt{2}\mathrm{e}^{\frac{11\pi i}{12}} = 8\sqrt{2}\left(\cos\frac{11\pi}{12} + i\sin\frac{11\pi}{12}\right) = -4\left(1+\sqrt{3}\right) + 4i\left(-1+\sqrt{3}\right)$$

$$z\omega^3 = 8\sqrt{2}\mathrm{e}^{\frac{\pi i}{4}} \times \mathrm{e}^{\frac{3\pi i}{3}} = 8\sqrt{2}\mathrm{e}^{\frac{5\pi i}{4}} = 8\sqrt{2}\left(\cos\frac{5\pi}{4} + i\sin\frac{5\pi}{4}\right) = -8-8i$$

$$z\omega^4 = 8\sqrt{2}\mathrm{e}^{\frac{\pi i}{4}} \times \mathrm{e}^{\frac{4\pi i}{3}} = 8\sqrt{2}\mathrm{e}^{\frac{19\pi i}{12}} = 8\sqrt{2}\left(\cos\frac{19\pi}{12} + i\sin\frac{19\pi}{12}\right) = 4\left(-1+\sqrt{3}\right) - 4i\left(1+\sqrt{3}\right)$$

$$z\omega^5 = 8\sqrt{2}\mathrm{e}^{\frac{\pi i}{4}} \times \mathrm{e}^{\frac{5\pi i}{3}} = 8\sqrt{2}\mathrm{e}^{\frac{23\pi i}{12}} = 8\sqrt{2}\left(\cos\frac{23\pi}{12} + i\sin\frac{23\pi}{12}\right) = 4\left(1+\sqrt{3}\right) + 4i\left(1-\sqrt{3}\right)$$



b When we square the vertices, the figure becomes an equilateral triangle since vertices that differ in argument by π get squared to the same value hence to find the vertices of the triangle we only need to square the first 3 vertices listed above which gives

$$z^2 = (8+8i)^2 = 128\mathrm{e}^{\frac{\pi i}{2}} = 128i$$

$$(z\omega)^2 = 128\mathrm{e}^{\frac{7\pi i}{6}} = 128\left(\cos\frac{7\pi}{6} + i\sin\frac{7\pi}{6}\right) = 128\left(-\frac{\sqrt{3}}{2} - \frac{i}{2}\right)$$

$$\left(z\omega^2\right) = 128\mathrm{e}^{\frac{11\pi i}{6}} = 128\left(\cos\frac{11\pi}{6} + i\sin\frac{11\pi}{6}\right) = 128\left(\frac{\sqrt{3}}{2} - \frac{i}{2}\right)$$

By geometrical considerations the side length of the equilateral triangle is $128 \times 2\cos\frac{\pi}{6} = 128\sqrt{3}$ hence the area is $\frac{1}{2} \times 128\sqrt{3} \times 128\sqrt{3}\sin\frac{\pi}{3} = 12288\sqrt{3}$

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6 We can represent the action of 'moving forward one and then turning to the right by $\frac{2\pi}{9}$' by the function that acts on complex numbers by $f(z) = \mathrm{e}^{-\frac{2\pi i}{9}}(z+1)$ without loss of generality we can assume the ant starts at the origin, then the final position of the ant after performing this 4 times is $f^4(0)$ which we shall now compute, we have

$f(0) = \mathrm{e}^{-\frac{2\pi i}{9}}$

So

$f^2(0) = f(\mathrm{e}^{-\frac{2\pi i}{9}}) = \mathrm{e}^{-\frac{2\pi i}{9}}(\mathrm{e}^{-\frac{2\pi i}{9}}+1) = \mathrm{e}^{-\frac{4\pi i}{9}} + \mathrm{e}^{-\frac{2\pi i}{9}}$

So

$$f^3(0) = f\left(\mathrm{e}^{-\frac{4\pi i}{9}} + \mathrm{e}^{-\frac{2\pi i}{9}}\right) = \mathrm{e}^{-\frac{2\pi i}{9}}\left(\mathrm{e}^{-\frac{4\pi i}{9}} + \mathrm{e}^{-\frac{2\pi i}{9}} + 1\right) = \mathrm{e}^{-\frac{6\pi i}{9}} + \mathrm{e}^{-\frac{4\pi i}{9}} + \mathrm{e}^{-\frac{2\pi i}{9}}$$

$$f^4(0) = \mathrm{e}^{-\frac{2\pi i}{9}}\left(\mathrm{e}^{-\frac{6\pi i}{9}} + \mathrm{e}^{-\frac{4\pi i}{9}} + \mathrm{e}^{-\frac{2\pi i}{9}} + 1\right) = \mathrm{e}^{-\frac{8\pi i}{9}} + \mathrm{e}^{-\frac{6\pi i}{9}} + \mathrm{e}^{-\frac{4\pi i}{9}} + \mathrm{e}^{-\frac{2\pi i}{9}}$$

Let $z = f^4(0)$ then this is just a geometric series which we can sum to give

$$z = \mathrm{e}^{-\frac{8\pi i}{9}} + \mathrm{e}^{-\frac{6\pi i}{9}} + \mathrm{e}^{-\frac{4\pi i}{9}} + \mathrm{e}^{-\frac{2\pi i}{9}} = \frac{\mathrm{e}^{-\frac{8\pi i}{9}}\left(1-\mathrm{e}^{\frac{8\pi i}{9}}\right)}{1-\mathrm{e}^{\frac{2\pi i}{9}}}$$

Hence the distance is the modulus of this complex number and we have

$$|z|^2 = \left|\frac{\mathrm{e}^{-\frac{8\pi i}{9}}\left(1-\mathrm{e}^{\frac{8\pi i}{9}}\right)}{1-\mathrm{e}^{\frac{2\pi i}{9}}}\right|^2 = \frac{\left|1-\mathrm{e}^{\frac{8\pi i}{9}}\right|^2}{\left|1-\mathrm{e}^{\frac{2\pi i}{9}}\right|^2} = \frac{\left(1-\cos\frac{8\pi}{9}\right)^2 + \sin^2\frac{8\pi}{9}}{\left(1-\cos\frac{2\pi}{9}\right)^2 + \sin^2\frac{2\pi}{9}} = \frac{2-2\cos\frac{8\pi}{9}}{2-2\cos\frac{2\pi}{9}} = \frac{\sin^2\frac{4\pi}{9}}{\sin^2\frac{\pi}{9}}$$

Hence the distance from the starting position is

$$|z| = \frac{\sin\frac{4\pi}{9}}{\sin\frac{\pi}{9}}$$

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Complex numbers – mixed exercise 1

1 a We have $e^{i\theta} = \cos\theta + i\sin\theta$ hence

$$\frac{1}{2}\left(e^{i\theta} + e^{-i\theta}\right) = \frac{1}{2}\left(\cos\theta + i\sin\theta + \cos{-\theta} + i\sin{-\theta}\right) = \frac{1}{2}\left(\cos\theta + i\sin\theta + \cos\theta - \sin\theta\right) = \cos\theta$$

Where we have used the fact that

$\cos\theta = \cos{-\theta}$

$\sin\theta = -\sin{-\theta}$

b We have

$$\cos A\cos B = \frac{1}{2}\left(e^{iA} + e^{-iA}\right) \times \frac{1}{2}\left(e^{iB} + e^{-iB}\right) = \frac{1}{4}\left(e^{i(A+B)} + e^{i(A-B)} + e^{i(B-A)} + e^{-i(A+B)}\right)$$

$$= \frac{1}{4}\left(\cos(A+B) + i\sin(A+B) + \cos(A-B) + i\sin(A-B) + \cos(B-A) + i\sin(B-A) + \cos(-(A+B)) + i\sin(-(A+B))\right)$$

$$= \frac{1}{4}\left(2\cos(A+B) + 2\cos(A-B)\right) = \frac{\cos(A+B) + \cos(A-B)}{2}$$

2 We want to prove by induction that if $z = r\left(\cos\theta + i\sin\theta\right)$ then for all $n \in \mathbb{N}$ $z^n = r^n\left(\cos n\theta + i\sin n\theta\right)$ clearly the base case $n = 1$ is true, now assume the statement is true for $n = k$ then we have

$$z^{k+1} = z \times z^k = r\left(\cos\theta + i\sin\theta\right) \times r^k\left(\cos k\theta + i\sin k\theta\right) = r^{k+1}\left(\cos\theta + i\sin\theta\right)\left(\cos k\theta + i\sin k\theta\right)$$

$$= r^{k+1}\left(\cos\theta\cos k\theta - \sin k\theta\sin\theta + i\left(\cos\theta\sin k\theta + \cos k\theta\sin\theta\right)\right)$$

$$= r^{k+1}\left(\cos(k+1)\theta + i\sin(k+1)\theta\right)$$

Hence the statement is true for $n = k+1$, this completes the induction.

3

$$\frac{(\cos 3x + \mathrm{i}\sin 3x)^2}{\cos x - \mathrm{i}\sin x}$$

$$= \frac{(\cos 3x + \mathrm{i}\sin 3x)^2}{\cos(-x) + \mathrm{i}\sin(-x)}$$

$$= \frac{\cos 6x + \mathrm{i}\sin 6x}{\cos(-x) + \mathrm{i}\sin(-x)}$$

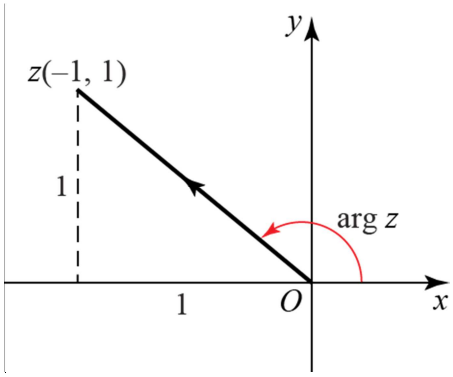
$$= \cos(6x - -x) + \mathrm{i}\sin(6x - -x)$$

$$= \cos 7x + \mathrm{i}\sin 7x$$

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4 a $(-1+\mathrm{i})^8$

If $z=-1+\mathrm{i}$, then



$r=\sqrt{(-1)^2+1^2}=\sqrt{2}$

$\theta=\arg z=\pi-\tan^{-1}\left(\frac{1}{1}\right)=\pi-\frac{\pi}{4}=\frac{3\pi}{4}$

So, $-1+\mathrm{i}=\sqrt{2}\left(\cos\frac{3\pi}{4}+\mathrm{i}\sin\frac{3\pi}{4}\right)$

$\therefore (-1+\mathrm{i})^8=\left[\sqrt{2}\left(\cos\frac{3\pi}{4}+\mathrm{i}\sin\frac{3\pi}{4}\right)\right]^8$

$=(\sqrt{2})^8\left(\cos\frac{24\pi}{4}+\mathrm{i}\sin\frac{24\pi}{4}\right)$

$=16(\cos 6\pi+\mathrm{i}\sin 6\pi)$

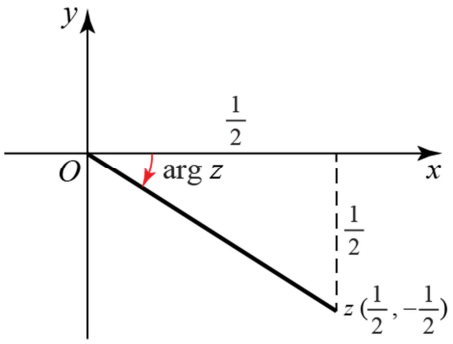
$=16(1+\mathrm{i}(0))$

Therefore, $(-1+\mathrm{i})^8=16$

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4 b $\dfrac{1}{\left(\frac{1}{2}-\frac{1}{2}\mathrm{i}\right)^{16}}=\left(\dfrac{1}{2}-\dfrac{1}{2}\mathrm{i}\right)^{-16}$

Let $z=\dfrac{1}{2}-\dfrac{1}{2}\mathrm{i}$, then



$$r=\sqrt{\left(\frac{1}{2}\right)^2+\left(-\frac{1}{2}\right)^2}=\sqrt{\frac{1}{4}+\frac{1}{4}}=\sqrt{\frac{1}{2}}=\frac{1}{\sqrt{2}}$$

$$\theta=\tan^{-1}\left(\frac{\frac{1}{2}}{\frac{1}{2}}\right)=-\frac{\pi}{4}$$

$$\text{So } \frac{1}{2}-\frac{1}{2}\mathrm{i}=\frac{1}{\sqrt{2}}\left[\cos\left(-\frac{\pi}{4}\right)+\mathrm{i}\sin\left(-\frac{\pi}{4}\right)\right]$$

$$\left(\frac{1}{2}-\frac{1}{2}\mathrm{i}\right)^{-16}=\left[\frac{1}{\sqrt{2}}\left(\cos\left(-\frac{\pi}{4}\right)+\mathrm{i}\sin\left(-\frac{\pi}{4}\right)\right)\right]^{-16}$$

$$=\left(2^{-\frac{1}{2}}\right)^{-16}\left(\cos\left(\frac{16\pi}{4}\right)+\mathrm{i}\sin\left(\frac{16\pi}{4}\right)\right)$$

$$=2^8(\cos 4\pi+\mathrm{i}\sin 4\pi)$$

$$=256(1+\mathrm{i}(0))$$

$$=256$$

Therefore, $\dfrac{1}{\left(\frac{1}{2}-\frac{1}{2}\mathrm{i}\right)^{16}}=256$

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5 a $z = \cos\theta + \mathrm{i}\sin\theta$

$z^n = (\cos\theta + \mathrm{i}\sin\theta)^{-n}$

$= \cos n\theta + \mathrm{i}\sin n\theta$

de Moivre's Theorem.

$\dfrac{1}{z^n} = z^{-n} = (\cos\theta + \mathrm{i}\sin\theta)^n$

$= \cos(-n\theta) + \mathrm{i}\sin(-n\theta)$

$= \cos n\theta - \mathrm{i}\sin n\theta$

de Moivre's Theorem.

$\cos(-n\theta) = \cos n\theta$
$\sin(-n\theta) = -\sin n\theta$

Therefore $z^n + \dfrac{1}{z^n} = \cos n\theta + \mathrm{i}\sin n\theta + \cos n\theta - \mathrm{i}\sin n\theta$

i.e. $z^n + \dfrac{1}{z^n} = 2\cos n\theta$ (as required)

b $\left(z^2 + \dfrac{1}{z^2}\right)^3 = (z^2)^3 + {}^3\mathrm{C}_1 (z^2)^2\left(\dfrac{1}{z^2}\right) + {}^3\mathrm{C}_2 (z^2)\left(\dfrac{1}{z^2}\right)^2 + \left(\dfrac{1}{z^2}\right)^3$

$= z^6 + 3z^4\left(\dfrac{1}{z^2}\right) + 3z^2\left(\dfrac{1}{z^4}\right) + \left(\dfrac{1}{z^6}\right)$

$= z^6 + 3z^2 + \dfrac{3}{z^2} + \dfrac{1}{z^6}$

$= \left(z^6 + \dfrac{1}{z^6}\right) = 3\left(z^2 + \dfrac{1}{z^2}\right)$

$= 2\cos 6\theta + 3(2)\cos 2\theta$

$= 2\cos 6\theta + 6\cos 2\theta$

Hence, $\left(z^2 + \dfrac{1}{z^2}\right)^3 = 2\cos 6\theta + 6\cos 2\theta$

c $\left(z^2 + \dfrac{1}{z^2}\right)^3 = (2\cos 2\theta)^3 = 8\cos^3 2\theta = 2\cos 6\theta + 6\cos 2\theta$

$\therefore \cos^3 2\theta = \dfrac{2}{8}\cos 6\theta + \dfrac{6}{8}\cos 2\theta$

Hence, $\cos^3 2\theta = \dfrac{1}{4}\cos 6\theta + \dfrac{3}{4}\cos 2\theta$

So $a = \dfrac{1}{4}, b = \dfrac{3}{4}$

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5 d $\int_0^{\frac{\pi}{6}} \cos^3 2\theta \, \mathrm{d}\theta = \int_0^{\frac{\pi}{6}} \frac{1}{4}\cos 6\theta + \frac{3}{4}\cos 2\theta \, \mathrm{d}\theta$

$$= \left[\frac{1}{24}\sin 6\theta + \frac{3}{8}\sin 2\theta\right]_0^{\frac{\pi}{6}}$$

$$= \left(\frac{1}{24}\sin \pi + \frac{3}{8}\sin\left(\frac{\pi}{3}\right)\right) - \left(\frac{1}{24}\sin 0 + \frac{3}{8}\sin 0\right)$$

$$= \left(\frac{1}{24}(0) + \frac{3}{8}\left(\frac{\sqrt{3}}{2}\right)\right) - (0)$$

$$= \frac{3}{16}\sqrt{3}$$

So, $\int_0^{\frac{\pi}{6}} \cos^3 2\theta \, \mathrm{d}\theta = \frac{3}{16}\sqrt{3}$ and $k = \frac{3}{16}$

6 a We wish to show that $\cos^5\theta = \frac{1}{16}(\cos 5\theta + 5\cos 3\theta + 10\cos\theta)$

Starting with $\cos\theta = \frac{1}{2}\left(\mathrm{e}^{i\theta} + \mathrm{e}^{-i\theta}\right)$ we have

$$\cos^5\theta = \frac{1}{32}\left(\mathrm{e}^{i\theta} + \mathrm{e}^{-i\theta}\right)^5 = \frac{1}{32}\left(\mathrm{e}^{5i\theta} + 5\mathrm{e}^{3i\theta} + 10\mathrm{e}^{i\theta} + 10\mathrm{e}^{-i\theta} + 5\mathrm{e}^{-3i\theta} + \mathrm{e}^{-5i\theta}\right)$$

$$= \frac{1}{32}(2\cos 5\theta + 10\cos 3\theta + 20\cos\theta) = \frac{1}{16}(\cos 5\theta + 5\cos 3\theta + 10\cos\theta)$$

b The area of the region is given by the integral

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^5 x \, \mathrm{d}x = 2\int_0^{\frac{\pi}{2}} \cos^5 x \, \mathrm{d}x$$

Using the first part of the question, this is equivalent to

$$\int_0^{\frac{\pi}{2}} \frac{1}{8}(\cos 5x + 5\cos 3x + 10\cos x)\,\mathrm{d}x$$

$$= \left[\frac{1}{8}\left(\frac{1}{5}\sin 5x + \frac{5}{3}\sin 3x + 10\sin x\right)\right]_0^{\frac{\pi}{2}} = \frac{1}{8}\left(\frac{1}{5}\sin\frac{5\pi}{2} + \frac{5}{3}\sin\frac{3\pi}{2} + 10\sin\frac{\pi}{2}\right) = \frac{1}{8}\left(\frac{1}{5} - \frac{5}{3} + 10\right) = \frac{16}{15}$$

7 a We have $\sin\theta = \frac{1}{2i}\left(\mathrm{e}^{i\theta} - \mathrm{e}^{-i\theta}\right)$ so that

$$\sin^6\theta = -\frac{1}{64}\left(\mathrm{e}^{i\theta} - \mathrm{e}^{-i\theta}\right)^6$$

$$= -\frac{1}{64}\left(\mathrm{e}^{6i\theta} - 6\mathrm{e}^{4i\theta} + 15\mathrm{e}^{2i\theta} - 20 + 15\mathrm{e}^{2i\theta} - 6\mathrm{e}^{4i\theta} + \mathrm{e}^{6i\theta}\right)$$

$$= -\frac{1}{64}(2\cos 6\theta - 12\cos 4\theta + 30\cos 2\theta - 20) = -\frac{1}{32}(\cos 6\theta - 6\cos 4\theta + 15\cos 2\theta - 10)$$

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7 b Let $\alpha = \frac{\pi}{2} - \theta$ then recall that we have $\cos\alpha = \sin\theta$ hence we have

$$\cos^6\alpha = \sin^6\theta = -\frac{1}{32}(\cos 6\theta - 6\cos 4\theta + 15\cos 2\theta - 10)$$

$$= -\frac{1}{32}\left(\cos 6\left(\frac{\pi}{2}-\alpha\right) - 6\cos 4\left(\frac{\pi}{2}-\alpha\right) + 15\cos 2\left(\frac{\pi}{2}-\alpha\right) - 10\right)$$

$$= -\frac{1}{32}(\cos(3\pi - 6\alpha) - 6\cos(2\pi - 4\alpha) + 15\cos(\pi - \alpha) - 10)$$

$$= -\frac{1}{32}(-\cos 6\alpha - 6\cos 4\alpha - 15\cos\alpha - 10) = \frac{1}{32}(\cos 6\alpha + 6\cos 4\alpha + 15\cos\alpha + 10)$$

c We have $\int_0^a \cos^6\theta + \sin^6\theta \,\mathrm{d}\theta = \frac{5\pi}{32}$ using the previous two parts we can evaluate the integral as

$$\int_0^a \cos^6\theta + \sin^6\theta \,\mathrm{d}\theta = \int_0^a \frac{1}{32}(12\cos 4\theta + 20)\,\mathrm{d}\theta = \frac{1}{32}[3\sin 4\theta + 20\theta]_0^a$$

$$= \frac{1}{32}(3\sin 4a + 20a)$$

Hence we require $3\sin 4a + 20a = 5\pi$ and observe that $a = \frac{\pi}{4}$ solves this and this is the only Solution since the function $3\sin 4a + 20a$ is increasing.

8 Let $z = \mathrm{e}^{i\theta}$ then we have

$$\sin 6\theta = \frac{1}{2i}\left(z^6 - z^{-6}\right) = \frac{1}{2i}\left((\cos\theta + i\sin\theta)^6 - (\cos\theta - i\sin\theta)^6\right)$$

When we expand the right hand side the only terms that survive are odd powers of sin hence we get

$$= \frac{1}{2i}\left(12i\cos^5\theta\sin\theta - 40i\cos^3\theta\sin^3 + 12i\cos\theta\sin^5\theta\right)$$

$$= 6\cos^5\theta\sin\theta - 20\cos^3\theta\sin^3 + 6\cos\theta\sin^5\theta$$

$$= 2\sin\theta\cos\theta\left(3\cos^4\theta - 10\cos^2\theta\sin^2\theta + 3\sin^4\theta\right)$$

$$= \sin 2\theta\left(3\cos^4\theta - 10\cos^2\theta\sin^2\theta + 3\sin^4\theta\right)$$

$$= \sin 2\theta\left(3\cos^4\theta - 10\cos^2\theta\left(1-\cos^2\theta\right) + 3\left(1-\cos^2\theta\right)^2\right)$$

$$= \sin 2\theta\left(3\cos^4\theta - 10\cos^2\theta\left(1-\cos^2\theta\right) + 3\left(1-2\cos^2\theta+\cos^4\theta\right)\right)$$

$$= \sin 2\theta\left(16\cos^4\theta - 16\cos^2\theta + 3\right)$$

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9 a Let $z = e^{i\theta}$ then we have

$$\cos 5\theta = \frac{1}{2i}\left(z^5 + z^{-5}\right) = \frac{1}{2i}\left(\left(\cos\theta + i\sin\theta\right)^5 + \left(\cos\theta - i\sin\theta\right)^5\right)$$

Now note the only terms that survive are even powers of sin so this becomes

$$\cos 5\theta = \frac{1}{2i}\left(z^5 + z^{-5}\right) = \frac{1}{2i}\left(\left(\cos\theta + i\sin\theta\right)^5 + \left(\cos\theta - i\sin\theta\right)^5\right)$$

$$= \frac{1}{2i}\left(2i\cos^5\theta - 20i\cos^3\theta\sin^2\theta + 10i\cos\theta\sin^4\theta\right)$$

$$= \cos^5\theta - 10\cos^3\theta\sin^2\theta + 5\cos\theta\sin^4\theta$$

$$= \cos\theta\left(\cos^4\theta - 10\cos^2\theta\sin^2\theta + 5\sin^4\theta\right)$$

$$= \cos\theta\left(\cos^4\theta - 10\cos^2\theta\left(1-\cos^2\theta\right) + 5\left(1-\cos^2\theta\right)^2\right)$$

$$\cos\theta\left(\cos^4\theta - 10\cos^2\theta + 10\cos^4\theta + 5\left(1 - 2\cos^2\theta + \cos^4\theta\right)\right)$$

$$\cos\theta\left(16\cos^4\theta - 20\cos^2\theta + 5\right)$$

b Now consider the equation

$16x^5 - 20x^3 + 5x + 1 = 0$

We make the substitution $x = \cos\theta$ so the equation becomes

$16\cos^5\theta - 20\cos^3\theta + 5\cos\theta = -1$

i.e. $\cos 5\theta = -1$ for which the general solution is $5\theta = \pi + 2k\pi$ for $k \in \mathbb{Z}$ we take 5 different values of θ

$$\theta_1 = \frac{\pi}{5}$$

$$\theta_2 = \frac{3\pi}{5}$$

$$\theta_3 = \pi$$

$$\theta_4 = \frac{7\pi}{5}$$

$$\theta_5 = \frac{9\pi}{5}$$

Let $x_i = \cos\theta_i$ so

$$x_1 = \cos\frac{\pi}{5} = \frac{1+\sqrt{5}}{4} = 0.809$$

$$x_2 = \cos\frac{3\pi}{5} = \frac{1-\sqrt{5}}{4} = -0.309$$

$$x_3 = \cos\pi = -1$$

$$x_4 = \cos\frac{7\pi}{5} = \frac{1-\sqrt{5}}{4} = -0.309$$

$$x_5 = \cos\frac{9\pi}{5} = \frac{1+\sqrt{5}}{4} = 0.809$$

Now we only have 3 distinct solutions so we still have to check this is all solutions, factorising out $x+1$ gives

$$16x^5 - 20x^3 + 5x + 1 = \left(x+1\right)\left(4x^2 - 2x - 1\right)^2 = 0$$

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9 b (Continued)

And observe that

$$\left(x-\left(\frac{1-\sqrt{5}}{4}\right)\right)\left(x-\left(\frac{1+\sqrt{5}}{4}\right)\right)=x^2-\frac{1}{2}x-\frac{1}{4}=\frac{1}{4}\left(4x^2-2x-1\right)$$

In other words the 3 distinct roots that we have obtained completely factorise the equation and so these three solutions are the only solutions.

Hence the three solutions are given by $-1, \frac{1+\sqrt{5}}{4}=0.809, \frac{1-\sqrt{5}}{4}=-0.309$

10 a Let $z=\mathrm{e}^{i\theta}=\cos\theta+i\sin\theta$ then $\sin\theta=\frac{1}{2i}\left(z-z^{-1}\right)$ so we have

$$\sin^5\theta=\frac{1}{32i}\left(z-z^{-1}\right)^5=\frac{1}{32i}\left(z^5-5z^3+10z-10z^{-1}+5z^{-3}-z^{-5}\right)$$

$$=\frac{1}{32i}\left(2i\sin 5\theta-10i\sin 3\theta+20i\sin\theta\right)=\frac{1}{16}\left(\sin 5\theta-5\sin 3\theta+10\sin\theta\right)$$

b We wish to solve

$\sin 5\theta-5\sin 3\theta+9\sin\theta=0$

Which we can rearrange to give

$\sin 5\theta-5\sin 3\theta+10\sin\theta=\sin\theta$

So $16\sin^5\theta=\sin\theta$

So we either have $\sin\theta=0$ in which case $\theta=0$ or $16\sin^4\theta=1$ hence $\sin\theta=\pm\frac{1}{2}$ and

on $0\le\theta<\pi$ the solutions correspond to $\theta=\frac{\pi}{6},\frac{5\pi}{6}$

11 a $(\cos\theta+\mathrm{i}\sin\theta)^5=\cos 5\theta+\mathrm{i}\sin 5\theta$ de Moivre's Theorem.

$$=\cos^5\theta+{}^5C_1\cos^4\theta(\mathrm{i}\sin\theta)+{}^5C_2\cos^3\theta(\mathrm{i}\sin\theta)^2$$
$$+{}^5C_3\cos^2\theta(\mathrm{i}\sin\theta)^3+{}^5C_4\cos^3\theta(\mathrm{i}\sin\theta)^4+(\mathrm{i}\sin\theta)^5$$

$$=\cos^5\theta+5\mathrm{i}\cos^4\theta\sin\theta+10\mathrm{i}^2\cos^3\theta\sin^2\theta$$ Binomial expansion.
$$+10\mathrm{i}^3\cos^2\theta\sin^3\theta+5\mathrm{i}^4\cos\theta\sin^4\theta+\mathrm{i}^5\sin^5\theta$$

Hence,

$$\cos 5\theta+\mathrm{i}\sin 5\theta=\cos^5\theta+5\mathrm{i}\cos^4\theta\sin\theta-10\cos^3\theta\sin\theta$$
$$-10\mathrm{i}\cos^2\theta\sin^3\theta+5\cos\theta\sin^4\theta+\mathrm{i}\sin^5\theta$$

Equating the real parts gives,

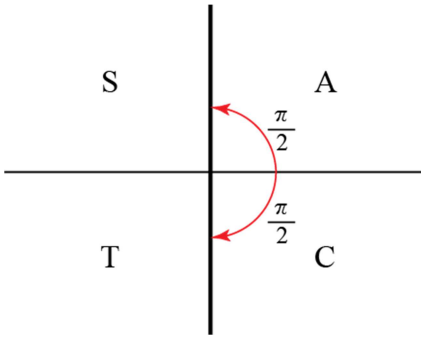
$$\cos 5\theta=\cos^5\theta-10\cos^3\theta\sin^2\theta+5\cos\theta\sin^4\theta$$
$$=\cos\theta(\cos^4\theta-10\cos^2\theta\sin^2\theta+5\sin^4\theta)$$
$$=\cos\theta(\cos^4\theta-10\cos^2\theta(1-\cos^2\theta)+5(1-\cos^2\theta)^2)$$ ← Applying $\sin^2\theta=1-\cos^2\theta$.
$$=\cos\theta(\cos^4\theta-10\cos^2\theta+10\cos^4\theta+5(1-2\cos^2\theta+\cos^4\theta))$$
$$=\cos\theta(\cos^4\theta-10\cos^2\theta+10\cos^4\theta+5-10\cos^2\theta+5\cos^4\theta)$$
$$=\cos\theta(16\cos^4\theta-20\cos^2\theta+5)$$

Hence, $\cos 5\theta=\cos\theta(16\cos^4\theta-20\cos^2\theta+5)$ (as required)

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11 b $\cos 5\theta = 0$

$$\alpha = \frac{\pi}{2}$$



So $5\theta = \left\{\frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2}, \frac{9\pi}{2}\right\}$

$$\theta = \left\{\frac{\pi}{10}, \frac{3\pi}{10}, \frac{5\pi}{10}, \frac{7\pi}{10}, \frac{9\pi}{10}\right\}$$

$$\theta = \left\{\frac{\pi}{10}, \frac{3\pi}{10}, \frac{5\pi}{10}, \frac{7\pi}{10}, \frac{9\pi}{10}\right\} \quad \text{for } 0 < \theta \leq \pi$$

$\cos 5\theta = 0 \Rightarrow \cos\theta(16\cos^4\theta - 20\cos^2\theta + 5) = 0$

Five solutions must come from: $\cos\theta(16\cos^4\theta - 20\cos^2\theta + 5) = 0$

Solution **(1)** $\cos\theta = 0$

$$\alpha = \frac{\pi}{2}$$

For $0 < \theta \leq \pi$, $\theta = \frac{\pi}{2}$ (as found earlier)

The final 4 solutions come from: $16\cos^4\theta - 20\cos^2\theta + 5 = 0$

$$\cos^2\theta = \frac{20 \pm \sqrt{400 - 4(16)(5)}}{32}$$

$$= \frac{20 \pm \sqrt{400 - 320}}{32}$$

$$= \frac{20 \pm \sqrt{80}}{32}$$

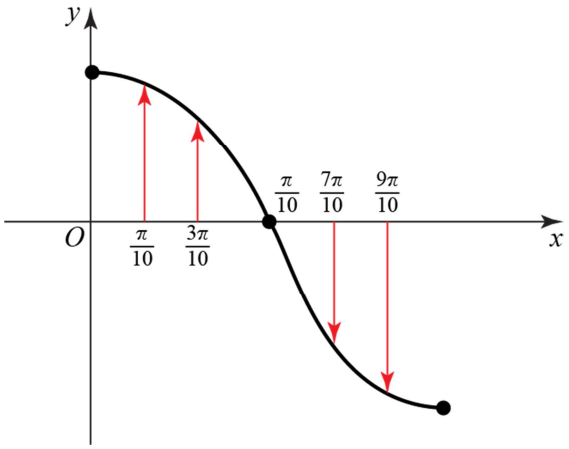
$$= \frac{20 \pm \sqrt{16}\sqrt{5}}{32}$$

$$= \frac{20 \pm 4\sqrt{5}}{32}$$

$$\therefore \cos^2\theta = \frac{5 + \sqrt{5}}{8}$$

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11 b (Continued)



Due to symmetry and as $\cos\left(\frac{\pi}{10}\right) > \cos\left(\frac{3\pi}{10}\right)$

$$\cos^2\left(\frac{\pi}{10}\right) = \cos^2\left(\frac{9\pi}{10}\right) > \cos^2\left(\frac{3\pi}{10}\right) = \cos^2\left(\frac{7\pi}{10}\right)$$

$$\therefore \cos^2\left(\frac{\pi}{10}\right) = \frac{5+\sqrt{5}}{8}$$

c $\cos^2\left(\frac{3\pi}{10}\right) = \frac{5-\sqrt{5}}{8}$

$$\cos^2\left(\frac{7\pi}{10}\right) = \cos^2\left(\frac{3\pi}{10}\right) = \frac{5-\sqrt{5}}{8}$$

$$\cos^2\left(\frac{9\pi}{10}\right) = \cos^2\left(\frac{\pi}{10}\right) = \frac{5+\sqrt{5}}{8}$$

Therefore, $\cos^2\left(\frac{3\pi}{10}\right) = \frac{5-\sqrt{5}}{8}$, $\cos^2\left(\frac{7\pi}{10}\right) = \frac{5-\sqrt{5}}{8}$, $\cos^2\left(\frac{9\pi}{10}\right) = \frac{5+\sqrt{5}}{8}$

12 a Let $z = e^{i\theta} = \cos\theta + i\sin\theta$ then we have

$$\tan 3\theta = \frac{\sin 3\theta}{\cos 3\theta} = \frac{\frac{1}{2i}\left(z^3 - z^{-3}\right)}{\frac{1}{2}\left(z^3 + z^{-3}\right)} = \frac{z^3 - z^{-3}}{i\left(z^3 + z^{-3}\right)} = \frac{\left(\cos\theta + i\sin\theta\right)^3 - \left(\cos\theta - i\sin\theta\right)^3}{i\left(\left(\cos\theta + i\sin\theta\right)^3 + \left(\cos\theta - i\sin\theta\right)^3\right)}$$

$$= \frac{6i\cos^2\theta\sin\theta - 2i\sin^3\theta}{i\left(2\cos^3\theta - 6\cos\theta\sin^2\theta\right)} = \frac{6\cos^2\theta\sin\theta - 2\sin^3\theta}{2\cos^3\theta - 6\cos\theta\sin^2\theta}$$

Now divide top and bottom by $\cos^3\theta$ to give

$$\tan 3\theta = \frac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta}$$

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12 b We have

$$\cot 3\theta = \frac{1}{\tan 3\theta} = \frac{1-3\tan^2\theta}{3\tan\theta - \tan^3\theta}$$

Now divide top and bottom by $\tan^3\theta$ to give

$$\cot 3\theta = \frac{\cot^3\theta - 3\cot\theta}{3\cot^2\theta - 1}$$

13 We have

$C = 1 + k\cos\theta + k^2\cos 2\theta + k^3\cos 3\theta + \ldots$

$S = k\sin\theta + k^2\sin 2\theta + k^3\sin 3\theta + \ldots$

So we have

$C + iS = 1 + ke^{i\theta} + k^2e^{2i\theta} + k^3e^{3i\theta} + \ldots$

Since $|k| < 1$ this is a geometric series which we can sum giving

$$C + iS = \frac{1}{1-ke^{i\theta}}$$

$$= \frac{1-ke^{-i\theta}}{(1-ke^{i\theta})(1-ke^{-i\theta})} = \frac{1-k\cos\theta + ki\sin\theta}{(1-k\cos\theta)^2 + k^2\sin^2\theta}$$

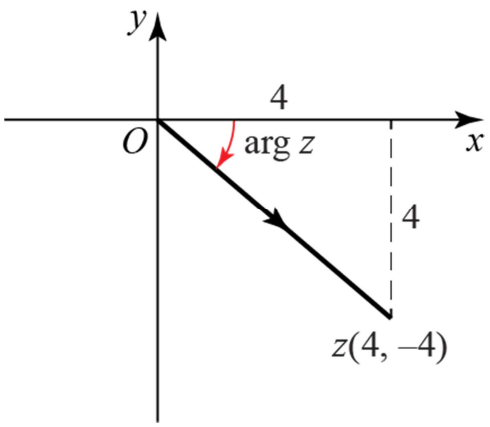
$$= \frac{1-k\cos\theta + ki\sin\theta}{1+k^2-2k\cos\theta}$$

Hence taking real and imaginary parts gives

$$C = \frac{1-k\cos\theta}{1+k^2-2k\cos\theta}$$

$$S = \frac{k\sin\theta}{1+k^2-2k\cos\theta}$$

14 a $4 - 4\text{i}$



$$\text{modulus } r = \sqrt{4^2 + (-4)^2} = \sqrt{16+16} = \sqrt{32} = \sqrt{16}\sqrt{2} = 4\sqrt{2}$$

$$\text{argument} = \theta = -\tan^{-1}\left(\frac{4}{4}\right) = -\frac{\pi}{4}$$

$$\therefore 4 - 4\text{i} = 4\sqrt{2}\left(\cos\left(-\frac{\pi}{4}\right) + \text{i}\sin\left(-\frac{\pi}{4}\right)\right)$$

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14 b $z^5 = 4 - 4\text{i}$

for $4 - 4\text{i}$, $r = 4\sqrt{2}$, $\theta = -\dfrac{\pi}{4}$

So, $z^5 = 4\sqrt{2}\,\text{e}^{\text{i}\left(-\frac{\pi}{4}\right)}$

$z^5 = 4\sqrt{2}\,\text{e}^{\text{i}\left(-\frac{\pi}{4}+2k\pi\right)}$, $k \in \mathbb{Z}$

Hence, $z = \left[4\sqrt{2}\,\text{e}^{\text{i}\left(-\frac{\pi}{4}+2k\pi\right)}\right]^{\frac{1}{5}}$

$= \left(4\sqrt{2}\right)^{\frac{1}{5}}\text{e}^{\text{i}\left(\frac{-\frac{\pi}{4}+2k\pi}{5}\right)}$

de Moivre's Theorem.

$= \sqrt{2}\,\text{e}^{\text{i}\left(-\frac{\pi}{20}+\frac{2k\pi}{5}\right)}$

$4\sqrt{2} = 2^{\frac{5}{2}}$

So, $(4\sqrt{2})^{\frac{1}{5}} = (2^{\frac{5}{2}})^{\frac{1}{5}}$

$= 2^{\frac{1}{2}} = \sqrt{2}$

$k = 0$, $z_1 = \sqrt{2}\,\text{e}^{\text{i}\left(-\frac{\pi}{20}\right)}$

$k = 1$, $z_2 = \sqrt{2}\,\text{e}^{\text{i}\left(\frac{7\pi}{20}\right)}$

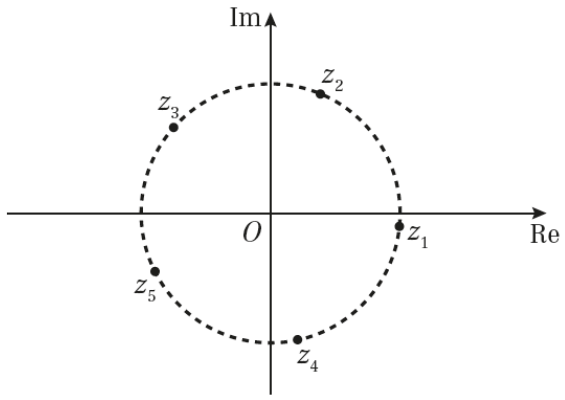
$k = 2$, $z_3 = \sqrt{2}\,\text{e}^{\text{i}\left(\frac{3\pi}{4}\right)}$

$k = -1$, $z_4 = \sqrt{2}\,\text{e}^{\text{i}\left(-\frac{9\pi}{20}\right)}$

$k = -2$, $z_5 = \sqrt{2}\,\text{e}^{\text{i}\left(-\frac{17\pi}{20}\right)}$

Therefore, $z = \sqrt{2}\,\text{e}^{-\frac{\pi i}{20}}, \sqrt{2}\,\text{e}^{\frac{7\pi i}{20}}, \sqrt{2}\,\text{e}^{\frac{3\pi i}{4}}, \sqrt{2}\,\text{e}^{-\frac{9\pi i}{20}}, \sqrt{2}\,\text{e}^{-\frac{17\pi i}{20}}$

c



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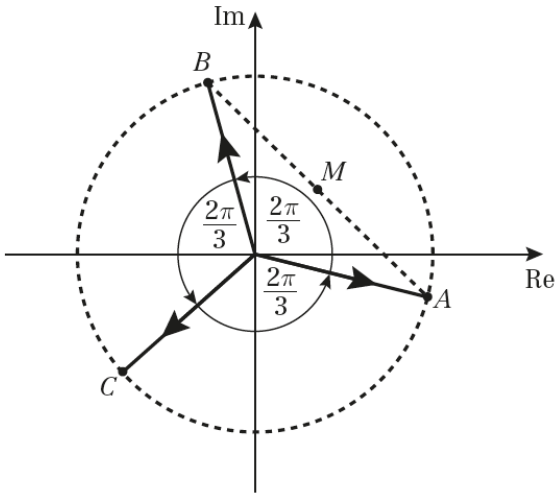
15 a We have $z = 2 - 2i = 2\sqrt{2}e^{-\frac{i\pi}{4}}$, now suppose $\omega^3 = z$ then let $\omega = re^{i\theta}$ then we have $r^3 = \sqrt{8}$ so $r = \left(\sqrt{8}\right)^{\frac{1}{3}} = \left(2^{\frac{3}{2}}\right)^{\frac{1}{3}} = \sqrt{2}$ and $3\theta = -\frac{\pi}{4} + 2k\pi$ hence three distinct roots are given by

$\omega_1 = \sqrt{2}\mathrm{e}^{-\frac{i\pi}{12}}$

$\omega_2 = \sqrt{2}\mathrm{e}^{\frac{7i\pi}{12}}$

$\omega_1 = \sqrt{2}\mathrm{e}^{-\frac{9i\pi}{12}}$

b



c We have $w = \frac{1}{2}\left(\sqrt{2}\mathrm{e}^{\frac{7i\pi}{12}} + \sqrt{2}\mathrm{e}^{-\frac{i\pi}{12}}\right)$ but by geometrical considerations the argument of w is just $\frac{1}{2}\left(\frac{-\pi}{12} + \frac{7\pi}{12}\right) = \frac{3\pi}{12} = \frac{\pi}{4}$ and by considering the triangle with vertices at the origin, A, B we have that the modulus is given by $\sqrt{2}\sin\frac{\pi}{6} = \frac{\sqrt{2}}{2}$ hence we have $w = \frac{\sqrt{2}}{2}\mathrm{e}^{\frac{i\pi}{4}}$

d We have

$$w^6 = \left(\frac{\sqrt{2}}{2}\mathrm{e}^{\frac{i\pi}{4}}\right)^6 = \frac{8}{64}\mathrm{e}^{\frac{6i\pi}{4}} = \frac{1}{8}\mathrm{e}^{\frac{3i\pi}{2}} = \frac{1}{8}\mathrm{e}^{\frac{-i\pi}{2}} = -\frac{i}{8}$$

16 a Let $z = 2 + i$ now let $\omega = \mathrm{e}^{\frac{2i\pi}{3}} = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$ be a primitive 3rd root of unity then the coordinates of the other two vertices are given by

$$z\omega = (2+i)\left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right) = \frac{-2-\sqrt{3}}{2} + i\frac{-1+2\sqrt{3}}{2}$$

$$z\omega^2 = (2+i)\left(-\frac{1}{2} - i\frac{\sqrt{3}}{2}\right) = \frac{-2+\sqrt{3}}{2} - i\frac{1+2\sqrt{3}}{2}$$

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16 b The length of one side is given by $|z - z\omega|$ we have

$$|z - z\omega| = |z||1 - \omega| = \sqrt{5}\left|\frac{3}{2} - i\frac{\sqrt{3}}{2}\right| = \frac{\sqrt{5}}{2}\sqrt{9+3} = \sqrt{5} \times \sqrt{3} = \sqrt{15}$$

Challenge

We consider the solutions of $\left(\frac{z+1}{z}\right)^6 = 1$, let $\omega = \frac{z+1}{z}$ then we have $\omega^6 = 1$ so $\omega = \mathrm{e}^{\frac{2ki\pi}{6}}$ for $k \in \mathbb{Z}$ hence we have

$$\frac{z+1}{z} = \mathrm{e}^{\frac{2ki\pi}{6}}$$

So

$$z+1 = z\mathrm{e}^{\frac{2ki\pi}{6}}$$

So

$$z\left(\mathrm{e}^{\frac{2ki\pi}{6}} - 1\right) = 1$$

Therefore

$$z = \frac{1}{\mathrm{e}^{\frac{ki\pi}{3}} - 1} = \frac{\mathrm{e}^{-\frac{ki\pi}{3}} - 1}{\left(\mathrm{e}^{\frac{ki\pi}{3}} - 1\right)\left(\mathrm{e}^{-\frac{ki\pi}{3}} - 1\right)} = \frac{\cos\frac{k\pi}{3} - 1 - i\sin\frac{k\pi}{3}}{\left(\cos\frac{k\pi}{3} - 1\right)^2 + \sin\frac{k\pi}{3}} = \frac{\cos\frac{k\pi}{3} - 1 - i\sin\frac{k\pi}{3}}{2 - 2\cos\frac{k\pi}{3}}$$

Assuming $\mathrm{e}^{\frac{2ki\pi}{6}} - 1 \neq 0$ i.e. $1 \leq k \leq 5$, when $k = 0$ the equation becomes $z = z + 1$ which has no solutions, however, for $1 \leq k \leq 5$ we get a different solution. So our six values of z are $z_0 = -\frac{1}{2}$

$$z_1 = \frac{\cos\frac{\pi}{3} - 1 - i\sin\frac{\pi}{3}}{2 - 2\cos\frac{\pi}{3}} = -\frac{1}{2} - \frac{i\sqrt{3}}{2}$$

$$z_2 = \frac{\cos\frac{2\pi}{3} - 1 - i\sin\frac{2\pi}{3}}{2 - 2\cos\frac{2\pi}{3}} = -\frac{1}{2} - \frac{i\sqrt{3}}{6}$$

$$z_3 = \frac{\cos\pi - 1 - i\sin\pi}{2 - 2\cos\pi} = -\frac{1}{2}$$

$$z_4 = \frac{\cos\frac{4\pi}{3} - 1 - i\sin\frac{4\pi}{3}}{1 - 2\cos\frac{4\pi}{3}} = -\frac{1}{2} + \frac{i\sqrt{3}}{6}$$

$$z_5 = \frac{\cos\frac{5\pi}{3} - 1 - i\sin\frac{5\pi}{3}}{1 - 2\cos\frac{5\pi}{3}} = -\frac{1}{2} + \frac{i\sqrt{3}}{2}$$

Which all lie on the straight line $\mathrm{Re}(z) = -\frac{1}{2}$

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Series 2A

1 a $\frac{1}{2}(r(r+1)-r(r-1))$

$=\frac{1}{2}(r^2+r-r^2+r)$

$=\frac{1}{2}(2r)$

$=r$

$=\text{LHS}$

Consider RHS.

Expand and simplify.

b $\sum_{r=1}^{n} r = \frac{1}{2}\sum_{r=1}^{n} r(r+1) - \frac{1}{2}\sum_{r=1}^{n} r(r-1)$

Use above.

Use method of differences.

$r=1$	$\frac{1}{2}\times 1\times 2$	$-\frac{1}{2}\times 1\times 0$
$r=2$	$\frac{1}{2}\times 2\times 3$	$-\frac{1}{2}\times 2\times 1$
$r=3$	$\frac{1}{2}\times 3\times 4$	$-\frac{1}{2}\times 3\times 2$
…	…	
$r=n-1$	$\frac{1}{2}(n-1)(n)$	$-\frac{1}{2}(n-1)(n-2)$
$r=n$	$\frac{1}{2}n(n+1)$	$-\frac{1}{2}n(n-1)$

When you add, all terms cancel except $\frac{1}{2}n(n+1)$.

Hence $\sum_{r=1}^{n} r = \frac{1}{2}n(n+1)$

2 $\sum_{r=1}^{n} \frac{1}{r(r+1)(r+2)} = \sum_{r=1}^{n} \frac{1}{2r(r+1)} - \sum_{r=1}^{n} \frac{1}{2(r+1)(r+2)}$

Use the information given and equate the summations.

Use method of differences.

All terms cancel except first and last.

Put $r=1$	$\frac{1}{2\times 1\times 2} - \frac{1}{2\times 2\times 3}$
$r=2$	$\frac{1}{2\times 2\times 3} - \frac{1}{2\times 3\times 4}$
$r=3$	$\frac{1}{2\times 3\times 4} - \frac{1}{2\times 4\times 5}$
⋮	
$r=n$	$\frac{1}{2n(n+1)} - \frac{1}{2(n+1)(n+2)}$

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2 Adding the first and last terms we have

$$\sum_{r=1}^{n}\frac{1}{r(r+1)(r+2)}=\frac{1}{4}-\frac{1}{2(n+1)(n+2)}$$

First and last from above.

$$=\frac{(n+1)(n+2)-2}{4(n+1)(n+2)}$$

Simplify.

$$=\frac{n^2+3n+2-2}{4(n+1)(n+2)}$$

$$=\frac{n(n+3)}{4(n+1)(n+2)}$$

3 a $\frac{1}{r(r+2)}\equiv\frac{A}{r}+\frac{B}{r+2}$ ← Set $\frac{1}{r(r+2)}$ identical to $\frac{A}{r}+\frac{B}{r+2}$.

$\equiv\frac{A(r+2)+Br}{r(r+2)}$ ← Add the two fractions.

$1\equiv A(r+2)+Br$

Put $r = 0$

$1 = 2A$

$A = \frac{1}{2}$

Put $r = 1$

$1 = \frac{1}{2}(3)+B$

$B = -\frac{1}{2}$

$\therefore \frac{1}{r(r+2)}=\frac{1}{2r}-\frac{1}{2(r+2)}$

b $\sum_{r=1}^{n}\frac{1}{r(r+2)}=\sum_{r=1}^{n}\frac{1}{2r}-\sum_{r=1}^{n}\frac{1}{2(r+2)}$

Use method of differences.

$r=1$ $\quad \frac{1}{2\times1}-\cancel{\frac{1}{2\times3}}$

$r=2$ $\quad \frac{1}{2\times2}-\cancel{\frac{1}{2\times4}}$

$r=3$ $\quad \cancel{\frac{1}{2\times3}}-\cancel{\frac{1}{2\times5}}$

$\vdots$

$r=n-1$ $\quad \cancel{\frac{1}{2(n-1)}}-\frac{1}{2(n+1)}$

All terms cancel except $\frac{1}{2}, \frac{1}{4}$

$\frac{1}{2(n+1)}$ and $\frac{1}{2(n+2)}$

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3 b $r = n$ $\quad \cancel{\frac{1}{2n}} \quad - \quad \frac{1}{2(n+2)}$

Adding the remaining terms we have

$$\sum_{r=1}^{n} \frac{1}{r(r+2)} = \frac{1}{2} + \frac{1}{4} - \frac{1}{2(n+1)} - \frac{1}{2(n+2)}$$

$$= \frac{2(n+1)(n+2) + (n+1)(n+2) - 2(n+2) - 2(n+1)}{4(n+1)(n+2)}$$

$$= \frac{2n^2 + 6n + 4 + n^2 + 3n + 2 - 2n - 4 - 2n - 2}{4(n+1)(n+2)}$$

$$= \frac{3n^2 + 5n}{4(n+1)(n+2)}$$

$$= \frac{n(3n+5)}{4(n+1)(n+2)}$$

4 a

$$\frac{1}{(r+2)(r+3)} \equiv \frac{A}{r+2} + \frac{B}{r+3}$$

Set $\frac{1}{(r+2)(r+3)}$ identical to $\frac{A}{r+2} + \frac{B}{r+3}$.

$$\equiv \frac{A(r+3) + B(r+2)}{(r+2)(r+3)}$$

Add the two fractions.

$$1 \equiv A(r+3) + B(r+2)$$

Compare numerators as they are equivalent.

$r = -3 \Rightarrow B = -1$

$r = -2 \Rightarrow A = 1$

Solve for A and B.

$$\therefore \frac{1}{(r+2)(r+3)} = \frac{1}{r+2} - \frac{1}{r+3}$$

b

$$\sum_{r=1}^{n} \frac{1}{(r+2)(r+3)} \equiv \sum_{r=1}^{n} \frac{1}{(r+2)} - \sum_{r=1}^{n} \frac{1}{(r+3)}$$

Use the method of differences.

$r = 1$ $\quad \frac{1}{3} \quad - \quad \cancel{\frac{1}{4}}$

$r = 2$ $\quad \cancel{\frac{1}{4}} \quad - \quad \cancel{\frac{1}{5}}$

$r = 3$ $\quad \cancel{\frac{1}{5}} \quad - \quad \cancel{\frac{1}{6}}$

$\vdots$

$r = n$ $\quad \cancel{\frac{1}{n+2}} \quad - \quad \frac{1}{n+3}$

All cancel except first and last.

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4 b Adding the remaining terms we have

$$\sum_{r=1}^{n}\frac{1}{(r+2)(r+3)}=\frac{1}{3}-\frac{1}{n+3}$$

$$=\frac{n+3-3}{3(n+3)}$$

$$=\frac{n}{3(n+3)}$$

5 a $\dfrac{1}{r!}-\dfrac{1}{(r+1)!}=\dfrac{r+1}{(r+1)!}-\dfrac{1}{(r+1)!}=\dfrac{r}{(r+1)!}$

b

$$\sum_{r=1}^{n}\frac{r}{(r+1)!} \equiv \sum_{r=1}^{n}\frac{1}{r!}-\sum_{r=1}^{n}\frac{1}{(r+1)!}$$

$r=1$ $\quad \dfrac{1}{1!}-\dfrac{\cancel{1}}{\cancel{2}!}$

$r=2$ $\quad \dfrac{\cancel{1}}{\cancel{2}!}-\dfrac{\cancel{1}}{\cancel{3}!}$

$r=3$ $\quad \dfrac{\cancel{1}}{\cancel{3}!}-\dfrac{\cancel{1}}{\cancel{4}!}$

$\vdots$

$r=n$ $\quad \dfrac{\cancel{1}}{\cancel{n}!}-\dfrac{1}{(n+1)!}$

$\therefore$

Use given.

Use method of differences.

All cancel except first and last term.

Adding the remaining terms we have

$$\sum_{r=1}^{n}\frac{r}{(r+1)!} = 1-\frac{1}{(n+1)!}$$

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6 $\displaystyle\sum_{r=1}^{n}\frac{2r+1}{r^2(r+1)^2} = \sum_{r=1}^{n}\frac{1}{r^2} - \sum_{r=1}^{n}\frac{1}{(r+1)^2}$ ← Use given.

$r=1$ $\quad \frac{1}{1} - \frac{1}{2^2}$

$r=2$ $\quad \frac{1}{2^2} - \frac{1}{3^2}$

$r=3$ $\quad \frac{1}{3^2} - \frac{1}{4^2}$

$\vdots$

$r=n$ $\quad \frac{1}{n^2} - \frac{1}{(n+1)^2}$

Use method of differences.

All terms cancel except first and last.

So adding the remaining terms we have

$$\sum_{r=1}^{n}\frac{2r+1}{r^2(r+1)^2} = 1 - \frac{1}{(n+1)^2} = \frac{(n+1)^2-1}{(n+1)^2}$$

$$= \frac{n^2+2n}{(n+1)^2}$$

$$= \frac{n(n+2)}{(n+1)^2}$$

Simplify.

7 a $\displaystyle\frac{1}{(2r+3)(2r+5)} = \frac{A}{2r+3} + \frac{B}{2r+5}$

$1 = A(2r+5) + B(2r+3)$

$1 = 2A = -2B$

Let $\mathrm{f}(r) = \dfrac{1}{2r+3}$

$$\sum_{r=1}^{n}\frac{1}{(2r+3)(2r+5)} = \frac{1}{2}\sum_{r=1}^{n}\left(\frac{1}{2r+3} - \frac{1}{2r+5}\right)$$

$$= \frac{1}{2}\sum_{r=1}^{n}\left(\mathrm{f}(r) - \mathrm{f}(r+1)\right) = \frac{1}{2}(\mathrm{f}(1) - \mathrm{f}(n+1))$$

$$= \frac{1}{2}\left(\frac{1}{2+3} - \frac{1}{2(n+1)+3}\right) = \frac{1}{2}\left(\frac{2n+5-5}{5(2n+5)}\right)$$

$$= \frac{n}{10n+25}$$

So $a = 10,\ b = 25$

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7 b $n=1$: RHS $=\dfrac{1}{10+25}=\dfrac{1}{35}=\dfrac{1}{5\times 7}=$LHS

Assume true for $n=k$

$$\sum_{r=1}^{k}\frac{1}{(2r+3)(2r+5)}=\frac{k}{10k+25}$$

$n=k+1$:

$$\sum_{r=1}^{k+1}\frac{1}{(2r+3)(2r+5)}=\frac{k}{10k+25}+\frac{1}{(2k+5)(2k+7)}$$

$$=\frac{1}{2k+5}\left(\frac{k}{5}+\frac{1}{2k+7}\right)=\frac{1}{2k+5}\left(\frac{2k^2+7k+5}{5(2k+7)}\right)$$

$$=\frac{k+1}{10(k+1)+25}$$

so true for $n=k+1$ if true for $n=k$

true for $n=1$ so true for all natural numbers

8 $$\frac{8}{(3r-2)(3r+4)}=\frac{A}{3r-2}+\frac{B}{3r+4}$$

$8=A(3r+4)+B(3r-2)$

$8=6A=-6B$

So $A=\dfrac{4}{3}$ and $B=\dfrac{-4}{3}$

Let $\mathrm{f}(r)=\dfrac{1}{3r-2}$

$$\sum_{r=1}^{n}\frac{8}{(3r-2)(3r+4)}=\frac{4}{3}\sum_{r=1}^{n}\left(\frac{1}{3r-2}-\frac{1}{3r+4}\right)$$

$$=\frac{4}{3}\sum_{r=1}^{n}\left(\mathrm{f}(r)-\mathrm{f}(r+2)\right)$$

$$=\frac{4}{3}(\mathrm{f}(1)+\mathrm{f}(2)-\mathrm{f}(n+1)-\mathrm{f}(n+2))$$

$$=\frac{4}{3}\left(1+\frac{1}{4}-\frac{1}{3n+1}-\frac{1}{3n+4}\right)$$

$$=\frac{4}{3}\left(\frac{\frac{5}{4}(3n+1)(3n+4)-(3n+1)-(3n+4)}{(3n+1)(3n+4)}\right)$$ **1**

$$=\frac{15n^2+25n+\frac{20}{3}-8n-\frac{20}{3}}{(3n+1)(3n+4)}=\frac{n(15n+17)}{(3n+1)(3n+4)}$$

So $a=15$ and $b=17$.

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9 Let $\mathrm{f}(r) = (r-1)^2$

$$\sum_{r=1}^{n} (r+1)^2 - (r-1)^2 = \mathrm{f}(n+2) + \mathrm{f}(n+1) - \mathrm{f}(2) - \mathrm{f}(1)$$

$$= (n+1)^2 + n^2 - 1 = 2n^2 + 2n = 2n(n+1)$$

So $a = 2$

10 a $\dfrac{3}{(3r+1)(3r+4)} = \dfrac{A}{3r+1} + \dfrac{B}{3r+4}$

$3 = A(3r+4) + B(3r+1)$

$3 = 3A = -3B$

Let $\mathrm{f}(r) = \dfrac{1}{3r+1}$

$$\sum_{r=1}^{n} \frac{3}{(3r+1)(3r+4)} = \sum_{r=1}^{n} \left(\frac{1}{3r+1} - \frac{1}{3r+4} \right)$$

$$= \sum_{r=1}^{n} \big(\mathrm{f}(r) - \mathrm{f}(r+1) \big) = \mathrm{f}(1) - \mathrm{f}(n+1)$$

$$= \frac{1}{4} - \frac{1}{3n+4} = \frac{3n}{12n+16}$$

So $a = 3,\ b = 12$ and $c = 16$

b $$\sum_{r=n}^{2n} \frac{3}{(3r+1)(3r+4)} = \frac{6n}{24n+16} - \frac{3(n-1)}{12(n-1)+16}$$

$$= \frac{3}{4} \left(\frac{n(3n+1) - (n-1)(3n+2)}{(3n+1)(3n+2)} \right)$$

$$= \frac{3}{4} \left(\frac{2n+2}{(3n+1)(3n+2)} \right) = \frac{3}{2} \frac{n+1}{(3n+1)(3n+2)}$$

11 $\dfrac{2r+1}{r(r+1)} = \dfrac{1}{r} + \dfrac{1}{r+1}, \mathrm{f}(r) = \dfrac{1}{r}$

$$\sum_{r=1}^{n} \frac{2r+1}{r(r+1)} = \mathrm{f}(1) + \mathrm{f}(2) + \mathrm{f}(2) + \mathrm{f}(3) + \ldots +$$

$\mathrm{f}(n-1) + \mathrm{f}(n) + \mathrm{f}(n) + \mathrm{f}(n+1)$

Terms don't cancel

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12 $\dfrac{1}{r(r+2)} = \dfrac{A}{r} + \dfrac{B}{r+2}$

$1 = A(r+2) + Br$

$1 = 2A = -2B$

Let $\mathrm{f}(r) = \dfrac{1}{r}$

$$\sum_{r=1}^{n} \frac{1}{r(r+2)} = \frac{1}{2}\sum_{r=1}^{n}\left(\frac{1}{r} - \frac{1}{r+2}\right)$$

$$= \frac{1}{2}(\mathrm{f}(1) + \mathrm{f}(2) - \mathrm{f}(n+1) - \mathrm{f}(n+2))$$

$$= \frac{1}{2}\left(1 + \frac{1}{2} - \frac{1}{n+1} - \frac{1}{n+2}\right) = \frac{3}{4} - \frac{1}{2}\frac{2n+3}{(n+1)(n+2)}$$

$$= \frac{3}{4} - \frac{1}{2}\frac{2n+3}{(n+1)(n+2)}$$

So $a = 2$ and $b = 3$

13 a $\dfrac{4}{(2r+1)(2r+5)} = \dfrac{A}{2r+1} + \dfrac{B}{2r+5}$

$4 = A(2r+5) + B(2r+1)$

$4 = 4A = -4B$

So $A = 1$ and $B = -1$

$$\frac{4}{(2r+1)(2r+5)} = \frac{1}{(2r+1)} - \frac{1}{(2r+5)}$$

b Let $\mathrm{f}(r) = \dfrac{1}{2r+1}$

$$\sum_{r=16}^{25} \frac{4}{r(2r+5)} = \sum_{r=16}^{25}\left(\frac{1}{2r+1} - \frac{1}{2r+5}\right)$$

$= \mathrm{f}(16) - \mathrm{f}(18) + \mathrm{f}(17) - \mathrm{f}(19) + \mathrm{f}(18) - \ldots$

$+\mathrm{f}(25) - \mathrm{f}(27)$

$= \mathrm{f}(16) + \mathrm{f}(17) - \mathrm{f}(26) - \mathrm{f}(27)$

$$= \frac{1}{33} + \frac{1}{35} - \frac{1}{53} - \frac{1}{55} = 0.0218 \text{ (4 d.p.)}$$

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Challenge

a $\log\left(1+\frac{1}{r+2}\right)=\log\left(\frac{r+3}{r+2}\right)=\log(r+3)-\log(r+2)$

Let $\mathrm{f}(r)=\log(r+2)$

$$\sum_{r=1}^{n}\log\left(1+\frac{1}{r+2}\right)=\mathrm{f}(n+1)-\mathrm{f}(1)$$

$$=\log(n+3)-\log 3=\log\frac{n+3}{3}$$

$$n=30:\sum_{r=1}^{30}\log\left(1+\frac{1}{r+2}\right)=\log 11$$

So $k=11$

b $\frac{18}{r(r+3)}=\frac{A}{r}+\frac{B}{r+3}$

$18=A(r+3)+Br$

$18=3A=-3B$

Let $\mathrm{f}(r)=\frac{1}{r}$

$$\sum_{r=1}^{n}\frac{18}{r(r+3)}=6\sum_{r=1}^{n}\left(\frac{1}{r}-\frac{1}{r+3}\right)$$

$$=6(\mathrm{f}(1)+\mathrm{f}(2)+\mathrm{f}(3)-\mathrm{f}(n+1)-\mathrm{f}(n+2)-\mathrm{f}(n+3))$$

$$=6\left(1+\frac{1}{2}+\frac{1}{3}-\frac{1}{n+1}-\frac{1}{n+2}-\frac{1}{n+3}\right)$$

$$=\frac{11(n+1)(n+2)(n+3)}{(n+1)(n+2)(n+3)}$$

$$-6\frac{(n+1)(n+2)+(n+1)(n+3)+(n+2)(n+3)}{(n+1)(n+2)(n+3)}$$

$$=\frac{11n^3+48n^2+49n}{(n+1)(n+2)(n+3)}$$

So $a=11$, $b=48$ and $c=49$

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Series 2B

1

	$f'(x)$	$f''(x)$	$f'''(x)$	$f^{(n)}(x)$
a	$2e^{2x}$	$2^2e^{2x} = 4e^{2x}$	$2^3e^{2x} = 8e^{2x}$	$2^n e^{2x}$
b	$n(1+x)^{n-1}$	$n(n-1)(1+x)^{n-2}$	$n(n-1)(n-2)(1+x)^{n-3}$	$n!$
c	$e^x + xe^x$	$e^x + (e^x + xe^x) = 2e^x + xe^x$	$2e^x + (e^x + xe^x) = 3e^x + xe^x$	$ne^x + xe^x$
d	$(1+x)^{-1}$	$-(1+x)^{-2}$	$(-1)(-2)(1+x)^{-3} = 2(1+x)^{-3}$	$(-1)^{n-1}(n-1)!(1+x)^{-n}$

2 a $y = e^{2+3x}$, so $\dfrac{dy}{dx} = 3e^{2+3x}$, $\dfrac{d^2y}{dx^2} = 3^2e^{2+3x}$, $\dfrac{d^3y}{dx^3} = 3^3e^{2+3x}$, and so on.

It follows that $\dfrac{d^ny}{dx^n} = 3^ne^{2+3x} = 3^n y$ as $y = e^{2+3x}$.

b

$y = e^{2+3x}$

$y' = 3e^{2+3x} = 3y$

$y'' = 3y' = 3^2y$

$\dfrac{d^ny}{dx^n} = 3^n y$

When $x = \log\dfrac{1}{9}$, $\dfrac{d^6y}{dx^6} = 3^6e^{2+3\log\frac{1}{9}}$

$= 3^6e^2\left(\dfrac{1}{9}\right)^3 = e^2$

3 a $y = \sin^2 3x = (\sin 3x)^2$, so $\dfrac{dy}{dx} = 2(\sin 3x)(3\cos 3x)$

$= 3(2\sin 3x\cos 3x)$

$= 3\sin 6x$

Use $\dfrac{du^n}{dx} = mu^{n-1}\dfrac{du}{dx}$.

Use $\sin 2A = 2\sin A\cos A$.

b $\dfrac{d^2y}{dx^2} = 18\cos 6x$, $\dfrac{d^3y}{dx^3} = -108\sin 6x$, $\dfrac{d^4y}{dx^4} = -648\cos 6x$

c $y'''' = -648\cos 6x$

When $x = \dfrac{\pi}{6}$, $\dfrac{d^4y}{dx^4} = -648\cos\pi = 648$

4 a $f'(x) = 2xe^{-x} - x^2e^{-x}$

$f''(x) = (2e^{-x} - 2xe^{-x}) - (2xe^{-x} - x^2e^{-x}) = e^{-x}(2 - 4x + x^2)$

$f'''(x) = e^{-x}(-4 + 2x) - e^{-x}(2 - 4x + x^2) = e^{-x}(-6 + 6x - x^2)$

b $f'''(x) = e^{-x}(6 - 2x) - e^{-x}(-6 + 6x - x^2) = e^{-x}(12 - 8x + x^2)$

so $f''''(2) = e^{-2}(12 - 16 + 4) = 0$

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5 a Given that $y = \sec x$, so $\dfrac{dy}{dx} = \sec x \tan x$

$$\frac{d^2y}{dx^2} = \sec x(\sec^2 x) + (\sec x \tan x)\tan x$$ ← Use the product rule.

$$= \sec x(\sec^2 x + \tan^2 x)$$

$$= \sec x(\sec^2 x + \sec^2 x - 1)$$ ← Use $1 + \tan^2 A = \sec^2 A$.

$$= 2\sec^3 x - \sec x$$

b $y'' = 2y^3 - y$

$y''' = 6y^2 y' - y'$

$= (6\sec^2 x - 1)\tan x \sec x$

When $x = \dfrac{\pi}{4}$, $y''' = (6\sec^2 \dfrac{\pi}{4} - 1)\tan \dfrac{\pi}{4} \sec \dfrac{\pi}{4}$

$= \left(6\left(\sqrt{2}\right)^2 - 1\right) \times 1 \times \sqrt{2} = 11\sqrt{2}$

6 a $\dfrac{d}{dx}(y^2) = \dfrac{d}{dx}(y^2)\dfrac{dy}{dx} = 2y\dfrac{dy}{dx}$ Use the chain rule.

$$\frac{d^2}{dx^2}(y^2) = \frac{d}{dx}\left(2y\frac{dy}{dx}\right) = 2y\frac{d^2y}{dx^2} + 2\frac{dy}{dx}\frac{dy}{dx} = 2y\frac{d^2y}{dx^2} + 2\left(\frac{dy}{dx}\right)^2$$ Use the product rule.

b $\dfrac{d^3}{dx^3}(y^2) = \dfrac{d}{dx}\left(2y\dfrac{d^2y}{dx^2} + 2\left(\dfrac{dy}{dx}\right)^2\right)$

$$= 2\left\{y\frac{d^3y}{dx^3} + \frac{dy}{dx} \times \frac{d^2y}{dx^2} + 2\frac{dy}{dx} \times \frac{d^2y}{dx^2}\right\}$$

$$= 2\left\{y\frac{d^3y}{dx^3} + 3\frac{dy}{dx} \times \frac{d^2y}{dx^2}\right\}$$

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7 $f(x) = \ln\left\{x + \sqrt{1+x^2}\right\}$

a $f'(x) = \dfrac{1}{x+\sqrt{(1+x^2)}} \times \left\{1 + \dfrac{x}{\sqrt{(1+x^2)}}\right\}$,

$= \dfrac{1}{x+\sqrt{(1+x^2)}} \times \left\{\dfrac{\sqrt{(1+x^2)}+x}{\sqrt{(1+x^2)}}\right\} = \dfrac{1}{\sqrt{(1+x^2)}}$

Use $\dfrac{d}{dx}(\ln u) = \dfrac{1}{u}\dfrac{du}{dx}$.

So $\sqrt{(1+x^2)}\ f'(x) = 1$

b Differentiating this equation w.r.t.x, using the product rule

$\sqrt{(1+x^2)}\, f''(x) + \dfrac{x}{\sqrt{(1+x^2)}} f'(x) = 0$

So $(1+x^2)f''(x) + xf'(x) = 0$ ← Multiply through by $\sqrt{(1+x^2)}$.

c Differentiating this results w.r.t. x

$\{(1+x^2)f'''(x) + 2xf''(x)\} + \{f'(x) + xf''(x)\} = 0$

giving

$(1+x^2)f'''(x) + 3xf''(x) + f'(x) = 0$

d $f'(0) = \dfrac{1}{\sqrt{1+0}} = 1$

Using $(1+x^2)f''(x) + xf'(x) = 0$ with $x = 0$ and $f'(0) = 1$

$f''(0) + (0)(1) = 0 \Rightarrow f''(0) = 0$

Using $(1+x^2)f'''(x) + 3xf''(x) + f'(x) = 0$ with $x = 0$, $f'(0) = 1$ and $f''(0) = 0$

$f'''(0) + (0)(0) + 1 = 0 \Rightarrow f'''(0) = -1$

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Series 2C

1 a $f(x) = (1-x)^{-1} \Rightarrow f(0) = 1$

$f'(x) = -1(1-x)^{-2}(-1) = (1-x)^{-2} \Rightarrow f'(0) = 1$

$f''(x) = -2(1-x)^{-3}(-1) = 2(1-x)^{-3} \Rightarrow f''(0) = 2$

$f'''(x) = -3.2(1-x)^{-4}(-1) = 3.2(1-x)^{-4} \Rightarrow f'''(0) = 3!$

General term: The pattern here is such that $f^{(r)}(x)$ can be written down

$$f^{(r)}(x) = r(r-1)\ldots 2(1-x)^{-(r+1)} = r!(1-x)^{-(r+1)} \quad \Rightarrow f^{(r)}(0) = r!$$

Using $f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \ldots + \frac{f^{(r)}(0)}{r!}x^r + \ldots$

$$(1-x)^{-1} = 1 + x + \frac{2}{2!}x^2 + \ldots + \frac{r!}{r!}x^r + \ldots = 1 + x + x^2 + \ldots + x^r + \ldots$$

b $f(x) = \sqrt{(1+x)} = (1+x)^{\frac{1}{2}} \Rightarrow f(0) = 1$

$f'(x) = \frac{1}{2}(1+x)^{-\frac{1}{2}} \Rightarrow f'(0) = \frac{1}{2}$

$f''(x) = \frac{1}{2}\left(-\frac{1}{2}\right)(1+x)^{-\frac{3}{2}} \Rightarrow f''(0) = -\frac{1}{4}$

$f'''(x) = \frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)(1+x)^{-\frac{5}{2}} \Rightarrow f'''(0) = \frac{3}{8}$

Using Maclaurin's expansion

$$\sqrt{(1+x)} = 1 + \frac{1}{2}x + \frac{\left(-\frac{1}{4}\right)}{2!}x^2 + \frac{\left(\frac{3}{8}\right)}{3!}x^3 - \ldots$$
$$= 1 + \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{16} - \ldots$$

2 $f(x) = e^{\sin x} \Rightarrow f(0) = 1$

$f'(x) = \cos x\, e^{\sin x} \Rightarrow f'(0) = 1$

$f''(x) = \cos^2 x\, e^{\sin x} - \sin x\, e^{\sin x} \Rightarrow f''(0) = 1$

Substituting into Maclaurin's expansion gives

$$e^{\sin x} = 1 + 1x + \frac{1}{2!}x^2 + \ldots$$
$$= 1 + x + \frac{1}{2}x^2 + \ldots$$

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3 a $f(x) = \cos x \quad \Rightarrow f(0) = 1$

$f'(x) = -\sin x \quad \Rightarrow f'(0) = 0$

$f''(x) = -\cos x \quad \Rightarrow f''(0) = -1$

$f'''(x) = \sin x \quad \Rightarrow f'''(0) = 0$

$f''''(x) = \cos x \quad \Rightarrow f''''(0) = 1$

The process repeats itself after every 4th derivative, like sin x does.
Using Maclaurin's expansion, only even powers of x are produced.

$$\cos x = 1 + \frac{(-1)}{2!}x^2 + \frac{1}{4!}x^4 + \ldots + \frac{(-1)^r}{(2r)!}x^{2r} + \ldots$$
$$= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \ldots + (-1)^r \frac{x^{2r}}{(2r)!} + \ldots$$

b Using $\cos x \approx 1 - \frac{x^2}{2!} + \frac{x^4}{4!}$ with $x = \frac{\pi}{6}$ (must be in radians)

$\cos \approx 1 - \frac{\pi^2}{72} + \frac{\pi^4}{31\,104} = 0.86605\ldots$ which is correct to 3 d.p.

4 a Substituting $x = 1$ into the Maclaurin expansion of e^x, gives

$$e = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} + \frac{1}{6!} + \frac{1}{7!} + \ldots$$

The approximations, to 4 d.p. where necessary, using n terms of the series are

n	1	2	3	4	5	6	7	8	9	10
Approx.	1	2	2.5	2.6667	2.7083	2.7167	2.7181	2.7183	2.7183	2.7183

So $e = 2.718$ (3 d.p.)

b Substituting $x = 0.2$ into the Maclaurin expansion of $\ln(1+x)$, gives

$$\ln\left(\frac{6}{5}\right) = 0.2 - \frac{(0.2)^2}{2} + \frac{(0.2)^3}{3} - \frac{(0.2)^4}{4} + \frac{(0.2)^5}{5} - \frac{(0.2)^6}{6} + \frac{(0.2)^7}{7} - \ldots$$

The approximations, to 4 d.p. where necessary, using n terms of the series are

n	1	2	3	4	5
Approximation	0.2	0.18	0.1827	0.1823	0.1823

So $\ln\left(\frac{6}{5}\right) = 0.182$ (3 d.p.)

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5 a $f(x) = e^{3x}, f^{(n)}(x) = 3^n e^{3x}$

So $f(0) = 1$, $f'(0) = 3$, $f''(0) = 3^2$, $f'''(0) = 3^3$, $f''''(0) = 3^4$

$$f(x) = e^{3x} = 1 + 3x + \frac{3^2}{2!}x^2 + \frac{3^3}{3!}x^3 + \frac{3^4}{4!}x^4 + \ldots$$

$$= 1 + 3x + \frac{9x^2}{2} + \frac{9x^3}{2} + \frac{27}{8}x^4 + \ldots \left[\text{Note : this is } 1 + 3x + \frac{(3x)^2}{2!} + \frac{(3x)^3}{3!} + \frac{(3x)^4}{4!} + \ldots\right]$$

b As $f(x) = \ln(1 + 2x)$, $f(0) = \ln 1 = 0$

$f'(x) = \frac{2}{1+2x} = 2(1+2x)^{-1}$, $f'(0) = 2$

$f''(x) = -4(1+2x)^{-2}$, $f''(0) = -4$

$f'''(x) = 16(1+2x)^{-3}$, $f'''(0) = 16$

$f''''(x) = -96(1+2x)^{-4}$, $f''''(0) = -96$

So $\ln(1+2x) = 0 + 2x + \frac{(-4)}{2!}x^2 + \frac{(16)}{3!}x^3 + \frac{(-96)}{4!}x^4 + \ldots$

$$= 2x - 2x^2 + \frac{8x^3}{3} - 4x^4 + \ldots \left[\text{Note : this is } 2x - \frac{(2x)^2}{2} + \frac{(2x)^3}{3} - \frac{(2x)^4}{4} + \ldots\right]$$

c $f(x) = \sin^2 x$ $f(0) = 0$

$f'(x) = 2\sin x \cos x = \sin 2x$ $f'(0) = 0$

$f''(x) = 2\cos 2x$ $f''(0) = 2$

$f'''(x) = -4\sin 2x$ $f'''(0) = 0$

$f''''(x) = -8\cos 2x$ $f''''(0) = -8$

So, $f(x) = \sin^2 x = 0 + 0x + \frac{2}{2!}x^2 + 0x^3 + \frac{(-8)}{4!}x^4 + \ldots = x^2 - \frac{x^4}{3} + \ldots$

6 $\cos\left(x - \frac{\pi}{4}\right) = \cos x \cos\left(\frac{\pi}{4}\right) + \sin x \sin\left(\frac{\pi}{4}\right)$

Use $\cos(A - B) = \cos A \cos B + \sin A \sin B$.

$$= \frac{1}{\sqrt{2}}(\cos x + \sin x)$$

$$= \frac{1}{\sqrt{2}}\left\{\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots\right) + \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots\right)\right\}$$

$$= \frac{1}{\sqrt{2}}\left(1 + x - \frac{x^2}{2} - \frac{x^3}{6} + \frac{x^4}{24} + \ldots\right)$$

7 a $f(x) = (1-x)^2 \ln(1-x)$

$f'(x) = (1-x^2) \times \frac{(-1)}{1-x} + 2(1-x)(-1)\ln(1-x)$

Use the product rule.

$= x - 1 - 2(1-x)\ln(1-x)$

$f''(x) = 1 - 2\left[(1-x) \times \frac{(-1)}{1-x} - \ln(1-x)\right] = 1 + 2 + 2\ln(1-x) = 3 + 2\ln(1-x)$

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7 b $f'''(x) = \dfrac{-2}{1-x}$

Substituting $x = 0$ in all the results gives

$f(0) = 0,\ f'(0) = -1,\ f''(0) = 3,\ f'''(0) = -2$

c $f(x) = (1-x)^2 \ln(1-x) = 0 + (-1)x + \dfrac{3}{2!}x^2 + \dfrac{(-2)}{3!}x^3 + \ldots$

$= -x + \dfrac{3x^2}{2} - \dfrac{1}{3}x^3$

8 a Using the series expansions for $\sin x$ and $\cos x$ as far as the term in x^5,

$\sin x = x - \dfrac{x^3}{3!} + \dfrac{x^5}{5!} - \ldots = x - \dfrac{1}{6}x^3 + \dfrac{1}{120}x^5 - \ldots$

$\cos x = 1 - \dfrac{x^2}{2!} + \dfrac{x^4}{4!} - \ldots = 1 - \dfrac{1}{2}x^2 + \dfrac{1}{24}x^4 - \ldots$

so $3\sin x - 4x\cos x + x = 3\left(x - \dfrac{1}{6}x^3 + \dfrac{1}{120}x^5 - \ldots\right) - 4x\left(1 - \dfrac{1}{2}x^2 + \dfrac{1}{24}x^4 - \ldots\right) + x$

$= 3x - \dfrac{1}{2}x^3 + \dfrac{1}{40}x^5 - 4x + 2x^3 - \dfrac{1}{6}x^5 + x + \ldots$

$3\sin x - 4x\cos x + x = \dfrac{3}{2}x^3 - \dfrac{17}{120}x^5 + \ldots$

b $\dfrac{3\sin x - 4x\cos x + x}{x^3} = \dfrac{3}{2} - \dfrac{17}{120}x^2 +$ higher powers in x using **a**

Hence, the limit, as $x \to 0$, is $\dfrac{3}{2}$.

9 a $f(x) = \ln\cos x \qquad \Rightarrow f(0) = 0$

$f'(x) = \dfrac{1}{\cos x} \times (-\sin x) \qquad \Rightarrow f'(0) = 0$

$= -\tan x$

b $f''(x) = -\sec^2 x \qquad \Rightarrow f''(0) - 1$

$f'''(x) = -2\sec x(\sec x \tan x) = -2\sec^2 x \tan x \qquad \Rightarrow f'''(0) = 0$

$f''''(x) = -2\{\sec^2 x(\sec^2 x) + \tan x(2\sec^2 x \tan x)\} \qquad \Rightarrow f''''(0) = -2$

c Substituting into Maclaurin's expansion

$\ln\cos x = 0 + 0x + \dfrac{(-1)}{2!}x^2 + 0x^3 + \dfrac{(-2)}{4!}x^4 + \ldots$

$= -\dfrac{x^2}{2} - \dfrac{x^4}{12} + \ldots$

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9 d Substituting $x = \frac{\pi}{4}$ gives $\ln\left(\frac{1}{\sqrt{2}}\right) = -\frac{1}{2}\left(\frac{\pi^2}{16}\right) - \frac{1}{12}\left(\frac{\pi^4}{256}\right)$

but but $\ln\left(\frac{1}{\sqrt{2}}\right) = \ln 2^{-\frac{1}{2}} = -\frac{1}{2}\ln 2$,

so $-\frac{1}{2}\ln 2 = -\frac{\pi^2}{2.16} - \frac{\pi^4}{12.256} + \ldots$

$\Rightarrow \ln 2 = \frac{\pi^2}{16} + \frac{\pi^4}{6.256}$, using only first two terms.

$= \frac{\pi^2}{16}\left(1 + \frac{\pi^2}{96}\right)$

10

$f(x) = \tan x \Rightarrow f(0) = 0$

$f'(x) = \sec^2 x \Rightarrow f'(0) = 1$

$f''(x) = 2\sec x(\sec x \tan x) = 2\sec^2 x \tan x \Rightarrow f''(0) = 0$

$f'''(x) = 2\sec^4 x + 4\sec^2 x \tan^2 x \Rightarrow f'''(0) = 2$

$f''''(x) = 16\sec^4 x \tan x + 8\sec^2 x \tan^3 x \Rightarrow f''''(0) = 0$ as $\tan(0) = 0$

$f'''''(x) = 16\sec^2 x \tan^4 x + 88\sec^4 x \tan^2 x + 16\sec^6 x \Rightarrow f'''''(0) = 16$ as $\tan(0) = 0$ and $\sec(0) = 1$

Substituting into Maclaurin's expansion gives

$\tan x = 0 + 1x + \frac{0}{2!}x^2 + \frac{2}{3!}x^3 + \frac{0}{4!}x^3 + \frac{16}{5!}x^5 + \ldots$

$= x + \frac{x^3}{3} + \frac{2}{15}x^5 + \ldots$

Challenge

a $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$

$\lim_{n\to\infty}\left|\frac{\frac{x^{n+1}}{(n+1)!}}{\frac{x^n}{n!}}\right| = \lim_{n\to\infty}\left|\frac{x}{n+1}\right| = 0, \forall x \in \mathbb{R}$

so the Maclaurin expansion of

e^x converges for all $x \in \mathbb{R}$

b $\ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}x^n}{n}$

$\lim_{n\to\infty}\left|\frac{\frac{(-1)^n x^{n+1}}{n+1}}{\frac{(-1)^{n-1}x^n}{n}}\right| = \lim_{n\to\infty}\left|-x\frac{n}{n+1}\right| = |x|\lim_{n\to\infty}\left|\frac{n}{n+1}\right| = |x|$

so the Maclaurin expansion of

$\ln(1+x)$ converges for $|x| < 1$

and diverges for $|x| > 1$

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Series 2D

1 a $\dfrac{1}{e^x} = e^{-x} = 1 + (-x) + \dfrac{(-x)^2}{2!} + \dfrac{(-x)^3}{3!} +$

$= 1 - x + \dfrac{x^2}{2} - \dfrac{x^3}{6} + \ldots$ valid for all values of x

b $\dfrac{e^{2x} \times e^{3x}}{e^x} = e^{4x} = 1 + (4x) + \dfrac{(4x)^2}{2!} + \dfrac{(4x)^3}{3!} +$

$= 1 + 4x + 8x^2 + \dfrac{32x^3}{3} + \ldots$ valid for all values of x

c $e^{1+x} = e \times e^x = e\left\{1 + x + \dfrac{x^2}{2} + \dfrac{x^3}{6} + \ldots\right\}$ valid for all values of x

d $\ln(1-x) = (-x) - \dfrac{(-x)^2}{2} + \dfrac{(-x)^3}{3} + \dfrac{(-x)^4}{4} + \ldots \quad [-1 < -x \leqslant 1]$

$\Rightarrow 1 > x \geqslant -1$

$= -x - \dfrac{x^2}{2} - \dfrac{x^3}{3} - \dfrac{x^4}{4} - \ldots \qquad -1 \leqslant x < 1$

e $\sin\left(\dfrac{x}{2}\right) = \left(\dfrac{x}{2}\right) - \dfrac{\left(\frac{x}{2}\right)^3}{3!} + \dfrac{\left(\frac{x}{2}\right)^5}{5!} - \dfrac{\left(\frac{x}{2}\right)^7}{7!} + \ldots$

$= \dfrac{x}{2} - \dfrac{x^3}{48} + \dfrac{x^5}{3840} - \dfrac{x^7}{645\,120} + \ldots$ valid for all values of x

f $\ln(2+3x) = \ln\left\{2\left(1 + \dfrac{3x}{2}\right)\right\} = \ln 2 + \ln\left(1 + \dfrac{3x}{2}\right)$

$= \ln 2 + \dfrac{3x}{2} - \dfrac{\left(\frac{3x}{2}\right)^2}{2} + \dfrac{\left(\frac{3x}{2}\right)^3}{3} + \qquad \left[-1 < \dfrac{3x}{2} \leqslant 1\right]$

$= \ln 2 + \dfrac{3x}{2} - \dfrac{9x^2}{8} + \dfrac{9x^3}{8} + \ldots \qquad -\dfrac{2}{3} < x \leqslant \dfrac{2}{3}$

2 a $\ln(1+x) = x - \dfrac{x^2}{2} + \dfrac{x^3}{3} - \dfrac{x^4}{4} + \dfrac{x^5}{5} - \ldots, \qquad -1 < x \leqslant 1$

$\ln(1-x) = -x - \dfrac{x^2}{2} - \dfrac{x^3}{3} - \dfrac{x^4}{4} - \dfrac{x^5}{5} - \ldots, \qquad -1 \leqslant x < 1$

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2 a

$$\ln\left(\frac{1+x}{1-x}\right) = \ln(1+x) - \ln(1-x)$$
$$= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \ldots - \left(-x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \frac{x^5}{5} - \ldots\right)$$
$$= 2x + \frac{2x^2}{3} + \frac{2x^5}{5} + \ldots$$
$$= 2\left(x + \frac{x^3}{3} + \frac{x^5}{5} + \ldots\right)$$

As x must be in both the intervals $-1 < x \leqslant 1$ and $-1 \leqslant x < 1$ this expansion requires x to be in the interval $-1 < x < 1$.

b $\ln\sqrt{\left(\frac{1+x}{1-x}\right)} = \ln\left(\frac{1+x}{1-x}\right)^{\frac{1}{2}} = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$

so $\ln\sqrt{\left(\frac{1+x}{1-x}\right)} = \left(x + \frac{x^3}{3} + \frac{x^5}{5} + \ldots\right), -1 < x < 1.$

c Solving $\left(\frac{1+x}{1-x}\right) = \frac{2}{3}$ gives $3 + 3x = 2 - 2x$

$5x = -1$

$x = -0.2$

This is a valid value of x.

So an approximation to $\ln\left(\frac{2}{3}\right)$ is $2\left(-0.2 - \frac{0.008}{3} - \frac{0.00032}{5}\right)$

$= 2(-0.2 - 0.0026666 - 0.000064)$

$= -0.4055$ (4 d.p.) This is accurate to 4 d.p.

d $\ln\sqrt{\left(\frac{1+x}{1-x}\right)}$ with $x = \frac{3}{5}$ gives $\ln\sqrt{4} = \ln 2$

so $\ln 2 \approx 0.6 + \frac{(0.6)^3}{3} + \frac{(0.6)^5}{5}$

$\approx 0.687552\ldots = 0.69$ (2 d.p.)

Use the result in **b**.

3 $e^{2x} = 1 + 2x + \frac{(2x)^2}{2!} + \frac{(2x)^3}{3!} + \ldots = 1 + 2x + 2x^2 + \frac{4x^3}{3} + \ldots$

$e^{-x} = 1 - x + \frac{(-x)^2}{2!} + \frac{(-x)^3}{3!} + \ldots = 1 - x + \frac{x^2}{2} - \frac{x^3}{6} + \ldots$

So $e^{2x} - e^{-x} \approx 3x + \frac{3}{2}x^2$, if terms x^3 and above may be neglected.

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4 a $3x\sin 2x = 3x\left\{(2x) - \frac{(2x)^3}{3!} + \ldots\right\} = 6x^2 - 4x^4 + \ldots$

$\cos 3x = \left\{1 - \frac{(3x)^2}{2!} + \frac{(3x)^4}{4!} - \right\} = 1 - \frac{9}{2}x^2 + \frac{27}{8}x^4 - \ldots$

So $3x\sin 2x - \cos 3x = 6x^2 - 4x^4 + \ldots - \left(1 - \frac{9}{2}x^2 + \frac{27}{8}x^4 - \ldots\right)$

$= -1 + \frac{21}{2}x^2 - \frac{59}{8}x^4 + \ldots$

b $\frac{3x\sin 2x - \cos 3x + 1}{x^2} = \frac{21}{2} - \frac{59}{8}x^2 + \text{terms in higher powers of } x$

As $x \to 0$, so $\frac{3x\sin 2x - \cos 3x + 1}{x^2}$ tends to $\frac{21}{2}$.

5 a $\ln(1 + x - 2x^2) = \ln(1 - x)(1 + 2x) = \ln(1 - x) + \ln(1 + 2x)$

$\ln(1 - x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \ldots, \qquad -1 \leqslant x < 1$

$\ln(1 + 2x) = (2x) - \frac{(2x)^2}{2} + \frac{(2x)^3}{3} - \frac{(2x)^4}{4} + \ldots, \qquad -\frac{1}{2} < x \leqslant \frac{1}{2}$

$= 2x - 2x^2 + \frac{8x^3}{3} - 4x^4$

So $\ln(1 + x - 2x^2) = \ln(1 - x) + \ln(1 + 2x)$

$= x - \frac{5x^2}{2} + \frac{7x^3}{3} - \frac{17x^4}{4} + \ldots, \qquad -\frac{1}{2} < x \leqslant \frac{1}{2}$ (smaller interval)

b $\ln(9 + 6x + x^2) = \ln(3 + x)^2 = 2\ln(3 + x) = 2\ln 3\left(1 + \frac{x}{3}\right) = 2\left\{\ln 3 + \ln\left(1 + \frac{x}{3}\right)\right\}$

The expansion of $\ln\left(1 + \frac{x}{3}\right)$ is $= \left(\frac{x}{3}\right) - \frac{\left(\frac{x}{3}\right)^2}{2} + \frac{\left(\frac{x}{3}\right)^3}{3} - \frac{\left(\frac{x}{3}\right)^4}{4} + \ldots, \quad \left[-1 < \frac{x}{3} \leqslant 1\right]$

$= \frac{x}{3} - \frac{x^2}{18} + \frac{x^3}{81} - \frac{x^4}{324} + \ldots, \qquad -3 < x \leqslant 3$

So $\ln(9 + 6x + x^2) = 2\left\{\ln 3 + \ln\left(1 + \frac{x}{3}\right)\right\}$

$= 2\ln 3 + \frac{2x}{3} - \frac{x^2}{9} + \frac{2x^3}{81} - \frac{x^4}{162} + \ldots, \qquad -3 < x \leqslant 3$

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6 a $\cos 2x = \left\{1 - \frac{(2x)^2}{2!} + \frac{(2x)^4}{4!} - \frac{(2x)^6}{6!} + \frac{(2x)^8}{8!} - \ldots\right\}$

$= 1 - 2x^2 + \frac{2x^4}{3} - \frac{4x^6}{45} + \frac{2x^8}{315} - \ldots$

b Using $\cos 2x = 1 - 2\sin^2 x$,

$2\sin^2 x = 1 - \cos 2x = 2x^2 - \frac{2x^4}{3} + \frac{4x^6}{45} - \frac{2x^8}{315} + \ldots$

So $\sin^2 x = x^2 - \frac{x^4}{3} + \frac{2x^6}{45} - \frac{x^8}{315} + \ldots$

[Alternative: write out expansion of sin x as far as term in x^7, square it, and collect together appropriate terms!]

7 $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots$

$(x-1)(e^x - 1) = (x-1)\left(x + \frac{x^2}{2} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots\right)$

$= x^2 + \frac{x^3}{2} + \frac{x^4}{6} \ldots - \left(x + \frac{x^2}{2} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots\right)$

$= -x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{8} + \ldots$

So $\ln(1+x) + (x-1)(e^x - 1) = \left(x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots\right) + \left(-x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{8} + \ldots\right)$

$= \frac{2x^3}{3} - \frac{x^4}{8} + \ldots \qquad \Rightarrow p = \frac{2}{3},\ q = -\frac{1}{8}$

8 a Only terms up to and including x^4 in the product are required, so using

$\sin x = x - \frac{x^3}{3!} + \ldots$ (next term is kx^5)

and the binomial expansion of $(1-x)^{-2}$, with terms up to and including x^3.

(It is not necessary to use the term in x^4, because it will be multiplied by expansion of sin x.)

$(1-x)^{-2} = 1 + (-2)(-x) + (-2)(-3)\frac{(-x)^2}{2!} + (-2)(-3)(-4)\frac{(-x)^3}{3!} + \ldots$

$= 1 + 2x + 3x^2 + 4x^3 + \ldots$

So $\frac{\sin x}{(1-x)^2} = \left(x - \frac{x^3}{6} + \ldots\right)(1 + 2x + 3x^2 + 4x^3 + \ldots)$

$= x + 2x^2 + 3x^3 + 4x^4 + \ldots - \left(\frac{x^3}{6} + \frac{x^4}{3} + \ldots\right)$

$= x + 2x^2 + \frac{17x^3}{6} + \frac{11x^4}{3} + \ldots$

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8 b $y = \dfrac{\sin x}{(1-x)^2} = x + 2x^2 + \dfrac{17x^3}{6} + \dfrac{11x^4}{3} + \ldots$

So $\dfrac{\mathrm{d}y}{\mathrm{d}x} = 1 + 4x +$ higher powers of $x \Rightarrow$ at the origin the gradient of tangent $= 1$.

9 a $(1-3x)\ln(1+2x) = (1-3x)\left(2 - 2x^2 + \dfrac{8x^3}{3} - 4x^4 + \ldots\right)$

$$= \left(2x - 2x^2 + \frac{8x^3}{3} - 4x^4 + \ldots\right) - (6x^2 - 6x^3 + 8x^4 - \ldots)$$

$$= 2x - 8x^2 + \frac{26}{3}x^3 - 12x^4 + \ldots$$

b $\mathrm{e}^{2x}\sin x = \left\{1 + (2x) + \dfrac{(2x)^2}{2!} + \dfrac{(2x)^3}{3!} + \dfrac{(2x)^4}{4!} + \ldots\right\}\left\{x - \dfrac{x^3}{3!} + \ldots\right\}$ [only terms up to x^4]

$$= \left(1 + 2x + 2x^2 + \frac{4x^3}{3} + \frac{2x^4}{3} + \ldots\right)\left(x - \frac{x^3}{6} + \ldots\right)$$

$$= \left(x + 2x^2 + 2x^3 + \frac{4x^4}{3}\right) + \left(-\frac{x^3}{6} - \frac{x^4}{3}\right) + \ldots$$

$$= x + 2x^2 + \frac{11}{6}x^3 + x^4 + \ldots$$

c $\sqrt{(1+x^2)}\mathrm{e}^{-x} = (1+x^2)^{\frac{1}{2}}\mathrm{e}^{-x}$

$$= \left\{1 + \frac{1}{2}x^2 + \left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\frac{(x^2)^2}{2!} + \ldots\right\}\left(1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots\right)$$

$$= \left(1 + \frac{x^2}{2} - \frac{x^4}{8} + \ldots\right)\left(1 - x + \frac{x^2}{2} - \frac{x^3}{6} + \frac{x^4}{24} + \ldots\right)$$

$$= \left\{1 - x + \left(\frac{1}{2} + \frac{1}{2}\right)x^2 + \left(-\frac{1}{2} - \frac{1}{6}\right)x^3 + \left(\frac{1}{24} + \frac{1}{4} - \frac{1}{8}\right)x^4 + \ldots\right\}$$

$$= 1 - x + x^2 - \frac{2}{3}x^3 + \frac{1}{6}x^4 + \ldots$$

10 a $\mathrm{e}^{-\frac{x^2}{2}} = 1 + \left(-\dfrac{x^2}{2}\right) + \dfrac{\left(-\frac{x^2}{2}\right)^2}{2!} + \dfrac{\left(-\frac{x^2}{2}\right)^3}{3!} + \dfrac{\left(-\frac{x^2}{2}\right)^4}{4!} + \ldots$

$$= 1 - \frac{x^2}{2} + \frac{x^4}{8} - \frac{x^6}{48} + \frac{x^8}{384} - \ldots$$

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10 b Area under the curve $= \int_{-1}^{1} e^{-\frac{x^2}{2}}\,dx = 2\int_{0}^{1} e^{-\frac{x^2}{2}}\,dx$

$$= 2\left[x - \frac{x^3}{6} + \frac{x^5}{40} - \frac{x^7}{336} + \frac{x^9}{3456} - \ldots\right]_0^1$$

Integrate the result from **a.**

$$\approx 2\left[1 - \frac{1}{6} + \frac{1}{40} - \frac{1}{336} + \frac{1}{3456}\right]$$

$$\approx 1.711 \text{ (3 d.p.)}$$

11 a $e^{px}\sin 3x = \left\{1 + (px) + \frac{(px)^2}{2!} + \frac{(px)^3}{3!} + \ldots\right\}\left\{(3x) - \frac{9x^3}{2} + \ldots\right\}$

$$= \left(1 + px + \frac{p^2x^2}{2} + \frac{p^3x^3}{6} + \ldots\right)\left(3x - \frac{9x^3}{2} + \ldots\right)$$

$$= \left(3x + 3px^2 + \frac{3p^2x^3}{2} + \ldots\right) + \left(-\frac{9x^3}{2} + \ldots\right)$$

$$= 3x + 3px^2 + \frac{3(p^2 - 3)x^3}{2} + \ldots$$

b $\ln(1+qx) = \left\{(qx) - \frac{(qx)^2}{2} + \frac{(qx)^3}{3} - \ldots\right\}$

So $e^{px}\sin 3x + \ln(1+qx) - x = 3x + 3px^2 + \frac{3(p^2-3)x^3}{2} + qx - \frac{q^2x^2}{2} + \frac{q^3x^3}{3} - x + \ldots$

$$= (2+q)x + \left(3p - \frac{q^2}{2}\right)x^2 + \left(\frac{3p^2}{2} + \frac{q^3}{3} - \frac{9}{2}\right)x^3 + \ldots$$

Coefficient of x is zero, so $q = -2$.

Coefficient of x^2 is zero, so $3p - 2 = 0 \Rightarrow p = \frac{2}{3}$

Coefficient of $x^3 = \frac{2}{3} - \frac{8}{3} - \frac{9}{2} = -\frac{13}{2}$, so $k = -\frac{13}{2}$

12 a $e^{x-\ln x} = e^x \times e^{-\ln x} = e^x \times e^{-\ln x^{-1}}$ Using $e^{a+b} = e^a \times e^b$

$= e^x \times x^{-1}$ using $e^{\ln k} = k$

$= \frac{e^x}{x}$

$e^{x-\ln x}\sin x = \frac{e^x \sin x}{x}$, and so, using the expansions of e^x and $\sin x$,

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12 a

$$f(x) = e^{x-\ln x}\sin x = \frac{\left(1+x+\frac{x^2}{2}+\frac{x^3}{6}+\ldots\right)\left(x-\frac{x^3}{6}+\ldots\right)}{x}, x>0$$

$$= \left(1+x+\frac{x^2}{2}+\frac{x^3}{6}+\ldots\right)\left(1-\frac{x^2}{6}+\ldots\right)$$

$$= \left(1+x+\frac{x^2}{2}+\frac{x^3}{6}\right)-\left(\frac{x^2}{6}+\frac{x^3}{6}\right) \quad \text{ignoring terms in } x^4 \text{ and above.}$$

$$= 1+x+\frac{x^2}{3} \qquad \text{There is no term in } x^3.$$

b $f(0.1) = \dfrac{e^{0.1}\sin 0.1}{0.1} = 1.103329\ldots$

The result in **a** gives an approximation for $f(0.1)$ of $1+0.1+0.00333333 = 1.103333\ldots$ which is correct to 6 s.f.

13 a

$$y = \sin 2x - \cos 2x$$
$$y' = 2\cos 2x + 2\sin 2x$$
$$y'' = -4\sin 2x + 4\cos 2x = -4y$$
$$y''' = -4y'$$
$$y'''' = -4y'' = 16y$$

b

$$\sin 2x = (2x) - \frac{(2x)^3}{3!} + \frac{(2x)^5}{5!} - \ldots$$

$$\cos 2x = 1 - \frac{(2x)^2}{2!} + \frac{(2x)^4}{4!} - \ldots$$

$$y = -1 + 2x + \frac{4x^2}{2!} - \frac{8x^3}{3!} - \frac{16x^4}{4!} + \ldots$$

$$= -1 + 2x + 2x^2 - \frac{4x^3}{3} - \frac{2x^4}{3} + \ldots$$

Challenge

a

$$y = (1-\beta^2)^{-\frac{1}{2}}$$
$$f(x) = (1+x)^a$$
$$f'(x) = a(1+x)^{a-1}, f''(x) = a(a-1)(1+x)^{a-2}$$
$$f(x) = 1 + ax + \frac{a(a-1)}{2!}x^2 + \ldots$$

$$\text{So } y = 1 + \left(-\frac{1}{2}\right)(-\beta^2) + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)(-\beta^2)^2}{2!} + \ldots$$

$$= 1 + \frac{1}{2}\beta^2 + \frac{3}{8}\beta^4 + \ldots$$

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Challenge

b $\beta = \frac{v}{c} = \frac{\frac{4.2c}{20}}{c} = 0.21$

$\lambda \approx 1 + \frac{1}{2}0.21^2 + \frac{3}{8}0.21^4 = 1.02278$ (5 d.p.)

Observed journey time is $\frac{20}{\lambda} = 19.55$ years

c $\gamma = \frac{1}{\sqrt{1-0.21^2}} = 1.02281$ (5 d.p.)

Percentage error $= \frac{\gamma - \lambda}{\gamma} \times 100$

$= \frac{1.022807 - 1.022779}{1.022807} \times 100$

$= 0.0027\%$

d The velocity will be larger and so $\beta = \frac{v}{c}$ will be larger making the error in γ larger. Hence the approximation would be less accurate.

if ship $3\times$ faster

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Series – mixed exercise 2

1 a $\dfrac{2}{(r+2)(r+4)} = \dfrac{A}{r+2} + \dfrac{B}{r+4}$

$2 = A(r+4) + B(r+2)$

$2 = 2A = -2B$

$\dfrac{2}{(r+2)(r+4)} = \dfrac{1}{r+2} - \dfrac{1}{r+4}$

b Let $\mathrm{f}(r) = \dfrac{1}{r+2}$

$$\sum_{r=1}^{n} \frac{2}{(r+2)(r+4)} = \sum_{r=1}^{n} \left(\frac{1}{r+2} - \frac{1}{r+4} \right)$$

$$= \sum_{r=1}^{n} \big(\mathrm{f}(r) - \mathrm{f}(r+2) \big)$$

$$= \mathrm{f}(1) + \mathrm{f}(2) - \mathrm{f}(n+1) - \mathrm{f}(n+2)$$

$$= \frac{1}{3} + \frac{1}{4} - \frac{1}{n+3} - \frac{1}{n+4}$$

$$= \frac{7(n+3)(n+4) - 12(n+3) - 12(n+4)}{12(n+3)(n+4)}$$

$$= \frac{7n^2 + 25n}{12(n+3)(n+4)}$$

2 a $\dfrac{4}{(4r-1)(4r+3)} = \dfrac{A}{4r-1} + \dfrac{B}{4r+3}$

$4 = A(4r+3) + B(4r-1)$

$4 = 4A = -4B$

$\dfrac{4}{(4r-1)(4r+3)} = \dfrac{1}{4r-1} - \dfrac{1}{4r+3}$

b Let $\mathrm{f}(r) = \dfrac{1}{4r-1}$

$$\sum_{r=1}^{n} \frac{4}{(4r-1)(4r+3)} = \sum_{r=1}^{n} \left(\frac{1}{4r-1} - \frac{1}{4r+3} \right)$$

$$= \sum_{r=1}^{n} \big(\mathrm{f}(r) - \mathrm{f}(r+1) \big) = \mathrm{f}(1) - \mathrm{f}(n+1)$$

$$= \frac{1}{3} - \frac{1}{4n+3}$$

$$= \frac{4n}{3(4n+3)}$$

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2 c $\displaystyle\sum_{r=100}^{200}\frac{4}{(4r-1)(4r+3)}$

$$=\sum_{r=1}^{200}\frac{4}{(4r-1)(4r+3)}-\sum_{r=1}^{99}\frac{4}{(4r-1)(4r+3)}$$

$$=\frac{800}{3(803)}-\frac{396}{3(399)}=\frac{404}{320397}=0.00126\ (3\text{ s.f.})$$

3 a $(r+1)^3-(r-1)^3$

$$=r^3+3r^2+3r+1-(r^3-3r^2+3r-1)$$

$$=6r^2+2$$

b $r^2=\frac{1}{6}\left((r+1)^3-(r-1)^3-2\right)$

$$\sum_{r=1}^{n}(r+1)^3-(r-1)^3=(n+1)^3+n^3-1$$

$$=2n^3+3n^2+3n$$

$$\sum_{r=1}^{n}r^2=\frac{1}{6}\left(2n^3+3n^2+3n-2\sum_{r=1}^{n}1\right)$$

$$=\frac{1}{6}\left(2n^3+3n^2+3n-2n\right)=\frac{1}{6}n(n+1)(2n+1)$$

4 $\dfrac{4}{(r+1)(r+3)}=\dfrac{A}{r+1}+\dfrac{B}{r+3}$

$4=A(r+3)+B(r+1)$

$4=2A=-2B$

Let $\mathrm{f}(r)=\dfrac{1}{r+1}$

$$\sum_{r=1}^{n}\frac{4}{(r+1)(r+3)}=2\sum_{r=1}^{n}\left(\frac{1}{r+1}-\frac{1}{r+3}\right)$$

$$=2(\mathrm{f}(1)+\mathrm{f}(2)-\mathrm{f}(n+1)-\mathrm{f}(n+2))$$

$$=2\left(\frac{1}{2}+\frac{1}{3}-\frac{1}{n+2}-\frac{1}{n+3}\right)=2\left(\frac{5}{6}-\frac{1}{n+2}-\frac{1}{n+3}\right)$$

$$=\frac{1}{3}\left(\frac{5(n+2)(n+3)-6(n+2)-6(n+3)}{(n+2)(n+3)}\right)$$

$$=\frac{5n^2+13n}{3(n+2)(n+3)}$$

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5 $\sum_{r=1}^{n}(r+1)^3-(r-1)^3=(n+1)^3+n^3-1$

$\sum_{r=n}^{2n}(r+1)^3-(r-1)^3$

$=(2n+1)^3+(2n)^3-1-(n^3+(n-1)^3-1)$

$=16n^3+12n^2+6n-(2n^3-3n^2+3n-2)$

$=14n^3+15n^2+3n+2$

So $a=14,\ b=15,\ c=3$ and d $=2$

6 a $y=e^{1-2x}$

$y'=-2e^{1-2x}=-2y$

$y''=-2y'=(-2)^2y$

$\frac{d^ny}{dx^n}=(-2)^ny$

b $\frac{d^8y}{dx^8}=(-2)^8y$

When $x=\log 32,\ \frac{d^8y}{dx^8}=2^8e^{1-2\log 32}$

$=2^8e(32)^{-2}=2^{8-2\times5}e=\frac{e}{4}$

7 a $f(x)=\ln(1+e^x)$ so $f(0)=\ln 2$

$f'(x)=\frac{e^x}{1+e^x}$ $=1-\frac{1}{1+e^x}=1-(1+e^x)^{-1}$ $f'(0)=\frac{1}{2}$

So $f''(x)=\frac{e^x}{(1+e^x)^2}$ or use the quotient rule $f''(0)=\frac{1}{4}$

b $f'''(x)=\frac{(1+e^x)^2e^x-e^x2(1+e^x)e^x}{(1+e^x)^4}$ Use the quotient rule and chain rule.

$=\frac{(1+e^x)e^x\{(1+e^x)-2e^x\}}{(1+e^x)^4}=\frac{e^x(1-e^x)}{(1+e^x)^3}$ $f'''(0)=0$

c Using Maclaurin's expansion

$\ln(1+e^x)=\ln 2+\frac{x}{2}+\frac{x^2}{8}+\ldots$

The expansion is valid for $-1<e^x\le 1\Rightarrow 0,\ e^x\le 1$ so for $x\le 0$.

8 a $\cos 4x=1-\frac{(4x)^2}{2!}+\frac{(4x)^4}{4!}-\frac{(4x)^6}{6!}+\ldots$

$=1-8x^2+\frac{32}{3}x^4-\frac{256}{45}x^6+\ldots$

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8 b $\cos 4x = 1 - 2\sin^2 2x,$

so $2\sin^2 2x = 1 - 2\cos 4x = 8x^2 - \frac{32}{3}x^4 + \frac{256}{45}x^6 + \ldots$

$\sin^2 2x = 4x^2 - \frac{16}{3}x^4 + \frac{128}{45}x^6 + \ldots$

9 Using $e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \ldots$ and $\cos x = 1 - \frac{x^2}{2} + \frac{x^4}{24} - \ldots$

$e^{\cos x} = e^{\left(1 - \frac{x^2}{2} + \frac{x^4}{24}\right)} = e \times e^{-\frac{x^2}{2}} \times e^{\frac{x^4}{24}}$

$= e\left\{1 + \left(-\frac{x^2}{2}\right) + \frac{1}{2}\left(-\frac{x^2}{2}\right)^2 + \ldots\right\}\left\{1 + \frac{x^4}{24} + \ldots\right\}$ no other terms required

$= e\left\{1 - \frac{x^2}{2} + \frac{x^4}{8} + \ldots\right\}\left\{1 + \frac{x^4}{24} + \ldots\right\}$

$= e\left\{1 - \frac{x^2}{2} + \frac{x^4}{8} + \frac{x^4}{24} + \ldots\right\} = e\left\{1 - \frac{x^2}{2} + \frac{x^4}{6} + \ldots\right\}$

10 $\ln[(1+x)^2(1-2x)] = 2\ln(1+x) + \ln(1-2x)$

$= 2\left\{x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots\right\} + \left\{(-2x) - \frac{(-2x)^2}{2} - \frac{(-2x)^3}{3} - \frac{(-2x)^4}{4} + \ldots\right\}$

$= 2x - x^2 + \frac{2}{3}x^3 - \frac{1}{2}x^4 - 2x - 2x^2 - \frac{8}{3}x^3 - 4x^4 + \ldots$

$= -3x^2 - 2x^3 - \ldots$

11 $f(x) = \ln(\sec x + \tan x)$ $\quad f(0) = \ln 1 = 0$

$f'(x) = \frac{\sec x \tan x + \sec^2 x}{\sec x + \tan x} = \frac{\sec x(\tan x + \sec x)}{\sec x + \tan x} = \sec x$ $\quad f'(0) = 1$

$f''(x) = \sec x \tan x$ $\quad f''(0) = 0$

$f'''(x) = \sec x \sec^2 x + \sec x \tan x \tan x$ $\quad f'''(0) = 1$

Substituting into Maclaurin's expansion gives $y = x + \frac{x^3}{6} + \ldots$

12 a $\frac{d}{dx}(e^x) = \frac{d}{dx}\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots + \frac{x^r}{r!} + \frac{x^{r+1}}{(r+1)!} + \ldots\right)$

$= 1 + \frac{2x}{2!} + \frac{3x^2}{3!} + \frac{4x^3}{4!} + \ldots + \frac{(r+1)x^r}{(r+1)!} + \ldots$

$= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots + \frac{x^r}{r!} + \ldots$

$= e^x$

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12 b $\dfrac{\mathrm{d}}{\mathrm{d}x}(\sin x)=\dfrac{\mathrm{d}}{\mathrm{d}x}\left(x-\dfrac{x^3}{3!}+\dfrac{x^5}{5!}-\ldots+(-1)^r\dfrac{x^{2r+1}}{(2r+1)!}+\ldots\right)$

$=1-\dfrac{3x^2}{3!}+\dfrac{5x^4}{5!}-\ldots+(-1)^r\dfrac{(2r+1)x^{2r}}{(2r+1)!}+\ldots$

$=1-\dfrac{x^2}{2!}+\dfrac{x^4}{4!}-\dfrac{x^6}{6!}+\ldots+(-1)^r\dfrac{x^{2r}}{(2r)!}+\ldots=\cos x$

c $\dfrac{\mathrm{d}}{\mathrm{d}x}(\cos x)=\dfrac{\mathrm{d}}{\mathrm{d}x}\left(1-\dfrac{x^2}{2!}+\dfrac{x^4}{4!}-\dfrac{x^6}{6!}+\ldots+(-1)^r\dfrac{x^{2r}}{(2r)!}+(-1)^{r+1}\dfrac{x^{2r+2}}{(2r+2)!}+\ldots\right)$

$=\left(-\dfrac{2x}{2!}+\dfrac{4x^3}{4!}-\dfrac{6x^5}{6!}+\ldots+(-1)^r\dfrac{2rx^{2r-1}}{(2r)!}+(-1)^{r+1}\dfrac{(2r+2)x^{2r+1}}{(2r+2)!}+\ldots\right)$

$=-x+\dfrac{x^3}{3!}-\dfrac{x^5}{5!}+\ldots+(-1)^{r+1}\dfrac{x^{2r+1}}{(2r+1)!}+\ldots$

$=-\left(x-\dfrac{1}{3!}x^3+\dfrac{1}{5!}x^5-\ldots+\dfrac{(-1)^r}{(2r+1)!}x^{2r+1}+\ldots\right)=-\sin x$

13 a You can write $\cos x=1-\left(\dfrac{x^2}{2}-\dfrac{x^4}{24}+\ldots\right)$; it is not necessary to have higher powers

$\sec x=\dfrac{1}{\cos x}=\dfrac{1}{1-\left(\frac{x^2}{2}-\frac{x^4}{24}+\ldots\right)}=\left\{1-\left(\dfrac{x^2}{2}-\dfrac{x^4}{24}+\ldots\right)\right\}^{-1}$

Using the binomial expansion but only requiring powers up to x^4

$\sec x=1+(-1)\left\{-\left(\dfrac{x^2}{2}-\dfrac{x^4}{24}\right)\right\}+\dfrac{(-1)(-2)}{2!}\left\{-\left(\dfrac{x^2}{2}-\dfrac{x^4}{24}\right)\right\}^2+\ldots$

$=1+\left(\dfrac{x^2}{2}-\dfrac{x^4}{24}\right)+\dfrac{x^4}{4}+\text{higher powers of } x$

$=1+\dfrac{x^2}{2}+\dfrac{5}{24}x^4+\ldots$

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13 b $\tan x = \dfrac{\sin x}{\cos x} = \sin x \times \sec x$

$$= \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots\right)\left(1 + \frac{x^2}{2} + \frac{5}{24}x^4 + \ldots\right)$$

$$= x + \frac{x^3}{2} + \frac{5}{24}x^5 - \frac{x^3}{3!} - \frac{1}{2(3!)}x^5 + \frac{x^5}{5!} + \ldots$$

$$= x + \left(\frac{1}{2} - \frac{1}{6}\right)x^3 + \left(\frac{5}{24} - \frac{1}{12} + \frac{1}{120}\right)x^5 + \ldots$$

$$= x + \frac{x^3}{3} + \frac{16}{120}x^5 + \ldots$$

$$= x + \frac{x^3}{3} + \frac{2}{15}x^5 + \ldots$$

14 Using $e^x = 1 + x + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + \ldots$ and $\cos 3x = 1 - \dfrac{(3x)^2}{2!} + \ldots$

$$e^x \cos 3x = \left(1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \ldots\right)\left(1 - \frac{9x^2}{2} + \ldots\right)$$

$$= \left\{1 + x + \left(\frac{x^2}{2} - \frac{9x^2}{2}\right) + \left(\frac{x^3}{6} - \frac{9x^3}{2}\right) + \ldots\right\}$$

$$= 1 + x - 4x^2 - \frac{13}{3}x^3 + \ldots$$

15 $f(x) = (1+x)^2 \ln(1+x)$.

$$f'(x) = (1+x)^2 \frac{1}{1+x} + 2(1+x)\ln(1+x) = (1+x)\{1 + 2\ln(1+x)\}$$

$$f''(x) = (1+x)\left(\frac{2}{1+x}\right) + \{1 + 2\ln(1+x)\} = 3 + 2\ln(1+x)$$

$$f'''(x) = \left(\frac{2}{1+x}\right)$$

$f(0) = 0,\ f'(0) = 1,\ f''(0) = 3,\ f'''(0) = 2$

Using Maclaurin's expansion

$$(1+x)^2 \ln(1+x) = 0 + (1)x + \frac{3}{2!}x^2 + \frac{2}{3!}x^3 + \ldots$$

$$= x + \frac{3}{2}x^2 + \frac{1}{3}x^3 + \ldots$$

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16 a $\ln(1+\sin x) = \ln\left\{1+\left(x-\frac{x^3}{3!}+\ldots\right)\right\}$

$$= \left(x-\frac{x^3}{3!}+\ldots\right)-\frac{1}{2}\left(x-\frac{x^3}{3!}+\ldots\right)^2+\frac{1}{3}\left(x-\frac{x^3}{3!}+\ldots\right)^3-\frac{1}{4}\left(x-\frac{x^3}{3!}+\ldots\right)^4+\ldots$$

$$= \left(x-\frac{x^3}{6}+\ldots\right)-\frac{1}{2}\left(x^2-\frac{x^4}{3}+\ldots\right)+\frac{1}{3}(x^3+\ldots)-\frac{1}{4}(x^4+\ldots) \text{ no other terms necessary}$$

$$= x-\frac{x^2}{2}+\frac{x^3}{6}-\frac{x^4}{12}+\ldots$$

b $\int_0^{\frac{\pi}{6}} \ln(1+\sin x)\mathrm{d}x \approx \int_0^{\frac{\pi}{6}}\left(x-\frac{x^2}{2}+\frac{x^3}{6}-\frac{x^4}{12}\right)\mathrm{d}x$

$$\approx \left[\frac{x^2}{2}-\frac{x^3}{6}+\frac{x^4}{24}-\frac{x^5}{60}\right]_0^{\frac{\pi}{6}} = \frac{\pi^2}{72}-\frac{\pi^3}{1296}+\frac{\pi^4}{31104}-\frac{\pi^5}{466560} = 0.116\,(3\text{d.p.})$$

17 a $\mathrm{f}(x) = \mathrm{e}^{\tan x} = \mathrm{e}^{x+\frac{x^3}{3}+\ldots} = \mathrm{e}^x \times \mathrm{e}^{\frac{x^3}{3}}$ (As only terms up to x^3 are required, only first two terms of $\tan x$ are needed.)

$$= \left(1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\ldots\right)\left(1+\frac{x^3}{3}+\ldots\right) \text{ no other terms required.}$$

$$= \left(1+\frac{x^3}{3}+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\ldots\right)$$

$$= 1+x+\frac{x^2}{2}+\frac{x^3}{2}+\ldots$$

b $\mathrm{e}^{-\tan x} = \mathrm{e}^{\tan(-x)}$, so replacing x by $-x$ in **a** gives

$$\mathrm{e}^{-\tan x} = 1-x+\frac{x^2}{2}-\frac{x^3}{2}+\ldots$$

18 a

$\mathrm{f}(x) = \ln\cos x$	$\mathrm{f}(0) = 0$
$\mathrm{f}'(x) = \frac{-\sin x}{\cos x} = -\tan x$	$\mathrm{f}'(0) = 0$
$\mathrm{f}''(x) = -\sec^2 x$	$\mathrm{f}''(0) = -1$
$\mathrm{f}'''(x) = -2\sec^2 x\tan x$	$\mathrm{f}'''(0) = 0$
$\mathrm{f}''''(x) = -2\sec^4 x - 4\sec^2 x\tan^2 x$	$\mathrm{f}''''(0) = -2$

Substituting into Maclaurin:

$$\ln\cos x = (-1)\frac{x^2}{2!}+(-2)\frac{x^4}{4!}+\ldots = -\frac{x^2}{2}-\frac{x^4}{12}-\ldots$$

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18 b Using $1+\cos x = 2\cos^2\left(\frac{x}{2}\right), \ln(1+\cos x) = \ln 2\cos^2\left(\frac{x}{2}\right) = \ln 2 + 2\ln\cos\left(\frac{x}{2}\right)$

so $\ln(1+\cos x) = \ln 2 + 2\left\{-\frac{1}{2}\left(\frac{x}{2}\right)^2 - \frac{1}{12}\left(\frac{x}{2}\right)^4 - \ldots\right\} = \ln 2 - \frac{x^2}{4} - \frac{x^4}{96} - \ldots$

19 a

$y = e^{3x} - e^{-3x}$

$y' = 3e^{3x} + 3e^{-3x}$

$y'' = 9e^{3x} - 9e^{-3x} = 9y$

$y''' = 9y', y'''' = 9y'' = 81y$

b When $x = 0$,

$y = 0$

$y' = 6$

$y'' = 9y = 0$

$y''' = 9y' = 54$

$y'''' = 81y = 0$

$y''''' = 81y' = 486$

so $y = 6x + \frac{54}{3!}x^3 + \frac{486}{5!}x^5 + \ldots$

$= 6x + 9x^3 + \frac{81}{20}x^5 + \ldots$

c

$e^{3x} = \sum_{n=0}^{\infty}\frac{(3x)^n}{n!}, e^{-3x} = \sum_{n=0}^{\infty}\frac{(-3x)^n}{n!}$

$y = e^{3x} - e^{-3x}$

$= \sum_{n=0}^{\infty}\frac{(3x)^n}{n!} - \frac{(-3x)^n}{n!} = \sum_{n=0}^{\infty}(1-(-1)^n)\frac{(3x)^n}{n!}$

n^{th} non-zero term is $(1-(-1)^{2n-1})\frac{3^{2n-1}x^{2n-1}}{(2n-1)!} = 2\frac{3^{2n-1}x^{2n-1}}{(2n-1)!}$

Challenge

$e^{ix} = \sum_{n=0}^{\infty}\frac{(ix)^n}{n!} = 1 + ix + \frac{(ix)^2}{2!} + \frac{(ix)^3}{3!} + \frac{(ix)^4}{4!} + \frac{(ix)^5}{5!} + \ldots$

$= 1 + ix - \frac{x^2}{2!} - i\frac{x^3}{3!} + \frac{x^4}{4!} + i\frac{x^5}{5!} + \ldots$

$= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \ldots + i\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} + \ldots\right)$

$= \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{(2n)!} + i\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{(2n+1)!} = \cos x + i\sin x$

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Methods in calculus 3A

1 a $\int_1^\infty \frac{1}{x^3}\,dx$

$= \lim_{t\to\infty} \int_1^t \frac{1}{x^3}\,dx$

$= \lim_{t\to\infty} \left[-\frac{1}{2x^2} \right]_1^t$

$= \lim_{t\to\infty} \left(-\frac{1}{2t^2} + \frac{1}{2} \right) = \frac{1}{2}$

So $\int_1^\infty \frac{1}{x^3}\,dx$ converges and $\int_1^\infty \frac{1}{x^3}\,dx = \frac{1}{2}$

b $\int_2^\infty x^{-\frac{3}{2}}\,dx$

$= \lim_{t\to\infty} \int_2^t x^{-\frac{3}{2}}\,dx$

$= \lim_{t\to\infty} \left[-2x^{-\frac{1}{2}} \right]_2^t$

$= \lim_{t\to\infty} \left(-2t^{-\frac{1}{2}} + \sqrt{2} \right) = \sqrt{2}$

So $\int_2^\infty x^{-\frac{3}{2}}\,dx$ converges and $\int_2^\infty x^{-\frac{3}{2}}\,dx = \sqrt{2}$

c $\int_0^\infty e^{-3x}\,dx$

$= \lim_{t\to\infty} \int_0^t e^{-3x}\,dx$

$= \lim_{t\to\infty} \left[-\frac{e^{-3x}}{3} \right]_0^t$

$= \lim_{t\to\infty} \left(-\frac{e^{-3t}}{3} + \frac{1}{3} \right) = \frac{1}{3}$

So $\int_0^\infty e^{-3x}\,dx$ converges and $\int_0^\infty e^{-3x}\,dx = \frac{1}{3}$

2 a $\int_0^\infty e^x\,dx$

$= \lim_{t\to\infty} \int_0^t e^x\,dx$

$= \lim_{t\to\infty} \left[e^x \right]_0^t$

$= \lim_{t\to\infty} \left(e^t - 1 \right)$

$e^t \to \infty$ as $t \to \infty$, so the integral diverges.

b $\int_1^\infty \frac{1}{\sqrt{x}}\,dx$

$= \lim_{t\to\infty} \int_1^t \frac{1}{\sqrt{x}}\,dx$

$= \lim_{t\to\infty} \left[2\sqrt{x} \right]_1^t$

$= \lim_{t\to\infty} \left(2\sqrt{t} - 2 \right)$

$\sqrt{t} \to \infty$ as $t \to \infty$, so the integral diverges.

c $\int_0^\infty \frac{8x}{\sqrt{1+x^2}}\,dx$

$= \lim_{t\to\infty} \int_0^t \frac{8x}{\sqrt{1+x^2}}\,dx$

$= \lim_{t\to\infty} \left[8\sqrt{1+x^2} \right]_0^t$

$= \lim_{t\to\infty} \left(8\sqrt{1+t^2} - 8 \right)$

$\sqrt{1+t^2} \to \infty$ as $t \to \infty$, so the integral diverges.

3 a $\int_0^1 \frac{1}{\sqrt{x}}\,dx$

$= \lim_{t\to 0} \int_t^1 \frac{1}{\sqrt{x}}\,dx$

$= \lim_{t\to 0} \left[2\sqrt{x} \right]_t^1$

$= \lim_{t\to 0} \left(2 - 2\sqrt{t} \right) = 2$

So $\int_0^1 \frac{1}{\sqrt{x}}\,dx$ converges and $\int_0^1 \frac{1}{\sqrt{x}}\,dx = 2$

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3 b $\int_0^{\frac{2}{3}} \frac{1}{\sqrt{2-3x}}\,dx$

$= \lim_{t\to\frac{2}{3}} \int_0^t \frac{1}{\sqrt{2-3x}}\,dx$

$= \lim_{t\to\frac{2}{3}} \left[-\frac{2}{3}\sqrt{2-3x}\right]_0^t$

$= \lim_{t\to\frac{2}{3}} \left(-\frac{2}{3}\sqrt{2-3t} + \frac{2\sqrt{2}}{3}\right) = \frac{2\sqrt{2}}{3}$

So $\int_0^{\frac{2}{3}} \frac{1}{\sqrt{2-3x}}\,dx$ converges and

$\int_0^{\frac{2}{3}} \frac{1}{\sqrt{2-3x}}\,dx = \frac{2\sqrt{2}}{3}$

c $\int_0^{\ln 3} \frac{e^x}{\sqrt{e^x-1}}\,dx$

Let $u = e^x - 1$ and $du = e^x dx$

$\int_0^{\ln 3} \frac{e^x}{\sqrt{e^x-1}}\,dx = \int_0^2 \frac{1}{\sqrt{u}}\,du$

$= \left[2\sqrt{u}\right]_0^2 = 2\sqrt{2}$

4 a $\int_{-1}^{2} \frac{1}{\sqrt{|x|}}\,dx$

Split $\int_{-1}^{2} \frac{1}{\sqrt{|x|}}\,dx$ as $\int_{-1}^{0} \frac{1}{\sqrt{|x|}}\,dx + \int_{0}^{2} \frac{1}{\sqrt{|x|}}\,dx$

To find $\int_{-1}^{0} \frac{1}{\sqrt{|x|}}\,dx$ consider $\int_{-1}^{t} \frac{1}{\sqrt{|x|}}\,dx$

$\int_{-1}^{t} \frac{1}{\sqrt{|x|}}\,dx = \int_{-1}^{t} \frac{1}{\sqrt{-x}}\,dx$

$= \left[-2\sqrt{-x}\right]_{-1}^{t} = -2\sqrt{-t} + 2$

So, $\lim_{t\to 0} \int_{-1}^{t} \frac{1}{\sqrt{|x|}}\,dx = \lim_{t\to 0}\left(-2\sqrt{-t}+2\right) = 2$

4 a Similarly, to find $\int_0^2 \frac{1}{\sqrt{|x|}}\,dx$ consider

$\int_t^2 \frac{1}{\sqrt{|x|}}\,dx = \int_t^2 \frac{1}{\sqrt{x}}\,dx$

$= \left[2\sqrt{x}\right]_t^2 = 2\sqrt{2} - 2\sqrt{t}$

So, $\lim_{t\to 0} \int_t^2 \frac{1}{\sqrt{|x|}}\,dx = \lim_{t\to 0}\left(2\sqrt{2} - 2\sqrt{t}\right)$

$= 2\sqrt{2}$

Since both integrals converge, we know that $\int_{-1}^{2} \frac{1}{\sqrt{|x|}}\,dx$ converges, and

$\int_{-1}^{2} \frac{1}{\sqrt{|x|}}\,dx = \int_{-1}^{0} \frac{1}{\sqrt{|x|}}\,dx + \int_{0}^{2} \frac{1}{\sqrt{|x|}}\,dx$

$= 2 + 2\sqrt{2} = 2\left(1+\sqrt{2}\right)$

b $\int_{-1}^{3} \frac{x-1}{\sqrt{3+2x-x^2}}\,dx$

$= \int_{-1}^{3} \frac{x-1}{\sqrt{(3-x)(x+1)}}\,dx$

Split $\int_{-1}^{3} \frac{x-1}{\sqrt{(3-x)(x+1)}}\,dx$ as

$\int_{-1}^{0} \frac{x-1}{\sqrt{(3-x)(x+1)}}\,dx + \int_{0}^{3} \frac{x-1}{\sqrt{(3-x)(x+1)}}\,dx$

To find $\int_{-1}^{0} \frac{x-1}{\sqrt{(3-x)(x+1)}}\,dx$ consider

$\int_{t}^{0} \frac{x-1}{\sqrt{(3-x)(x+1)}}\,dx = \left[-\sqrt{(3-x)(x+1)}\right]_t^0$

$= -3 + \sqrt{(3-t)(t+1)}$

So,

$\lim_{t\to -1} \int_t^0 \frac{x-1}{\sqrt{(3-x)(x+1)}}\,dx$

$= \lim_{t\to -1}\left(-3 + \sqrt{(3-t)(t+1)}\right)$

$= -3$

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4 b Similarly, to find $\int_0^3 \frac{x-1}{\sqrt{(3-x)(x+1)}}\,dx$ consider $\int_0^t \frac{x-1}{\sqrt{(3-x)(x+1)}}\,dx$

$$\int_0^t \frac{x-1}{\sqrt{(3-x)(x+1)}}\,dx = \left[-\sqrt{(3-x)(x+1)}\right]_0^t$$
$$= -\sqrt{(3-t)(t+1)} + 3$$

So,

$$\lim_{t\to 3}\int_0^t \frac{x-1}{\sqrt{(3-x)(x+1)}}\,dx$$
$$= \lim_{t\to -1}\left(-\sqrt{(3-t)(t+1)}+3\right)$$
$$= 3$$

Since both integrals converge, we know that $\int_{-1}^3 \frac{x-1}{\sqrt{3+2x-x^2}}\,dx$ converges, and

$$\int_{-1}^3 \frac{x-1}{\sqrt{3+2x-x^2}}\,dx$$
$$= \left(\begin{array}{l}\int_{-1}^0 \frac{x-1}{\sqrt{(3-x)(x+1)}}\,dx + \\ \int_0^3 \frac{x-1}{\sqrt{(3-x)(x+1)}}\,dx\end{array}\right)$$
$$= -3+3 = 0$$

c $\int_0^{\pi} \tan x\,dx$

Split $\int_0^{\pi} \tan x\,dx$ as $\int_0^{\frac{\pi}{2}} \tan x\,dx + \int_{\frac{\pi}{2}}^{\pi} \tan x\,dx$

To find $\int_0^{\frac{\pi}{2}} \tan x\,dx$ consider $\int_0^t \tan x\,dx$

$$\int_0^t \tan x\,dx = \left[\ln|\sec x|\right]_0^t = \ln|\sec t|$$

So $\lim_{t\to\frac{\pi}{2}}\int_0^t \tan x\,dx = \lim_{t\to\frac{\pi}{2}} \ln|\sec t|$

$\ln|\sec t| \to \infty$ as $t \to \frac{\pi}{2}$. Therefore, the integral diverges.

5 a $\int \frac{1}{(7-3x)^2}\,dx$

Substitute $u = 7-3x$, so $du = -3dx$

$$\int \frac{1}{(7-3x)^2}\,dx = -\frac{1}{3}\int \frac{1}{u^2}\,du$$
$$= \frac{1}{3u} + C = \frac{1}{3(7-3x)} + C$$

b $\int_{-\infty}^2 \frac{1}{(7-3x)^2}\,dx$

Consider $\int_t^2 \frac{1}{(7-3x)^2}\,dx$

$$\int_t^2 \frac{1}{(7-3x)^2}\,dx = \left[\frac{1}{3(7-3x)}\right]_t^2$$
$$= \frac{1}{3} - \frac{1}{3(7-3t)}$$

$$\lim_{t\to-\infty}\int_t^2 \frac{1}{(7-3x)^2}\,dx$$
$$= \lim_{t\to-\infty}\left(\frac{1}{3} - \frac{1}{3(7-3t)}\right) = \frac{1}{3}$$

Therefore, the integral converges, and

$$\int_{-\infty}^2 \frac{1}{(7-3x)^2}\,dx = \frac{1}{3}$$

6 a $\int x^2 e^{x^3}\,dx$

Substitute $u = x^3$, so $du = 3x^2dx$

$$\int x^2 e^{x^3}\,dx = \frac{1}{3}\int e^u\,du = \frac{1}{3}e^u + c$$

Therefore, $\int x^2 e^{x^3}\,dx = \frac{1}{3}e^{x^3} + C$

b $\int_{-\infty}^1 x^2 e^{x^3}\,dx = \lim_{t\to-\infty}\int_t^1 x^2 e^{x^3}\,dx$

$$= \lim_{t\to-\infty}\left[\frac{1}{3}e^{x^3}\right]_t^1 = \lim_{t\to-\infty}\left(\frac{1}{3}e - \frac{1}{3}e^{t^3}\right)$$
$$= \frac{e}{3}$$

Therefore, the integral converges, and

$$\int_{-\infty}^1 x^2 e^{x^3}\,dx = \frac{e}{3}$$

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7 a $\int \frac{\ln x}{x}\,dx$

Substitute $u = \ln x$, so $du = \frac{1}{x}dx$

$\int \frac{\ln x}{x}dx = \int u\,du = \frac{u^2}{2} + C$

Therefore, $\int \frac{\ln x}{x}dx = \frac{(\ln x)^2}{2} + C$

b $\int_1^\infty \frac{\ln x}{x}dx = \lim_{t\to\infty}\int_1^t \frac{\ln x}{x}dx$

$= \lim_{t\to\infty}\left[\frac{(\ln x)^2}{2}\right]_1^t = \lim_{t\to\infty}\left(\frac{(\ln t)^2}{2}\right)$

$\ln t \to \infty$ as $t \to \infty$, so the integral diverges.

8 a $\int (\ln x)^2\,dx$

Integrating by parts,

$\int (\ln x)^2\,dx = x(\ln x)^2 - \int 2\ln x$

Integrating the second term using integration by parts again,

$\int \ln x\,dx = x\ln x - \int x\frac{1}{x}dx = x\ln x - x$

Therefore,

$\int (\ln x)^2\,dx = x\left((\ln x)^2 - 2\ln x + 2\right) + C$

b $\int_0^1 (\ln x)^2\,dx = \lim_{t\to 0}\int_t^1 (\ln x)^2\,dx$

$= \lim_{t\to 0}\left[x\left((\ln x)^2 - 2\ln x + 2\right)\right]_t^1$

$= \lim_{t\to 0}\left(2 - t\left((\ln t)^2 - 2\ln t + 2\right)\right) = 2$

Since $t(\ln t)^2 \to 0$ and $t\ln t \to 0$ as $t \to 0$, the integral converges.

8 c $\int_0^\infty (\ln x)^2\,dx = \int_0^1 (\ln x)^2\,dx + \int_1^\infty (\ln x)^2\,dx$

We know that the first integral converges.

$\int_1^\infty (\ln x)^2\,dx = \lim_{t\to\infty}\int_1^\infty (\ln x)^2\,dx$

$= \lim_{t\to\infty}\left[x\left((\ln x)^2 - 2\ln x + 2\right)\right]_1^\infty$

$= \lim_{t\to\infty}\left(t\left((\ln t)^2 - 2\ln t + 2\right) - 2\right)$

Since $t(\ln t)^2 \to \infty$ as $t \to \infty$, the integral diverges.

9 $\int_0^2 \frac{6x}{\sqrt[3]{4-x^2}}dx = \lim_{t\to 2}\int_0^t \frac{6x}{\sqrt[3]{4-x^2}}dx$

Consider $\int \frac{6x}{\sqrt[3]{4-x^2}}dx$

Substitute $u = 4 - x^2$, so $du = -2x\,dx$

$\int \frac{6x}{\sqrt[3]{4-x^2}}dx = \int -\frac{3}{\sqrt[3]{u}}du = -\frac{9}{2}u^{\frac{2}{3}} + C$

$\lim_{t\to 2}\int_0^t \frac{6x}{\sqrt[3]{4-x^2}}dx$

$= \lim_{t\to 2}\left[-\frac{9}{2}\sqrt[3]{\left(4-x^2\right)^2}\right]_0^t$

$= \lim_{t\to 2}\left(-\frac{9}{2}\sqrt[3]{\left(4-t^2\right)^2} + 9\sqrt[3]{2}\right) = 9\sqrt[3]{2}$

10 $\int_{-2}^2 \frac{\sqrt{2-x} - 3\sqrt{2+x}}{\sqrt{4-x^2}}dx$

$= \int_{-2}^2 \frac{\sqrt{2-x} - 3\sqrt{2+x}}{\sqrt{(2-x)(2+x)}}dx$

$= \int_{-2}^2 \left(\frac{1}{\sqrt{2+x}} - \frac{3}{\sqrt{2-x}}\right)dx$

$= \int_{-2}^2 \frac{1}{\sqrt{2+x}}dx - \int_{-2}^2 \frac{3}{\sqrt{2-x}}dx$

$\int_{-2}^2 \frac{1}{\sqrt{2+x}}dx = \lim_{t\to -2}\int_t^2 \frac{1}{\sqrt{2+x}}dx$

$= \lim_{t\to -2}\left[2\sqrt{2+x}\right]_t^2$

$= \lim_{t\to -2}\left(4 - 2\sqrt{2+t}\right) = 4$

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10 $\int_{-2}^{2} \frac{3}{\sqrt{2-x}}\,dx = \lim_{t\to 2}\int_{-2}^{t} \frac{1}{\sqrt{2-x}}\,dx$

$= \lim_{t\to 2}\left[-6\sqrt{2-x}\right]_{-2}^{t}$

$= \lim_{t\to 2}\left(-6\sqrt{2-t}+12\right) = 12$

So,

$\int_{-2}^{2} \frac{\sqrt{2-x}-3\sqrt{2+x}}{\sqrt{4-x^2}}\,dx$

$= \int_{-2}^{2} \frac{1}{\sqrt{2+x}}\,dx - \int_{-2}^{2} \frac{3}{\sqrt{2-x}}\,dx$

$= 4-12 = -8$

11 The area enclosed by the curve in the figure is given by $\left|\int_{0}^{1} \ln x\right|$ which may be written as

$\left|\lim_{t\to 0}\int_{t}^{1} \ln x\,dx\right|$

Integrating by parts,

$\left|\lim_{t\to 0}\int_{t}^{1} \ln x\,dx\right| = \left|\lim_{t\to 0}\left[x\ln x - x\right]_{t}^{1}\right|$

$= \left|\lim_{t\to 0}\left(-1 - t\ln t + t\right)\right| = 1$

12 a Since $\tan x$ is undefined in the upper limit $x = \frac{\pi}{2}$, $\int_{0}^{\frac{\pi}{2}} \tan x\,dx$ is an improper integral.

b $\int \tan x\,dx = \int \frac{\sin x}{\cos x}\,dx$

Substitute $u = \cos x$, so $du = -\sin x\,dx$

$\int \frac{\sin x}{\cos x}\,dx = \int -\frac{1}{u}\,du$

$= -\ln u + C = \ln\frac{1}{u} + C = \ln\left(\sec x\right) + C$

$\int_{0}^{\frac{\pi}{2}} \tan x\,dx = \lim_{t\to\frac{\pi}{2}}\int_{0}^{t} \tan x\,dx$

$= \lim_{t\to\frac{\pi}{2}}\left[-\ln\left(\cos x\right)\right]_{0}^{t} = \lim_{t\to\frac{\pi}{2}}\left(-\ln\left(\cos t\right)\right)$

$-\ln\left(\cos t\right) \to \infty$ as $t \to \frac{\pi}{2}$ so the integral diverges.

13 a $\sec^2 x$ is undefined at $x = \frac{\pi}{2}$ so the integral should be split up at that point.

b $\int_{0}^{\pi} \sec^2 x\,dx = \int_{0}^{\frac{\pi}{2}} \sec^2 x\,dx + \int_{\frac{\pi}{2}}^{\pi} \sec^2 x\,dx$

$\int_{0}^{\frac{\pi}{2}} \sec^2 x\,dx = \lim_{t\to\frac{\pi}{2}}\int_{0}^{t} \sec^2 x\,dx$

$= \lim_{t\to\frac{\pi}{2}}\left[\tan x\right]_{0}^{t} = \lim_{t\to\frac{\pi}{2}} \tan t$

$\tan t \to \infty$ as $t \to \frac{\pi}{2}$ so the integral diverges.

14 $\int_{1}^{\infty} \frac{1}{x^a}\,dx$

If $a = 1$,

$\int_{1}^{\infty} \frac{1}{x^a}\,dx = \lim_{t\to\infty}\int_{1}^{t} \frac{1}{x}\,dx$

$= \lim_{t\to\infty}\left[\ln x\right]_{1}^{t} = \lim_{t\to\infty} \ln t$

$\ln t \to \infty$ as $t \to \infty$ so the integral diverges.

If $a \neq 1$,

$\int_{1}^{\infty} \frac{1}{x^a}\,dx = \lim_{t\to\infty}\int_{1}^{t} \frac{1}{x^a}\,dx$

$= \lim_{t\to\infty}\left[\frac{x^{-a+1}}{-a+1}\right]_{1}^{t}$

$= \lim_{t\to\infty}\left(\frac{t^{-a+1}}{-a+1} - \frac{1}{-a+1}\right)$

If $a < 1$, $\frac{t^{-a+1}}{-a+1} \to \infty$ as $t \to \infty$ so the integral diverges.

If $a > 1$, $\frac{t^{-a+1}}{-a+1} \to 0$ as $t \to \infty$ so the integral converges, and

$\lim_{t\to\infty}\left(\frac{t^{-a+1}}{-a+1} - \frac{1}{-a+1}\right) = \frac{1}{a-1}$

Therefore, $a > 1$, and $\int_{1}^{\infty} \frac{1}{x^a}\,dx = \frac{1}{a-1}$

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15 a $\int_0^k \frac{1}{2x^2+3x+1}\,dx = \int_0^k \frac{1}{(x+1)(2x+1)}\,dx$

$$= \int_0^k \left(\frac{2}{2x+1} - \frac{1}{x+1}\right)dx$$

$$= \left[\ln(2x+1) - \ln(x+1)\right]_0^k$$

$$= \left[\ln\left(\frac{2x+1}{x+1}\right)\right]_0^k$$

$$= \ln\left(\frac{2k+1}{k+1}\right)$$

b $\int_0^\infty \frac{1}{2x^2+3x+1}\,dx = \lim_{k\to\infty} \ln\left(\frac{2k+1}{k+1}\right)$

$$= \lim_{k\to\infty} \ln\left(\frac{2+\frac{1}{k}}{1+\frac{1}{k}}\right) = \ln 2$$

Challenge

$\int e^{-x}\sin^2 x\,dx$

Using trigonometric identities,

$$\int e^{-x}\sin^2 x\,dx = \int e^{-x}\left(\frac{1-\cos 2x}{2}\right)dx$$

$$= -\frac{1}{2}e^{-x} - \frac{1}{2}\int e^{-x}\cos 2x\,dx$$

Integrating by parts,

$$\int e^{-x}\cos 2x\,dx = \begin{pmatrix} -e^{-x}\cos 2x - \\ 2\int e^{-x}\sin 2x\,dx \end{pmatrix}$$

Integrating by parts again,

$$\int e^{-x}\sin 2x\,dx = \begin{pmatrix} -e^{-x}\sin 2x + \\ 2\int e^{-x}\cos 2x\,dx \end{pmatrix}$$

Adding the above two equations gives,

$$\int e^{-x}\cos 2x\,dx = \begin{pmatrix} -e^{-x}\cos 2x + \\ 2e^{-x}\sin 2x - \\ 4\int e^{-x}\cos 2x\,dx \end{pmatrix}$$

$$5\int e^{-x}\cos 2x\,dx = -e^{-x}\cos 2x + 2e^{-x}\sin 2x$$

$$\int e^{-x}\cos 2x\,dx = \frac{1}{5}\begin{pmatrix} -e^{-x}\cos 2x + \\ 2e^{-x}\sin 2x \end{pmatrix}$$

Therefore,

$\int e^{-x}\sin^2 x\,dx$

$$= -\frac{1}{2}e^{-x} - \frac{1}{10}\left(-e^{-x}\cos 2x + 2e^{-x}\sin 2x\right)$$

$$= \frac{1}{10}e^{-x}\left(-2\sin 2x + \cos 2x - 5\right)$$

So,

$$\int_0^\infty e^{-x}\sin^2 x\,dx = \lim_{t\to\infty}\int_0^t e^{-x}\sin^2 x\,dx$$

$$= \lim_{t\to\infty}\left[\frac{1}{10}e^{-x}\left(-2\sin 2x + \cos 2x - 5\right)\right]_0^t$$

$$= \lim_{t\to\infty}\left(\frac{1}{10}e^{-t}\left(-2\sin 2t + \cos 2t - 5\right) - \left(-\frac{2}{5}\right)\right)$$

$e^{-t}\sin 2t \to 0$ and $e^{-t}\cos 2t \to 0$ as $t \to \infty$.

Therefore,

$$\int_0^\infty e^{-x}\sin^2 x\,dx = \frac{2}{5}$$

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Methods in calculus 3B

1 a $f(x)=1$

The mean value of $f(x)$ on $[0,1]$ is:

$$\frac{1}{1-0}\int_0^1 1\,dx=\left[x\right]_0^1=1$$

b $f(x)=\frac{1}{x+1}$

The mean value of $f(x)$ on $[0,1]$ is:

$$\frac{1}{1-0}\int_0^1 \frac{1}{x+1}dx=\left[\ln(x+1)\right]_0^1$$
$$=\ln 2-\ln 1=\ln 2$$

c $f(x)=e^x+1$

The mean value of $f(x)$ on $[0,1]$ is:

$$\frac{1}{1-0}\int_0^1 e^x+1\,dx=\left[e^x+x\right]_0^1$$
$$=e+1-1=e$$

2 a $f(x)=\frac{e^{3x}}{e^{3x}+1}$

Consider $\int\frac{e^{3x}}{e^{3x}+1}dx$

Let $u=e^{3x}$ and $du=3e^{3x}dx$

$$\int\frac{e^{3x}}{e^{3x}+1}dx=\frac{1}{3}\int\frac{1}{u+1}du.$$
$$=\frac{1}{3}\ln(u+1)+C$$
$$=\frac{1}{3}\ln(e^{3x}+1)+C$$

The mean value of $f(x)$ on $[0,2]$ is:

$$\frac{1}{2}\int_0^2\frac{e^{3x}}{e^{3x}+1}dx$$
$$=\frac{1}{6}\left[\ln(e^{3x}+1)\right]_0^2$$
$$=\frac{1}{6}\left(\ln(e^6+1)-\ln 2\right)$$
$$=\frac{1}{6}\ln\left(\frac{e^6+1}{2}\right)$$

2 b $f(x)=\cos^3 x\sin^2 x$

Consider $\cos^3 x\sin^2 x$

$$=\cos x(\cos x\sin x)^2$$
$$=\cos x\left(\frac{1}{2}\sin 2x\right)^2$$
$$=\frac{1}{4}\cos x(\sin 2x\sin 2x)$$
$$=\frac{1}{4}\cos x\left(\frac{1-\cos 4x}{2}\right)$$
$$=\frac{1}{8}(\cos x-\cos x\cos 4x)$$
$$=\frac{1}{8}\left(\cos x-\frac{1}{2}\cos 5x-\frac{1}{2}\cos 3x\right)$$
$$=\frac{1}{16}(2\cos x-\cos 5x-\cos 3x)$$

Therefore,

$$\int_0^{\frac{\pi}{2}}\cos^3 x\sin^2 x\,dx$$
$$=\frac{1}{16}\int_0^{\frac{\pi}{2}}(2\cos x-\cos 5x-\cos 3x)dx$$
$$=\frac{1}{16}\left[2\sin x-\frac{1}{5}\sin 5x-\frac{1}{3}\sin 3x\right]_0^{\frac{\pi}{2}}$$
$$=\frac{1}{16}\left(2-\frac{1}{5}+\frac{1}{3}\right)=\frac{2}{15}$$

So the mean value of $f(x)$ on $\left[0,\frac{\pi}{2}\right]$ is:

$$\frac{2}{\pi}\int_0^{\frac{\pi}{2}}\cos^3 x\sin^2 x\,dx=\frac{4}{15\pi}$$

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2 c $f(x)=xe^{-x}$

Consider $\int xe^{-x}$

Integrating by parts,

$\int xe^{-x}\mathrm{d}x=-xe^{-x}+\int e^{-x}\mathrm{d}x$

$=-xe^{-x}-e^{-x}$

So the mean value of $\mathrm{f}(x)$ on $[1,3]$ is:

$\frac{1}{3-1}\int_1^3 xe^{-x}\mathrm{d}x$

$=\frac{1}{2}\left[-xe^{-x}-e^{-x}\right]_1^3$

$=\frac{1}{2}\left(-3e^{-3}-e^{-3}+e^{-1}+e^{-1}\right)$

$=\frac{1}{2}\left(-\frac{4}{e^3}+\frac{2}{e^1}\right)=\frac{e^2-2}{e^3}$

d $\mathrm{f}(x)=\frac{5}{(x+2)(2x+1)}$

Consider $\int_0^3 \frac{5}{(x+2)(2x+1)}\mathrm{d}x$

$=\int_0^3\left(-\frac{5}{3(x+2)}+\frac{10}{3(2x+1)}\right)\mathrm{d}x$

$=\frac{5}{3}\left[-\ln\left(|x+2|\right)+\ln\left(|2x+1|\right)\right]_0^3$

$=\frac{5}{3}\left[\ln\left(\left|\frac{2x+1}{x+2}\right|\right)\right]_0^3$

$=\frac{5}{3}\left(\ln\frac{7}{5}-\ln\frac{1}{2}\right)=\frac{5}{3}\ln\frac{14}{5}$

So the mean value of $\mathrm{f}(x)$ on $[0,3]$ is:

$\frac{1}{3-0}\int_0^3 \frac{5}{(x+2)(2x+1)}\mathrm{d}x$

$=\frac{5}{9}\ln\frac{14}{5}$

2 e $\mathrm{f}(x)=(\sec x-\cos x)^2$

Consider $\int(\sec x-\cos x)^2\,\mathrm{d}x$

$=\int\left(\sec^2 x-2\sec x\cos x+\cos^2 x\right)\mathrm{d}x$

$=\int\sec^2 x\mathrm{d}x-2\int\mathrm{d}x+\int\left(\frac{\cos 2x+1}{2}\right)\mathrm{d}x$

$=\tan x-2x+\frac{1}{4}\sin 2x+\frac{1}{2}x+C$

$=\tan x+\frac{1}{4}\sin 2x-\frac{3}{2}x+C$

$\int_0^{\frac{\pi}{4}}(\sec x-\cos x)^2\,\mathrm{d}x$

$=\left[\tan x+\frac{1}{4}\sin 2x-\frac{3}{2}x\right]_0^{\frac{\pi}{4}}$

$=1+\frac{1}{4}-\frac{3\pi}{8}=\frac{5}{4}-\frac{3\pi}{8}$

So the mean value of $\mathrm{f}(x)$ on $\left[0,\frac{\pi}{4}\right]$ is:

$\frac{4}{\pi}\int_0^{\frac{\pi}{4}}(\sec x-\cos x)^2\,\mathrm{d}x=\frac{5}{\pi}-\frac{3}{2}$

3 a For turning points, $\mathrm{f}'(x)=0$

$\mathrm{f}'(x)=3x^2-6x-24=0$

$3(x^2-2x-8)=0$

$3(x-4)(x+2)=0$

$x=4,-2$

When $x=4$,

$\mathrm{f}(x)=4^3-3\times4^2-24\times4+100$

$\mathrm{f}(x)=20$

When $x=-2$,

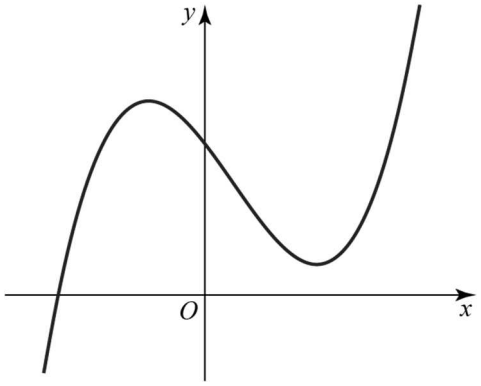
$\mathrm{f}(x)=(-2)^3-3\times(-2)^2-24\times(-2)+100$

$\mathrm{f}(x)=128$

Therefore, the turning points are $(4,20)$ and $(-2,128)$.

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3 b The graph of $y = \mathrm{f}(x) = x^3 - 3x^2 - 24x + 100$ is:



c The lower bound on the mean value is 20 and the upper bound is 128 since these are the minimum and maximum values of the function on the interval $[-2, 4]$.

d The mean value of $\mathrm{f}(x)$ over the interval $[-2, 4]$ is:

$$\frac{1}{4-(-2)}\int_{-2}^{4}\mathrm{f}(x)\,\mathrm{d}x$$

$$= \frac{1}{6}\int_{-2}^{4}\left(x^3 - 3x^2 - 24x + 100\right)\mathrm{d}x$$

$$= \frac{1}{6}\left[\frac{x^4}{4} - \frac{3x^3}{3} - \frac{24x^2}{2} + 100x\right]_{-2}^{4}$$

$$= \frac{1}{6}\left(\begin{matrix}64 - 64 - 192 + 400 - \\ 4 - 8 + 48 + 200\end{matrix}\right)$$

$$= 74$$

4 $\mathrm{f}(x) = \dfrac{\sin x \cos x}{\cos 2x + 2}$

Consider $\displaystyle\int \frac{\sin x \cos x}{\cos 2x + 2}\,\mathrm{d}x$

$$= \frac{1}{2}\int \frac{\sin 2x}{\cos 2x + 2}\,\mathrm{d}x$$

Let $u = \cos 2x + 2$ and $du = -2\sin 2x\,\mathrm{d}x$. So,

$$\frac{1}{2}\int \frac{\sin 2x}{\cos 2x + 2}\,\mathrm{d}x = -\frac{1}{4}\int \frac{1}{u}\,du$$

$$= -\frac{1}{4}\ln u + C$$

$$= -\frac{1}{4}\ln|\cos 2x + 2| + C$$

So the mean value of $\mathrm{f}(x)$ on $\left[0, \dfrac{\pi}{2}\right]$ is:

$$\frac{2}{\pi}\int_{0}^{\frac{\pi}{2}}\frac{\sin x \cos x}{\cos 2x + 2}\,\mathrm{d}x$$

$$= \frac{2}{\pi}\left[-\frac{1}{4}\ln|\cos 2x + 2|\right]_{0}^{\frac{\pi}{2}}.$$

$$= \frac{\ln 3}{2\pi}$$

5 $\mathrm{f}(x) = x\sqrt{x+4}$

Consider $\displaystyle\int x\sqrt{x+4}\,\mathrm{d}x$

$$= \int (x + 4 - 4)\sqrt{x+4}\,\mathrm{d}x$$

$$= \int (x+4)^{\frac{3}{2}}\,\mathrm{d}x - 4\int \sqrt{x+4}\,\mathrm{d}x$$

$$= \frac{2}{5}(x+4)^{\frac{5}{2}} - \frac{8}{3}(x+4)^{\frac{3}{2}} + C$$

So the mean value of on $[0, 5]$ is:

$$\frac{1}{5}\int_{0}^{5} x\sqrt{x+4}\,\mathrm{d}x$$

$$= \frac{1}{5}\left[\frac{2}{5}(x+4)^{\frac{5}{2}} - \frac{8}{3}(x+4)^{\frac{3}{2}}\right]_{0}^{5}$$

$$= \frac{1}{5}\left(\frac{486}{5} - 72 - \frac{64}{5} + \frac{64}{3}\right)$$

$$= \frac{506}{75}$$

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6 $f(x) = x\sin 2x$

Consider $\int x\sin 2x\,dx$

Integrating by parts,

$$= -x\frac{\cos 2x}{2} + \int \frac{\cos 2x}{2}\,dx$$

$$= -\frac{1}{2}x\cos 2x + \frac{1}{4}\sin 2x + C$$

So the mean value of $f(x)$ on $\left[0, \frac{\pi}{3}\right]$ is:

$$\frac{3}{\pi}\int_0^{\frac{\pi}{3}} x\sin 2x\,dx$$

$$= \frac{3}{\pi}\left[-\frac{1}{2}x\cos 2x + \frac{1}{4}\sin 2x\right]_0^{\frac{\pi}{3}}$$

$$= \frac{3}{\pi}\left(\frac{\pi}{12} + \frac{\sqrt{3}}{8}\right)$$

$$= \frac{1}{4} + \frac{3\sqrt{3}}{8\pi}$$

7 a $f(x) = \dfrac{5x}{(2x-1)(x+2)}$

Consider $\int \dfrac{5x}{(2x-1)(x+2)}\,dx$

$$= \int\left(\frac{1}{2x-1} + \frac{2}{x+2}\right)dx$$

$$= \int \frac{1}{2x-1}\,dx + \int \frac{2}{x+2}\,dx$$

$$= \frac{1}{2}\ln|2x-1| + 2\ln|x+2| + C$$

$$= \ln\left|\sqrt{2x+1}(x+2)^2\right| + C$$

So the mean value of $f(x)$ on $[1,5]$ is:

$$\frac{1}{4}\int_1^5 \frac{5x}{(2x-1)(x+2)}\,dx$$

$$= \frac{1}{4}\left[\ln\left|\sqrt{2x+1}(x+2)^2\right|\right]_1^5$$

$$= \frac{1}{4}\left(\ln(3\times 49) - \ln 9\right)$$

$$= \frac{1}{4}\ln\frac{49}{3}$$

7 b The mean value of $f(x) + \ln k$ is equal to the mean value of $f(x)$ plus the mean value of $\ln k$.

The mean value of $\ln k$ over the interval $[1,5]$ is:

$$\frac{1}{4}\int_1^5 \ln k\,dx = \frac{1}{4}\ln k\int_1^5 dx$$

$$= \frac{1}{4}\ln k\left[x\right]_1^5 = \frac{1}{4}\ln k^4$$

Therefore, the mean value of $f(x) + \ln k$ is $\frac{1}{4}\ln\frac{49}{3} + \frac{1}{4}\ln k^4 = \frac{1}{4}\ln\frac{49k^4}{3}$

8 a $f(x) = x\left(x^2-4\right)^4$

Consider $\int x\left(x^2-4\right)^4 dx$

Let $u = x^2 - 4$ and $du = 2x\,dx$

$$= \frac{1}{2}\int u^4\,dx$$

$$= \frac{u^5}{10} + C$$

$$= \frac{\left(x^2-4\right)^5}{10} + C$$

So the mean value of $f(x)$ on $[0,2]$ is:

$$\frac{1}{2}\int_0^2 x\left(x^2-4\right)^4 dx$$

$$= \frac{1}{2}\left[\frac{\left(x^2-4\right)^5}{10}\right]_0^2$$

$$= \frac{256}{5}$$

8 b The mean value of $-2f(x)$ is equal to the mean value of $f(x)$ over the interval $[0,2]$ multiplied by -2.

Therefore, the mean value of $-2f(x)$ on $[0,2]$ is $-\frac{512}{5}$.

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9 $f(x) = \ln(kx)$

Consider $\int \ln(kx)\,dx$

Integrating by parts,

$= x\ln(kx) - \int x\frac{1}{x}\,dx$

$= x\ln(kx) - x + C$

So the mean value of $f(x)$ on $[0,2]$ is:

$\frac{1}{2}\int_0^2 \ln(kx)\,dx = -2$

$\frac{1}{2}\left[x\ln(kx) - x\right]_0^2 = -2$

$\frac{1}{2}(2\ln 2k - 2) = -2$

$\ln 2k = -1$

$2k = \frac{1}{e}$

$k = \frac{1}{2e}$

10 Given that the mean value of $f(x)$ on the interval $[a,b]$ is m:

$\frac{1}{b-a}\int_a^b f(x)\,dx = m$

The mean value of $f(x)+c$ on the interval $[a,b]$ is:

$\frac{1}{b-a}\int_a^b (f(x)+c)\,dx$

$= \frac{1}{b-a}\int_a^b f(x)\,dx + \frac{1}{b-a}\int_a^b c\,dx$

$= \frac{1}{b-a}\int_a^b f(x)\,dx + \frac{c}{b-a}\int_a^b dx$

$= m + \frac{c}{b-a}(b-a) = m + c$

11 $f(x) = \frac{1}{\sqrt{2-x}}$

Consider $\int \frac{1}{\sqrt{2-x}}\,dx$

$= -2\sqrt{2-x} + C$

So the mean value of $f(x)$ on $[0,2]$ is:

$\frac{1}{2}\int_0^2 \frac{1}{\sqrt{2-x}}\,dx$

$= \frac{1}{2}\left[-2\sqrt{2-x}\right]_0^2$

$= \sqrt{2}$

12 The graph of $y = f(x) = \sin^5 x$ on the interval $[0,\pi]$ is the negative of the graph graph of $y = \sin^5 x$ on the interval $[\pi,2\pi]$. Therefore, on evaluating the integral $\int_0^{2\pi} \sin^5 x\,dx = \int_0^{\pi} \sin^5 x\,dx + \int_{\pi}^{2\pi} \sin^5 x\,dx$, the two integrals on the right hand side cancel each other out and the mean value of $f(x)$ on the interval $[0,2\pi]$ is zero.

13 a $\int f(x)\,dx = \int \frac{\cos x}{(2+\sin x)^2}\,dx$

Let $u = 2+\sin x$ and $du = \cos x\,dx$. So,

$\int \frac{\cos x}{(2+\sin x)^2}\,dx = \int \frac{1}{u^2}\,du$

$= -\frac{1}{u} + C = -\frac{1}{2+\sin x} + C$

b The mean value of $f(x)$ over the interval $\left[0, \frac{5\pi}{3}\right]$ is:

$\frac{3}{5\pi}\int_0^{\frac{5\pi}{3}} f(x)\,dx$

$= \frac{3}{5\pi}\left[-\frac{1}{2+\sin x}\right]_0^{\frac{5\pi}{3}}$

$= \frac{3}{5\pi}\left(-\frac{8+2\sqrt{3}}{13} + \frac{1}{2}\right)$

$= -\frac{3}{130\pi}\left(3+4\sqrt{3}\right)$

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13 c The mean value of $f(x)+3x$ is equal to the mean value of $f(x)$ plus the mean value of $3x$ on the given interval.
The mean value of $3x$ over the interval $\left[0,\frac{5\pi}{3}\right]$ is:

$$\frac{3}{5\pi}\int_0^{\frac{5\pi}{3}} 3x\,dx$$

$$=\frac{3}{5\pi}\left[\frac{3x^2}{2}\right]_0^{\frac{5\pi}{3}}$$

$$=\frac{5\pi}{2}$$

Therefore, the mean value of $f(x)+3x$ is

$$-\frac{3}{130\pi}\left(3+4\sqrt{3}\right)+\frac{5\pi}{2}$$

14 a For turning points, $f'(x)=0$

$$f'(x)=-3-4x=0$$

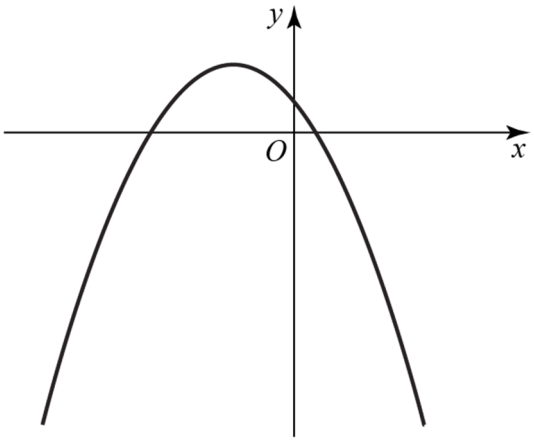
$$x=-\frac{3}{4}$$

At $x=-\frac{3}{4}$,

$$f(x)=1-3\left(-\frac{3}{4}\right)-2\left(-\frac{3}{4}\right)^2=\frac{17}{8}$$

So the turning point is $\left(-\frac{3}{4},\frac{17}{8}\right)$

The graph of $y=f(x)=1-3x-2x^2$ is:



14 b

$$\int_a^{a+1} f(x)\,dx=\int_a^{a+1}\left(1-3x-2x^2\right)dx$$

$$=\left[x-\frac{3x^2}{2}-\frac{2x^3}{3}\right]_a^{a+1}$$

$$=\left(\begin{array}{c}a+1-\frac{3(a+1)^2}{2}-\frac{2(a+1)^3}{3}\\ -a+\frac{3a^2}{2}+\frac{2a^3}{3}\end{array}\right)$$

$$=1-3a-\frac{3}{2}-2a^2-2a-\frac{2}{3}$$

$$=-\frac{7}{6}-5a-2a^2$$

b The mean value over an interval of length 1, say $[a,a+1]$, is given by

$$\int_a^{a+1} f(x)\,dx=-\frac{7}{6}-5a-2a^2$$

c To find the maximum value,

$$\frac{d}{da}\left(-\frac{7}{6}-5a-2a^2\right)=0$$

$$-5-4a=0$$

$$a=-\frac{5}{4}$$

The maximum value occurs when $a=-\frac{5}{4}$ and the maximum mean value is

$$=-\frac{7}{6}-5\left(-\frac{5}{4}\right)-2\left(-\frac{5}{4}\right)^2=\frac{47}{24}$$

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Methods in calculus 3C

1 a Let $y = \arctan x$

then $\tan y = x$

$$\sec^2 y \frac{dy}{dx} = 1$$

$$\frac{dy}{dx} = \frac{1}{\sec^2 y} = \cos^2 y$$

But $\tan y = x$, so $\cos y = \frac{1}{\sqrt{1+x^2}}$

Therefore,

$$\frac{dy}{dx} = \frac{1}{1+x^2}$$

b Let $y = \arccos x$

then $\cos y = x$

$$-\sin y \frac{dy}{dx} = 1$$

$$\frac{dy}{dx} = -\frac{1}{\sin y}$$

But $\cos y = x$, so $\sin y = \sqrt{1-x^2}$

Therefore,

$$\frac{dy}{dx} = -\frac{1}{\sqrt{1-x^2}}$$

c Let $y = \arccos x^2$

then $\cos y = x^2$

$$-\sin y \frac{dy}{dx} = 2x$$

$$\frac{dy}{dx} = -\frac{2x}{\sin y}$$

But $\cos y = x^2$, so $\sin y = \sqrt{1-x^4}$

Therefore,

$$\frac{dy}{dx} = -\frac{2x}{\sqrt{1-x^4}}$$

d Let $y = \arctan\left(x^3 + 3x\right)$

then $\tan y = x^3 + 3x$

$$\sec^2 y \frac{dy}{dx} = 3x^2 + 3$$

$$\frac{dy}{dx} = \frac{3x^2+3}{\sec^2 y}$$

$$= \left(3x^2+3\right)\cos^2 y$$

But $\tan y = x^3 + 3x$

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1 d So, $\cos y = \dfrac{1}{\sqrt{1+\left(x^3+3x\right)^2}}$

Therefore,

$\dfrac{dy}{dx} = \dfrac{3x^2+3}{1+\left(x^3+3x\right)^2}$

e Let $y = \arcsin\left(\dfrac{1}{x}\right)$

then $\sin y = \dfrac{1}{x}$

$\cos y \dfrac{dy}{dx} = -\dfrac{1}{x^2}$

$\dfrac{dy}{dx} = -\dfrac{1}{x^2 \cos y}$

But $\sin y = \dfrac{1}{x}$

So, $\cos y = \sqrt{1-\left(\dfrac{1}{x}\right)^2} = \dfrac{\sqrt{x^2-1}}{x}$

Therefore,

$\dfrac{dy}{dx} = -\dfrac{1}{x\sqrt{x^2-1}}$

2 $y = (\arccos x)(\arcsin x)$

Using the product rule for differentiating, and the standard results for $\dfrac{d}{dx}(\arccos x)$ and $\dfrac{d}{dx}(\arcsin x)$,

$\dfrac{dy}{dx} = \begin{pmatrix} (\arccos x)\dfrac{d}{dx}(\arcsin x) + \\ (\arcsin x)\dfrac{d}{dx}(\arccos x) \end{pmatrix}$

$\dfrac{dy}{dx} = \arccos x . \dfrac{1}{\sqrt{1-x^2}} - \arcsin x . \dfrac{1}{\sqrt{1-x^2}}$

$\dfrac{dy}{dx} = \dfrac{\arccos x - \arcsin x}{\sqrt{1-x^2}}$

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3 $y = \dfrac{1+\arctan x}{1-\arctan x}$

$y(1-\arctan x) = 1+\arctan x$

$(1+y)\arctan x = y-1$

$\arctan x = \dfrac{y-1}{y+1}$

$\tan\left(\dfrac{y-1}{y+1}\right) = x$

$\sec^2\left(\dfrac{y-1}{y+1}\right)\left(\dfrac{(y+1)-(y-1)}{(y+1)^2}\right)\dfrac{dy}{dx} = 1$

$\left(1+\tan^2\left(\dfrac{y-1}{y+1}\right)\right)\left(\dfrac{2}{(y+1)^2}\right)\dfrac{dy}{dx} = 1$

$(1+x^2)\dfrac{(1-\arctan x)^2}{2}\dfrac{dy}{dx} = 1$

$\dfrac{dy}{dx} = \dfrac{2}{(1+x^2)(1-\arctan x)^2}$

4 $f(x) = \arccos x + \arcsin x$

$f'(x) = \dfrac{d}{dx}(\arccos x) + \dfrac{d}{dx}(\arcsin x)$

$f'(x) = \dfrac{1}{\sqrt{1-x^2}} - \dfrac{1}{\sqrt{1-x^2}} = 0$

$f(x) = \int f'(x)\,dx = \int 0\,dx = C$

Consider

$f(0) = \arccos 0 + \arcsin 0$

$= \dfrac{\pi}{2} + 0 = \dfrac{\pi}{2} = C$

Therefore, $f(x) = \dfrac{\pi}{2}$ for all values of x.

5 a Let $y = \arccos 2x$

Let $t = 2x$ $y = \arccos t$

then $\dfrac{dt}{dx} = 2$ $\dfrac{dy}{dt} = \dfrac{-1}{\sqrt{1-t^2}}$

$\dfrac{dy}{dt} = \dfrac{-1}{\sqrt{1-t^2}} \times 2$

$= \dfrac{-2}{\sqrt{1-4x^2}}$

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5 b Let $y = \arctan\frac{x}{2}$

Let $t = \frac{x}{2}$ $y = \arctan t$

$\frac{dt}{dx} = \frac{1}{2}$ $\frac{dy}{dt} = \frac{1}{1+t^2}$

$\frac{dy}{dx} = \frac{1}{1+t^2} \times \frac{1}{2} = \frac{1}{2\left(1+\frac{x^2}{4}\right)} = \frac{2}{4+x^2}$ or $\frac{2}{x^2+4}$

c Let $y = \arcsin 3x$

$\sin y = 3x$

$\cos y \frac{dy}{dx} = 3$

$\frac{dy}{dx} = \frac{3}{\cos y} = \frac{3}{\sqrt{1-\sin^2 y}}$

$= \frac{3}{\sqrt{1-9x^2}}$

$= \frac{3}{\sqrt{1-9x^2}}$

d Let $y = \operatorname{arccot}(x+1)$

Let $t = x+1$ and $\frac{dt}{dx} = 1$

$y = \operatorname{arccot} t$

$\cot y = t$

$-\operatorname{cosec}^2 y \frac{dy}{dt} = 1$

$\frac{dy}{dt} = -\sin^2 y$

$= -\frac{1}{1+(x+1)^2}$

Using chain rule,

$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$

$= -\frac{1}{1+(x+1)^2}$

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5 e Let $y = \arcsin\left(1-x^2\right)$

Let $t = 1-x^2$ and $\dfrac{\mathrm{d}t}{\mathrm{d}x} = -2x$

$y = \arcsin t$

$\sin y = t$

$\cos y \dfrac{\mathrm{d}y}{\mathrm{d}t} = 1$

$\dfrac{\mathrm{d}y}{\mathrm{d}t} = \dfrac{1}{\cos y}$

$= \dfrac{1}{\sqrt{1-\sin^2 y}}$

$= \dfrac{1}{\sqrt{1-\left(1-x^2\right)^2}}$

Using chain rule,

$\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{\mathrm{d}y}{\mathrm{d}t} \times \dfrac{\mathrm{d}t}{\mathrm{d}x}$

$= -\dfrac{2x}{\sqrt{1-\left(1-x^2\right)^2}}$

$= -\dfrac{2x}{\sqrt{x^2\left(2-x^2\right)}}$

f Let $y = \arccos x^2$

Let

$t = x^2 \qquad y = \arccos t$

$\dfrac{\mathrm{d}t}{\mathrm{d}x} = 2x \quad \dfrac{\mathrm{d}y}{\mathrm{d}t} = \dfrac{-1}{\sqrt{1-t^2}}$

$\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{-1}{\sqrt{1-t^2}} \times 2x$

$= \dfrac{-2x}{\sqrt{1-x^4}}$

g Let $y = \mathrm{e}^x \arccos x$

$\dfrac{\mathrm{d}y}{\mathrm{d}x} = \mathrm{e}^x \arccos x - \mathrm{e}^x \dfrac{1}{\sqrt{1-x^2}}$

$= \mathrm{e}^x\left(\arccos x - \dfrac{1}{\sqrt{1-x^2}}\right)$

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5 h Let $y = \arcsin x \cos x$

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}\cos x + \arcsin x \times -\sin x$$

$$= \frac{\cos x}{\sqrt{1-x^2}} - \sin x \arcsin x$$

i Let $y = x^2 \arccos x$

$$\frac{dy}{dx} = 2x \arccos x - x^2 \times \frac{1}{\sqrt{1-x^2}}$$

$$= 2x \arccos x - \frac{x^2}{\sqrt{1-x^2}}$$

$$= x\left(2\arccos x - \frac{x}{\sqrt{1-x^2}}\right)$$

j Let $y = e^{\arctan x}$

$$\frac{dy}{dx} = \frac{e^{\arctan x}}{1+x^2}$$

6

$$\tan y = x \arctan x$$

$$\sec^2 y \frac{dy}{dx} = \arctan x + \frac{x}{1+x^2}$$

$$\frac{dy}{dx} = \frac{1}{\sec^2 y}\left(\arctan x + \frac{x}{1+x^2}\right)$$

$$= \frac{1}{1+x^2(\arctan x)^2}\left(\arctan x + \frac{x}{1+x^2}\right)$$

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7 $y = \arcsin x$

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}$$

$$\frac{d^2y}{dx^2} = \frac{0 - \frac{1}{2}(1-x^2)^{-\frac{1}{2}} \times -2x}{\left(\sqrt{1-x^2}\right)^2}$$

$$= \frac{x(1-x^2)^{-\frac{1}{2}}}{(1-x^2)}$$

$$= \frac{x}{\sqrt{1-x^2}(1-x^2)}$$

$$(1-x^2)\frac{d^2y}{dx^2} = x\frac{dy}{dx}$$

$$(1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} = 0$$

8 $y = \arcsin 2x \quad x = \frac{1}{4} \quad y = \arcsin x\left(\frac{2}{4}\right) = \frac{\pi}{6}$

$$\frac{dy}{dx} = \frac{2}{\sqrt{1-4x^2}} = \frac{2}{\sqrt{1-\frac{1}{4}}} = \frac{4}{\sqrt{3}}$$

Tangent is

$$\left(y - \frac{\pi}{6}\right) = \frac{4}{\sqrt{3}}\left(x - \frac{1}{4}\right)$$

$$\sqrt{3}y - \frac{\pi\sqrt{3}}{6} = 4x - 1$$

$$y = \frac{4}{\sqrt{3}}x + \frac{\pi}{6} - \frac{1}{\sqrt{3}}$$

9 a $y = (\arctan x)^2$

Let $t = \arctan x$

So, $\frac{dt}{dx} = \frac{1}{1+x^2}$

$y = t^2$

$$\frac{dy}{dt} = 2t = 2\arctan x$$

$$\frac{dy}{dx} = \frac{dy}{dt} \times, \frac{dt}{dx}$$

$$= \frac{2\arctan x}{1+x^2}$$

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9 b $y = \dfrac{1}{\arcsin x}$

Let $t = \arcsin x$

So, $\dfrac{dt}{dx} = \dfrac{1}{\sqrt{1-x^2}}$

$y = \dfrac{1}{t}$

$\dfrac{dy}{dt} = -\dfrac{1}{t^2} = -\dfrac{1}{(\arcsin x)^2}$

$\dfrac{dy}{dx} = \dfrac{dy}{dt} \times, \dfrac{dt}{dx}$

$= -\dfrac{1}{\sqrt{1-x^2}\,(\arcsin x)^2}$.

c $y = \arctan(\arctan x)$

Let $t = \arctan x$

So, $\dfrac{dt}{dx} = \dfrac{1}{1+x^2}$

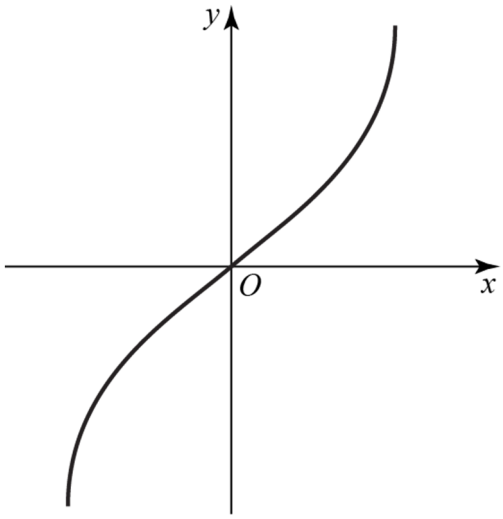
$y = \arctan t$

$\dfrac{dy}{dt} = \dfrac{1}{1+t^2} = \dfrac{1}{1+(\arctan x)^2}$

$\dfrac{dy}{dx} = \dfrac{dy}{dt} \times \dfrac{dt}{dx}$

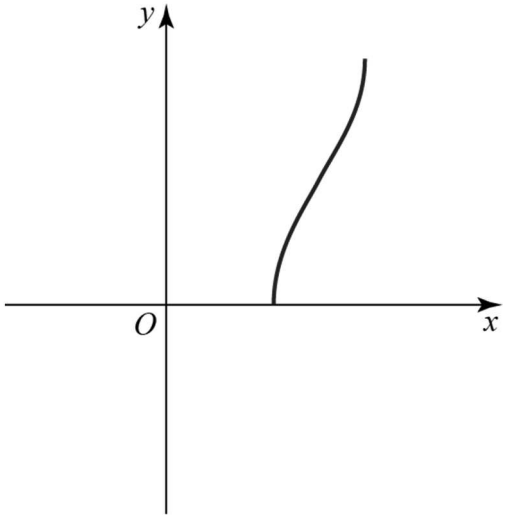
$= \dfrac{1}{(1+x^2)\left(1+(\arctan x)^2\right)}$

10 a The graph of $y = \arcsin(\arcsin x)$ is:

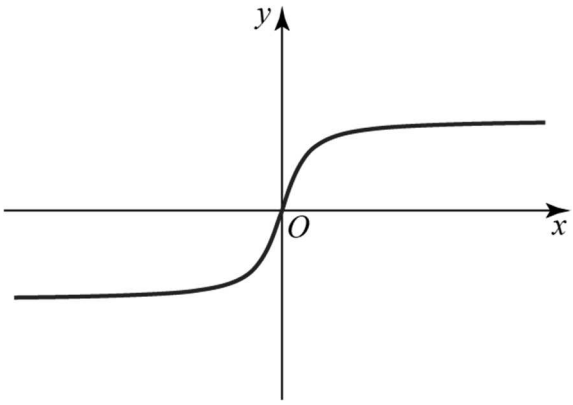


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10 b The graph of $y = \arccos(\arccos x)$ is:



c The graph of $y = \arctan(\arctan x)$ is:



11 a Let $t = \arccos x$, so $\cos t = x$

$$\sin(\arccos x) = \sin t$$
$$= \pm\sqrt{1-\cos^2 t}$$

Since $\arccos x$ has the range $[0, \pi]$, and $\sin t$ is positive on this domain, we have

$$\sin(\arccos x) = \sqrt{1-x^2}$$

b Let $t = \arctan x$, so $\tan t = x$

Therefore,

$$\cos(\arctan x) = \cos t$$
$$= \frac{1}{\sec t}$$
$$= \pm\frac{1}{\sqrt{1+x^2}}$$

Since $\arctan x$ has the range $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, and $\cos t$ is positive on this domain, we have

$$\cos(\arctan x) = \frac{1}{\sqrt{1+x^2}}$$

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11 c Let $t = \arccos x$, so $\cos t = x$
Therefore,

$$\sec(\arccos x) = \sec t$$
$$= \frac{1}{\cos t}$$
$$= \frac{1}{x}$$

d Let $t = \text{arcsec}x$, so $\sec t = x$
Therefore,

$$\sin(\text{arcsec}x) = \sin t$$
$$= \pm\sqrt{1-\cos^2 t}\ .$$
$$= \pm\sqrt{1-\left(\frac{1}{x}\right)^2}$$
$$= \pm\frac{\sqrt{x^2-1}}{x}$$

Since $\text{arcsec}x$ has the range $\left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right]$, and $\sin t$ is positive on this domain, we have

$$\sin(\text{arcsec}x) = \sqrt{1-\frac{1}{x^2}}$$

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Methods in calculus 3D

1 Using $x = a\tan\theta, \mathrm{d}x = a\sec^2\mathrm{d}\theta$

$$\text{so } \int \frac{1}{a^2+x^2}\mathrm{d}x = \int \frac{1}{a^2+a^2\tan^2\theta} a\sec^2\theta\ \mathrm{d}\theta$$

$$= \int \frac{a\sec^2\theta}{a^2\sec^2\theta}\mathrm{d}\theta$$

$$= \frac{1}{a}\int \mathrm{d}\theta$$

$$= \frac{1}{a}\theta + C$$

$$= \frac{1}{a}\arctan\left(\frac{x}{a}\right) + C$$

$x = a\tan\theta \Rightarrow \arctan\left(\frac{x}{a}\right)$

2 Using $x = \cos\theta,\ \mathrm{d}x = -\sin\theta\,\mathrm{d}\theta$

$$\text{so } \int \frac{1}{\sqrt{1-x^2}}\mathrm{d}x = \int \frac{1}{\sqrt{1-\cos^2\theta}}(-\sin\theta)\mathrm{d}\theta$$

$$= -\int \mathrm{d}\theta$$

$$= -\theta + C$$

$$= -\arccos x + C$$

3 a Let $x = 2\sin\theta$, so $\mathrm{d}x = 2\cos\theta\,\mathrm{d}\theta$

$$\int \frac{3}{\sqrt{4-x^2}}\mathrm{d}x = \int \frac{3}{\sqrt{4-4\sin^2\theta}} 2\cos\theta\,\mathrm{d}\theta$$

$$= \int \frac{6\cos\theta}{2\cos\theta}\mathrm{d}\theta$$

$$= 3\int \mathrm{d}\theta$$

$$= 3\theta + C$$

$$= 3\arcsin\left(\frac{x}{2}\right) + C$$

b Let $x = \sqrt{5}\tan\theta, \text{so } \mathrm{d}x = \sqrt{5}\sec^2\theta\,\mathrm{d}\theta$

$$\int \frac{4}{5+x^2}\mathrm{d}x = \int \frac{4}{5+5\tan^2\theta}\sqrt{5}\sec^2\theta\,\mathrm{d}\theta$$

$5 + 5\tan^2\theta = 5(1+\tan^2\theta) = 5\sec^2\theta$

$$= \int \frac{4\sqrt{5}\sec^2\theta}{5\sec^2\theta}\mathrm{d}\theta$$

$$= \frac{4\sqrt{5}}{5}\int \mathrm{d}\theta$$

$$= \frac{4\sqrt{5}}{5}\theta + C$$

$$= \frac{4\sqrt{5}}{5}\arctan\left(\frac{x}{\sqrt{5}}\right) + C$$

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3 c $\int \frac{1}{\sqrt{25-x^2}}\,\mathrm{d}x = \arcsin\left(\frac{x}{5}\right)+C$

Using $\int \frac{1}{\sqrt{a^2-x^2}}\,\mathrm{d}x = \arcsin\left(\frac{x}{a}\right)+C$

4 $\int \frac{1}{4+3x^2}\,\mathrm{d}x$

$= \frac{1}{3}\int \frac{1}{\frac{4}{3}+x^2}\,\mathrm{d}x$

$= \frac{1}{3}\left(\frac{1}{\sqrt{\frac{4}{3}}}\arctan\left(\frac{x}{\sqrt{\frac{4}{3}}}\right)\right)+c$

$= \frac{\sqrt{3}}{6}\arctan\left(\frac{\sqrt{3}x}{2}\right)+c$

Therefore, $A = \frac{\sqrt{3}}{6}$ and $B = \frac{\sqrt{3}}{2}$

5 $\int \frac{1}{\sqrt{3-4x^2}}\,\mathrm{d}x$

$= \frac{1}{\sqrt{4}}\int \frac{1}{\sqrt{\frac{3}{4}-x^2}}\,\mathrm{d}x$

$= \frac{1}{2}\arcsin\left(\frac{x}{\sqrt{\frac{3}{4}}}\right)$

$= \frac{1}{2}\arcsin\left(\frac{2x}{\sqrt{3}}\right)+c$

Therefore, $P = \frac{1}{2}$ and $Q = \frac{2}{\sqrt{3}}$

6 a $\int_1^3 \frac{2}{1+x^2}\,\mathrm{d}x = 2\left[\arctan x\right]_1^3$

$= 2(\arctan 3 - \arctan 1)$

$= 2\left(\arctan 3 - \frac{\pi}{4}\right)$

Remember that you need to be in radian mode.

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6 b $\int_{0.25}^{0.5} \frac{3}{\sqrt{1-4x^2}}\,\mathrm{d}x = \frac{3}{\sqrt{4}}\int_{0.25}^{0.5} \frac{1}{\sqrt{\frac{1}{4}-x^2}}\,\mathrm{d}x$

$$= \frac{3}{2}\left[\arcsin\frac{x}{\sqrt{\frac{1}{4}}}\right]_{0.25}^{0.5}$$

$$= \frac{3}{2}\left[\arcsin 2x\right]_{0.25}^{0.5}$$

$$= \frac{3}{2}(\arcsin 0.5 - \arcsin 0.25)$$

$$= \frac{3}{2}\left(\frac{\pi}{2} - \frac{\pi}{6}\right)$$

$$= \frac{\pi}{2}$$

c $\int_{-1}^{2} \frac{1}{\sqrt{21-3x^2}}\,\mathrm{d}x = \frac{1}{\sqrt{3}}\int_{-1}^{2} \frac{1}{\sqrt{7-x^2}}\,\mathrm{d}x$

$$= \frac{1}{\sqrt{3}}\left[\arcsin\left(\frac{x}{\sqrt{7}}\right)\right]_{-1}^{2}$$

$$= \frac{1}{\sqrt{3}}\left[\arcsin\left(\frac{2}{\sqrt{7}}\right) - \arcsin\left(-\frac{1}{\sqrt{7}}\right)\right]$$

$$= \frac{1}{\sqrt{3}}[0.85707\ldots - (-0.38759\ldots)]$$ ← You need to be in radian mode

$$= 0.719 \quad (3\text{ s.f.})$$

7 $\int_{\sqrt{2}}^{\sqrt{3}} \frac{1}{\sqrt{4-x^2}}\,\mathrm{d}x$

$$= \left[\arcsin\left(\frac{x}{2}\right)\right]_{\sqrt{2}}^{\sqrt{3}}$$

$$= \arcsin\left(\frac{\sqrt{3}}{2}\right) - \arcsin\left(\frac{1}{\sqrt{2}}\right)$$

$$= \frac{\pi}{3} - \frac{\pi}{4} = \frac{\pi}{12}$$

8 $\mathrm{f}(x) = \frac{2+3x}{1+3x^2}$

$$\int \mathrm{f}(x)\,\mathrm{d}x = \int \frac{2+3x}{1+3x^2}\,\mathrm{d}x$$

$$= 2\int \frac{1}{1+3x^2}\,\mathrm{d}x + 3\int \frac{x}{1+3x^2}\,\mathrm{d}x$$

Consider $\int \frac{1}{1+3x^2}\,\mathrm{d}x$

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$$= \frac{1}{3}\int \frac{1}{\frac{1}{3}+x^2}\,\mathrm{d}x$$

8 $$= \frac{1}{3}\left(\frac{1}{\sqrt{\frac{1}{3}}}\arctan\left(\frac{x}{\sqrt{\frac{1}{3}}}\right)\right)+c_1$$

$$= \frac{\sqrt{3}}{3}\arctan\left(\sqrt{3}x\right)+c_1$$

Similarly, consider $\int \frac{x}{1+3x^2}\,\mathrm{d}x$

Let $u = 1+3x^2$ and $du = 6x\mathrm{d}x$

$$\int \frac{x}{1+3x^2}\,\mathrm{d}x = \frac{1}{6}\int \frac{1}{u}\,du$$

$$= \frac{1}{6}\ln u + c_2$$

$$= \frac{1}{6}\ln\left(1+3x^2\right)+c_2$$

Therefore,

$$\int \mathrm{f}\left(x\right)\mathrm{d}x$$

$$= 2\int \frac{1}{1+3x^2}\,\mathrm{d}x + 3\int \frac{x}{1+3x^2}\,\mathrm{d}x$$

$$= \frac{2\sqrt{3}}{3}\arctan\left(\sqrt{3}x\right)+\frac{1}{2}\ln\left(1+3x^2\right)+c$$

Therefore, $A = \frac{2\sqrt{3}}{3}$ and $B = \frac{1}{2}$

9 $\mathrm{f}\left(x\right) = \frac{2x-1}{\sqrt{2-x^2}}$

$$\int \mathrm{f}\left(x\right)\mathrm{d}x = \int \frac{2x-1}{\sqrt{2-x^2}}\,\mathrm{d}x$$

$$= \int \frac{2x}{\sqrt{2-x^2}}\,\mathrm{d}x - \int \frac{1}{\sqrt{2-x^2}}\,\mathrm{d}x$$

Consider $\int \frac{2x}{\sqrt{2-x^2}}\,\mathrm{d}x$

Let $u = 2-x^2$ and $du = -2x\mathrm{d}x$

$$\int \frac{2x}{\sqrt{2-x^2}}\,\mathrm{d}x = -\int \frac{1}{\sqrt{u}}\,du$$

$$= -2\sqrt{u}+c_1$$

$$= -2\sqrt{2-x^2}+c_1$$

Consider $\int \frac{1}{\sqrt{2-x^2}}\,\mathrm{d}x$

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$= \arcsin\left(\frac{x}{\sqrt{2}}\right) + c_2$

9 Therefore,

$\int f(x)\,dx$

$= \int \frac{2x}{\sqrt{2-x^2}}\,dx - \int \frac{1}{\sqrt{2-x^2}}\,dx$

$= -2\sqrt{2-x^2} - \arcsin\left(\frac{x}{\sqrt{2}}\right) + c$

Therefore, $A = -1$ and $B = -2$

10 $f(x) = \frac{8x-3}{4+x^2}$

$\int f(x)\,dx = \int \frac{8x-3}{4+x^2}\,dx$

$= \int \frac{8x}{4+x^2}\,dx - 3\int \frac{1}{4+x^2}\,dx$

Consider $\int \frac{1}{4+x^2}\,dx$

$= \frac{1}{2}\arctan\left(\frac{x}{2}\right) + c_1$

Similarly, consider $\int \frac{8x}{4+x^2}\,dx$

Let $u = 4+x^2$ and $du = 2x\,dx$

$\int \frac{8x}{4+x^2}\,dx = 4\int \frac{1}{u}\,du$

$= 4\ln u + c_2$

$= 4\ln\left(4+x^2\right) + c_2$

Therefore,

$\int f(x)\,dx$

$= \int \frac{8x}{4+x^2}\,dx - 3\int \frac{1}{4+x^2}\,dx$

$= 4\ln\left(4+x^2\right) - \frac{3}{2}\arctan\left(\frac{x}{2}\right) + c$

Therefore, $A = 4$ and $B = -\frac{3}{2}$

11 $f(x) = \frac{4x-1}{\sqrt{6-5x^2}}$

$\int f(x)\,dx = \int \frac{4x-1}{\sqrt{6-5x^2}}\,dx$

$= \int \frac{4x}{\sqrt{6-5x^2}}\,dx - \int \frac{1}{\sqrt{6-5x^2}}\,dx$

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11 Consider $\int \frac{1}{\sqrt{6-5x^2}}\mathrm{d}x$

$$= \frac{1}{\sqrt{5}}\int \frac{1}{\sqrt{\frac{6}{5}-x^2}}\mathrm{d}x$$

$$= \frac{1}{\sqrt{5}}\arcsin\left(\frac{x}{\sqrt{\frac{6}{5}}}\right)+c_1$$

$$= \frac{1}{\sqrt{5}}\arcsin\left(\frac{\sqrt{5}x}{\sqrt{6}}\right)+c_1$$

Similarly, consider $\int \frac{4x}{\sqrt{6-5x^2}}\mathrm{d}x$

Let $u = 6-5x^2$ and $du = -10x\mathrm{d}x$

$$\int \frac{4x}{\sqrt{6-5x^2}}\mathrm{d}x = -\frac{2}{5}\int \frac{1}{\sqrt{u}}du$$

$$= -\frac{4}{5}\sqrt{u}+c_2$$

$$= -\frac{4}{5}\sqrt{6-5x^2}+c_2$$

Therefore,

$\int \mathrm{f}(x)\mathrm{d}x$

$$= \int \frac{4x}{\sqrt{6-5x^2}}\mathrm{d}x - \int \frac{1}{\sqrt{6-5x^2}}\mathrm{d}x$$

$$= -\frac{4}{5}\sqrt{6-5x^2} - \frac{1}{\sqrt{5}}\arcsin\left(\frac{\sqrt{5}x}{\sqrt{6}}\right)+c$$

Therefore, $P = -\frac{4}{5}$ and $Q = -\frac{1}{\sqrt{5}}$

12 a $\mathrm{f}(x) = \frac{x+5}{x^2+16}$

$$\int f(x)dx = \int \frac{x+5}{x^2+16}dx$$

$$= \int \frac{x}{x^2+16}dx + 5\int \frac{1}{x^2+16}dx$$

Consider $\int \frac{x}{x^2+16}dx$

Let $u = x^2+16$ and $du = 2xdx$

$$\int \frac{x}{x^2+16}dx = \frac{1}{2}\int \frac{1}{u}du$$

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$= \frac{1}{2}\ln u + c_1$

$= \frac{1}{2}\ln\left(x^2+16\right) + c_1$

12 a Similarly, consider $\int \frac{1}{x^2+16}\,\mathrm{d}x$

$= \frac{1}{4}\arctan\left(\frac{x}{4}\right) + c_1$

Therefore,

$\int \mathrm{f}\left(x\right)\mathrm{d}x$

$= \int \frac{x}{x^2+16}\,\mathrm{d}x + 5\int \frac{1}{x^2+16}\,\mathrm{d}x$

$= \frac{1}{2}\ln\left(x^2+16\right) + \frac{5}{4}\arctan\left(\frac{x}{4}\right) + c$

Therefore, $A = \frac{1}{2}$ and $B = \frac{5}{4}$

b The mean value of $\mathrm{f}\left(x\right)$ over the interval $\left[0,4\right]$ is:

$\frac{1}{4}\int_0^4 \mathrm{f}\left(x\right)\mathrm{d}x$

$= \frac{1}{4}\left[\frac{1}{2}\ln\left(x^2+16\right) + \frac{5}{4}\arctan\left(\frac{x}{4}\right)\right]_0^4$

$= \frac{1}{4}\left(\frac{1}{2}\ln 2 + \frac{5\pi}{16}\right)$

c The mean value of $-4\mathrm{f}\left(x\right)$ over the interval $\left[0,4\right]$ is -4 times the mean value of $\mathrm{f}\left(x\right)$ over the interval $\left[0,4\right]$

$= -\frac{1}{2}\ln 2 - \frac{5\pi}{16}$

13 With $x = \frac{2}{3}\tan\theta$ and $\mathrm{d}x = \frac{2}{3}\sec^2\theta\,\mathrm{d}\theta$,

$9x^2 + 4 = 9\left(\frac{4}{9}\tan^2\theta\right) + 4 = 4\tan^2\theta + 4 = 4(\tan^2\theta + 1) = 4\sec^2\theta$

and $\frac{x^2}{9x^2+4} = \frac{\frac{4}{9}\tan^2\theta}{4\sec^2\theta} = \frac{\tan^2\theta}{9\sec^2\theta}$

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13 so $\int \frac{x^2}{9x^2+4}\,\mathrm{d}x = \int \frac{\tan^2\theta}{9\sec^2\theta} \times \frac{2}{3}\sec^2\theta\,\mathrm{d}\theta$

$$= \frac{2}{27}\int \tan^2\theta\,\mathrm{d}\theta$$

$$= \frac{2}{27}\int (\sec^2\theta - 1)\,\mathrm{d}\theta$$

$$= \frac{2}{27}(\tan\theta - \theta) + C$$

$$= \frac{2}{27}\left(\frac{3x}{2} - \arctan\frac{3x}{2}\right) + C$$

$$= \frac{x}{9} - \frac{2}{27}\arctan\frac{3x}{2} + C$$

14 With $x = \frac{1}{2}\sin\theta$, $\mathrm{d}x = \frac{1}{2}\cos\theta\,\mathrm{d}\theta$

$1 - 4x^2 = 1 - \sin^2\theta = \cos^2\theta$ and so $\frac{x^2}{\sqrt{1-4x^2}} = \frac{\sin^2\theta}{4\cos\theta}$

So $\int_0^{\frac{1}{4}} \frac{x^2}{\sqrt{1-4x^2}}\,\mathrm{d}x = \int_0^{\frac{\pi}{6}} \frac{\sin^2\theta}{4\cos\theta} \times \frac{1}{2}\cos\theta\,\mathrm{d}\theta$

$$= \frac{1}{8}\int_0^{\frac{\pi}{6}} \sin^2\theta\,\mathrm{d}\theta$$

$$= \frac{1}{16}\int_0^{\frac{\pi}{6}} (1 - 2\cos 2\theta)\,\mathrm{d}\theta$$

$$= \frac{1}{16}\left[\theta - \frac{\sin 2\theta}{2}\right]_0^{\frac{\pi}{6}}$$

$$= \frac{1}{16}\left[\frac{\pi}{6} - \frac{\sqrt{3}}{4}\right]$$

$$= \frac{1}{192}(2\pi - 3\sqrt{3})$$

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Challenge

a $\int \frac{1}{x\sqrt{x^2-1}}\mathrm{d}x$

$x=\sec\theta$, so $\mathrm{d}x=\sec\theta\tan\theta\mathrm{d}\theta$

$\sqrt{x^2-1}=\sqrt{\sec^2\theta-1}=\tan\theta$

$\int \frac{1}{x\sqrt{x^2-1}}\mathrm{d}x=\int \frac{\sec\theta\tan\theta}{\sec\theta\tan\theta}d\theta$

$=\int d\theta=\theta+c=\text{arcsec}x+c$

b $\int \frac{\sqrt{x^2-1}}{x}\mathrm{d}x$

$x=\sec\theta$, so $\mathrm{d}x=\sec\theta\tan\theta d\theta$

Substituting gives

$x=\sec\theta$ so $\cos\theta=\frac{1}{x}$ and $\tan\theta=\sqrt{x^2-1}$

Therefore

$\tan\theta-\theta+c=\sqrt{x^2-1}-\text{arcsec}\,x+c$

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Methods in calculus 3E

1 a $\dfrac{1}{(x^2+1)(x+3)} = \dfrac{Ax+B}{x^2+1} + \dfrac{C}{x+3}$

$1 \equiv (Ax+B)(x+3) + C(x^2+1)$

If $x=-3$, $C=\dfrac{1}{10}$

$A+C=0 \Rightarrow A=-\dfrac{1}{10}$

$3B+C=1 \Rightarrow B=\dfrac{3}{10}$

Therefore,

$$\frac{1}{(x^2+1)(x+3)} = \frac{-x+3}{10(x^2+1)} + \frac{1}{10(x+3)}$$

b $\dfrac{1}{(x^2+2)(x-1)} = \dfrac{Ax+B}{x^2+2} + \dfrac{C}{x-1}$

$1 \equiv (Ax+B)(x-1) + C(x^2+2)$

If $x=1$, $C=\dfrac{1}{3}$

$A+C=0 \Rightarrow A=-\dfrac{1}{3}$

$-B+2C=1 \Rightarrow B=-\dfrac{1}{3}$

Therefore,

$$\frac{1}{(x^2+2)(x-1)} = -\frac{x+1}{3(x^2+2)} + \frac{1}{3(x-1)}$$

c $\dfrac{x-4}{x(x^2+7)} = \dfrac{Ax+B}{x^2+7} + \dfrac{C}{x}$

$x-4 \equiv (Ax+B)(x) + C(x^2+7)$

$B=1$

Set $x=0$, so $C=-\dfrac{4}{7}$

$A+C=0 \Rightarrow A=\dfrac{4}{7}$

Therefore,

$$\frac{x-4}{x(x^2+7)} = \frac{4x+7}{7(x^2+7)} - \frac{4}{7x}$$

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2 $\int \frac{x^2-3x}{(x^2+6)(x+2)}\,dx = \int \frac{Ax+B}{x^2+6} + \frac{C}{x+2}\,dx$

$x^2-3x \equiv (Ax+B)(x+2)+C(x^2+6)$

Set $x=-2$, so $C=1$

$A+C=1 \Rightarrow A=0$

$2B+6C=0 \Rightarrow B=-3$

Therefore,

$\int \frac{x^2-3x}{(x^2+6)(x+2)}\,dx$

$= -3\int \frac{1}{x^2+6}\,dx + \int \frac{1}{x+2}\,dx$

$= -\frac{3}{\sqrt{2}}\arctan\left(\frac{x}{\sqrt{6}}\right) + \ln|x+2| + c$

Therefore, $A=1$ and $B=-\frac{3}{\sqrt{2}}$

3 **a** Writing $x^4-x^3-4x^2-2x-12=(x+2)(x^3+ax^2+bx-6)$ and solving we obtain

$f(x)=(x-3)(x+2)(x^2+2)$

b $\int \frac{x^3-20x^2+4x-24}{x^4-x^3-4x^2-2x-12}\,dx$

$= \int \frac{x^3-20x^2+4x-24}{(x+2)(x-3)(x^2+2)}\,dx$

$= \int \frac{A}{x+2} + \frac{B}{x-3} + \frac{Cx+D}{x^2+2}\,dx$

$$x^3-20x^2+4x-24 \equiv \begin{pmatrix} A(x-3)(x^2+2) \\ +B(x+2)(x^2+2)+ \\ Cx(x+2)(x-3)+ \\ D(x+2)(x-3) \end{pmatrix}$$

Set $x=3$, so $B=-3$

Set $x=-2$, so $A=4$

$A+B+C=1 \Rightarrow C=0$

$-6A+4B-6D=-24 \Rightarrow D=-2$

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3 b $\int \frac{x^3-20x^2+4x-24}{(x+2)(x-3)(x^2+2)}\,dx$

$= 4\int \frac{1}{x+2}\,dx - 3\int \frac{1}{x-3}\,dx - 2\int \frac{1}{x^2+2}\,dx$

$= \begin{pmatrix} 4\ln|x+2| - 3\ln|x-3| \\ -\sqrt{2}\arctan\left(\frac{x}{\sqrt{2}}\right) + c \end{pmatrix}$

$= \left(\ln\left|\frac{(x+2)^4}{(x-3)^3}\right| - \sqrt{2}\arctan\left(\frac{x}{\sqrt{2}}\right) + c\right)$

Therefore, $A = 4$, $B = -3$, and $D = -\sqrt{2}$

4 $\int \frac{2-x}{x^3+4x}\,dx = \int \frac{2-x}{x(x^2+4)}\,dx$

$= \frac{Ax+B}{x^2+4} + \frac{C}{x}$

$2 - x \equiv (Ax+B)(x) + C(x^2+4)$

$B = -1$

Set $x = 0$, so $C = \frac{1}{2}$

$A + C = 0 \Rightarrow A = -\frac{1}{2}$

Therefore,

$\int \frac{2-x}{x^3+4x}\,dx$

$= -\frac{1}{2}\int \frac{x}{x^2+4}\,dx - \int \frac{1}{x^2+4}\,dx + \frac{1}{2}\int \frac{1}{x}\,dx$

$= -\frac{1}{4}\ln|x^2+4| - \frac{1}{2}\arctan\left(\frac{x}{2}\right) + \frac{1}{2}\ln|x| + c$

$= \frac{1}{4}\ln\left|\frac{x^2}{x^2+4}\right| - \frac{1}{2}\arctan\left(\frac{x}{2}\right) + c$

Therefore, $A = \frac{1}{4}$, $B = -\frac{1}{2}$

5 $\int \frac{x^2+1}{4x^4+9x^2}\,dx = \int \frac{x^2+1}{x^2(4x^2+9)}\,dx$

$= \frac{Ax+B}{x^2} + \frac{Cx+D}{4x^2+9}$

$x^2 + 1 \equiv (Ax+B)(4x^2+9) + (Cx+D)(x^2)$

$B = \frac{1}{9}$

$4B + D = 1 \Rightarrow D = \frac{5}{9}$

$A = 0$

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5 $C=0$

Therefore,

$$\int\frac{x^2+1}{4x^4+9x^2}\,\mathrm{d}x$$

$$=\frac{1}{9}\int\frac{1}{x^2}\,\mathrm{d}x+\frac{5}{9}\int\frac{1}{4x^2+9}\,\mathrm{d}x$$

$$=\frac{1}{9}\int\frac{1}{x^2}\,\mathrm{d}x+\frac{5}{36}\int\frac{1}{x^2+\frac{9}{4}}\,\mathrm{d}x$$

$$=-\frac{1}{9x}+\frac{5}{36}\left(\frac{1}{\frac{3}{2}}\arctan\left(\frac{x}{\frac{3}{2}}\right)\right)+c$$

$$=-\frac{1}{9x}+\frac{5}{54}\arctan\left(\frac{2x}{3}\right)+c$$

Therefore, $A=-\frac{1}{9}$, $B=\frac{5}{54}$

6 $\int\frac{x^3+9x^2+x+1}{x^4-1}\,\mathrm{d}x$

$$=\int\frac{x^3+9x^2+x+1}{(x-1)(x+1)(x^2+1)}\,\mathrm{d}x$$

$$=\int\frac{A}{x-1}+\frac{B}{x+1}+\frac{Cx+D}{x^2+1}\,\mathrm{d}x$$

$$x^3+9x^2+x+1\equiv\begin{pmatrix}A(x+1)(x^2+1)+\\B(x-1)(x^2+1)+\\Cx(x-1)(x+1)+\\D(x-1)(x+1)\end{pmatrix}$$

Set $x=-1$, so $B=-2$

Set $x=1$, so $A=3$

$A+B+C=1\Rightarrow C=0$

$A-B-D=1\Rightarrow D=4$

Therefore,

$$\int\frac{x^3+9x^2+x+1}{(x-1)(x+1)(x^2+1)}\,\mathrm{d}x$$

$$=3\int\frac{1}{x-1}\,\mathrm{d}x-2\int\frac{1}{x+1}\,\mathrm{d}x+4\int\frac{1}{x^2+1}\,\mathrm{d}x$$

$$=3\ln|x-1|-2\ln|x+1|+4\arctan x+c$$

$$=\ln\left|\frac{(x-1)^3}{(x+1)^2}\right|+4\arctan x+c$$

Therefore, $A=3$, $B=2$, and $D=4$

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7 a $\mathrm{f}(x) = x^3 - 4x^2 + 6x - 24$

Since $\mathrm{f}(4) = 0$, $(x-4)$ is a factor. Using long division, we find that

$\mathrm{f}(x) = (x-4)(x^2+6)$

b $\dfrac{2x^2-3x+24}{x^3-4x^2+6x-24} = \dfrac{2x^2-3x+24}{(x-4)(x^2+6)}$

$= \dfrac{A}{x-4} + \dfrac{Bx+C}{x^2+6}$ $\qquad 2x^2-3x+24 \equiv \begin{pmatrix} A(x^2+6)+ \\ (Bx+C)(x-4) \end{pmatrix}$

Set $x = -2$, so $A = 2$

$A + B = 2 \Rightarrow B = 0$

$6A - 4C = 24 \Rightarrow C = -3$

So,

$\dfrac{2x^2-3x+24}{x^3-4x^2+6x-24} = \dfrac{2}{x-4} - \dfrac{3}{x^2+6}$

c $\displaystyle\int \frac{2x^2-3x+24}{x^3-4x^2+6x-24}\,\mathrm{d}x$

$= \displaystyle\int \frac{2}{x-4} - \frac{3}{x^2+6}\,\mathrm{d}x$

$= 2\displaystyle\int \frac{1}{x-4}\,\mathrm{d}x - 3\int \frac{1}{x^2+6}\,\mathrm{d}x$

$= 2\ln|x-4| - \dfrac{3}{\sqrt{6}}\arctan\left(\dfrac{x}{\sqrt{6}}\right) + c$

$= 2\ln|x-4| - \sqrt{\dfrac{3}{2}}\arctan\left(\dfrac{x}{\sqrt{6}}\right) + c$

8 a $\displaystyle\int \frac{1}{(x-2)(2x-1)}\,\mathrm{d}x = \int \frac{A}{x-2} + \frac{B}{2x-1}\,\mathrm{d}x$

$1 \equiv A(2x-1) + B(x-2)$

Set $x = 2$, so $A = \dfrac{1}{3}$

Set $x = \dfrac{1}{2}$, so $B = -\dfrac{2}{3}$

$\displaystyle\int \frac{1}{(x-2)(2x-1)}\,\mathrm{d}x$

$= \dfrac{1}{3}\displaystyle\int \frac{1}{x-2}\,\mathrm{d}x - \frac{2}{3}\int \frac{1}{2x-1}\,\mathrm{d}x$

$= \dfrac{1}{3}\ln|x-2| - \dfrac{1}{3}\ln|2x-1| + c$

$= \dfrac{1}{3}\ln\left|\dfrac{x-2}{2x-1}\right| + c$

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8 b $\int_{\frac{1}{2}}^{2} \frac{1}{(x-2)(2x-1)}\,dx$

Split the integral as

$\int_{\frac{1}{2}}^{1} \frac{1}{(x-2)(2x-1)}\,dx + \int_{1}^{2} \frac{1}{(x-2)(2x-1)}\,dx$

Consider $\int_{1}^{2} \frac{1}{(x-2)(2x-1)}\,dx$

$= \lim_{t\to 2} \int_{1}^{t} \frac{1}{(x-2)(2x-1)}\,dx$

$= \lim_{t\to 2} \left[\frac{1}{3}\ln\left|\frac{x-2}{2x-1}\right|\right]_1^t$

$= \lim_{t\to 2} \left(\frac{1}{3}\ln\left|\frac{t-2}{2t-1}\right|\right)$

$\ln\left|\frac{t-2}{2t-1}\right| \to \infty$ as $t \to 2$, so the integral diverges.

9 a $\frac{x^4+5x^2+2x}{x^4+10x^2+24} = \frac{x^4+5x^2+2x}{(x^2+4)(x^2+6)}$

$= A + \frac{Bx+C}{x^2+4} + \frac{Dx+E}{x^2+6}$

$x^4+5x^2+2x \equiv \begin{pmatrix} A(x^2+4)(x^2+6)+ \\ (Bx+C)(x^2+6)+ \\ (Dx+E)(x^2+4) \end{pmatrix}$

$A = 1$

$B + D = 0$ and $6B + 4D = 2$

So $B = 1$ and $D = -1$

$24A + 6C + 4E = 0$

$10A + C + E = 5$

So $C = -2$ and $E = -3$

Therefore,

$\frac{x^4+5x^2+2x}{x^4+10x^2+24} = 1 + \frac{x-2}{x^2+4} - \frac{x+3}{x^2+6}$

b $\int \frac{x^4+5x^2+2x}{x^4+10x^2+24}\,dx$

$= \int 1 + \frac{x-2}{x^2+4} - \frac{x+3}{x^2+6}\,dx$

$= \int dx + \int \frac{x}{x^2+4}\,dx - 2\int \frac{1}{x^2+4}\,dx$

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9 b $-\int \frac{x}{x^2+6}\,dx - 3\int \frac{1}{x^2+6}\,dx$

$= x + \frac{1}{2}\ln\left|x^2+4\right| - \arctan\left(\frac{x}{2}\right)$

$-\frac{1}{2}\ln\left|x^2+6\right| - \frac{3}{\sqrt{6}}\arctan\left(\frac{x}{\sqrt{6}}\right) + c$

$= x + \frac{1}{2}\ln\left|\frac{x^2+4}{x^2+6}\right| - \arctan\left(\frac{x}{2}\right)$

$-\sqrt{\frac{3}{2}}\arctan\left(\frac{x}{\sqrt{6}}\right) + c$

10 Setting up the model $\frac{x^2+4x+10}{x(x^2+5)} = \frac{A}{x} + \frac{Bx+C}{x^2+5}$

$\Rightarrow x^2+4x+10 \equiv A(x^2+5) + (Bx+C)x$

$x = 0 \quad \Rightarrow 10 = 5A \quad \Rightarrow \quad A = 2$

Coefficient of $x \quad \Rightarrow \quad 4 = C$

Coefficient of $x^2 \quad \Rightarrow \quad 1 = A + B \quad \Rightarrow \quad B = -1$

So $\int \frac{x^2+4x+10}{x^3+5x}\,dx = \int\left(\frac{2}{x} + \frac{-x+4}{x^2+5}\right)dx$

$= \int\left(\frac{2}{x} + \frac{4}{x^2+5} - \frac{1}{2}\frac{2x}{x^2+5}\right)dx$

$= 2\ln x + \frac{4}{\sqrt{5}}\arctan\left(\frac{x}{\sqrt{5}}\right) - \frac{1}{2}\ln(x^2+5) + C$

$= \frac{1}{2}\ln\left(\frac{x^4}{x^2+5}\right) + \frac{4}{\sqrt{5}}\arctan\left(\frac{x}{\sqrt{5}}\right) + C$

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11 Setting up the model $\dfrac{2}{(x+1)(x^2+1)} = \dfrac{A}{x+1} + \dfrac{Bx+C}{x^2+1}$

$$\Rightarrow 2 \equiv A(x^2+1) + (Bx+C)(x+1)$$

$$x = -1 \Rightarrow 2 = 2A \Rightarrow A = -1$$

Coefficient of $x^2 \Rightarrow 0 = A + B \Rightarrow B = -1$

Coefficient of $x \Rightarrow 0 = B + C \Rightarrow C = 1$

So $\displaystyle\int_0^1 \frac{2}{(x+1)(x^2+1)}\,\mathrm{d}x = \int_0^1 \frac{1}{(x+1)}\,\mathrm{d}x + \int_0^1 \frac{1-x}{(x^2+1)}\,\mathrm{d}x$

$$= \int_0^1 \frac{1}{(x+1)}\,\mathrm{d}x + \int_0^1 \frac{1}{(x^2+1)}\,\mathrm{d}x - \int_0^1 \frac{x}{(x^2+1)}\,\mathrm{d}x$$

$$= [\ln(1+x)]_0^1 + [\arctan x]_0^1 - \left[\frac{1}{2}\ln(1+x^2)\right]_0^1$$

$$= \ln 2 + \arctan 1 - \frac{1}{2}\ln 2$$

$$= \frac{\pi}{4} + \frac{1}{2}\ln 2$$

$$= \frac{1}{4}(\pi + 2\ln 2)$$

12 a $\dfrac{x^4+1}{x\left(x^2+2\right)^2} = \dfrac{A}{x} + \dfrac{Bx+C}{x^2+2} + \dfrac{Dx+E}{\left(x^2+2\right)^2}$

$$x^4+1 \equiv \begin{pmatrix} A\left(x^2+2\right)^2 + \\ (Bx+C)x\left(x^2+2\right) + \\ (Dx+E)x \end{pmatrix}$$

Set $x = 0$, so $A = \dfrac{1}{4}$

$$A + B = 1 \Rightarrow B = \frac{3}{4}$$

$$4A + 2B + D = 0 \Rightarrow D = -\frac{5}{2}$$

$$C = 0$$

$$2C + E = 0 \Rightarrow E = 0$$

Therefore,

$$\frac{x^4+1}{x\left(x^2+2\right)^2} = \frac{1}{4x} + \frac{3x}{4\left(x^2+2\right)} - \frac{5x}{2\left(x^2+2\right)^2}$$

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12 b $\int \frac{x^4+1}{x\left(x^2+2\right)^2}\,\mathrm{d}x$

$$=\int \frac{1}{4x}+\frac{3x}{4\left(x^2+2\right)}-\frac{5x}{2\left(x^2+2\right)^2}\,\mathrm{d}x$$

$$=\frac{1}{4}\int \frac{1}{x}\,\mathrm{d}x+\frac{3}{4}\int \frac{x}{x^2+2}\,\mathrm{d}x-\frac{5}{2}\int \frac{x}{\left(x^2+2\right)^2}\,\mathrm{d}x$$

Consider $\int \frac{x}{\left(x^2+2\right)^2}\,\mathrm{d}x$

Let $u=x^2+2$ and $du=2x\mathrm{d}x$

$$\int \frac{x}{\left(x^2+2\right)^2}\,\mathrm{d}x=\frac{1}{2}\int \frac{1}{u^2}\,du$$

$$=-\frac{1}{2u}+c=-\frac{1}{2\left(x^2+2\right)}+c_1$$

So,

$$\frac{1}{4}\int \frac{1}{x}\,\mathrm{d}x+\frac{3}{4}\int \frac{x}{x^2+2}\,\mathrm{d}x-\frac{5}{2}\int \frac{x}{\left(x^2+2\right)^2}\,\mathrm{d}x$$

$$=\frac{1}{4}\ln|x|+\frac{3}{8}\ln\left|x^2+2\right|+\frac{5}{4\left(x^2+2\right)}+c$$

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Challenge

a $\int \frac{1}{x^2-8x+8}\,\mathrm{d}x$

$= \int \frac{1}{(x-4)^2-8}\,\mathrm{d}x$

$= \int \frac{1}{(x-4)^2-(2\sqrt{2})^2}\,\mathrm{d}x$

$= \int \frac{1}{(x-4+2\sqrt{2})(x-4-2\sqrt{2})}\,\mathrm{d}x$

$= \int \frac{A}{x-4+2\sqrt{2}} + \frac{B}{x-4-2\sqrt{2}}\,\mathrm{d}x$

$1 \equiv A(x-4-2\sqrt{2}) + B(x-4+2\sqrt{2})$

Set $x = 4-2\sqrt{2}$, so $A = -\frac{1}{4\sqrt{2}}$

$A + B = 0 \Rightarrow B = \frac{1}{4\sqrt{2}}$

So,

$\int \frac{1}{x^2-8x+8}\,\mathrm{d}x$

$= \frac{1}{4\sqrt{2}}\left(\begin{array}{l} -\int \frac{1}{x-4+2\sqrt{2}}\,\mathrm{d}x + \\ \int \frac{1}{x-4-2\sqrt{2}}\,\mathrm{d}x \end{array}\right)$

$= \frac{1}{4\sqrt{2}}\left(\begin{array}{l} -\ln|x-4+2\sqrt{2}| + \\ \ln|x-4+2\sqrt{2}| \end{array}\right) + c$

$= \frac{1}{4\sqrt{2}} \ln\left|\frac{x-4-2\sqrt{2}}{x-4+2\sqrt{2}}\right| + c$

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Challenge

b $\int \frac{1}{2x^2+4x+11}\,\mathrm{d}x$

$$= \frac{1}{2}\int \frac{1}{(x+1)^2+\frac{9}{2}}\,\mathrm{d}x$$

$$= \frac{1}{2}\int \frac{1}{(x+1)^2+\left(\frac{3}{\sqrt{2}}\right)^2}\,\mathrm{d}x$$

(Let $u = x+1$, so $du = \mathrm{d}x$)

$$= \frac{1}{2}\int \frac{1}{u^2+\left(\frac{3}{\sqrt{2}}\right)^2}\,du$$

$$= \frac{1}{3\sqrt{2}}\arctan\left(\frac{u}{\frac{3}{\sqrt{2}}}\right)+c$$

$$= \frac{1}{3\sqrt{2}}\arctan\left(\frac{\sqrt{2}(x+1)}{3}\right)+c$$

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Methods in calculus – mixed exercise 3

1 a $u = e^x$, so $du = e^x dx$

$$\int \frac{1}{e^x + e^{-x}} dx = \int \frac{1}{e^x + \frac{1}{e^x}} dx$$

$$= \int \frac{e^x}{e^{2x} + 1} dx$$

$$= \int \frac{1}{u^2 + 1} du$$

$$= \arctan u + c$$

$$= \arctan e^x + c$$

b $$\int_{-\infty}^{\infty} \frac{1}{e^x + e^{-x}} dx$$

$$= \int_{-\infty}^{0} \frac{1}{e^x + e^{-x}} dx + \int_{0}^{\infty} \frac{1}{e^x + e^{-x}} dx$$

Consider $\int_{-\infty}^{0} \frac{1}{e^x + e^{-x}} dx$

$$= \lim_{t \to -\infty} \int_{t}^{0} \frac{1}{e^x + e^{-x}} dx$$

$$= \lim_{t \to -\infty} \left[\arctan e^x \right]_t^0$$

$$= \lim_{t \to -\infty} \left(\frac{\pi}{4} - \arctan e^t \right)$$

$\arctan e^t \to 0$ as $t \to -\infty$, so

$$\int_{-\infty}^{0} \frac{1}{e^x + e^{-x}} dx = \frac{\pi}{4}$$

Similarly, consider $\int_{0}^{\infty} \frac{1}{e^x + e^{-x}} dx$

$$= \lim_{t \to \infty} \int_{0}^{t} \frac{1}{e^x + e^{-x}} dx$$

$$= \lim_{t \to \infty} \left[\arctan e^x \right]_0^t$$

$$= \lim_{t \to \infty} \left(\arctan e^t - \frac{\pi}{4} \right)$$

$\arctan e^t \to \frac{\pi}{2}$ as $t \to \infty$, so

$$\int_{0}^{\infty} \frac{1}{e^x + e^{-x}} dx = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}$$

1 b So, $\int_{-\infty}^{\infty} \frac{1}{e^x + e^{-x}} dx$

$$= \int_{-\infty}^{0} \frac{1}{e^x + e^{-x}} dx + \int_{0}^{\infty} \frac{1}{e^x + e^{-x}} dx$$

$$= \frac{\pi}{4} + \frac{\pi}{4} = \frac{\pi}{2}$$

2 $$f(x) = \frac{1 - \cos x}{\sin^2 x} = \frac{1}{\sin^2 x} - \frac{\cos x}{\sin^2 x}$$

$$\int f(x) dx = \int \frac{1}{\sin^2 x} - \frac{\cos x}{\sin^2 x} dx$$

Consider $\int \frac{1}{\sin^2 x} dx$

$$= \int \csc^2 x dx$$

$$= -\cot x + c_1$$

Similarly, consider $\int \frac{\cos x}{\sin^2 x} dx$

Let $u = \sin x$, so $du = \cos x dx$

$$\int \frac{\cos x}{\sin^2 x} dx = \int \frac{1}{u^2} du$$

$$= -\frac{1}{u} + c_2$$

$$= -\frac{1}{\sin x} + c_2$$

$$= -\csc x + c_2$$

Therefore,

$$\int f(x) dx = \int \frac{1}{\sin^2 x} - \frac{\cos x}{\sin^2 x} dx$$

$$= -\cot x + \csc x + c$$

The mean value of $f(x)$ over the interval $\left[\frac{\pi}{6}, \frac{\pi}{3} \right]$ is:

$$\frac{1}{\frac{\pi}{3} - \frac{\pi}{6}} \int_{\frac{\pi}{3}}^{\frac{\pi}{6}} f(x) dx$$

$$= \frac{6}{\pi} \left[-\cot x + \csc x \right]_{\frac{\pi}{3}}^{\frac{\pi}{6}}$$

$$= \frac{6}{\pi} \left(-\frac{1}{\sqrt{3}} + \frac{2}{\sqrt{3}} + \sqrt{3} - 2 \right)$$

$$= \frac{6}{\pi} \left(\frac{4}{\sqrt{3}} - 2 \right)$$

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3 $f(x) = x\sin 2x$

Consider $\int x\sin 2x\,dx$

Integrating by parts,

$$= -\frac{x\cos 2x}{2} + \int \frac{\cos 2x}{2}\,dx$$

$$= -\frac{x\cos 2x}{2} + \frac{\sin 2x}{4} + c$$

The mean value of $f(x)$ over the interval $\left[0, \frac{\pi}{2}\right]$ is:

$$\frac{2}{\pi}\int_0^{\frac{\pi}{2}} f(x)\,dx$$

$$= \frac{2}{\pi}\left[-\frac{x\cos 2x}{2} + \frac{\sin 2x}{4}\right]_0^{\frac{\pi}{2}}$$

$$= \frac{2}{\pi}\left(\frac{\pi}{4}\right) = \frac{1}{2}$$

4 a $y = \arccos x^2$

Let $t = x^2$, so $\frac{dt}{dx} = 2x$

So, $y = \arccos t$

$\cos y = t$

$$-\sin y\frac{dy}{dt} = 1$$

$$\frac{dy}{dt} = -\frac{1}{\sin y}$$

$$= -\frac{1}{\sqrt{1-\cos^2 y}}$$

$$= -\frac{1}{\sqrt{1-x^4}}$$

Using chain rule,

$$\frac{dy}{dx} = \frac{dy}{dt}\times\frac{dt}{dx} = -\frac{2x}{\sqrt{1-x^4}}$$

b $\int \frac{3x}{\sqrt{16-x^4}}\,dx$

$$= \frac{1}{4}\int \frac{3x}{\sqrt{1-\left(\frac{x}{2}\right)^4}}\,dx$$

4 b (Let $u = \frac{x}{2}$, so $du = \frac{1}{2}dx$)

$$= \frac{3}{2}\int \frac{2u}{\sqrt{1-u^4}}\,du$$

$$= -\frac{3}{2}\arccos u^2 + c$$

$$= -\frac{3}{2}\arccos\left(\frac{x^2}{4}\right) + c$$

5 a $y = f(x) = \arctan\left(\frac{2x+3}{x-1}\right)$

Let $t = \frac{2x+3}{x-1}$

$$\frac{dt}{dx} = \frac{(x-1)2-(2x+3)}{(x-1)^2}$$

$$= -\frac{5}{(x-1)^2}$$

Also, $y = \arctan t$

$$\frac{dy}{dt} = \frac{1}{1+t^2}$$

$$= \frac{1}{1+\left(\frac{2x+3}{x-1}\right)^2}$$

$$= \frac{(x-1)^2}{(x-1)^2+(2x+3)^2}$$

$$= \frac{(x-1)^2}{5(x^2+2x+2)}$$

Using chain rule,

$$\frac{dy}{dx} = \frac{dy}{dt}\times\frac{dt}{dx}$$

$$= \frac{(x-1)^2}{5(x^2+2x+2)}\times -\frac{5}{(x-1)^2}$$

$$= -\frac{1}{x^2+2x+2}$$

b $x^2+2x+2 = (x+1)^2+1$

So $(x+1)^2+1 \geq 1$

Hence,

$$|f'(x)| = \left|-\frac{1}{x^2+2x+2}\right| \leqslant 1$$

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6 a We say an integral $\int_a^b f(x)\,dx$ is improper if one or both of the limits are infinite or if $f(x)$ is undefined at some point in the domain $[a,b]$.

b $\int_0^\infty \frac{1}{(x+1)\sqrt{x}}\,dx$

The given interval is improper because the upper limit is infinite and also because the integrand is undefined at the lower limit $x=0$.

c $y=\arctan\sqrt{x}$

Let $t=\sqrt{x}$, so $\frac{dt}{dx}=\frac{1}{2\sqrt{x}}$

So, $y=\arctan t$

$\tan y=t$

$\sec^2 y\frac{dy}{dt}=1$

$\frac{dy}{dt}=\frac{1}{\sec^2 y}$

$=\frac{1}{1+\tan^2 y}$

$=\frac{1}{1+x}$

Using chain rule,

$\frac{dy}{dx}=\frac{dy}{dt}\times\frac{dt}{dx}=\frac{1}{2(x+1)\sqrt{x}}$

$\int_0^\infty \frac{1}{(x+1)\sqrt{x}}=\left(\int_0^1 \frac{1}{(x+1)\sqrt{x}}\,dx+\int_1^\infty \frac{1}{(x+1)\sqrt{x}}\,dx\right)$

Consider $\int_0^1 \frac{1}{(x+1)\sqrt{x}}\,dx$

$=\lim_{t\to 0}\int_t^1 \frac{1}{(x+1)\sqrt{x}}\,dx$

$=\lim_{t\to 0}\left[2\arctan\sqrt{x}\right]_t^1$

$=\lim_{t\to 0}\left(\frac{\pi}{2}-2\arctan\sqrt{t}\right)=\frac{\pi}{2}$

6 c $\arctan\sqrt{t}\to 0$ as $t\to 0$, so the integral converges.

Consider $\int_1^\infty \frac{1}{(x+1)\sqrt{x}}\,dx$

$=\lim_{t\to\infty}\int_1^t \frac{1}{(x+1)\sqrt{x}}\,dx$

$=\lim_{t\to\infty}\left[2\arctan\sqrt{x}\right]_1^t$

$=\lim_{t\to 0}\left(2\arctan\sqrt{t}-\frac{\pi}{2}\right)=\frac{\pi}{2}$

$\arctan\sqrt{t}\to\frac{\pi}{2}$ as $t\to\infty$, so the integral converges.

Since both integrals converge,

$\int_0^\infty \frac{1}{(x+1)\sqrt{x}}$ converges, and

$\int_0^\infty \frac{1}{(x+1)\sqrt{x}}=\frac{\pi}{2}+\frac{\pi}{2}=\pi$

7 $f(x)=\frac{1+5x}{\sqrt{1-5x^2}}$

$\int f(x)\,dx=\int\frac{1+5x}{\sqrt{1-5x^2}}\,dx$

$=\int\frac{1}{\sqrt{1-5x^2}}\,dx+\int\frac{5x}{\sqrt{1-5x^2}}\,dx$

Consider $\int\frac{1}{\sqrt{1-5x^2}}\,dx$

$=\frac{1}{\sqrt{5}}\int\frac{1}{\sqrt{\left(\frac{1}{\sqrt{5}}\right)^2-x^2}}\,dx$

$=\frac{1}{\sqrt{5}}\arcsin\left(\frac{x}{\frac{1}{\sqrt{5}}}\right)+c_1$

$=\frac{1}{\sqrt{5}}\arcsin\left(\sqrt{5}x\right)+c_1$

Consider $\int\frac{5x}{\sqrt{1-5x^2}}\,dx$

Let $u=1-5x^2$ and $du=-10x\,dx$

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7 $\int \frac{5x}{\sqrt{1-5x^2}}\,\mathrm{d}x = -\frac{1}{2}\int \frac{1}{\sqrt{u}}\,du$

$= -\sqrt{u} + c_2$

$= -\sqrt{1-5x^2} + c_2$

Therefore,

$\int \mathrm{f}(x)\,\mathrm{d}x$

$= \int \frac{1}{\sqrt{1-5x^2}}\,\mathrm{d}x + \int \frac{5x}{\sqrt{1-5x^2}}\,\mathrm{d}x$

$= \frac{1}{\sqrt{5}}\arcsin\left(\sqrt{5}x\right) - \sqrt{1-5x^2} + c$

Therefore, $A = -1$ and $B = \frac{1}{\sqrt{5}}$

8 a Consider $\int \frac{1}{x^2+1}\,\mathrm{d}x$

Let $x = \tan\theta$, $\mathrm{d}x = \sec^2\theta\,\mathrm{d}\theta$

$1 + x^2 = 1 + \tan^2\theta = \sec^2\theta$

So,

$\int \frac{1}{x^2+1}\,\mathrm{d}x$

$= \int \frac{\sec^2\theta}{\sec^2\theta}\,d\theta$

$= \theta + c$

$= \arctan x + c$

Therefore,

$\int_0^t \frac{1}{1+x^2}\,\mathrm{d}x$

$= \left[\arctan x\right]_0^t$

$= \arctan t$

b i $\int_0^\infty \frac{1}{1+x^2}\,\mathrm{d}x$

$= \lim_{t\to\infty}\int_0^t \frac{1}{1+x^2}\,\mathrm{d}x$

$= \lim_{t\to\infty}\left[\arctan x\right]_0^t$

$= \lim_{t\to\infty}\arctan t$

$\arctan t \to \frac{\pi}{2}$ as $t \to \infty$, so

$\int_0^\infty \frac{1}{1+x^2}\,\mathrm{d}x = \frac{\pi}{2}$

8 b ii $\int_{-\infty}^\infty \frac{1}{1+x^2}\,\mathrm{d}x$

$= \int_{-\infty}^0 \frac{1}{1+x^2}\,\mathrm{d}x + \int_0^\infty \frac{1}{1+x^2}\,\mathrm{d}x$

Consider $\int_{-\infty}^0 \frac{1}{1+x^2}\,\mathrm{d}x$

$= \lim_{t\to-\infty}\int_t^0 \frac{1}{1+x^2}\,\mathrm{d}x$

$= \lim_{t\to-\infty}\left[\arctan x\right]_t^0$

$= \lim_{t\to-\infty}\left(-\arctan t\right)$

$\arctan t \to -\frac{\pi}{2}$ as $t \to -\infty$, so

$\int_{-\infty}^0 \frac{1}{1+x^2}\,\mathrm{d}x = \frac{\pi}{2}$

Therefore,

$\int_{-\infty}^\infty \frac{1}{1+x^2}\,\mathrm{d}x = \frac{\pi}{2} + \frac{\pi}{2} = \pi$

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9 a $\mathrm{f}(x)=\dfrac{1+2x}{1+4x^2}$

$\int \mathrm{f}(x)\,\mathrm{d}x=\int \dfrac{1+2x}{1+4x^2}\,\mathrm{d}x$

$=\int \dfrac{1}{1+4x^2}\,\mathrm{d}x+\int \dfrac{2x}{1+4x^2}\,\mathrm{d}x$

Consider $\int \dfrac{1}{1+4x^2}\,\mathrm{d}x$

$=\dfrac{1}{4}\int \dfrac{1}{\left(\frac{1}{2}\right)^2+x^2}\,\mathrm{d}x$

$=\dfrac{1}{4}\left(2\arctan\left(\dfrac{x}{\frac{1}{2}}\right)\right)+c_1$

$=\dfrac{1}{2}\arctan 2x+c_1$

Similarly, consider $\int \dfrac{2x}{1+4x^2}\,\mathrm{d}x$

Let $u=1+4x^2$ and $du=8x\mathrm{d}x$

$\int \dfrac{2x}{1+4x^2}\,\mathrm{d}x=\dfrac{1}{4}\int \dfrac{1}{u}\,du$

$=\dfrac{1}{4}\ln u+c_2$

$=\dfrac{1}{4}\ln\left(1+4x^2\right)+c_2$

Therefore,

$\int \mathrm{f}(x)\,\mathrm{d}x$

$=\int \dfrac{1}{1+4x^2}\,\mathrm{d}x+\int \dfrac{2x}{1+4x^2}\,\mathrm{d}x$

$=\dfrac{1}{2}\arctan 2x+\dfrac{1}{4}\ln\left(1+4x^2\right)+c$

Therefore $A=\dfrac{1}{4}$ and $B=\dfrac{1}{2}$

b $\displaystyle\int_0^{0.5} f(x)\,\mathrm{d}x$

$=\left[\dfrac{1}{2}\arctan 2x+\dfrac{1}{4}\ln\left(1+4x^2\right)\right]_0^{0.5}$

$=\dfrac{\pi}{8}+\dfrac{1}{4}\ln 2=\dfrac{1}{8}(\pi+2\ln 2)$

10 a $\displaystyle\int \frac{1}{\sqrt{4-9x^2}}\,\mathrm{d}x$

$=\dfrac{1}{\sqrt{9}}\displaystyle\int \frac{1}{\sqrt{\left(\sqrt{\frac{9}{4}}\right)^2-x^2}}\,\mathrm{d}x$

$=\dfrac{1}{3}\arcsin\left(\dfrac{x}{\frac{2}{3}}\right)+c$

$=\dfrac{1}{3}\arcsin\left(\dfrac{3}{2}x\right)+c$

Hence $P=\dfrac{1}{3}$ and $Q=\dfrac{3}{2}$

b $\displaystyle\int_0^{\frac{2}{3}} \frac{1}{\sqrt{4-9x^2}}\,\mathrm{d}x$

$=\lim_{t\to\frac{2}{3}}\displaystyle\int_0^{t} \frac{1}{\sqrt{4-9x^2}}\,\mathrm{d}x$

$=\lim_{t\to\frac{2}{3}}\left[\dfrac{1}{3}\arcsin\left(\dfrac{3}{2}x\right)\right]_0^{t}$

$=\lim_{t\to\frac{2}{3}}\dfrac{1}{3}\arcsin\left(\dfrac{3}{2}t\right)=\dfrac{\pi}{6}$

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11 Consider $\int_0^{\frac{1}{2}} \frac{x^4}{\sqrt{1-x^2}}\,dx$

$x = \sin\theta,\ dx = \cos\theta d\theta$

$1 - x^2 = 1 - \sin^2\theta = \cos^2\theta$

So,

$$\int_0^{\frac{1}{2}} \frac{x^4}{\sqrt{1-x^2}}\,dx$$

$$= \int_0^{\frac{\pi}{6}} \sin^4\theta\, d\theta$$

Expanding $\sin^4\theta$ using de Moivre's theorem,

$$= \int_0^{\frac{\pi}{6}} \frac{\cos 4\theta - 4\cos 2\theta + 3}{8}\,d\theta$$

$$= \frac{1}{8}\left[\frac{1}{4}\sin 4\theta - 2\sin 2\theta + 3\theta\right]_0^{\frac{\pi}{6}}$$

$$= \frac{1}{8}\left(\frac{\sqrt{3}}{8} - \sqrt{3} + \frac{\pi}{2}\right)$$

$$= \frac{1}{64}\left(4\pi - 7\sqrt{3}\right)$$

12 a $f(x) = \frac{x}{1+x^4}$

$$\int_0^1 f(x)\,dx = \int_0^1 \frac{x}{1+x^4}\,dx$$

$u = x^2$ and $du = 2x dx$

So,

$$\int_0^1 \frac{x}{1+x^4}\,dx = \frac{1}{2}\int_0^1 \frac{1}{1+u^2}\,du$$

$$= \frac{1}{2}\left[\arctan u\right]_0^1 = \frac{\pi}{8}$$

12 b $\int_0^\infty f(x)\,dx = \lim_{t\to\infty}\int_0^t \frac{x}{1+x^4}\,dx$

$$= \lim_{t\to\infty}\left(\frac{1}{2}\left[\arctan x^2\right]_0^t\right)$$

$$= \frac{1}{2}\lim_{t\to\infty}\arctan t^2$$

$\arctan t^2 \to \frac{\pi}{2}$ as $t \to \infty$, so the integral converges, and

$$\int_0^\infty f(x)\,dx = \frac{\pi}{4}$$

13 $\int \frac{2x^3 - 2x^2 + 18x + 9}{x^4 + 9x^2}$

$$= \int \frac{2x^3 - 2x^2 + 18x + 9}{x^2\left(x^2+9\right)}\,dx$$

$$= \int \frac{Ax+B}{x^2} + \frac{Cx+D}{x^2+9}\,dx$$

$2x^3 - 2x^2 + 18x + 9 \equiv$

$(Ax+B)(x^2+9) + (Cx+D)(x^2)$

Set $x = 0$, so $B = 1$

$B + D = -2 \Rightarrow D = -3$

$9A = 18 \Rightarrow A = 2$

$A + C = 2 \Rightarrow C = 0$

Therefore,

$$\int \frac{2x^3 - 2x^2 + 18x + 9}{x^4 + 9x^2}$$

$$= \int \frac{2}{x}\,dx + \int \frac{1}{x^2}\,dx - 3\int \frac{1}{x^2+9}\,dx$$

$$= 2\ln|x| - \frac{1}{x} - \arctan\left(\frac{x}{3}\right) + c$$

14 a $f(x) = \frac{x^2 - 3x + 14}{x^3 - 4x^2 + 2x - 8}$

$$= \frac{P}{x-4} + \frac{Q}{x^2+2}$$

$x^2 - 3x + 14 \equiv P(x^2+2) + Q(x-4)$

Set $x = 4$, so $P = 1$

$2P - 4Q = 14 \Rightarrow Q = -3$

Therefore,

$$\frac{x^2 - 3x + 14}{x^3 - 4x^2 + 2x - 8} = \frac{1}{x-4} - \frac{3}{x^2+2}$$

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14 b $\int f(x)\,dx$

$= \int \frac{1}{x-4}\,dx - 3\int \frac{1}{x^2+2}\,dx$

$= \ln|x-4| - \frac{3}{\sqrt{2}}\arctan\left(\frac{x}{\sqrt{2}}\right) + c$

Therefore, $A = 1$ and $B = -\frac{3}{\sqrt{2}}$

c $\int_4^\infty f(x)\,dx = \int_4^5 f(x)\,dx + \int_5^\infty f(x)\,dx$

Consider $\int_4^5 f(x)\,dx$

$= \lim_{t\to 4}\int_t^5 \frac{1}{x-4} - \frac{3}{x^2+2}\,dx$

$= \lim_{t\to 4}\left[\ln|x-4| - \frac{3}{\sqrt{2}}\arctan\left(\frac{x}{\sqrt{2}}\right)\right]_t^5$

$= \lim_{t\to 4}\left(\begin{array}{l} -\frac{3}{\sqrt{2}}\arctan\left(\frac{5}{\sqrt{2}}\right) - \\ \ln|x-4| + \frac{3}{\sqrt{2}}\arctan\left(\frac{x}{\sqrt{2}}\right)\end{array}\right)$

$\ln|t-4| \to -\infty$ as $t \to 4$, so the integral diverges.

15 a $\int \frac{2}{x^3+x}\,dx$

$= \int \frac{2}{x(x^2+1)}\,dx$

$= \int \frac{A}{x} + \frac{Bx+C}{x^2+1}\,dx$

$2 \equiv A(x^2+1) + (Bx+C)(x)$

Set $x = 0$, so $A = 2$

$A + B = 0 \Rightarrow B = -2$

$C = 0$

Therefore,

$\int \frac{2}{x^3+x}\,dx$

$= 2\int \frac{1}{x}\,dx - 2\int \frac{x}{x^2+1}\,dx$

$= 2\ln|x| - \ln|x^2+1| + c$

$= \ln\left|\frac{x^2}{x^2+1}\right| + c$

15 b The mean value of $f(x)$ over the interval $[1,2]$ is:

$\frac{1}{2-1}\int_1^2 f(x)\,dx$

$= \left[\ln\left|\frac{x^2}{x^2+1}\right|\right]_1^2$

$= \ln\left(\frac{4}{5}\right) - \ln\left(\frac{1}{2}\right) = \ln\left(\frac{8}{5}\right).$

c The mean value of $-\frac{6}{x}$ over the interval $[1,2]$ is:

$\frac{1}{2-1}\int_1^2 -\frac{6}{x}\,dx.$

$= [-6\ln x]_1^2$

$= -6\ln 2$

$= -\ln 64.$

The mean value of $2f(x) - \frac{6}{x}$ over the interval $[1,2]$ is:

$2\ln\left(\frac{8}{5}\right) - \ln 64$

$= 2\ln 8 - 2\ln 5 - 2\ln 8$

$= -2\ln 5$

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Challenge

a $f(x) = x^3 - 2x + 4$

The mean value of $f(x)$ over the interval $[0, 2]$ is:

$$\frac{1}{2-0}\int_1^2 x^3 - 2x + 4 \mathrm{d}x$$

$$= \frac{1}{2}\left[\frac{x^4}{4} - x^2 + 4x\right]_0^2$$

$$= 4$$

The mean value $f(c) = 4$

$$c^3 - 2c + 4 = 4$$

$$c\left(c^2 - 2\right) = 0$$

$$c = 0, \pm\sqrt{2}$$

Since c has to lie in the domain $[0, 2]$,

$c = 0$ or $c = \sqrt{2}$.

b One example is $f(x) = 0$ for $x \leqslant 1$ and $f(x) = 1$ for $x > 1$. This has mean value $\frac{1}{2}$ lie on the domain $[0, 2]$, but the function only attains values 0 and 1.

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Volumes of revolution 4A

1 a Volume $= \pi\int_0^2 \left(\frac{2}{x+1}\right)^2 \mathrm{d}x$

$= \pi\int_0^2 \frac{4}{(x+1)^2}\,\mathrm{d}x$

$= \pi\left[-\frac{4}{x+1}\right]_0^2$

$= \pi\left(\left(-\frac{4}{3}\right)-(-4)\right)$

$= \frac{8\pi}{3}$

b Volume $= \pi\int_0^{\frac{\pi}{2}} \left(\frac{4\sin x}{1+\cos x}\right)\mathrm{d}x$

Let $u = 1+\cos x \Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x} = -\sin x$

Volume $= \pi\int_2^1 \left(-\frac{4}{u}\right)\mathrm{d}u$

$= \pi\int_1^2 \frac{4}{u}\mathrm{d}u$

$= \pi\left[4\ln u\right]_1^2$

$= 4\pi\ln 2$

c Volume $= \pi\int_0^{\frac{\pi}{4}} \left(\sqrt{x}\sec x\right)^2 \mathrm{d}x$

$= \pi\int_0^{\frac{\pi}{4}} x\sec^2 x\,\mathrm{d}x$

Let $u = x$ and $\frac{\mathrm{d}v}{\mathrm{d}x} = \sec^2 x$

So $\frac{\mathrm{d}u}{\mathrm{d}x} = 1$ and $v = \tan x$

Integrating by parts

Volume $= \pi\left(\left[x\tan x\right]_0^{\frac{\pi}{4}} - \int_0^{\frac{\pi}{4}} \tan x\,\mathrm{d}x\right)$

$= \pi\left(\left(\frac{\pi}{4}-0\right)-\left[\ln(\sec x)\right]_0^{\frac{\pi}{4}}\right)$

$= \pi\left(\frac{\pi}{4}-\left(\ln\sqrt{2}-\ln 1\right)\right)$

$= \pi\left(\frac{\pi}{4}-\ln\sqrt{2}\right)$

$= \frac{\pi}{4}(\pi-\ln 4)$

1 d Volume $= \pi\int_0^2 \frac{5x}{10x^2+1}\mathrm{d}x$

$= \pi\left[\frac{1}{4}\ln\left(10x^2+1\right)\right]_0^2$

$= \frac{\pi}{4}(\ln 41-\ln 1)$

$= \frac{\pi}{4}\ln 41$

e Volume $= \pi\int_1^2 \frac{\ln x}{x^2}\,\mathrm{d}x$

Let $u = \ln x$ and $\frac{\mathrm{d}v}{\mathrm{d}x} = \frac{1}{x^2}$

So $\frac{\mathrm{d}u}{\mathrm{d}x} = \frac{1}{x}$ and $v = -\frac{1}{x}$

Integrating by parts

Volume $= \pi\left(\left[-\frac{\ln x}{x}\right]_1^2 + \int_1^2 \frac{1}{x^2}\,\mathrm{d}x\right)$

$= \pi\left(-\frac{\ln 2}{2}+\left[-\frac{1}{x}\right]_1^2\right)$

$= \pi\left(-\frac{\ln 2}{2}+\left(-\frac{1}{2}+1\right)\right)$

$= \frac{\pi}{2}(1-\ln 2)$

f Volume $= \pi\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} (\operatorname{cosec} x+\cot x)^2\mathrm{d}x$

$= \pi\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \left(\operatorname{cosec}^2 x+2\operatorname{cosec} x\cot x+\cot^2 x\right)\mathrm{d}x$

$= \pi\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \left(2\operatorname{cosec}^2 x+2\operatorname{cosec} x\cot x-1\right)\mathrm{d}x$

$= \pi\left[-2\cot x-2\operatorname{cosec} x-x\right]_{\frac{\pi}{3}}^{\frac{\pi}{2}}$

$= \pi\left(\left(-2-\frac{\pi}{2}\right)-\left(-\frac{2}{\sqrt{3}}-\frac{4}{\sqrt{3}}-\frac{\pi}{3}\right)\right)$

$= \pi\left(2\sqrt{3}-2-\frac{\pi}{6}\right)$

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2 Volume $= \pi\int_0^{\frac{\pi}{2}} \cos^2 x \sin 2x \, dx$

$= \pi\int_0^{\frac{\pi}{2}} 2\cos^3 x \sin x \, dx$

$= \pi\left[-\frac{1}{2}\cos^4 x\right]_0^{\frac{\pi}{2}}$

$= \pi\left(0-\left(-\frac{1}{2}\right)\right)$

$= \frac{\pi}{2}$

3 $y = \ln x$

When $y = 0$, $\ln a = 0 \Rightarrow a = 1$

Volume $= \pi\int_1^3 (\ln x)^2 \, dx$

Let $u = (\ln x)^2$ and $\frac{dv}{dx} = 1$

So $\frac{du}{dx} = \frac{2}{x}\ln x$ and $v = x$

Integrating by parts

Volume $= \pi\left(\left[x(\ln x)^2\right]_1^3 - 2\int_1^3 \ln x \, dx\right)$

$= \pi\left(3(\ln 3)^2 - 2\int_1^3 \ln x \, dx\right)$

Now let $u = \ln x$ and $\frac{dv}{dx} = 1$

So $\frac{du}{dx} = \frac{1}{x}$ and $v = x$

Integrating by parts again, Volume

$= \pi\left(3(\ln 3)^2 - 2[x\ln x]_1^3 + 2\int_1^3 dx\right)$

$= \pi\left(3(\ln 3)^2 - 6\ln 3 + 2[x]_1^3\right)$

$= \pi\left(3(\ln 3)^2 - 6\ln 3 + 4\right)$

4 a $I = \int_{\frac{3}{2}}^{\frac{3\sqrt{2}}{2}} \frac{1}{x^2\sqrt{9-x^2}} \, dx$

Let $x = 3\sin\theta$

$\frac{dx}{d\theta} = 3\cos\theta$

When $x = \frac{3\sqrt{2}}{2}$, $\theta = \frac{\pi}{4}$

When $x = \frac{3}{2}$, $\theta = \frac{\pi}{6}$

$I = \int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \frac{3\cos\theta}{9\sin^2\theta\sqrt{9-9\sin^2\theta}} \, d\theta$

$= \frac{1}{9}\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \frac{3\cos\theta}{3\sin^2\theta\sqrt{1-\sin^2\theta}} \, d\theta$

$= \frac{1}{9}\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \frac{1}{\sin^2\theta} \, d\theta$

$= \frac{1}{9}\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \operatorname{cosec}^2\theta \, d\theta$

$= \frac{1}{9}\left[-\cot\theta\right]_{\frac{\pi}{6}}^{\frac{\pi}{4}}$

$= \frac{1}{9}\left(-1-\left(-\sqrt{3}\right)\right)$

$= \frac{1}{9}\left(\sqrt{3}-1\right)$

b Volume $= \pi\int_{\frac{3}{2}}^{\frac{3\sqrt{2}}{2}} y^2 \, dx$

$= \pi\int_{\frac{3}{2}}^{\frac{3\sqrt{2}}{2}} \frac{81}{x^2\sqrt{9-x^2}} \, dx$

$= 81\pi \times I$ from part a above

$= 81\pi \times \frac{1}{9}\left(\sqrt{3}-1\right)$

$= 9\pi\left(\sqrt{3}-1\right)$

5 $y^2 = \frac{4x+3}{(x+2)(2x-1)}$

$\frac{4x+3}{(x+2)(2x-1)} = \frac{A}{x+2} + \frac{B}{2x-1}$

$4x+3 = A(2x-1) + B(x+2)$

$x = \frac{1}{2} \Rightarrow 5 = \frac{5B}{2} \Rightarrow B = 2$

$x = -2 \Rightarrow -5 = -5A \Rightarrow A = 1$

So $y^2 = \frac{1}{x+2} + \frac{2}{2x-1}$

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5 Volume $= \pi\int_1^4 y^2\ \mathrm{d}x$

$= \pi\int_1^4 \left(\frac{1}{x+2} + \frac{2}{2x-1}\right)\mathrm{d}x$

$= \pi\left[\ln(x+2) + \ln(2x-1)\right]_1^4$

$= \pi(\ln 6 + \ln 7 - \ln 3 - \ln 1)$

$= \pi \ln 14$

6 a $2y^2 = x\sin x + x$

$y = 0 \Rightarrow x\sin x + x = 0$

So $x(\sin x + 1) = 0$

$x = 0$ or $\sin x = -1$

$x = 0$ or $x = \frac{3\pi}{2}$

Coordinates of point A are $\left(\frac{3\pi}{2}, 0\right)$

b Volume $= \pi\int_0^{\frac{3\pi}{2}} y^2\ \mathrm{d}x$

$= \frac{\pi}{2}\int_0^{\frac{3\pi}{2}} (x\sin x + x)\ \mathrm{d}x$

$= \frac{\pi}{2}\left(\int_0^{\frac{3\pi}{2}} x\sin x\ \mathrm{d}x + \int_0^{\frac{3\pi}{2}} x\ \mathrm{d}x\right)$

Let $u = x$ and $\frac{\mathrm{d}v}{\mathrm{d}x} = \sin x$

So $\frac{\mathrm{d}u}{\mathrm{d}x} = 1$ and $v = -\cos x$

Volume

$= \frac{\pi}{2}\left(\left[-x\cos x\right]_0^{\frac{3\pi}{2}} + \int_0^{\frac{3\pi}{2}} \cos x\ \mathrm{d}x + \left[\frac{x^2}{2}\right]_0^{\frac{3\pi}{2}}\right)$

$= \frac{\pi}{2}\left(0 + \left[\sin x\right]_0^{\frac{3\pi}{2}} + \frac{9\pi^2}{8}\right)$

$= \frac{\pi}{2}\left(\frac{9\pi^2}{8} - 1\right)$

7 a Volume $= \pi\int_{-1}^{2}\left(\frac{10}{3(5+2x)}\right)^2\ \mathrm{d}x$

$= \frac{100\pi}{9}\int_{-1}^{2}\frac{1}{(5+2x)^2}\ \mathrm{d}x$

$= -\frac{100\pi}{18}\left[\frac{1}{5+2x}\right]_{-1}^{2}$

$= -\frac{100\pi}{18}\left(\frac{1}{9} - \frac{1}{3}\right)$

$= \frac{100\pi}{18} \times \frac{2}{9}$

$= \frac{100\pi}{81}$

b Volume generated by $y = \frac{20}{3(5+2x)}$

$= \frac{400\pi}{9}\int_{-1}^{2}\frac{1}{(5+2x)^2}\ \mathrm{d}x$

$= 4 \times \frac{100\pi}{81}$ (answer to part a)

Volume generated by the area between the two curves will be

$= \frac{400\pi}{81} - \frac{100\pi}{81}$

$= \frac{300\pi}{81}$

$= \frac{100\pi}{27}$

8 $y = xe^{-x}$ crosses the line $y = \frac{1}{4}x$

When $xe^{-x} = \frac{1}{4}x$

$xe^{-x} - \frac{1}{4}x = 0$

$x\left(e^{-x} - \frac{1}{4}\right) = 0$

$x = 0$ or $e^{-x} = \frac{1}{4}$

$x = 0$ or $x = \ln 4$

Volume V generated by the curve

$= \pi\int_0^{\ln 4} x^2 e^{-2x}\ \mathrm{d}x$

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8 Let $u = x^2$ and $\frac{dv}{dx} = e^{-2x}$

So $\frac{du}{dx} = 2x$ and $v = -\frac{1}{2}e^{-2x}$

Integrating by parts

$$V = \pi\left(\left[-\frac{x^2e^{-2x}}{2}\right]_0^{\ln 4} + \frac{1}{2}\int_0^{\ln 4} 2xe^{-2x}\,dx\right)$$

$$= \pi\left(-\frac{(\ln 4)^2}{32} + \int_0^{\ln 4} xe^{-2x}\,dx\right)$$

Now let $u = x$ and $\frac{dv}{dx} = e^{-2x}$

So $\frac{du}{dx} = 1$ and $v = -\frac{1}{2}e^{-2x}$

Integrating by parts again

$$V = \pi\left(-\frac{(\ln 4)^2}{32} - \left[\frac{xe^{-2x}}{2}\right]_0^{\ln 4} + \frac{1}{2}\int_0^{\ln 4} e^{-2x}\,dx\right)$$

$$= \pi\left(-\frac{(\ln 4)^2}{32} - \frac{\ln 4}{32} - \frac{1}{4}\left[e^{-2x}\right]_0^{\ln 4}\right)$$

$$= \pi\left(-\frac{(\ln 4)^2}{32} - \frac{\ln 4}{32} - \frac{1}{4}\left(\frac{1}{16} - 1\right)\right)$$

$$= \pi\left(\frac{15}{64} - \frac{(\ln 4)^2}{32} - \frac{\ln 4}{32}\right)$$

$$= 0.4115$$

Volume generated by $y = \frac{1}{4}x$ between $x = 0$ and $x = \ln 4$ is the volume of a cone of height $\ln 4$ and radius $\frac{1}{4}\ln 4$

$$= \frac{1}{3}\pi\left(\frac{1}{4}\ln 4\right)^2 \times \ln 4$$

$$= \frac{\pi}{48}(\ln 4)^3$$

$$= 0.1744$$

Volume generated by

$R = 0.4115 - 0.1744 = 0.237$ (to 3 s.f.)

Challenge

The x coordinates of the points of intersection between the curve and the line are given by

$$\sin x = \frac{1}{\sqrt{2}}$$

$$x = \frac{\pi}{4} \text{ and } \frac{3\pi}{4}$$

Volume generated when R is rotated around $y = \frac{1}{\sqrt{2}}$ will be

$$= \pi\int_{\frac{\pi}{4}}^{\frac{3\pi}{4}}\left(\sin x - \frac{1}{\sqrt{2}}\right)^2 dx$$

$$= \pi\int_{\frac{\pi}{4}}^{\frac{3\pi}{4}}\left(\sin^2 x - \frac{2}{\sqrt{2}}\sin x + \frac{1}{2}\right)dx$$

$$= \pi\int_{\frac{\pi}{4}}^{\frac{3\pi}{4}}\left(\frac{1}{2}(1 - \cos 2x) - \frac{2}{\sqrt{2}}\sin x + \frac{1}{2}\right)dx$$

$$= \pi\int_{\frac{\pi}{4}}^{\frac{3\pi}{4}}\left(-\frac{1}{2}\cos 2x - \frac{2}{\sqrt{2}}\sin x + 1\right)dx$$

$$= \pi\left[-\frac{1}{4}\sin 2x + \sqrt{2}\cos x + x\right]_{\frac{\pi}{4}}^{\frac{3\pi}{4}}$$

$$= \pi\left(\left(\frac{1}{4} + \sqrt{2}\left(-\frac{1}{\sqrt{2}}\right) + \frac{3\pi}{4}\right) - \left(-\frac{1}{4} + \sqrt{2}\left(\frac{1}{\sqrt{2}}\right) + \frac{\pi}{4}\right)\right)$$

$$= \pi\left(\frac{\pi}{2} - \frac{3}{2}\right)$$

$$= \frac{\pi(\pi - 3)}{2}$$

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Volumes of revolution 4B

1 a Volume $= \pi\int_0^1 x^2 \, dy$

$= \pi\int_0^1 \left(e^{2y} - e^{-y}\right)^2 \, dy$

$= \pi\int_0^1 \left(e^{4y} - 2e^{y} + e^{-2y}\right) dy$

$= \pi\left[\frac{1}{4}e^{4y} - 2e^{y} - \frac{1}{2}e^{-2y}\right]_0^1$

$= \pi\left(\left(\frac{1}{4}e^{4} - 2e - \frac{1}{2}e^{-2}\right) - \left(\frac{1}{4} - 2 - \frac{1}{2}\right)\right)$

$= \pi\left(\frac{1}{4}e^{4} - 2e - \frac{1}{2}e^{-2} + \frac{9}{4}\right)$

b Volume $= \pi\int_0^1 x^2 \, dy$

$= \pi\int_0^1 ye^{2y^2} \, dy$

$= \pi\left[\frac{1}{4}e^{2y^2}\right]_0^1$

$= \pi\left(\frac{1}{4}e^{2} - \frac{1}{4}\right)$

$= \frac{\pi}{4}\left(e^{2} - 1\right)$

c Volume $= \pi\int_1^5 x^2 \, dy$

$= \pi\int_1^5 \left(\frac{5 - \ln y}{y^2}\right) dy$

$= \pi\int_1^5 \left(\frac{5}{y^2} - \frac{\ln y}{y^2}\right) dy$

$= \pi\left(\left[-\frac{5}{y}\right]_1^5 - \int_1^5 \frac{\ln y}{y^2} \, dy\right)$

$= \pi\left(\left(-1 - (-5)\right) - \int_1^5 \frac{\ln y}{y^2} \, dy\right)$

$= \pi\left(4 - \int_1^5 \frac{\ln y}{y^2} \, dy\right)$

Let $u = \ln y$ and $\frac{dv}{dy} = \frac{1}{y^2}$

So $\frac{du}{dy} = \frac{1}{y}$ and $v = -\frac{1}{y}$

1 c Integrating by parts

Volume $= \pi\left(4 - \left(\left[-\frac{\ln y}{y}\right]_1^5 + \int_1^5 \frac{1}{y^2} \, dy\right)\right)$

$= \pi\left(4 + \frac{\ln 5}{5} - \left[-\frac{1}{y}\right]_1^5\right)$

$= \pi\left(4 + \frac{\ln 5}{5} - \left(-\frac{1}{5} + 1\right)\right)$

$= \frac{\pi}{5}\left(\ln 5 + 16\right)$

d Volume $= \pi\int_0^1 x^2 \, dy$

$= \pi\int_{e^4}^{e^9} \frac{1}{y \ln y} \, dy$

$= \pi\left[\ln\left(\ln y\right)\right]_{e^4}^{e^9}$

$= \pi\left(\ln 9 - \ln 4\right)$

$= \pi\left(\ln\frac{9}{4}\right)$

2 a $y = \frac{1}{x} - 1 \Rightarrow x = \frac{1}{y+1}$

Volume $= \pi\int_0^1 \frac{1}{(y+1)^2} \, dy$

$= \pi\left[-\frac{1}{y+1}\right]_0^1$

$= \pi\left(-\frac{1}{2} + 1\right)$

$= \frac{\pi}{2}$

b $y = \frac{5 - 2x^2}{x^2 - 1} \Rightarrow x^2 = \frac{y+5}{y+2}$

Volume $= \pi\int_{-1}^1 \frac{y+5}{y+2} \, dy$

$= \pi\int_{-1}^1 \left(1 + \frac{3}{y+2}\right) dy$

$= \pi\left[y + 3\ln\left(y+2\right)\right]_{-1}^1$

$= \pi\left(\left(1 + 3\ln 3\right) - \left(-1\right)\right)$

$= \pi\left(2 + 3\ln 3\right)$

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2 c $y = 2e^{x^2} \Rightarrow x^2 = \ln\left(\frac{y}{2}\right)$

$$\text{Volume} = \pi\int_2^4 \ln\left(\frac{y}{2}\right)\mathrm{d}y$$

$$= \pi\int_2^4 (\ln y - \ln 2)\,\mathrm{d}y$$

Let $u = \ln y$ and $\frac{\mathrm{d}v}{\mathrm{d}y} = 1$

So $\frac{\mathrm{d}u}{\mathrm{d}y} = \frac{1}{y}$ and $v = y$

Integrating by parts

$$\text{Volume} = \pi\left([y\ln y]_2^4 - \int_2^4 \mathrm{d}y - [y\ln 2]_2^4\right)$$

$$= \pi\left((4\ln 4 - 2\ln 2) - [y]_2^4 - (4\ln 2 - 2\ln 2)\right)$$

$$= \pi(6\ln 2 - 2 - 2\ln 2)$$

$$= \pi(4\ln 2 - 2)$$

d $y = \arccos\sqrt{x} \Rightarrow x^2 = \cos^4 y$

$$\text{Volume} = \pi\int_0^{\frac{\pi}{2}} \cos^4 y\,\mathrm{d}y$$

$$\cos^4 y = \left(\cos^2 y\right)^2$$

$$= \left(\frac{1}{2}(\cos 2y + 1)\right)^2$$

$$= \frac{1}{4}(\cos^2 2y + 2\cos 2y + 1)$$

$$= \frac{1}{4}\left(\frac{1}{2}(\cos 4y + 1) + 2\cos 2y + 1\right)$$

$$= \frac{1}{8}(\cos 4y + 4\cos 2y + 3)$$

$$\text{Volume} = \frac{\pi}{8}\int_0^{\frac{\pi}{2}} (\cos 4y + 4\cos 2y + 3)\,\mathrm{d}y$$

$$= \frac{\pi}{8}\left[\frac{1}{4}\sin 4y + 2\sin 2y + 3y\right]_0^{\frac{\pi}{2}}$$

$$= \frac{\pi}{8}\left(\frac{3\pi}{2}\right)$$

$$= \frac{3\pi^2}{16}$$

3 $\text{Volume} = \pi\int_1^b \frac{1}{(2y+1)^2}\,\mathrm{d}y$

$$= \pi\left[-\frac{1}{2}\left(\frac{1}{2y+1}\right)\right]_1^b$$

$$= -\frac{\pi}{2}\left(\frac{1}{2b+1} - \frac{1}{3}\right)$$

$$= \frac{\pi}{2}\left(\frac{1}{3} - \frac{1}{2b+1}\right)$$

So $\frac{\pi}{2}\left(\frac{1}{3} - \frac{1}{2b+1}\right) = \frac{\pi}{10}$

$$\frac{1}{3} - \frac{1}{2b+1} = \frac{1}{5}$$

$$\frac{1}{2b+1} = \frac{2}{15}$$

$$2b + 1 = \frac{15}{2}$$

$$2b = \frac{13}{2}$$

$$b = \frac{13}{4}$$

4 a $x = \sqrt{y}\sin y$

When $x = 0$, $\sin y = 0 \Rightarrow y = 0, \pi, 2\pi, \ldots$

So $b = \pi$

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4 b Volume $= \pi\int_0^{\pi} x^2 \, dy$

$= \pi\int_0^{\pi} y\sin^2 y \, dy$

$= \frac{\pi}{2}\int_0^{\pi} y(1-\cos 2y)\, dy$

$= \frac{\pi}{2}\left(\left[\frac{y^2}{2}\right]_0^{\pi} - \int_0^{\pi} y\cos 2y \, dy\right)$

$= \frac{\pi}{2}\left(\frac{\pi^2}{2} - \int_0^{\pi} y\cos 2y \, dy\right)$

Let $u = y$ and $\frac{dv}{dy} = \cos 2y$

So $\frac{du}{dy} = 1$ and $v = \frac{1}{2}\sin 2y$

Integrating by parts

Volume

$= \frac{\pi}{2}\left(\frac{\pi^2}{2} - \left(\left[\frac{y}{2}\sin 2y\right]_0^{\pi} - \frac{1}{2}\int_0^{\pi}\sin 2y \, dy\right)\right)$

$= \frac{\pi}{2}\left(\frac{\pi^2}{2} - \frac{1}{4}\left[\cos 2y\right]_0^{\pi}\right)$

$= \frac{\pi^3}{4}$

5 $y = 3\ln(x-1)$

$\frac{y}{3} = \ln(x-1)$

$x - 1 = e^{\frac{y}{3}}$

$x = e^{\frac{y}{3}} + 1$

Volume $= \pi\int_0^{5} x^2 \, dy$

$= \pi\int_0^{5}\left(e^{\frac{y}{3}} + 1\right)^2 dy$

$= \pi\int_0^{5}\left(e^{\frac{2y}{3}} + 2e^{\frac{y}{3}} + 1\right)dy$

$= \pi\left[\frac{3}{2}e^{\frac{2y}{3}} + 6e^{\frac{y}{3}} + y\right]_0^{5}$

$= \pi\left(\left(\frac{3}{2}e^{\frac{10}{3}} + 6e^{\frac{5}{3}} + 5\right) - \left(\frac{3}{2} + 6\right)\right)$

$= \pi\left(\frac{3}{2}e^{\frac{10}{3}} + 6e^{\frac{5}{3}} - \frac{5}{2}\right)$

6 a $\cos y + \sqrt{3}\sin y = 2\left(\frac{1}{2}\cos y + \frac{\sqrt{3}}{2}\sin y\right)$

$= 2\left(\cos y\cos\alpha + \sin y\sin\alpha\right)$ where $\alpha = \frac{\pi}{3}$

$= 2\cos\left(y - \frac{\pi}{3}\right)$

b Volume $= \pi\int_0^{\frac{\pi}{3}} x^2 \, dy$

$= \pi\int_0^{\frac{\pi}{3}}\left(\frac{1}{\cos y + \sqrt{3}\sin y}\right)^2 dy$

$= \pi\int_0^{\frac{\pi}{3}}\frac{1}{4\cos^2\left(y - \frac{\pi}{3}\right)}dy$

$= \frac{\pi}{4}\int_0^{\frac{\pi}{3}}\sec^2\left(y - \frac{\pi}{3}\right)dy$

$= \frac{\pi}{4}\left[\tan\left(y - \frac{\pi}{3}\right)\right]_0^{\frac{\pi}{3}}$

$= \frac{\pi}{4}\left(\tan 0 - \tan\left(-\frac{\pi}{3}\right)\right)$

$= \frac{\pi\sqrt{3}}{4}$

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7 a $I = \int_0^1 \frac{2^y}{\left(2^y+1\right)^2}\,\mathrm{d}y$

Let $u = 2^y \Rightarrow \frac{\mathrm{d}u}{\mathrm{d}y} = 2^y \ln 2$

$y = 1 \Rightarrow u = 2$

$y = 0 \Rightarrow u = 1$

$I = \int_1^2 \frac{u}{(u+1)^2 u \ln 2}\,\mathrm{d}u$

$= \frac{1}{\ln 2}\int_1^2 \frac{1}{(u+1)^2}\,\mathrm{d}u$

$= \frac{1}{\ln 2}\left[-\frac{1}{u+1}\right]_1^2$

$= \frac{1}{\ln 2}\left(-\frac{1}{3}+\frac{1}{2}\right)$

$= \frac{1}{6\ln 2}$

b Volume $= \pi\int_0^1 x^2\,\mathrm{d}y$

$= \pi\int_0^1 \frac{2^y}{\left(2^y+1\right)^2}\,\mathrm{d}y$

$= \pi \times \frac{1}{6\ln 2}$

$= \frac{\pi}{6\ln 2}$

8 a By De Moivre's Theorem

$\sin 3\theta = 3\cos^2\theta\sin\theta - \sin^3\theta$

$= 3\left(1-\sin^2\theta\right)\sin\theta - \sin^3\theta$

$= 3\sin\theta - 4\sin^3\theta$

$\sin^3\theta = \frac{1}{4}(3\sin\theta - \sin 3\theta)$

and

$\sin 5\theta = 5\cos^4\sin\theta - 10\cos^2\theta\sin^3\theta + \sin^5\theta$

$= 5\left(1-\sin^2\theta\right)^2\sin\theta - 10\left(1-\sin^2\theta\right)\sin^3\theta + \sin^5\theta$

$= 5\sin\theta - 20\sin^3\theta + 16\sin^5\theta$

$= 5\sin\theta - 5(3\sin\theta - \sin 3\theta) + 16\sin^5\theta$

$= -10\sin\theta + 5\sin 3\theta + 16\sin^5\theta$

$\sin^5\theta = \frac{1}{16}(10\sin\theta - 5\sin 3\theta + \sin 5\theta)$

8 b $x = \sin^2 y \sin\sqrt{y} \Rightarrow x^2 = \sin^5 y$

Volume $= \pi\int_{\frac{\pi}{4}}^{\pi} \sin^5 y\,\mathrm{d}y$

$= \frac{\pi}{16}\int_{\frac{\pi}{4}}^{\pi}(10\sin y - 5\sin 3y + \sin 5y)\,\mathrm{d}y$

$= \frac{\pi}{16}\left[-10\cos y + \frac{5}{3}\cos 3y - \frac{1}{5}\cos 5y\right]_{\frac{\pi}{4}}^{\pi}$

$= \frac{\pi}{16}\left(\left(10-\frac{5}{3}+\frac{1}{5}\right)-\left(-\frac{10}{\sqrt{2}}-\frac{5}{3\sqrt{2}}+\frac{1}{5\sqrt{2}}\right)\right)$

$= \frac{\pi}{16}\left(\frac{150-25+3}{15}+\frac{150+25-3}{15\sqrt{2}}\right)$

$= \frac{\pi}{16}\left(\frac{128}{15}+\frac{172}{15\sqrt{2}}\right)$

$= \frac{\pi}{16}\left(\frac{128}{15}+\frac{86\sqrt{2}}{15}\right)$

$= \frac{\pi}{15}\left(8+\frac{43\sqrt{2}}{8}\right)$

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Volumes of revolution 4C

1 $y = t^2 \Rightarrow y^2 = t^4$

$x = t^3 \Rightarrow \dfrac{\mathrm{d}x}{\mathrm{d}t} = 3t^2$

$x = 0 \Rightarrow t = 0$

$x = 8 \Rightarrow t = 2$

$\text{Volume} = \pi\int_0^2 y^2 \dfrac{\mathrm{d}x}{\mathrm{d}t}\,\mathrm{d}t$

$= \pi\int_0^2 3t^6\,\mathrm{d}t$

$= \pi\left[\dfrac{3t^7}{7}\right]_0^2$

$= \dfrac{384\pi}{7}$

2 a $x = e^t$

$x = e^2 \Rightarrow t = 2$

$x = e^3 \Rightarrow t = 3$

b $x = e^t \Rightarrow \dfrac{\mathrm{d}x}{\mathrm{d}t} = e^t$

$y = \sqrt{t-1} \Rightarrow y^2 = t - 1$

$\text{Volume} = \pi\int_2^3 (t-1)e^t\,\mathrm{d}t$

$= \pi\int_2^3 \left(te^t - e^t\right)\mathrm{d}t$

$= \pi\left(\int_2^3 te^t\,dt - \int_2^3 e^t\,\mathrm{d}t\right)$

$= \pi\left(\int_2^3 te^t\,dt - e^3 + e^2\right)$

Let $u = t$ and $\dfrac{\mathrm{d}v}{\mathrm{d}t} = e^t$

So $\dfrac{\mathrm{d}u}{\mathrm{d}t} = 1$ and $v = e^t$

Integrating by parts

$\text{Volume} = \pi\left(\left[te^t\right]_2^3 - \int_2^3 e^t\,\mathrm{d}t - e^3 + e^2\right)$

$= \pi\left(\left(3e^3 - 2e^2\right) - \left[e^t\right]_2^3 - e^3 + e^2\right)$

$= \pi\left(\left(3e^3 - 2e^2\right) - \left(e^3 - e^2\right) - e^3 + e^2\right)$

$= \pi e^3$

2 c $y^2 = t - 1 \Rightarrow t = y^2 + 1$

$x = e^t \Rightarrow t = \ln x$

$y^2 + 1 = \ln x$

$y^2 = \ln x - 1$

d $I = \pi\int_{e^2}^{e^3} (\ln x - 1)\,\mathrm{d}x$

$= \pi\left(\int_{e^2}^{e^3} \ln x\,\mathrm{d}x - \int_{e^2}^{e^3} \mathrm{d}x\right)$

$= \pi\left(\int_{e^2}^{e^3} \ln x\,\mathrm{d}x - e^3 + e^2\right)$

Let $u = \ln x$ and $\dfrac{\mathrm{d}v}{\mathrm{d}x} = 1$

So $\dfrac{\mathrm{d}u}{\mathrm{d}x} = \dfrac{1}{x}$ and $v = x$

Integrating by parts

$I = \pi\left(\left[x\ln x\right]_{e^2}^{e^3} - \int_{e^2}^{e^3} \mathrm{d}x - e^3 + e^2\right)$

$= \pi\left(3e^3 - 2e^2 - e^3 + e^2 - e^3 + e^2\right)$

$= \pi e^3$

3 a $x = \sqrt{1 - \sin\theta} \Rightarrow x^2 = 1 - \sin\theta$

$\sin\theta = 1 - x^2$

$\sin^2\theta = 1 - 2x^2 + x^4$

$y = \cos\theta \Rightarrow \cos^2\theta = y^2$

$\sin^2\theta + \cos^2\theta = 1$

$\Rightarrow 1 - 2x^2 + x^4 + y^2 = 1$

$y^2 = 2x^2 - x^4$

b $y = 0 \Rightarrow 2x^2 - x^4 = 0$

$x^2(2 - x^2) = 0$

$x = 0$ or $\sqrt{2}$

P has coordinates $\left(\sqrt{2}, 0\right)$

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3 c $\text{Volume} = \pi\int_0^{\sqrt{2}}\left(2x^2 - x^4\right)\mathrm{d}x$

$$= \pi\left[\frac{2x^3}{3} - \frac{x^5}{5}\right]_0^{\sqrt{2}}$$

$$= \pi\left(\left(\frac{4\sqrt{2}}{3} - \frac{4\sqrt{2}}{5}\right) - 0\right)$$

$$= \frac{8\sqrt{2}}{15}\pi$$

4 a $x = \tan\theta,\ y = \sec^3\theta$

$y = 1 \Rightarrow \sec\theta = 1 \Rightarrow \theta = 0$

$y = 8 \Rightarrow \sec\theta = 2 \Rightarrow \theta = \frac{\pi}{3}$

b $\frac{\mathrm{d}y}{\mathrm{d}\theta} = 3\sec^3\theta\tan\theta$

$\text{Volume} = \pi\int_0^{\frac{\pi}{3}} x^2\frac{\mathrm{d}y}{\mathrm{d}\theta}\mathrm{d}\theta$

$$= 3\pi\int_0^{\frac{\pi}{3}}\sec^3\theta\tan^3\theta\,\mathrm{d}\theta$$

$$= 3\pi\int_0^{\frac{\pi}{3}}\sec^3\theta\tan\theta\left(\sec^2\theta - 1\right)\mathrm{d}\theta$$

$$= 3\pi\int_0^{\frac{\pi}{3}}\left(\sec^5\theta\tan\theta - \sec^3\theta\tan\theta\right)\mathrm{d}\theta$$

$$= 3\pi\left[\frac{1}{5}\sec^5\theta - \frac{1}{3}\sec^3\theta\right]_0^{\frac{\pi}{3}}$$

$$= 3\pi\left(\left(\frac{32}{5} - \frac{8}{3}\right) - \left(\frac{1}{5} - \frac{1}{3}\right)\right)$$

$$= 3\pi\left(\frac{56}{15} + \frac{2}{15}\right)$$

$$= \frac{58\pi}{5}$$

c $\tan^2\theta = \sec^2\theta - 1$

So $x^2 = y^{\frac{2}{3}} - 1$

4 d $\pi\int_1^8 x^2\,\mathrm{d}y = \pi\int_1^8\left(y^{\frac{2}{3}} - 1\right)\mathrm{d}y$

$$= \pi\left[\frac{3}{5}y^{\frac{5}{3}} - y\right]_1^8$$

$$= \pi\left(\left(\frac{96}{5} - 8\right) - \left(\frac{3}{5} - 1\right)\right)$$

$$= \pi\left(\frac{56}{5} + \frac{2}{5}\right)$$

$$= \frac{58\pi}{5}$$

5 a $x = \sin^4\theta\sqrt{\cos\theta}$

$y = \cos\theta \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}\theta} = -\sin\theta$

Lower bound for y is when $\theta = \frac{\pi}{2}$

Upper bound for y is when $\theta = 0$

$\text{Volume} = \pi\int_{\frac{\pi}{2}}^0 -\sin^9\theta\cos\theta\,\mathrm{d}\theta$

$$= \frac{\pi}{10}\left[-\sin^{10}\theta\right]_{\frac{\pi}{2}}^0$$

$$= \frac{\pi}{10}$$

6 $y = t^2 \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}t} = 2t$

$t = \pm 2 \Rightarrow a = 4$

Required volume is that generated by the curve for $0 \le t \le 2$.

$\text{Volume} = \pi\int_0^2 x^2\frac{\mathrm{d}y}{\mathrm{d}t}\mathrm{d}t$

$$= \pi\int_0^2 8t^3\,\mathrm{d}t$$

$$= \pi\left[2t^4\right]_0^2$$

$$= 32\pi$$

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7 a $\int \cos^2\theta \,\mathrm{d}\theta = \frac{1}{2}\int (\cos 2\theta + 1)\,\mathrm{d}\theta$

$= \frac{1}{2}\left(\frac{1}{2}\sin 2\theta + \theta\right) + c$

$= \frac{1}{4}\sin 2\theta + \frac{1}{2}\theta + c$

b $x = \cot\theta \Rightarrow \frac{\mathrm{d}x}{\mathrm{d}\theta} = -\mathrm{cosec}^2\theta$

$x = \frac{1}{\sqrt{3}} \Rightarrow \theta = \frac{\pi}{3}$

$x = \sqrt{3} \Rightarrow \theta = \frac{\pi}{6}$

$\text{Volume} = \pi\int_{\frac{\pi}{3}}^{\frac{\pi}{6}} 16\sin^2 2\theta\left(-\mathrm{cosec}^2\theta\right)\mathrm{d}\theta$

$= -\pi\int_{\frac{\pi}{3}}^{\frac{\pi}{6}} \frac{16\sin^2 2\theta}{\sin^2\theta}\,\mathrm{d}\theta$

$= -16\pi\int_{\frac{\pi}{3}}^{\frac{\pi}{6}} \frac{4\sin^2\theta\cos^2\theta}{\sin^2\theta}\,\mathrm{d}\theta$

$= -64\pi\int_{\frac{\pi}{3}}^{\frac{\pi}{6}} \cos^2\theta\,\mathrm{d}\theta$

$k = -64,\ a = \frac{\pi}{3},\ b = \frac{\pi}{6}$

c $\text{Volume} = 64\pi\left[\frac{1}{4}\sin 2\theta + \frac{1}{2}\theta\right]_{\frac{\pi}{6}}^{\frac{\pi}{3}}$

$= 64\pi\left(\left(\frac{\sqrt{3}}{8} + \frac{\pi}{6}\right) - \left(\frac{\sqrt{3}}{8} + \frac{\pi}{12}\right)\right)$

$= 64\pi\left(\frac{\pi}{6} - \frac{\pi}{12}\right)$

$= \frac{64}{12}\pi^2$

$= \frac{16}{3}\pi^2$

8 $y = \ln 2t \Rightarrow t = \frac{1}{2}e^y$

$x = \frac{1}{2t} = e^{-y}$

$\text{Volume} = \pi\int_0^a e^{-2y}\,\mathrm{d}y$

$= \pi\left[-\frac{1}{2}e^{-2y}\right]_0^a$

$= \pi\left(-\frac{1}{2}e^{-2a} + \frac{1}{2}\right)$

$= \frac{\pi}{2}\left(1 - e^{-2a}\right) = \frac{24\pi}{49}$

$1 - e^{-2a} = \frac{48}{49}$

$e^{-2a} = \frac{1}{49}$

$e^{2a} = 49$

$2a = \ln 49$

$a = \frac{1}{2}\ln 49 = \ln 7$

9 $y = t^2 \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}t} = 2t$

$\text{Volume} = \pi\int_0^{\pi} 8t\sin^2 t\,\mathrm{d}t$

$= 4\pi\int_0^{\pi} t\left(1 - \cos 2t\right)\mathrm{d}t$

$= 4\pi\left(\left[\frac{t^2}{2}\right]_0^{\pi} - \int_0^{\pi} t\cos 2t\,\mathrm{d}t\right)$

$= 4\pi\left(\frac{\pi^2}{2} - \int_0^{\pi} t\cos 2t\,\mathrm{d}t\right)$

Let $u = t$ and $\frac{\mathrm{d}v}{\mathrm{d}t} = \cos 2t$

So $\frac{\mathrm{d}u}{\mathrm{d}t} = 1$ and $v = \frac{1}{2}\sin 2t$

Integrating by parts

$\text{Volume} = 4\pi\left(\frac{\pi^2}{2} - \left[\frac{t}{2}\sin 2t\right]_0^{\pi} + \frac{1}{2}\int_0^{\pi}\sin 2t\,\mathrm{d}t\right)$

$= 4\pi\left(\frac{\pi^2}{2} + \frac{1}{2}\int_0^{\pi}\sin 2t\,\mathrm{d}t\right)$

$= 4\pi\left(\frac{\pi^2}{2} + \frac{1}{2}\left[-\frac{1}{2}\cos 2t\right]_0^{\pi}\right)$

$= 2\pi^3$

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10 $x = t^2 - 2t \Rightarrow \dfrac{\mathrm{d}x}{\mathrm{d}t} = 2t - 2$

Volume generated by curve

$= \pi\int_1^{-1} (1-t^2)^2 (2t-2)\,\mathrm{d}t$

$= 2\pi\int_1^{-1} (1-2t^2+t^4)(t-1)\,\mathrm{d}t$

$= 2\pi\int_1^{-1} (t-2t^3+t^5-1+2t^2-t^4)\,\mathrm{d}t$

$= 2\pi\int_1^{-1} (t^5-t^4-2t^3+2t^2+t-1)\,\mathrm{d}t$

$= 2\pi\left[\dfrac{t^6}{6} - \dfrac{t^5}{5} - \dfrac{t^4}{2} + \dfrac{2t^3}{3} + \dfrac{t^2}{2} - t\right]_1^{-1}$

$= 2\pi\left(\left(\dfrac{1}{6}+\dfrac{1}{5}-\dfrac{1}{2}-\dfrac{2}{3}+\dfrac{1}{2}+1\right) - \left(\dfrac{1}{6}-\dfrac{1}{5}-\dfrac{1}{2}+\dfrac{2}{3}+\dfrac{1}{2}-1\right)\right)$

$= \dfrac{32}{15}\pi$

P coordinates are $(-1, 0)$

Q coordinates are $(0, 1)$

S coordinates are $(3, 0)$

Volume of cone generated by line PQ

$= \dfrac{1}{3}\pi \times 1^2 \times 1 = \dfrac{\pi}{3}$

Volume of cone generated by line QS

$= \dfrac{1}{3}\pi \times 1^2 \times 3 = \pi$

Volume generated by R

$= \left(\dfrac{32}{15} - \dfrac{1}{3} - 1\right)\pi$

$= \dfrac{12}{15}\pi$

$= \dfrac{4}{5}\pi$

11 a $x = e^t \quad y = e^{-2t}$

$\dfrac{\mathrm{d}y}{\mathrm{d}t} = -2e^{-2t}$

$y = 1 \Rightarrow t = 0$

$y = 6 \Rightarrow t = -\dfrac{1}{2}\ln 6$

Volume $= \pi\int_0^{-\frac{1}{2}\ln 6} x^2 \dfrac{\mathrm{d}y}{\mathrm{d}t}\,\mathrm{d}t$

$= \pi\int_0^{-\frac{1}{2}\ln 6} e^{2t}\left(-2e^{-2t}\right)\mathrm{d}t$

$= -2\pi\int_0^{-\frac{1}{2}\ln 6} \mathrm{d}t$

$= -2\pi\left[t\right]_0^{-\frac{1}{2}\ln 6}$

$= -2\pi\left(-\dfrac{1}{2}\ln 6 - 0\right)$

$= \pi \ln 6$

b Find the gradient of the tangent at (1, 1).

$\dfrac{\mathrm{d}y}{\mathrm{d}t} = -2e^{-2t}$ and $\dfrac{\mathrm{d}x}{\mathrm{d}t} = e^t$

$\dfrac{\mathrm{d}y}{\mathrm{d}x} = -\dfrac{2e^{-2t}}{e^t} = -2e^{-3t}$

When $y = 1$, $t = 0$ and $\dfrac{\mathrm{d}y}{\mathrm{d}x} = -2$

Equation of tangent is

$y - 1 = -2(x - 1)$

$y = -2x + 3$

So the tangent cuts the y axis at (0, 3).

Volume generated by S is the volume generated by R minus the volume of the cone of base 1 and height 2.

Volume of cone $= \dfrac{1}{3}\pi \times 1^2 \times 2 = \dfrac{2}{3}\pi$

Volume generated by S

$= \pi \ln 6 - \dfrac{2}{3}\pi$

$= \pi\left(\ln 6 - \dfrac{2}{3}\right)$

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Volumes of revolution 4D

1 a $y = 0 \Rightarrow a = \dfrac{2000}{20} = 100$

Diameter $= 2a = 200$ metres.

b Volume $= \pi\int_0^{120}\left(\dfrac{2000}{20+y}\right)^2 \mathrm{d}y$

$= 4\,000\,000\pi\int_0^{120}\dfrac{1}{(20+y)^2}\mathrm{d}y$

$= 4\,000\,000\pi\left[-\dfrac{1}{20+y}\right]_0^{120}$

$= 4\,000\,000\pi\left(-\dfrac{1}{140}+\dfrac{1}{20}\right)$

$= 4\,000\,000\pi\times\dfrac{6}{140}$

$= \dfrac{1\,200\,000\pi}{7}\ \mathrm{m}^3$

2 Volume of water $= \pi\int_0^{20}\dfrac{100y}{10y^2+1}\mathrm{d}y$

$= \pi\left[5\ln\left(10y^2+1\right)\right]_0^{20}$

$= 5\pi\ln 4001\,\mathrm{cm}^3$

So $p = 5, q = 4001$

3 a $\cos 3\theta = \cos(2\theta+\theta)$

$= \cos 2\theta\cos\theta - \sin 2\theta\sin\theta$

$= \left(2\cos^2\theta - 1\right)\cos\theta - 2\sin^2\theta\cos\theta$

$= 2\cos^3\theta - \cos\theta - 2\left(1-\cos^2\theta\right)\cos\theta$

$= 4\cos^3\theta - 3\cos\theta$

$4\cos^3\theta = 3\cos\theta + \cos 3\theta$

$\cos^3\theta = \dfrac{3}{4}\cos\theta + \dfrac{1}{4}\cos 3\theta$

b $x = 50\cos\theta \Rightarrow x^2 = 2500\cos^2\theta$

$y = 30\sin\theta \Rightarrow \dfrac{dy}{d\theta} = 30\cos\theta$

3 b Volume of tent

$= \pi\int_0^{\frac{\pi}{2}}\left(2500\cos^2\theta\times 30\cos\theta\right)\mathrm{d}\theta$

$= 75\,000\pi\int_0^{\frac{\pi}{2}}\cos^3\theta\,\mathrm{d}\theta$

$= \dfrac{75\,000}{4}\pi\int_0^{\frac{\pi}{2}}\left(3\cos\theta+\cos 3\theta\right)\mathrm{d}\theta$

$= \dfrac{75\,000}{4}\pi\left[3\sin\theta+\dfrac{1}{3}\sin 3\theta\right]_0^{\frac{\pi}{2}}$

$= \dfrac{75\,000}{4}\pi\left(3-\dfrac{1}{3}\right)$

$= 50\,000\pi\ \mathrm{m}^3$

4 a $x = \sin y\sqrt{\sin 2y} \Rightarrow x^2 = 2\sin^3 y\cos y$

Volume $= 2\pi\int_0^{\frac{\pi}{2}}\sin^3 y\cos y\,\mathrm{d}y$

$= 2\pi\left[\dfrac{1}{4}\sin^4 y\right]_0^{\frac{\pi}{2}}$

$= \dfrac{\pi}{2}\ \mathrm{m}^3$

b Linear scale factor $= 6\pi\Big/\dfrac{\pi}{2} = 12$

Volume scale factor $= 12^3 = 1728$

Volume of real hot-air balloon $= 1728\times\dfrac{\pi}{2}$

$= 864\pi\ \mathrm{m}^3$

5 a $x = 2\sin 2\theta \quad y = 3\sin\theta$

$x^2 = 4\sin^2 2\theta = 16\sin^2\theta\cos^2\theta$

$= 16\sin^2\theta(1-\sin^2\theta)$

$= 16\sin^2\theta - 16\sin^4\theta$

$= 16\left(\dfrac{y}{3}\right)^2 - 16\left(\dfrac{y}{3}\right)^4$

$= 16\dfrac{y^2}{9} - 16\dfrac{y^4}{81}$

$= \dfrac{16}{81}y^2\left(9-y^2\right)$

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5 b For the model, when $x = 0,\ y = 3$

$$\text{Volume} = \frac{16}{81}\pi\int_0^3 \left(9y^2 - y^4\right)\mathrm{d}y$$
$$= \frac{16}{81}\pi\left[3y^3 - \frac{y^5}{5}\right]_0^3$$
$$= \frac{16}{81}\pi\left(81 - \frac{243}{5}\right)$$
$$= \frac{16}{81}\pi \times \frac{162}{5}$$
$$= \frac{32}{5}\pi \text{ mm}^3$$

Maximum number of earrings
= integer value of $300 \Big/ \frac{32}{5}\pi = 14$

c i e.g. patterned earring may mean that earring requires less material.

ii e.g. wasted material upon transfer to mould.

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Volumes of revolution – mixed exercise 4

1 a $I = \int x\cos 2x\,\mathrm{d}x$

Let $u = x$ and $\dfrac{\mathrm{d}v}{\mathrm{d}x} = \cos 2x$

So $\dfrac{\mathrm{d}u}{\mathrm{d}x} = 1$ and $v = \dfrac{1}{2}\sin 2x$

Integrating by parts

$$I = \frac{x}{2}\sin 2x - \frac{1}{2}\int \sin 2x\,\mathrm{d}x$$
$$= \frac{1}{2}x\sin 2x + \frac{1}{4}\cos 2x + c$$

b $y = 2x^{\frac{1}{2}}\sin x$

$y^2 = 4x\sin^2 x = 2x(1-\cos 2x)$

$$\text{Volume} = 2\pi\int_0^{\frac{\pi}{2}} x\left(1-\cos 2x\right)\mathrm{d}x$$
$$= 2\pi\left[\frac{x^2}{2} - \frac{1}{2}x\sin 2x - \frac{1}{4}\cos 2x\right]_0^{\frac{\pi}{2}}$$
$$= 2\pi\left(\left(\frac{\pi^2}{8} + \frac{1}{4}\right) + \frac{1}{4}\right)$$
$$= \pi\left(\frac{\pi^2}{4} + 1\right)$$

2 a $I = \int_0^{\frac{\pi}{4}} x\sec^2 x\,\mathrm{d}x$

Let $u = x$ and $\dfrac{\mathrm{d}v}{\mathrm{d}x} = \sec^2 x$

So $\dfrac{\mathrm{d}u}{\mathrm{d}x} = 1$ and $v = \tan x$

$$I = \left[x\tan x\right]_0^{\frac{\pi}{4}} - \int_0^{\frac{\pi}{4}} \tan x\,\mathrm{d}x$$
$$= \left(\frac{\pi}{4} - 0\right) + \left[\ln\left(\cos x\right)\right]_0^{\frac{\pi}{4}}$$
$$= \frac{\pi}{4} + \ln\frac{1}{\sqrt{2}}$$
$$= \frac{\pi}{4} - \ln\sqrt{2}$$
$$= \frac{\pi}{4} - \frac{1}{2}\ln 2$$

2 b $\text{Volume} = \pi\int_0^{\frac{\pi}{4}} y^2\,\mathrm{d}x$

$$= \pi\int_0^{\frac{\pi}{4}} x\sec^2 x\,\mathrm{d}x$$
$$= \pi\left(\frac{1}{4}\pi - \frac{1}{2}\ln 2\right)$$

3 Volume generated by area under C

$$= \pi\int_1^{\frac{5}{2}}\left(\frac{x+2}{x}\right)^2\mathrm{d}x$$
$$= \pi\int_1^{\frac{5}{2}}\frac{x^2+4x+4}{x^2}\mathrm{d}x$$
$$= \pi\int_1^{\frac{5}{2}}\left(1 + \frac{4}{x} + \frac{4}{x^2}\right)\mathrm{d}x$$
$$= \pi\left[x + 4\ln x - \frac{4}{x}\right]_1^{\frac{5}{2}}$$
$$= \pi\left(\left(\frac{5}{2} + 4\ln\frac{5}{2} - \frac{8}{5}\right) - \left(1-4\right)\right)$$
$$= \pi\left(\frac{39}{10} + 4\ln\frac{5}{2}\right)$$

Volume generated by T is a cone of radius 3 and height 1.5.

$$\text{Volume of cone} = \frac{1}{3}\pi\times 9\times 1.5 = \frac{9}{2}\pi$$

So volume generated by R

$$= \pi\left(\frac{39}{10} + 4\ln\frac{5}{2}\right) - \frac{9}{2}\pi$$
$$= \pi\left(4\ln\frac{5}{2} - \frac{3}{5}\right)$$

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4 $x = \frac{\pi}{4} \Rightarrow \sec x - \cos x = \sqrt{2} - \frac{1}{\sqrt{2}}$

Also $\operatorname{cosec} x - \sin x = \sqrt{2} - \frac{1}{\sqrt{2}}$

So the curves meet at $x = \frac{\pi}{4}$

Volume generated by $y = \sec x - \cos x$ between $x = 0$ and $x = \frac{\pi}{4}$ is given by

$$\pi\int_0^{\frac{\pi}{4}} (\sec x - \cos x)^2 \, dx$$

$$= \pi\int_0^{\frac{\pi}{4}} \left(\sec^2 x - 2 + \cos^2 x\right) dx$$

$$= \pi\int_0^{\frac{\pi}{4}} \left(\sec^2 x - 2 + \frac{1}{2}(1 + \cos 2x)\right) dx$$

$$= \pi\left[\tan x - \frac{3}{2}x + \frac{1}{4}\sin 2x\right]_0^{\frac{\pi}{4}}$$

$$= \pi\left(1 - \frac{3\pi}{8} + \frac{1}{4}\right)$$

$$= \frac{\pi}{8}(10 - 3\pi)$$

Due to the symmetry of the curves, the total volume generated by R will be

$$2 \times \frac{\pi}{8}(10 - 3\pi)$$

$$= \frac{\pi}{4}(10 - 3\pi)$$

5 $x = e^{\frac{1}{2}y} - 2 \Rightarrow x^2 = e^y - 4e^{\frac{1}{2}y} + 4$

$$\text{Volume} = \pi\int_2^4 \left(e^y - 4e^{\frac{1}{2}y} + 4\right) dy$$

$$= \pi\left[e^y - 8e^{\frac{1}{2}y} + 4y\right]_2^4$$

$$= \pi\left(\left(e^4 - 8e^2 + 16\right) - \left(e^2 - 8e + 8\right)\right)$$

$$= \pi\left(e^4 - 9e^2 + 8e + 8\right)$$

6 a $x = 2\sin^2 t \Rightarrow \frac{dx}{dt} = 4\sin t \cos t$

$y = 2\cos t \Rightarrow \frac{dy}{dt} = -2\sin t$

$$\frac{dy}{dx} = \frac{-2\sin t}{4\sin t \cos t} = -\frac{1}{2\cos t}$$

At point $P\left(\frac{3}{2}, 1\right)$ $t = \frac{\pi}{3}$ and $\frac{dy}{dx} = -1$

Equation of l is

$$y - 1 = -1\left(x - \frac{3}{2}\right)$$

$$y = -x + \frac{5}{2}$$

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6 b $x = 2\sin^2 t \Rightarrow \sin^2 t = \frac{x}{2}$

$y = 2\cos t \Rightarrow \cos^2 t = \frac{y^2}{4}$

$\frac{y^2}{4} + \frac{x}{2} = 1$

$x = 2 - \frac{y^2}{2}$

Volume generated by the curve C

$= \pi\int_0^2 x^2 \,\mathrm{d}y$

$= \pi\int_0^2 \left(2 - \frac{y^2}{2}\right)^2 \mathrm{d}y$

$= \pi\int_0^2 \left(4 - 2y^2 + \frac{y^4}{4}\right)\mathrm{d}y$

$= \pi\left[4y - \frac{2y^3}{3} + \frac{y^5}{20}\right]_0^2$

$= \pi\left(8 - \frac{16}{3} + \frac{32}{20}\right)$

$= \frac{\pi}{15}(120 - 80 + 24)$

$= \frac{64\pi}{15}$

Volume generated by line l

$= \frac{1}{3}\pi \times \left(\frac{5}{2}\right)^2 \times \frac{5}{2}$

$= \frac{125}{24}\pi$

Volume generated by region R

$= \frac{125}{24}\pi - \frac{64}{15}\pi$

$= \frac{\pi}{120}(625 - 512)$

$= \frac{113}{120}\pi$

7 a $x = (t+1)^2, \quad y = \frac{1}{2}t^3 + 3$

$\frac{\mathrm{d}x}{\mathrm{d}t} = 2(t+1)$

Area of $R = \int_0^2 y\frac{\mathrm{d}x}{\mathrm{d}t}\mathrm{d}t$

$= 2\int_0^2 \left(\frac{1}{2}t^3 + 3\right)(t+1)\,\mathrm{d}t$

$= 2\int_0^2 \left(\frac{1}{2}t^4 + \frac{1}{2}t^3 + 3t + 3\right)\mathrm{d}t$

$= 2\left[\frac{1}{10}t^5 + \frac{1}{8}t^4 + \frac{3}{2}t^2 + 3t\right]_0^2$

$= 2\left(\left(\frac{32}{10} + 2 + 6 + 6\right) - 0\right)$

$= \frac{172}{5}$

b Volume generated by R by rotating about the y axis

$= \pi\int_0^2 x^2 \frac{\mathrm{d}y}{\mathrm{d}t}\mathrm{d}t$

$= \frac{3}{2}\pi\int_0^2 (t+1)^4 t^2 \,\mathrm{d}t$

$= \frac{3}{2}\pi\int_0^2 \left(t^4 + 4t^3 + 6t^2 + 4t + 1\right)t^2 \,\mathrm{d}t$

$= \frac{3}{2}\pi\int_0^2 \left(t^6 + 4t^5 + 6t^4 + 4t^3 + t^2\right)\mathrm{d}t$

$= \frac{3}{2}\pi\left[\frac{1}{7}t^7 + \frac{2}{3}t^6 + \frac{6}{5}t^5 + t^4 + \frac{1}{3}t^3\right]_0^2$

$= \frac{3}{2}\pi\left(\frac{128}{7} + \frac{128}{3} + \frac{192}{5} + 16 + \frac{8}{3}\right)$

$= \frac{3}{2}\pi\left(\frac{1920 + 4480 + 4032 + 1680 + 280}{105}\right)$

$= \frac{6196}{35}\pi$

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8 Parameters are $x = \cos t,\ y = \sin t$

A unit sphere can be generated by rotating the semi-circle around the y axis between the limits $-\frac{\pi}{2}$ and $\frac{\pi}{2}$.

i.e. Volume $= \pi\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} x^2 \frac{\mathrm{d}y}{\mathrm{d}t}\mathrm{d}t$

$= \pi\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^2 t \cos t\,\mathrm{d}t$

$= \pi\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(1-\sin^2 t\right)\cos t\,\mathrm{d}t$

$= \pi\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(\cos t - \cos t \sin^2 t\right)\mathrm{d}t$

$= \pi\left[\sin t - \frac{1}{3}\sin^3 t\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}}$

$= \pi\left(\left(1-\frac{1}{3}\right)-\left(-1+\frac{1}{3}\right)\right)$

$= \frac{4\pi}{3}$

9 a $\sin 3\theta = \sin\left(2\theta+\theta\right)$

$= \sin 2\theta\cos\theta + \cos 2\theta \sin\theta$

$= 2\sin\theta\cos^2\theta + \left(1-2\sin^2\theta\right)\sin\theta$

$= 2\sin\theta(1-\sin^2\theta) + \sin\theta - 2\sin^3\theta$

$= 3\sin\theta - 4\sin^3\theta$

$\sin^3\theta = \frac{3}{4}\sin\theta - \frac{1}{4}\sin 3\theta$

B $x = 15\cos\theta,\ y = 10\sin\theta$

When $\theta = 0,\ x = 15$, when $\theta = \pi,\ x = -15$

So lower limit is $\theta = \pi$ and upper limit is $\theta = 0$

Volume $= \pi\int_{\pi}^{0} \left(10\sin\theta\right)^2\left(-15\sin\theta\right)\mathrm{d}\theta$

$= -1500\pi\int_{\pi}^{0} \sin^3\theta\,\mathrm{d}\theta$

$= -1500\pi\int_{\pi}^{0} \left(\frac{3}{4}\sin\theta - \frac{1}{4}\sin 3\theta\right)\mathrm{d}\theta$

$= -1500\pi\left[-\frac{3}{4}\cos\theta + \frac{1}{12}\cos 3\theta\right]_{\pi}^{0}$

$= -1500\pi\left(\left(-\frac{3}{4}+\frac{1}{12}\right)-\left(\frac{3}{4}-\frac{1}{12}\right)\right)$

$= 2000\pi\ \mathrm{cm}^3$

10 $x = 2\sin 2t,\ y = 4\cos t$

Note that R is defined by $0 \le t \le \frac{\pi}{2}$

When $t = 0,\ y = 4$, when $t = \frac{\pi}{2},\ y = 0$

So lower limit is $t = \frac{\pi}{2}$ and upper limit is $t = 0$

So volume $= \pi\int_{\frac{\pi}{2}}^{0} x^2 \frac{\mathrm{d}y}{\mathrm{d}t}\mathrm{d}t$

$= -16\pi\int_{\frac{\pi}{2}}^{0} \sin^2 2t \sin t\,\mathrm{d}t$

$= -16\pi\int_{\frac{\pi}{2}}^{0} 4\sin^2 t\cos^2 t \sin t\,\mathrm{d}t$

$= -64\pi\int_{\frac{\pi}{2}}^{0} \sin^3 t\cos^2 t\,\mathrm{d}t$

$= -64\pi\int_{\frac{\pi}{2}}^{0} \left(1-\cos^2 t\right)\sin t\cos^2 t\,\mathrm{d}t$

$= -64\pi\int_{\frac{\pi}{2}}^{0} \left(\sin t\cos^2 t - \sin t\cos^4 t\right)\mathrm{d}t$

$= -64\pi\left[-\frac{1}{3}\cos^3 t + \frac{1}{5}\cos^5 t\right]_{\frac{\pi}{2}}^{0}$

$= -64\pi\left(\left(-\frac{1}{3}+\frac{1}{5}\right)-0\right)$

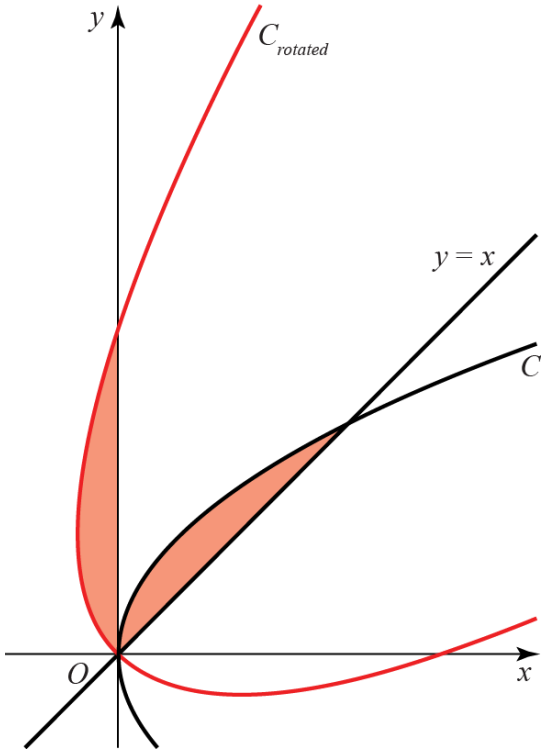
$= -64\pi \times -\frac{2}{15}$

$= \frac{128\pi}{15}$

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Challenge

If the curve C and he line $y = x$ are rotated $\pi/4$ radians anticlockwise about the origin, then the required volume will be given by the rotated curve about the y axis between the points at which it crosses the y axis.



The parametric equations of the rotated curve are found using

$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ where $\theta = \frac{\pi}{4}$

$$\text{i.e.} \begin{pmatrix} x_{rotated} \\ y_{rotated} \end{pmatrix} = \begin{pmatrix} \cos\frac{\pi}{4} & -\sin\frac{\pi}{4} \\ \sin\frac{\pi}{4} & \cos\frac{\pi}{4} \end{pmatrix} \begin{pmatrix} t^2 \\ 2t \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} t^2 \\ 2t \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{\sqrt{2}}(t^2 - 2t) \\ \frac{1}{\sqrt{2}}(t^2 + 2t) \end{pmatrix}$$

$$\frac{dy_{rotated}}{dt} = \frac{1}{\sqrt{2}}(2t+2)$$

$x_{rotated} = 0 \Rightarrow t^2 - 2t = 0$

So $t = 0$ or 2

$$\text{Volume} = \pi\int_0^2 \left(\frac{1}{2}(t^2-2t)^2 \times \frac{1}{\sqrt{2}}(2t+2)\right) dt$$

$$= \frac{\pi}{2\sqrt{2}}\int_0^2 (t^4 - 4t^3 + 4t^2)(2t+2)\,dt$$

$$= \frac{\pi}{2\sqrt{2}}\int_0^2 (2t^5 - 8t^4 + 8t^3 + 2t^4 - 8t^3 + 8t^2)\,dt$$

$$= \frac{\pi}{2\sqrt{2}}\int_0^2 (2t^5 - 6t^4 + 8t^2)\,dt$$

$$= \frac{\pi}{2\sqrt{2}}\left[\frac{1}{3}t^6 - \frac{6}{5}t^5 + \frac{8}{3}t^3\right]_0^2$$

$$= \frac{\pi}{2\sqrt{2}}\left(\frac{64}{3} - \frac{192}{5} + \frac{64}{3}\right)$$

$$= \frac{\pi}{2\sqrt{2}}\left(\frac{320 - 576 + 320}{15}\right)$$

$$= \frac{\pi}{2\sqrt{2}} \times \frac{64}{15}$$

$$= \frac{64\pi}{30\sqrt{2}}$$

$$\left(= \frac{32\pi}{15\sqrt{2}}\right)$$

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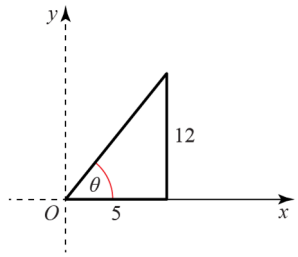
Polar coordinates 5A

1 a

$$\arctan\left(\frac{12}{5}\right) = 1.176$$

$$r = \sqrt{5^2 + 12^2} = 13$$

$\therefore$ point is (13, 1.176)

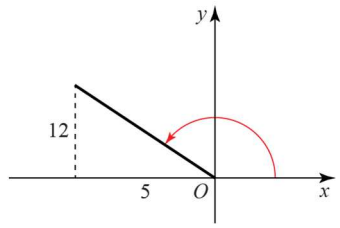


b

$$r = \sqrt{(-5)^2 + 12^2} = 13$$

$$\theta = \pi - \arctan\left(\frac{12}{5}\right) = 1.966$$

$\therefore$ point is (13, 1.966)



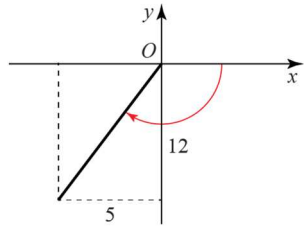
c

$$\theta = -\left(\pi - \arctan\frac{12}{5}\right)$$

$$= -1.966$$

$$r = \sqrt{(-5)^2 + (-12)^2} = 13$$

$\therefore$ point is (13, -1.966)

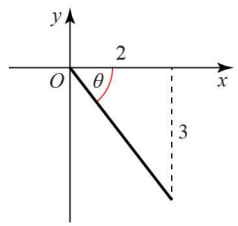


d

$$\theta = -\arctan\frac{3}{2} = -0.983$$

$$r = \sqrt{2^2 + (-3)^2} = \sqrt{13}$$

$\therefore$ point is $(\sqrt{13}, -0.983)$

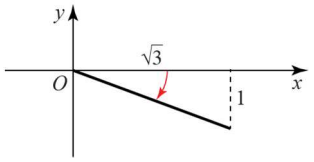


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1 e $\theta = -\arctan\dfrac{1}{\sqrt{3}} = -\dfrac{\pi}{6}$

$r = \sqrt{\sqrt{3^2} + (-1)^2} = \sqrt{4} = 2$

$\therefore$ point is $(2, -\dfrac{\pi}{6})$



2 a $x = 6\cos\left(\dfrac{\pi}{6}\right) = \dfrac{6\sqrt{3}}{2} = 3\sqrt{3}$

$y = 6\sin\left(\dfrac{\pi}{6}\right) = 3$ $\therefore$ point is $(3\sqrt{3}, 3)$

b $x = 6\cos\left(-\dfrac{\pi}{6}\right) = \dfrac{6\sqrt{3}}{2} = 3\sqrt{3}$

$y = 6\sin\left(-\dfrac{\pi}{6}\right) = -3$ $\therefore$ point is $(3\sqrt{3}, -3)$

c $x = 6\cos\left(\dfrac{3\pi}{4}\right) = -\dfrac{6}{\sqrt{2}}$ or $-3\sqrt{2}$

$y = 6\sin\left(\dfrac{3\pi}{4}\right) = \dfrac{6}{\sqrt{2}} = 3\sqrt{2}$ $\therefore$ point is $(-3\sqrt{2}, 3\sqrt{2})$

d $x = 10\cos\left(\dfrac{5\pi}{4}\right) = -\dfrac{10}{\sqrt{2}} = -5\sqrt{2}$

$y = 10\sin\left(\dfrac{5\pi}{4}\right) = -\dfrac{10}{\sqrt{2}} = -5\sqrt{2}$ $\therefore$ point is $(-5\sqrt{2}, -5\sqrt{2})$

e $x = 2\cos(\pi) = -2$

$y = 2\sin(\pi) = 0$ $\therefore$ point is $(-2, 0)$

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3 a $r = 2$ is $x^2 + y^2 = 4$

b $r = 3\sec\theta$

$\Rightarrow r\cos\theta = 3$ i.e. $x = 3$

c $r = 5\operatorname{cosec}\theta$

$\Rightarrow r\sin\theta = 5$ i.e. $y = 5$

d $r = 4a\tan\theta\sec\theta$

$r = \dfrac{4a\sin\theta}{\cos^2\theta}$

$r\cos^2\theta = 4a\sin\theta$ ← Multiply by r.

$r^2\cos^2\theta = 4ar\sin\theta$

$\therefore \quad x^2 = 4ay$ or $y = \dfrac{x^2}{4a}$

e $r = 2a\cos\theta$

$r^2 = 2ar\cos\theta$

$\therefore\ x^2 + y^2 = 2ax$ or $(x-a)^2 + y^2 = a^2$

f $r = 3a\sin\theta$ ← Multiply by r.

$r^2 = 3ar\sin\theta$

$x^2 + y^2 = 3ay$ or $x^2 + \left(y - \dfrac{3a}{2}\right)^2 = \dfrac{9a^2}{4}$

g $r = 4(1 - \cos 2\theta)$ ← Use $\cos 2\theta = 1 - 2\sin^2\theta$, $\therefore\ 2\sin^2\theta = 1 - \cos 2\theta$

$r = 4 \times 2\sin^2\theta$

$r^3 = 8r^2\sin^2\theta$ ← $\times r^2$

$\therefore (x^2 + y^2)^{\frac{3}{2}} = 8y^2$

h $r = 2\cos^2\theta$

$r^3 = 2r^2\cos^2\theta$ ← $\times r^2$

$(x^2 + y^2)^{\frac{3}{2}} = 2x^2$

i $r^2 = 1 + \tan^2\theta$

$\therefore \quad r^2 = \sec^2\theta$ ← Use $\sec^2\theta = 1 + \tan^2\theta$.

$\therefore r^2\cos^2\theta = 1$

i.e. $x^2 = 1$ or $x = \pm 1$

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4 a $x^2 + y^2 = 16$

$\Rightarrow \quad r^2 = 16 \quad$ or $\quad r = 4$

b $xy = 4$

$\Rightarrow r\cos\theta \; r\sin\theta = 4$

$$r^2 = \frac{4}{\cos\theta\sin\theta} = \frac{8}{2\cos\theta\sin\theta}$$

i.e. $r^2 = 8 \operatorname{cosec} 2\theta$

c $(x^2 + y^2)^2 = 2xy$

$\Rightarrow \quad (r^2)^2 = 2r\cos\theta \; r\sin\theta$

$r^4 = 2r^2\cos\theta\sin\theta$

$r^2 = \sin 2\theta$

d $x^2 + y^2 - 2x = 0$

$\Rightarrow \quad r^2 - 2r\cos\theta = 0$

$r^2 = 2r\cos\theta$

$r = 2\cos\theta$

e $(x + y)^2 = 4$

$\Rightarrow \quad x^2 + y^2 + 2xy = 4$

$\Rightarrow \quad r^2 + 2r\cos\theta \; r\sin\theta = 4$

$\Rightarrow \quad r^2(1 + \sin 2\theta) = 4$

$$r^2 = \frac{4}{1 + \sin 2\theta}$$

f $x - y = 3$

$r\cos\theta - r\sin\theta = 3$

$r(\cos\theta - \sin\theta) = 3$

$$r\left(\frac{1}{\sqrt{2}}\cos\theta - \frac{1}{\sqrt{2}}\sin\theta\right) = \frac{3}{\sqrt{2}}$$

$$r\cos\left(\theta + \frac{\pi}{4}\right) = \frac{3}{\sqrt{2}}$$

$\therefore \quad r = \frac{3}{\sqrt{2}}\sec\left(\theta + \frac{\pi}{4}\right)$

g $y = 2x$

$\Rightarrow \quad r\sin\theta = 2r\cos\theta$

$\tan\theta = 2 \quad$ or $\quad \theta = \arctan 2$

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4 h

$$y = -\sqrt{3}x + a$$
$$r\sin\theta = -\sqrt{3}\,r\cos\theta + a$$
$$r(\sin\theta + \sqrt{3}\cos\theta) = a$$
$$r\left(\frac{1}{2}\sin\theta + \frac{\sqrt{3}}{2}\cos\theta\right) = \frac{a}{2}$$
$$r\sin\left(\theta + \frac{\pi}{3}\right) = \frac{a}{2}$$
$$\therefore \quad r = \frac{a}{2}\operatorname{cosec}\left(\theta + \frac{\pi}{3}\right)$$

i

$$y = x(x-a)$$
$$r\sin\theta = r\cos\theta\,(r\cos\theta - a)$$
$$\tan\theta = r\cos\theta - a$$
$$r\cos\theta = \tan\theta + a$$
$$r = \tan\theta\sec\theta + a\sec\theta$$

Challenge

First we convert
(r_1, θ_1) and (r_2, θ_2) to their Cartesian coordinate representation by using the relations
$r\cos\theta = x$ and $r\sin\theta = y$
This gives the Cartesian points
$(x_1, y_1) = (r_1\cos\theta_1, r_1\sin\theta_1)$ and $(x_2, y_2) = (r_2\cos\theta_2, r_2\sin\theta_2)$
In Cartesian coordinates, the distance between two points,
(x_1, y_1) and (x_2, y_2) is $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$
Substituting $(x_1, y_1) = (r_1\cos\theta_1, r_1\sin\theta_1)$ and $(x_2, y_2) = (r_2\cos\theta_2, r_2\sin\theta_2)$

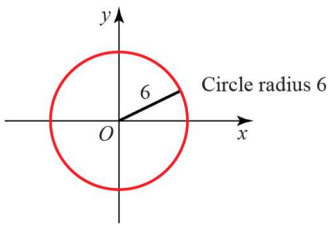
into this expression for d, we obtain

$$d = \sqrt{(r_2\cos\theta_2 - r_1\cos\theta_1)^2 + (r_2\sin\theta_2 - r_1\sin\theta_1)^2}$$
$$= \sqrt{\begin{matrix}(r_2^2\cos^2\theta_2 - 2r_2r_1\cos\theta_2\cos\theta_1 + r_1^2\cos^2\theta_1) \\ +(r_2^2\sin^2\theta_2 - 2r_2r_1\sin\theta_2\sin\theta_1 + r_1^2\sin^2\theta_1)\end{matrix}}$$
$$= \sqrt{r_2^2 + r_1^2 - 2r_2r_1(\cos\theta_2\cos\theta_1 + \sin\theta_2\sin\theta_1)}$$
$$= \sqrt{r_1^2 + r_2^2 - 2r_2r_1\cos(\theta_1 - \theta_2)}$$

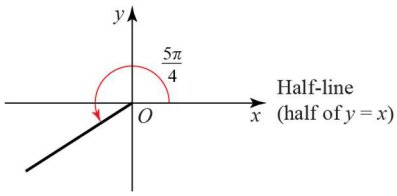
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Polar coordinates 5B

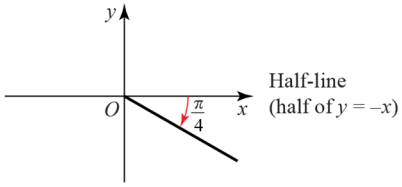
1 a $r = 6$



b $\theta = \frac{5\pi}{4}$



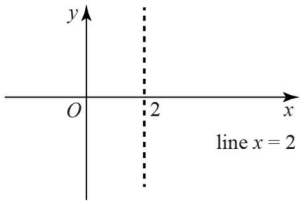
c $\theta = -\frac{\pi}{4}$



d $r = 2\sec\theta$

$\Rightarrow \quad r\cos\theta = 2$

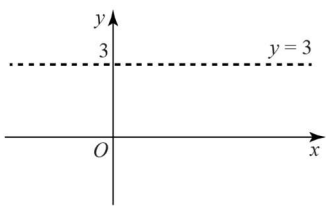
i.e. $x = 2$



e $r = 3\operatorname{cosec}\theta$

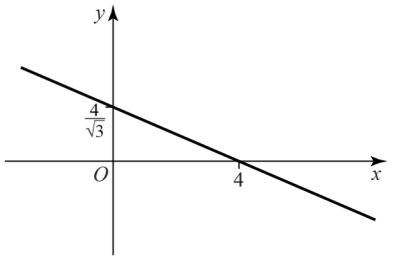
$\Rightarrow \quad r\sin\theta = 3$

i.e. $y = 3$



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1 f $r = 2\sec\left(\theta - \frac{\pi}{3}\right)$



$$r\cos\left(\theta - \frac{\pi}{3}\right) = 2$$

$$\Rightarrow \quad r\cos\theta\cos\frac{\pi}{3} + r\sin\theta\sin\frac{\pi}{3} = 2$$

$$\Rightarrow \quad \frac{x}{2} + y\frac{\sqrt{3}}{2} = 2$$

$$x + y\sqrt{3} = 4$$

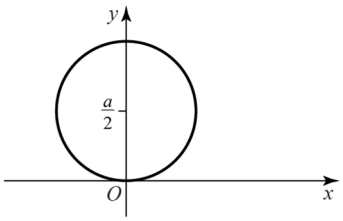
or $$y = \frac{4}{\sqrt{3}} - \frac{1}{\sqrt{3}}x$$

g $r = a\sin\theta$

$$\Rightarrow \quad r^2 = ar\sin\theta$$

$$x^2 + y^2 = ay$$

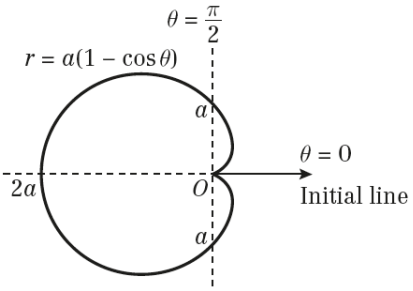
$$x^2 + \left(y - \frac{a}{2}\right)^2 = \frac{a^2}{4}$$



Circle centre $\left(0, \frac{a}{2}\right)$ radius $\frac{a}{2}$

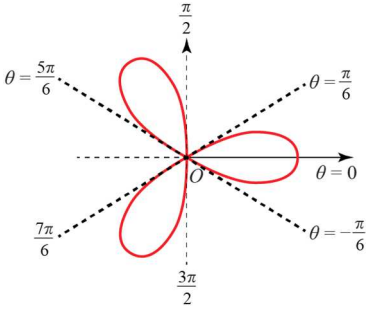
h $r = a(1 - \cos\theta)$

θ	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
r	0	a	$2a$	a	0



i $r = a\cos 3\theta$

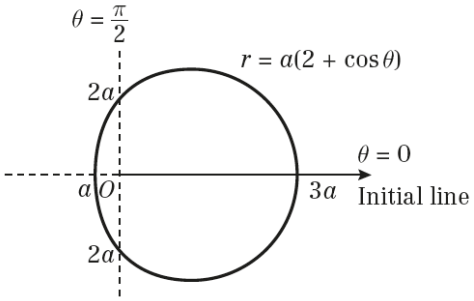
θ	0	$\frac{\pi}{6}$	$-\frac{\pi}{6}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$	$\frac{7\pi}{6}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$
r	a	0	0	0	a	0	0	a	0



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1 j $r = a(2+\cos\theta)$

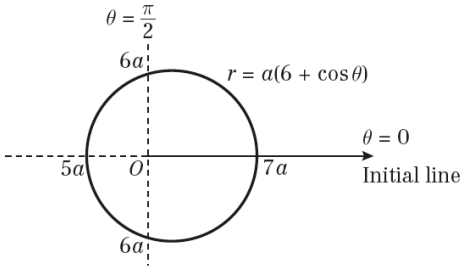
θ	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
r	$3a$	$2a$	a	$2a$	$3a$



$2 = 2 \times 1$ ∴ no dimple.

k $r = a(6+\cos\theta)$

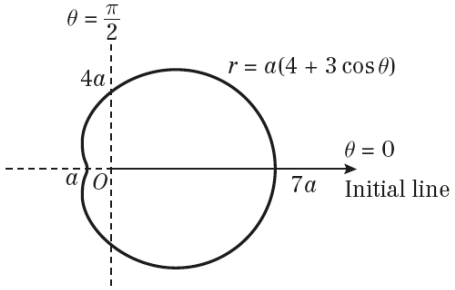
θ	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
r	$7a$	$6a$	$5a$	$6a$	$7a$



$6 > 2 \times 1$ ∴ no dimple.

l $r = a(4+3\cos\theta)$

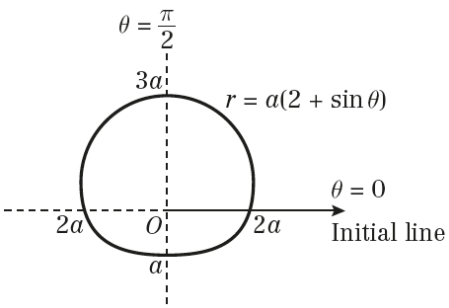
θ	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
r	$7a$	$4a$	a	$4a$	$7a$



$4 < 2 \times 3$ ∴ a dimple at $\theta = \pi$.

m $r = a(2+\sin\theta)$

θ	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
r	$2a$	$3a$	$2a$	a	$2a$

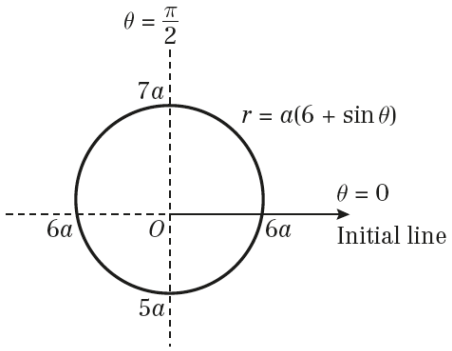


$2 = 2 \times 1$ so no dimple

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1 n $r = a(6+\sin\theta)$

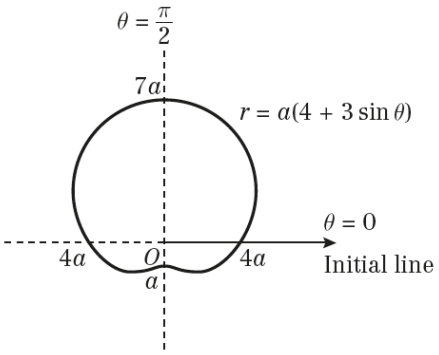
θ	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
r	$6a$	$7a$	$6a$	$5a$	$6a$



$6 > 2 \times 1$ so no dimple

o $r = a(4+3\sin\theta)$

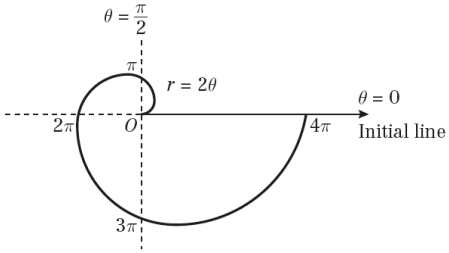
θ	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
r	$4a$	$7a$	$4a$	a	$4a$



$4 < 2 \times 3 \therefore$ there is a dimple at $\theta = \frac{3\pi}{2}$

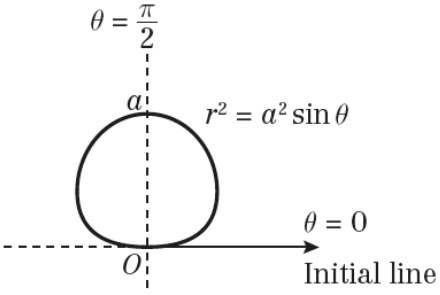
p $r = 2\theta$

θ	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
r	0	π	2π	3π	4π



q $r^2 = a^2\sin\theta$

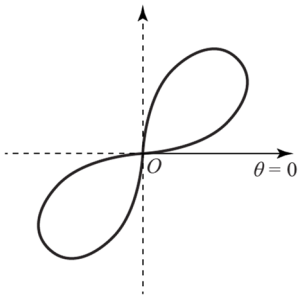
θ	0	$\frac{\pi}{2}$	π
r	0	a	0



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1 r $r^2 = a^2 \sin 2\theta$

θ	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	π	$\frac{5\pi}{4}$	$\frac{3\pi}{2}$
r	0	a	0	0	a	0



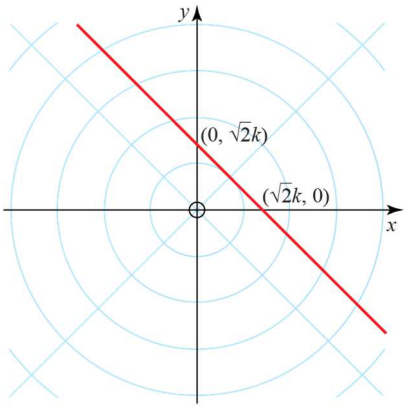
2 First we rearrange by multiplying both sides of $r = k\sec\left(\frac{\pi}{4} - \theta\right)$ by $\cos\left(\frac{\pi}{4} - \theta\right)$ to obtain

$r\cos\left(\frac{\pi}{4} - \theta\right) = k$.

We then use the compound angle formula and obtain $r\cos\left(\frac{\pi}{4}\right)\cos\theta + r\sin\left(\frac{\pi}{4}\right)\sin\theta = k$,

which is equivalent to $\frac{r\cos\theta}{\sqrt{2}} + \frac{r\sin\theta}{\sqrt{2}} = k$.

Setting $r\cos\theta = x$ and $r\sin\theta = y$, we obtain $x + y = \sqrt{2}k$, a Cartesian coordinate representation of the equation. Now we plot $y = \sqrt{2}k - x$.

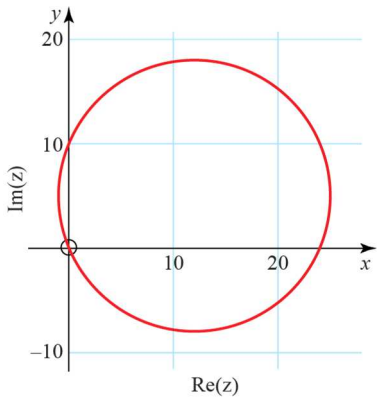


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3 a $|z-12-5i|=13$

So $|z-(12+5i)|=13$

This is a circle centred at $z=12+5i$ with radius 13.



b Letting $z=x+iy$ denote a complex number, the Cartesian equation for a circle centred at $(12,5)$ with radius 13 is $(x-12)^2+(y-5)^2=169$.

Converting this equation to polar coordinates we get

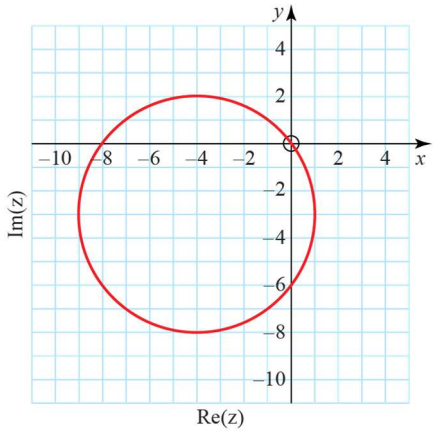
$(r\cos\theta-12)^2+(r\sin\theta-5)^2=169$

Then we expand and simplify to get $r^2-24r\cos\theta-10r\sin\theta=0$ which can be written as $r=24\cos\theta+10\sin\theta$ when $r\neq 0$.

4 a $|z+4+3i|=5$

So $|z-(-4-3i)|=5$

This is a circle centred at $z=-4-3i$ with radius 5.



b Letting $z=x+iy$ denote a complex number, the Cartesian equation for a circle centred at $(-4,-3)$ with radius 5 is $(x+4)^2+(y+3)^2=25$.

Converting this equation to polar coordinates we get

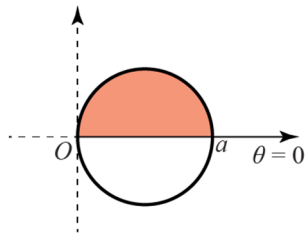
$(r\cos\theta+4)^2+(r\sin\theta+3)^2=25$.

Then we expand and simplify to get $r^2+8r\cos\theta+6r\sin\theta=0$ which can be written as $r=-8\cos\theta-6\sin\theta$ when $r\neq 0$.

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Polar coordinates 5C

1 a



$$r = a\cos\theta$$

$$\text{Area} = \frac{1}{2}a^2\int_0^{\frac{\pi}{2}}\cos^2\theta\,\mathrm{d}\theta$$

$\cos 2\theta = 2\cos^2\theta - 1$

$$= \frac{a^2}{4}\int_0^{\frac{\pi}{2}}(\cos 2\theta + 1)$$

$$= \frac{a^2}{4}\left[\frac{1}{2}\sin 2\theta + \theta\right]_0^{\frac{\pi}{2}}$$

$$= \frac{a^2}{4}\left[\left(0 + \frac{\pi}{2}\right) - (0)\right]$$

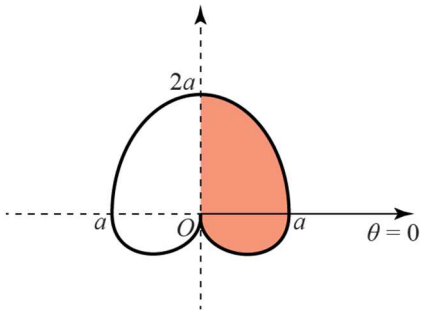
$$= \frac{\pi a^2}{8}$$

$r = a\cos\theta$ is a circle centre $\left(\frac{a}{2}, 0\right)$ and radius $\frac{a}{2}$.

The area of the semicircle is $\therefore \frac{1}{2}\pi\frac{a^2}{4} = \frac{a^2\pi}{8}$.

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1 b



$$r = a(1+\sin\theta)$$

$$\text{Area} = \frac{1}{2}a^2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}(1+2\sin\theta+\sin^2\theta)\,\mathrm{d}\theta$$

$$= \frac{1}{2}a^2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\left(1+2\sin\theta+\frac{1}{2}-\frac{1}{2}\cos 2\theta\right)\mathrm{d}\theta$$

Use $\cos 2\theta = 1-2\sin^2\theta$.

$$= \frac{1}{2}a^2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\left(\frac{3}{2}+2\sin\theta-\frac{1}{2}\cos 2\theta\right)\mathrm{d}\theta$$

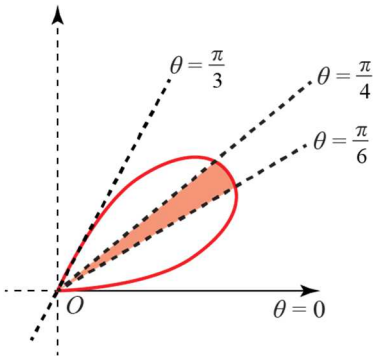
$$= \frac{1}{2}a^2\left[\frac{3}{2}\theta-2\cos\theta-\frac{1}{4}\sin 2\theta\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}}$$

$$= \frac{1}{2}a^2\left[\left(\frac{3\pi}{4}-0-0\right)-\left(-\frac{3\pi}{4}-0-0\right)\right]$$

$$= \frac{3\pi a^2}{4}$$

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1 c



$$r = a\sin 3\theta$$

$$\text{Area} = \frac{1}{2}a^2\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \sin^2 3\theta \,\mathrm{d}\theta$$

$$= \frac{a^2}{4}\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} (1-\cos 6\theta)\,\mathrm{d}\theta$$

Use $\cos 6\theta = 1 - 2\sin^2 3\theta$.

$$= \frac{a^2}{4}\left[\theta - \frac{1}{6}\sin 6\theta\right]_{\frac{\pi}{6}}^{\frac{\pi}{4}}$$

$$= \frac{a^2}{4}\left[\left(\frac{\pi}{4} - \frac{1}{6}\sin\frac{3\pi}{2}\right) - \left(\frac{\pi}{6} - \frac{1}{6}\sin\pi\right)\right]$$

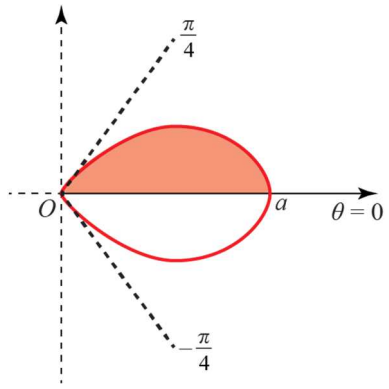
$$= \frac{a^2}{4}\left(\frac{\pi}{4} + \frac{1}{6} - \frac{\pi}{6}\right)$$

$$= \frac{a^2}{4}\left(\frac{\pi}{12} + \frac{2}{12}\right)$$

$$= \frac{(\pi+2)a^2}{48}$$

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1 d



$$r = a^2 \cos 2\theta$$

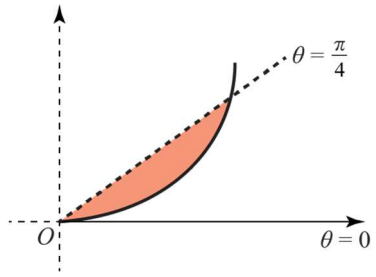
$$\text{Area} = \frac{1}{2}a^2 \int_0^{\frac{\pi}{4}} \cos 2\theta \, \mathrm{d}\theta$$

$$= \left[\frac{a^2}{4}\sin 2\theta\right]_0^{\frac{\pi}{4}}$$

$$= \left(\frac{a^2}{4}\sin\frac{\pi}{2}\right) - (0)$$

$$= \frac{a^2}{4}$$

e



$$r^2 = a^2 \tan\theta$$

$$\text{Area} = \frac{1}{2}a^2 \int_0^{\frac{\pi}{4}} \tan\theta \, \mathrm{d}\theta$$

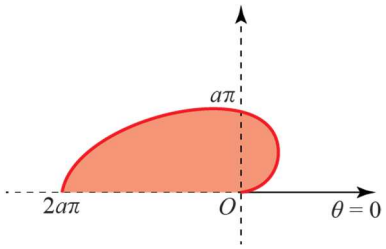
$$= \left[\frac{1}{2}a^2 \ln\sec\theta\right]_0^{\frac{\pi}{4}}$$

$$= \left(\frac{1}{2}a^2 \ln\sqrt{2}\right) - (0)$$

$$= \frac{a^2 \ln\sqrt{2}}{2} \quad \text{or} \quad \frac{a^2 \ln 2}{4}$$

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1 f



$$r = 2a\theta$$

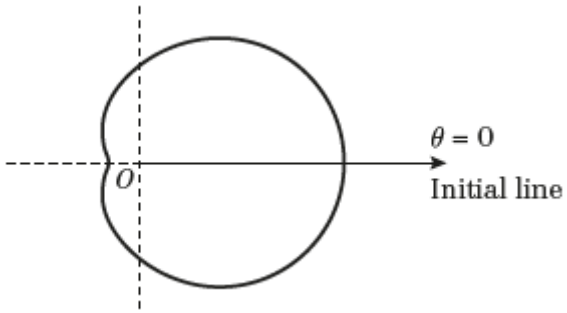
$$\text{Area} = \frac{1}{2}\int_0^{\pi} 4a^2\theta^2 \,\mathrm{d}\theta$$

$$= 2a^2\left[\frac{\theta^3}{3}\right]_0^{\pi}$$

$$= 2a^2\left[\left(\frac{\pi^3}{3}\right) - (0)\right]$$

$$= \frac{2a^2\pi^3}{3}$$

g



$$r = a(3 + 2\cos\theta)$$

$$\text{Area} = \frac{a^2}{2}\int_0^{\frac{\pi}{2}} (9 + 12\cos\theta + 4\cos^2\theta)\,\mathrm{d}\theta$$

$$= \frac{a^2}{2}\int_0^{\frac{\pi}{2}} (11 + 12\cos\theta + 2\cos 2\theta)\,\mathrm{d}\theta$$

Use $\cos 2\theta = 2\cos^2\theta - 1$.

$$= \frac{a^2}{2}\left[11\theta + 12\sin\theta + \sin 2\theta\right]_0^{\frac{\pi}{2}}$$

$$= \frac{a^2}{2}\left[\left(\frac{11\pi}{2} + 12 + 0\right) - (0)\right]$$

$$= \frac{a^2}{4}(11\pi + 24)$$

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2 $\text{Area} = \frac{1}{2}a^2\int_0^{2\pi}(p^2 + 2pq\cos\theta + q^2\cos^2\theta)\,\mathrm{d}\theta$

$= \frac{1}{2}a^2\int_0^{2\pi}\left(p^2 + 2pq\cos\theta + \frac{q^2}{2}\cos 2\theta + \frac{q^2}{2}\right)\mathrm{d}\theta$

Use $\cos 2\theta = 2\cos^2\theta - 1$.

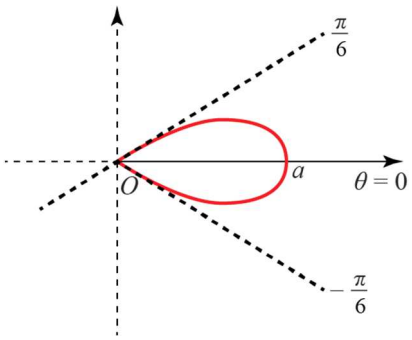
$= \frac{1}{2}a^2\int_0^{2\pi}\left(\left[\frac{2p^2+q^2}{2}\right] + 2pq\cos\theta + \frac{q^2}{2}\cos 2\theta\right)\mathrm{d}\theta$

$= \frac{1}{2}a^2\left[\left[\frac{2p^2+q^2}{2}\right]\theta + 2pq\sin\theta + \frac{q^2}{4}\sin 2\theta\right]_0^{2\pi}$

$= \frac{1}{2}a^2\left[\left(\left[\frac{2p^2+q^2}{2}\right]\pi \times 2 + 0 + 0\right) - (0)\right]$

$= \frac{a^2(2p^2+q^2)\pi}{2}$

3



$\text{Area} = \frac{1}{2}a^2\int_{-\frac{\pi}{6}}^{\frac{\pi}{6}}\cos^2 3\theta\,\mathrm{d}\theta = 2\times\frac{1}{2}a^2\int_0^{\frac{\pi}{6}}\cos^2 3\theta\,\mathrm{d}\theta$

$= \frac{a^2}{2}\int_0^{\frac{\pi}{6}}(1+\cos 6\theta)\,\mathrm{d}\theta$

Use $\cos 6\theta = 2\cos^2 3\theta - 1$.

$= \frac{a^2}{2}\left[\theta + \frac{1}{6}\sin 6\theta\right]_0^{\frac{\pi}{6}}$

$= \frac{a^2}{2}\left[\left(\frac{\pi}{6} + 0\right) - (0)\right]$

$= \frac{\pi a^2}{12}$

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4 In order to find the area enclosed by a single loop, we calculate

$$\frac{1}{2}\int_0^{2\pi} r^2 \,\mathrm{d}\theta = \frac{1}{2}\int_0^{2\pi} (a+5\sin\theta)^2 \,\mathrm{d}\theta$$

$$= \frac{1}{2}\int_0^{2\pi} \left(a^2 + 10a\sin\theta + 25\sin^2\theta\right) \mathrm{d}\theta$$

$$= \frac{1}{2}\int_0^{2\pi} \left(a^2 + 10a\sin\theta + \frac{25}{2}(1-\cos 2\theta)\right) \mathrm{d}\theta$$

$$= \frac{1}{2}\left[a^2\theta - 10a\cos\theta + \frac{25}{2}\theta - \frac{25}{4}\sin 2\theta\right]_0^{2\pi}$$

$$= \frac{1}{2}\left(2\pi a^2 + 25\pi\right).$$

In order to have $\frac{1}{2}\left(2\pi a^2 + 25\pi\right) = \frac{187\pi}{2}$ and $a > 0$, we solve to obtain $a = 9$

5 First we find the intersection points of these two curves.

$a\sin 4\theta = a\sin 2\theta$

$\sin 4\theta = \sin 2\theta$

$4\theta = 2\theta + 2k\pi$ or $4\theta = \pi - 2\theta + 2k\pi$

$2\theta = 2k\pi$ or $6\theta = \pi(1+2k)$

$\theta = k\pi$ or $\theta = \frac{\pi}{6}(1+2k)$

$(k \in \mathbb{Z})$.

Since we are working in the range $0 \leqslant \theta \leqslant \frac{\pi}{2}$, we only care about the intersections occurring at $\theta = 0$, $\theta = \frac{\pi}{6}$ and $\theta = \frac{\pi}{2}$. In fact we only have a positive value for $a\sin 4\theta$ between $0 \leqslant \theta \leqslant \frac{\pi}{4}$ and so the loop is only defined in this range. Note that $a\sin 4\theta = 0$ when $\theta = \frac{\pi}{4}$, which means that it intersects the origin.

Now we calculate the area to be

$$A = \frac{1}{2}\int_0^{\frac{\pi}{6}} (a\sin 2\theta)^2 \,\mathrm{d}\theta + \frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} (a\sin 4\theta)^2 \,\mathrm{d}\theta$$

$$= \frac{a^2}{4}\left(\int_0^{\frac{\pi}{6}} (1-\cos 4\theta)\,\mathrm{d}\theta + \int_{\frac{\pi}{6}}^{\frac{\pi}{4}} (1-\cos 8\theta)\,\mathrm{d}\theta\right)$$

$$= \frac{a^2}{4}\left(\left[\theta - \frac{\sin 4\theta}{4}\right]_0^{\frac{\pi}{6}} + \left[\theta - \frac{\sin 8\theta}{8}\right]_{\frac{\pi}{6}}^{\frac{\pi}{4}}\right)$$

$$= \frac{a^2}{4}\left(\left(\frac{\pi}{6} - \frac{\sqrt{3}}{8} - 0\right) + \left(\frac{\pi}{4} - \left(\frac{\pi}{6} + \frac{\sqrt{3}}{16}\right)\right)\right)$$

$$= \frac{a^2}{4}\left(\frac{\pi}{4} - \frac{3\sqrt{3}}{16}\right).$$

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6 The points of intersection are given by $1+\sin\theta = 3\sin\theta$

So $2\sin\theta = 1$ which means $\theta = \arcsin\left(\frac{1}{2}\right) = \frac{\pi}{6}$ and $\frac{5\pi}{6}$.

So the polar coordinates of intersection are $\left(\frac{3}{2}, \frac{\pi}{6}\right)$ and $\left(\frac{3}{2}, \frac{5\pi}{6}\right)$.

Since there is symmetry about the vertical axis, we may compute the area as

$$\text{Area} = 2\times\left(\frac{1}{2}\int_0^{\frac{\pi}{6}}(3\sin\theta)^2\,\mathrm{d}\theta + \frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{\pi}{2}}(1+\sin\theta)^2\,\mathrm{d}\theta\right)$$

$$= \int_0^{\frac{\pi}{6}}\left(9\sin^2\theta\right)\mathrm{d}\theta + \int_{\frac{\pi}{6}}^{\frac{\pi}{2}}\left(1+2\sin\theta+\sin^2\theta\right)\mathrm{d}\theta$$

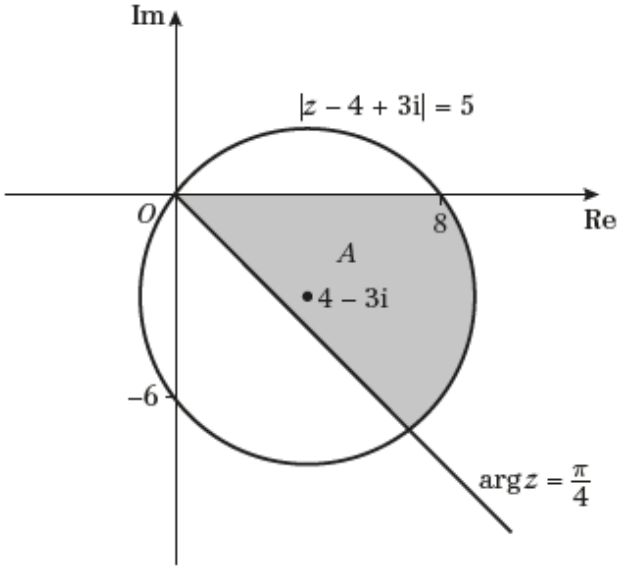
$$= \int_0^{\frac{\pi}{6}}\frac{9}{2}(1-\cos 2\theta)\,\mathrm{d}\theta + \int_{\frac{\pi}{6}}^{\frac{\pi}{2}}\left(1+2\sin\theta+\frac{1}{2}(1-\cos 2\theta)\right)\mathrm{d}\theta$$

$$= \left[\frac{9}{2}\theta - \frac{9\sin 2\theta}{4}\right]_0^{\frac{\pi}{6}} + \left[\frac{3}{2}\theta - 2\cos\theta - \frac{\sin 2\theta}{4}\right]_{\frac{\pi}{6}}^{\frac{\pi}{2}}$$

$$= \frac{3\pi}{4} - \frac{9\sqrt{3}}{8} + \frac{\pi}{2} + \frac{9\sqrt{3}}{8}$$

$$= \frac{5\pi}{4}$$

7 a The set of points $A = \left\{z : -\frac{\pi}{4} \leqslant \arg z \leqslant 0\right\} \cap \left\{z : |z-4+3i| \leqslant 5\right\}$ define a segment of a circle, centred at $(4,-3)$ with radius 5.



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7 b The equation for a circle centred at $(4,-3)$ with radius 5, in Cartesian coordinates is $(x-4)^2+(y+3)^2=25$. To find the area of the region bounded by A, $\theta=-\frac{\pi}{4}$ and $\theta=0$, we convert into polar coordinates, obtaining $(r\cos\theta-4)^2+(r\sin\theta+3)^2=25$. This simplifies to $r=8\cos\theta-6\sin\theta$ when $r\neq 0$. Now we calculate

$$A=\frac{1}{2}\int_{-\frac{\pi}{4}}^{0}(8\cos\theta-6\sin\theta)^2\,\mathrm{d}\theta$$

$$=\frac{1}{2}\int_{-\frac{\pi}{4}}^{0}(64\cos^2\theta-96\cos\theta\sin\theta+36\sin^2\theta)\mathrm{d}\theta$$

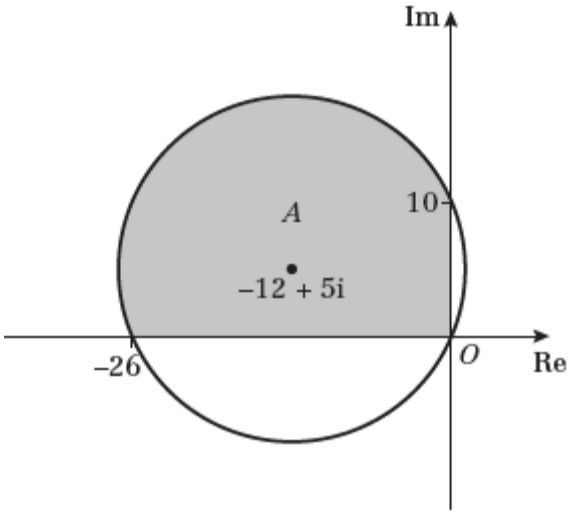
$$=\frac{1}{2}\int_{-\frac{\pi}{4}}^{0}(32(1+\cos2\theta)-48\sin2\theta+18(1-\cos2\theta))\mathrm{d}\theta$$

$$=\frac{1}{2}\left[(32(\theta+\tfrac{1}{2}\sin2\theta)+24\cos2\theta+18(\theta-\tfrac{1}{2}\sin2\theta))\right]_{-\frac{\pi}{4}}^{0}$$

$$=\frac{1}{2}\left(24+7+\frac{25\pi}{2}\right)$$

$$\approx 35.1$$

8 a The set of points $A=\left\{z:\frac{\pi}{2}\leqslant\arg z\leqslant\pi\right\}\cap\{z:|z+12-5i|\leqslant 13\}$ define a region of a circle, centred at $(-12,5)$ with radius 13.



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8 b To find the area of the region bounded by A, $\theta = \frac{\pi}{2}$ and $\theta = \pi$, we use the polar form $r = -24\cos\theta + 10\sin\theta$ to calculate

$$A = \frac{1}{2}\int_{\frac{\pi}{2}}^{\pi} (-24\cos\theta + 10\sin\theta)^2 \, \mathrm{d}\theta$$

$$= \frac{1}{2}\int_{\frac{\pi}{2}}^{\pi} (576\cos^2\theta - 480\cos\theta\sin\theta + 100\sin^2\theta) \mathrm{d}\theta$$

$$= \frac{1}{2}\int_{\frac{\pi}{2}}^{\pi} (288(1+\cos 2\theta) - 240\sin 2\theta + 50(1-\cos 2\theta)) \mathrm{d}\theta$$

$$= \frac{1}{2}\left[(288(\theta + \tfrac{1}{2}\sin 2\theta) + 120\cos 2\theta + 50(\theta - \tfrac{1}{2}\sin 2\theta))\right]_{\frac{\pi}{2}}^{\pi}$$

$$= \frac{1}{2}(338\pi + 120 - 169\pi + 120)$$

$$\approx 385.$$

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9 In order to find the area of the shaded region, we must find the area of the sector bounded by the curve and the line OA, then subtract the area of the triangle OAB. The value of θ at the point A can be found by solving $r = 1 + \cos 3\theta = \frac{2+\sqrt{2}}{2}$ in order to get $\theta = \frac{\pi}{12}$.

We now find the area of the sector bounded by the curve and the line OA.

$$A_{sector} = \frac{1}{2}\int_0^{\frac{\pi}{12}} (1+\cos 3\theta)^2 \, \mathrm{d}\theta$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{12}} (1 + 2\cos 3\theta + \cos^2 3\theta)\mathrm{d}\theta$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{12}} (1 + 2\cos 3\theta + \tfrac{1}{2}(\cos 6\theta + 1))\mathrm{d}\theta$$

$$= \frac{1}{2}\left[(\theta + \tfrac{2}{3}\sin 3\theta + \tfrac{1}{2}(\tfrac{1}{6}\sin 6\theta + \theta))\right]_0^{\frac{\pi}{12}}$$

$$= \frac{1}{2}\left(\frac{\pi}{8} + \frac{\sqrt{2}}{3} + \frac{1}{12}\right)$$

$$\approx 0.47372.$$

Now we find the area of the triangle OAB by using the formula

$Area_{OAB} = \frac{1}{2}ab\sin C$, where

a is the length of OA,
b is the length of OB and
C is the angle between OA and OB.

$$Area_{OAB} = \frac{1}{2}ab\sin C$$

$$= \frac{1}{2} \times \frac{2+\sqrt{2}}{2} \times \frac{2+\sqrt{2}}{2} \times \sin\left(\frac{\pi}{12}\right)$$

$$\approx 0.37713.$$

Thus, the area of the shaded region is found to be

$$A = Area_{sector} - Area_{OAB}$$

$$\approx 0.47372 - 0.37713$$

$$= 0.0966 \text{ (3sf)}$$

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10 Note that there is symmetry about the vertical axis and so we may compute one side and multiply by 2. In order to find the area of the shaded region, we first must find the area of the sector of $r = 1 + \sin\theta$ between $\theta = -\pi$ and the right hand side intersection point of the two curves.

This intersection point occurs when $1 + \sin\theta = 3\sin\theta$ i.e. when $\theta = \frac{\pi}{6}$.

So now we find the area of the sector of $r = 1 + \sin\theta$ for $-\pi \leqslant \theta \leqslant \frac{\pi}{6}$. We will denote this area A_1.

$$A_1 = \frac{1}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{6}} (1+\sin\theta)^2 \,\mathrm{d}\theta$$

$$= \frac{1}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{6}} (1+2\sin\theta+\sin^2\theta)\mathrm{d}\theta$$

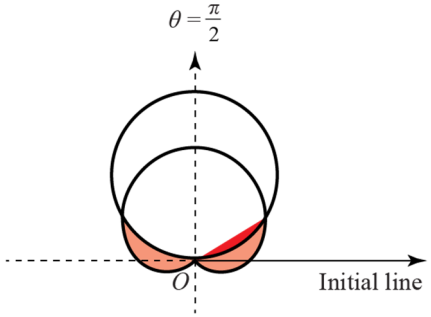
$$= \frac{1}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{6}} (1+2\sin\theta+\tfrac{1}{2}(1-\cos 2\theta))\mathrm{d}\theta$$

$$= \frac{1}{2}\left[(\theta - 2\cos\theta + \tfrac{1}{2}(\theta - \tfrac{1}{2}\sin 2\theta))\right]_{-\frac{\pi}{2}}^{\frac{\pi}{6}}$$

$$= \frac{1}{2}\left(\pi - \frac{9\sqrt{3}}{8}\right)$$

$$\approx 0.59652.$$

This integral has included a small extra region we do not want (the red region in the image)



We find the value of the unwanted regions area (which we will denote A_2) by

$$A_2 = \frac{1}{2}\int_{0}^{\frac{\pi}{6}} (3\sin\theta)^2 \,\mathrm{d}\theta$$

$$= \frac{1}{2}\int_{0}^{\frac{\pi}{6}} (9\sin^2\theta)\mathrm{d}\theta$$

$$= \frac{9}{4}\int_{0}^{\frac{\pi}{6}} (1-\cos 2\theta)\mathrm{d}\theta$$

$$= \frac{9}{4}\left[(\theta - \tfrac{1}{2}\sin 2\theta)\right]_{0}^{\frac{\pi}{6}}$$

$$= \frac{9}{4}\left(\frac{\pi}{6} - \frac{\sqrt{3}}{4}\right)$$

$$\approx 0.20382.$$

So the right hand side of the shaded region has area

$$A_{right} \approx 0.59652 - 0.20382$$

$$= 0.3927.$$

Remember that this is only half of the total region we want to find, so that means we just need to double A_{right} in order to find $A_{total} = 0.79$ (2 d.p.)

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Challenge

a The far left hand side of the shell is a point which occurs when $\theta = 3\pi$ and so we can set $r_{left} = 3k\pi$.

Similarly for the far right hand side of the shell, which occurs at $\theta = 4\pi$, giving $r_{right} = 4k\pi$.

Thus the horizontal diameter is

$d = 7k\pi = 3$ cm, so we conclude that $k = \dfrac{3}{7\pi}$

b In order to find the total area of the cross section, we must ensure that all angles are covered exactly once. So we choose to integrate between $2\pi \leqslant \theta \leqslant 4\pi$.

$$A = \frac{1}{2}\int_{2\pi}^{4\pi}\left(\frac{3\theta}{7\pi}\right)^2 \mathrm{d}\theta$$

$$= \frac{1}{2}\int_{2\pi}^{4\pi}\left(\frac{9\theta^2}{49\pi^2}\right)\mathrm{d}\theta$$

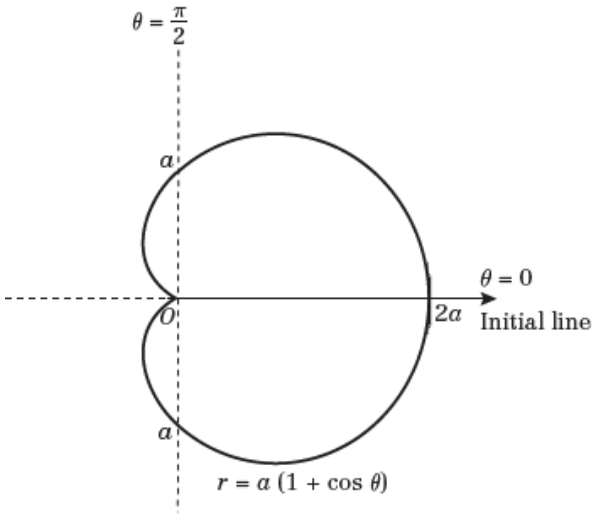
$$= \frac{1}{2}\left[\frac{3\theta^3}{49\pi^2}\right]_{2\pi}^{4\pi}$$

$$= \frac{3}{98}(64\pi - 8\pi)$$

$$= \frac{12\pi}{7}$$

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Polar coordinates 5D



1 $r = a(1+\cos\theta)$

Require $\dfrac{\mathrm{d}}{\mathrm{d}\theta}(r\cos\theta) = 0$

i.e. $\dfrac{\mathrm{d}}{\mathrm{d}\theta}(a\cos\theta + a\cos^2\theta) = a\left(-\sin\theta - 2\cos\theta\sin\theta\right)$

So $0 = -a\sin\theta\left(1+2\cos\theta\right)$

$\sin\theta = 0 \Rightarrow \quad \theta = 0,\ \pi$ (from sketch π is not allowed)

$\cos\theta = -\dfrac{1}{2} \Rightarrow \theta = \pm\dfrac{2\pi}{3} \Rightarrow r = a\left(1-\dfrac{1}{2}\right) = \dfrac{a}{2}$

$\therefore$ points are $(2a,\ 0)$ and $\left(\dfrac{a}{2}, \dfrac{2\pi}{3},\right)\left(\dfrac{a}{2}, \dfrac{-2\pi}{3}\right)$

2 $r = \mathrm{e}^{2\theta}$

a $x = r\cos\theta = \mathrm{e}^{2\theta}\cos\theta$

$\dfrac{\mathrm{d}x}{\mathrm{d}\theta} = 0 \Rightarrow 0 = 2\mathrm{e}^{2\theta}\cos\theta - \mathrm{e}^{2\theta}\sin\theta$

$0 = \mathrm{e}^{2\theta}(2\cos\theta - \sin\theta)$

$\Rightarrow \quad \tan\theta = 2$

$\therefore \quad \theta = 1.107$ (rads)

$r = \mathrm{e}^{2\times 1.107} = 9.1549\ldots$

So at (9.15, 1.11) the tangent is perpendicular to initial line.

b $y = r\sin\theta = \mathrm{e}^{2\theta}\sin\theta$

$\dfrac{\mathrm{d}y}{\mathrm{d}\theta} = 0 \Rightarrow 0 = 2\mathrm{e}^{2\theta}\sin\theta + \mathrm{e}^{2\theta}\cos\theta$

$0 = \mathrm{e}^{2\theta}(2\sin\theta + \cos\theta)$

$\Rightarrow \quad \tan\theta = -\dfrac{1}{2}$

$\therefore \quad \theta = 2.6779\ldots$

$r = \mathrm{e}^{2\times 2.6779\ldots} = 211.852\ldots$

So at (212, 2.68) the tangent is parallel to initial line.

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3 $r = a\cos 2\theta$

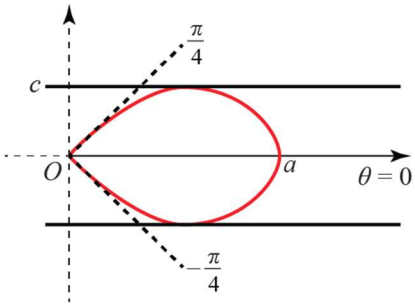
a $y = r\sin\theta = a\sin\theta\cos 2\theta$

$$\frac{dy}{d\theta} = 0 \Rightarrow 0 = a[\cos\theta\cos 2\theta - 2\sin 2\theta\sin\theta]$$

$$0 = a\cos\theta[\cos 2\theta - 4\sin^2\theta]$$

$$0 = a\cos\theta[\cos^2\theta - 5\sin^2\theta]$$

$\cos\theta = \Rightarrow \theta = \frac{\pi}{2}$ (outside range)

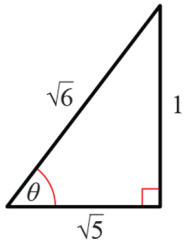


$\therefore \quad \tan^2\theta = \frac{1}{5} \Rightarrow \tan\theta = \pm\frac{1}{\sqrt{5}}$

$\theta = \pm 0.42053\ldots$

$$r = a\left(\cos^2\theta - \sin^2\theta\right) = a\left(\frac{5}{6} - \frac{1}{6}\right) = \frac{2a}{3}$$

$\therefore$ points are $\left(\frac{2a}{3}, \pm 0.421\right)$



b The lines are $y = \pm c$ where $c = r\sin(0.42053\ldots)$

$$= \frac{2a}{3} \times \frac{1}{\sqrt{6}} = \frac{a\sqrt{6}}{9}$$

The line $y = c$ is $r\sin\theta = \frac{a\sqrt{6}}{9}$

$\therefore$ Tangents have equations $r = \pm\frac{a\sqrt{6}}{9}\operatorname{cosec}\theta$

4 $r = a(7 + 2\cos\theta)$

$y = r\sin\theta = a(7\sin\theta + 2\cos\theta\sin\theta)$

$y = a(7\sin\theta + \sin 2\theta)$

$$\frac{dy}{d\theta} = 0 \Rightarrow 0 = a(7\cos\theta + 2\cos 2\theta)$$

$\Rightarrow \quad 0 = 4\cos^2\theta + 7\cos\theta - 2$

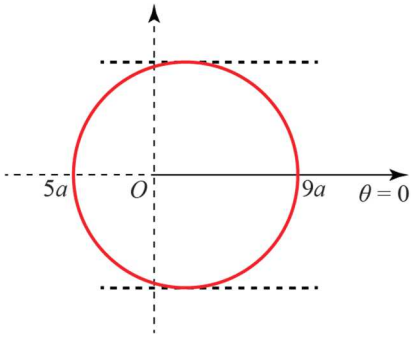
$0 = (4\cos\theta - 1)(\cos\theta + 2)$

$\cos\theta = \frac{1}{4}$ (or -2)

$\Rightarrow \quad \theta = \pm 1.318\ldots$

$$r = a\left(7 + \frac{2}{4}\right) = \frac{15}{2}a$$

$\therefore$ tangents are parallel at $\left(\frac{15}{2}a, \pm 1.32\right)$



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5 $r = 2 + \cos\theta$

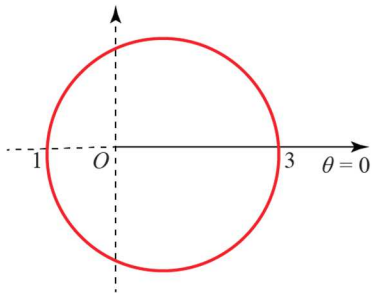
$x = r\cos\theta = 2\cos\theta + \cos^2\theta$

$\dfrac{dx}{d\theta} = 0 \Rightarrow 0 = -2\sin\theta - 2\cos\theta\sin\theta$

$0 = -2\sin\theta(1 + \cos\theta)$

$\sin\theta = 0 \quad \Rightarrow \theta = 0,\ \pi$

$\cos\theta = -1 \Rightarrow \theta = \pi$



$\therefore$ tangents are perpendicular to the initial line at:

$(3, 0)$ and $(1, \pi)$

The equations are

$r\cos\theta = 3$ $\qquad$ $r\cos\theta = -1$

$r = 3\sec\theta$ $\qquad$ $r = -\sec\theta$

6 $r = a(1 + \tan\theta)$

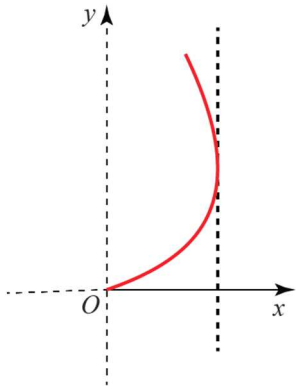
$x = r\cos\theta = a(\cos\theta + \sin\theta)$

$\dfrac{dx}{d\theta} = 0 \Rightarrow \quad 0 = a(-\sin\theta + \cos\theta)$

$\Rightarrow \tan\theta = 1$

$\Rightarrow \quad \theta = \dfrac{\pi}{4}$

$\therefore$ point is $\left(2a, \dfrac{\pi}{4}\right)$



7 In order to find the length of the line OA, first we must find where the point is.

We find an expression for

$y = r\sin\theta$

$= \sin\theta + 3\cos\theta\sin\theta.$

Now differentiating with respect to θ,

$\dfrac{dy}{d\theta} = \cos\theta + 3\cos^2\theta - 3\sin^2\theta.$

Rearrange in terms of $\cos\theta$, then set equal to 0 and solve

$6\cos^2\theta + \cos\theta - 3 = 0$

$\cos\theta = \dfrac{-1 + \sqrt{73}}{12}$

We neglect the negative term since $0 \leqslant \theta \leqslant \dfrac{\pi}{2}$.

Substituting this back into the given expression for r, we find that

$r = 1 + 3\left(\dfrac{-1 + \sqrt{73}}{12}\right)$

$= \dfrac{3 + \sqrt{73}}{4}$

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8 First we need to find the points for which the tangent to the curve is perpendicular to the initial line. We form an expression for x and differentiate with respect to θ.

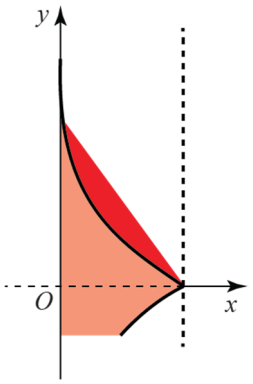
$$x = r\cos\theta$$
$$= 2\cos\theta + 2\cos^2\theta$$
$$\frac{\mathrm{d}x}{\mathrm{d}\theta} = -2\sin\theta - 4\cos\theta\sin\theta$$
$$= -2\sin\theta(1 + 2\cos\theta).$$

We now solve equal to 0 in order to find our required θ values. We choose to neglect the solutions coming from $\sin\theta = 0$ factor, although $\theta = k\pi$ are clearly tangent to the curve and perpendicular to the initial line, they are not the tangents we are looking for. This can be clearly seen by looking at the diagram.

So, $1 + 2\cos\theta = 0$ gives $\theta = \pm\frac{2\pi}{3}$.

Since we have symmetry about the horizontal axis, we may compute the top half of the region and then double it later on.

To find the area of the top region, we will need to find the area of the sector that lies between $\frac{2\pi}{3} \leqslant \theta \leqslant \pi$ as shown in the diagram (red region).



$$A_{sector} = \frac{1}{2}\int_{\frac{2\pi}{3}}^{\pi} (2(1+\cos\theta))^2\,\mathrm{d}\theta$$
$$= 2\int_{\frac{2\pi}{3}}^{\pi} (1 + 2\cos\theta + \cos^2\theta)\mathrm{d}\theta$$
$$= 2\int_{\frac{2\pi}{3}}^{\pi} (1 + 2\cos\theta + \tfrac{1}{2}(1+\cos 2\theta))\mathrm{d}\theta$$
$$= 2\left[\theta + 2\sin\theta + \tfrac{1}{2}(\theta + \tfrac{1}{2}\sin 2\theta)\right]_{\frac{2\pi}{3}}^{\pi}$$
$$= \pi - \frac{7\sqrt{3}}{4}$$

Now we find the area of the right-angle triangle bounded by the horizontal axis, the tangent and the line $\theta = \frac{2\pi}{3}$.

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8 Continued

Using the formula

$$A_{tri} = \frac{1}{2} \times Base \times Height$$

$$= \frac{1}{2} |x||y|$$

$$= \frac{1}{2} r^2 |\cos\theta||\sin\theta|$$

$$= 2(1+\cos\theta)^2 |\cos\theta||\sin\theta|$$

and substituting in $\theta = \frac{2\pi}{3}$, we find that

$$A_{tri} = 2\left(1-\tfrac{1}{2}\right)^2 \left(\tfrac{1}{2}\right)\left(\tfrac{\sqrt{3}}{2}\right)$$

$$= \frac{\sqrt{3}}{8}$$

So, our upper shaded region is

$$A_{top} = A_{tri} - A_{sector}$$

$$= \frac{\sqrt{3}}{8} - \left(\pi - \frac{7\sqrt{3}}{4}\right)$$

$$= \frac{15\sqrt{3}}{8} - \pi.$$

Which gives our total shaded region as

$$A = 2A_{top}$$

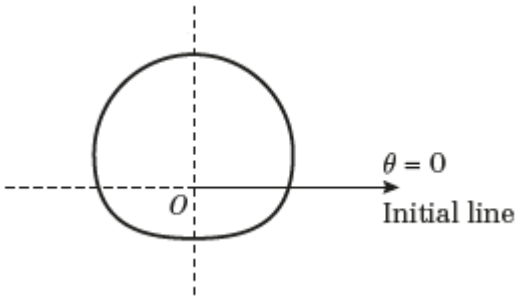
$$= \frac{15\sqrt{3}}{4} - 2\pi$$

$$\approx 0.212$$

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Polar coordinates – mixed exercise 5

1



$$r = a\left(1+\frac{1}{2}\sin\theta\right)$$

$$\text{Area} = \frac{1}{2}a^2\int_0^{2\pi}\left(1+\frac{1}{2}\sin\theta\right)^2 \mathrm{d}\theta$$

$$= \frac{a^2}{2}\int_0^{2\pi}\left(1+\sin\theta+\frac{1}{4}\sin^2\theta\right)\mathrm{d}\theta$$

$$= \frac{a^2}{2}\int_0^{2\pi}\left(\frac{9}{8}+\sin\theta-\frac{\cos 2\theta}{8}\right)\mathrm{d}\theta$$

Use $\cos 2\theta = 1-2\sin^2\theta$.

$$= \frac{a^2}{2}\left[\frac{9}{8}\theta-\cos\theta-\frac{\sin 2\theta}{16}\right]_0^{2\pi}$$

$$= \frac{a^2}{2}\left[\left(\frac{9\pi}{4}-1-0\right)-(0-1-0)\right]$$

$$= \frac{9\pi a^2}{8}$$

2 $OB = 2a\sec\alpha$

$OA = a(1+\cos\alpha)$

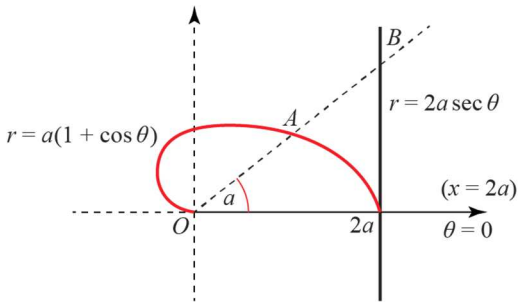
$2OA = OB \Rightarrow 1+\cos\alpha = \sec\alpha$

$$\cos^2\alpha+\cos\alpha-1=0$$

$$\cos\alpha = \frac{-1\pm\sqrt{1+4}}{2}$$

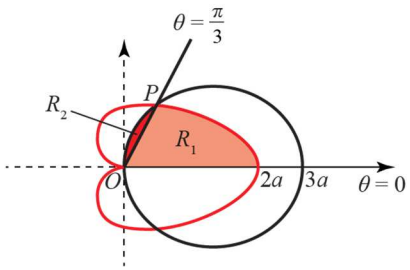
$\therefore$ α is acute.

$$\cos\alpha = \frac{\sqrt{5}-1}{2}$$



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3 First find P:



$$1+\cos\theta = 3\cos\theta$$
$$1 = 2\cos\theta$$
$$\Rightarrow \quad \theta = \arccos\frac{1}{2} = \frac{\pi}{3}$$

By symmetry the required area $= 2(R_1 + R_2)$

$$R_1 = \frac{1}{2}\int_0^{\frac{\pi}{3}} (1+\cos\theta)^2 \,\mathrm{d}\theta = \frac{1}{2}\int_0^{\frac{\pi}{3}} (1+2\cos\theta+\cos^2\theta)\,\mathrm{d}\theta$$

$$R_1 = \frac{1}{2}\int_0^{\frac{\pi}{3}} \left(\frac{3}{2}+2\cos\theta+\frac{\cos 2\theta}{2}\right)\mathrm{d}\theta$$

Use $\cos 2\theta = 2\cos^2\theta - 1$.

$$= \frac{1}{2}\left[\frac{3}{2}\theta + 2\sin\theta + \frac{1}{4}\sin 2\theta\right]_0^{\frac{\pi}{3}}$$
$$= \frac{1}{2}\left[\left(\frac{\pi}{2}+2\sin\frac{\pi}{3}+\frac{1}{4}\sin\frac{2\pi}{3}\right)-(0)\right]$$
$$= \frac{1}{2}\left[\frac{\pi}{2}+\sqrt{3}+\frac{\sqrt{3}}{8}\right] = \frac{\pi}{4}+\frac{9\sqrt{3}}{16}$$

$$R_2 = \frac{9}{2}\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \cos^2\theta\,\mathrm{d}\theta = \frac{9}{4}\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} (1+\cos 2\theta)\,\mathrm{d}\theta$$

$$= \frac{9}{4}\left[\theta+\frac{1}{2}\sin 2\theta\right]_{\frac{\pi}{3}}^{\frac{\pi}{2}} = \frac{9}{4}\left[\left(\frac{\pi}{2}+0\right)-\left(\frac{\pi}{3}+\frac{\sqrt{3}}{4}\right)\right]$$
$$= \frac{3\pi}{8}-\frac{9\sqrt{3}}{16}$$

$$\therefore \text{ Area required} = 2\left(\frac{3\pi}{8}+\frac{\pi}{4}\right) = \frac{5\pi}{4}$$

4 $r^2 = a^2\sin 2\theta$ (must have $\sin 2\theta \geq 0$)

$$r = a\sqrt{\sin 2\theta}$$
$$x = r\cos\theta = a\cos\theta\sqrt{\sin 2\theta}$$
$$\frac{\mathrm{d}x}{\mathrm{d}\theta} = 0 \Rightarrow 0 = -\sin\theta\sqrt{\sin 2\theta} + \frac{1}{2}\cos\theta\frac{1}{\sqrt{\sin 2\theta}}2\cos 2\theta$$

i.e. $\quad 0 = -\sin\theta\times\sin 2\theta + \cos\theta\cos 2\theta$

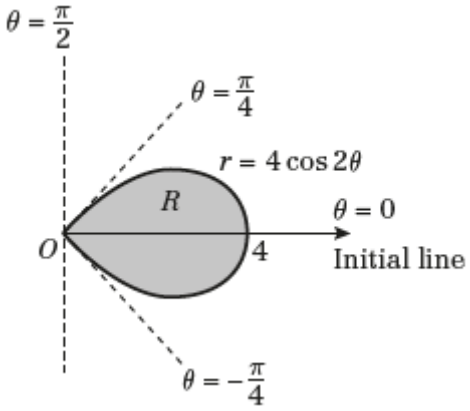
i.e. $\quad 0 = \cos 3\theta$

$$\therefore \quad 3\theta = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{7\pi}{2}$$
$$\therefore \quad \theta = \frac{\pi}{6}, \frac{\pi}{2}, \frac{7\pi}{6}$$

So $\left(a\sqrt{\frac{\sqrt{3}}{2}}, \frac{\pi}{6}\right), \left(a\sqrt{\frac{\sqrt{3}}{2}}, \frac{7\pi}{6}\right)$ and $\left(0, \frac{\pi}{2}\right)$

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5 a



b $\text{Area} = 2 \times \frac{1}{2}\int_0^{\frac{\pi}{4}} 16\cos^2 2\theta \, \mathrm{d}\theta$

$= \int_0^{\frac{\pi}{4}} (8 + 8\cos 4\theta)\mathrm{d}\theta$

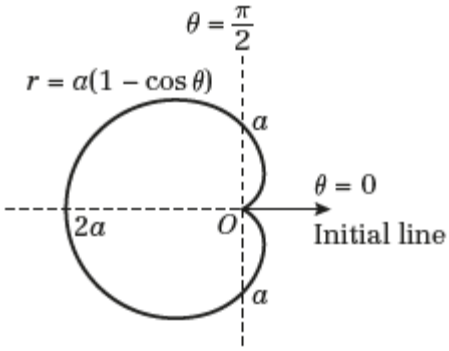
$= \left[8\theta + 2\sin 4\theta\right]_0^{\frac{\pi}{4}}$

$= 2\pi + 0 - 0$

$= 2\pi$

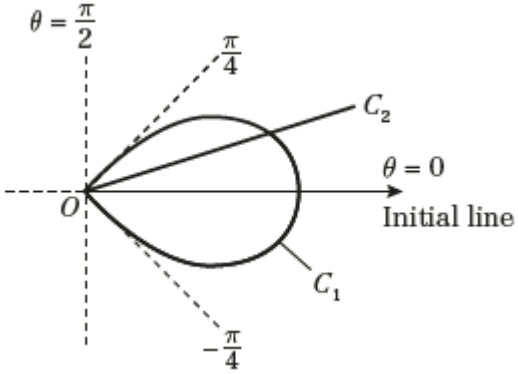
$2\cos^2\theta = 1 + \cos 2\theta.$

6



Max r is $2a$ at point $(2a, \pi)$

7 a



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7 b $\text{Area} = \frac{1}{2}\int_{\frac{\pi}{12}}^{\frac{\pi}{4}} 4\cos^2 2\theta$

$= \int_{\frac{\pi}{12}}^{\frac{\pi}{4}} (1+\cos 4\theta)\,\mathrm{d}\theta$

$= \left[\theta + \frac{1}{4}\sin 4\theta\right]_{\frac{\pi}{12}}^{\frac{\pi}{4}}$

$= \left(\frac{\pi}{4}+0\right) - \left(\frac{\pi}{12}+\frac{1}{4}\sin\frac{\pi}{3}\right)$

$= \frac{\pi}{6} - \frac{1}{4}\times\frac{\sqrt{3}}{2}$

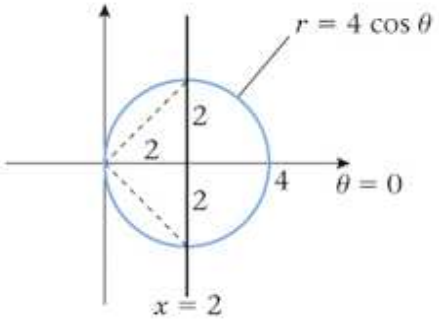
$= \frac{\pi}{6} - \frac{\sqrt{3}}{8}$

$\cos 4\theta = 2\cos^2 2\theta - 1$

8 a $r = 2\sec\theta$

$r\cos\theta = 2$

$x = 2$



b $x = 2$ is a diameter

$r = \sqrt{2^2+2^2} = 2\sqrt{2}$

So polar coordinates are

$\left(2\sqrt{2}, \frac{\pi}{4}\right)$ $\left(2\sqrt{2}, -\frac{\pi}{4}\right)$

9 a $a(1+\cos\theta) = 3a\cos\theta$

$1 = 2\cos\theta$

$\cos\theta = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{3}$

So P is $\left(\frac{3}{2}a, \frac{\pi}{3}\right)$

b $\text{Area} = \frac{a^2}{2}\int_0^{\frac{\pi}{3}}\left(\frac{3}{2}+2\cos\theta+\frac{\cos 2\theta}{2}\right)\mathrm{d}\theta + \frac{9}{2}a^2\int_{\frac{\pi}{3}}^{\frac{\pi}{2}}\cos^2\theta\,\mathrm{d}\theta$

$= \frac{a^2}{2}\left[\frac{3}{2}\theta + 2\sin\theta + \frac{1}{4}\sin 2\theta\right]_0^{\frac{\pi}{3}} + \frac{9}{4}a^2\left[\theta+\frac{1}{2}\sin 2\theta\right]_{\frac{\pi}{3}}^{\frac{\pi}{2}}$

$= \frac{a^2}{2}\left[\frac{\pi}{2}+\sqrt{3}+\frac{\sqrt{3}}{8}\right] + \frac{9}{4}a^2\left[\frac{\pi}{6}-\frac{\sqrt{3}}{4}\right]$

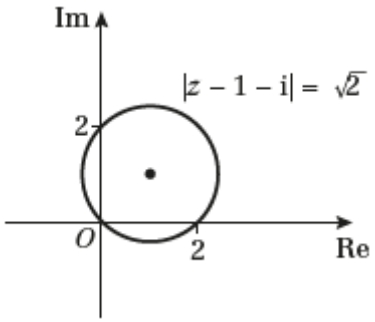
$= \frac{5\pi}{8}a^2$

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10 a
$$r^2 = \sec 2\theta$$
$$r^2 \cos 2\theta = 1$$
$$r^2(2\cos^2\theta - 1) = 1$$
$$2r^2\cos^2\theta = 1 + r^2$$
$$2x^2 = 1 + x^2 + y^2$$
$$\therefore \quad y^2 = x^2 - 1$$

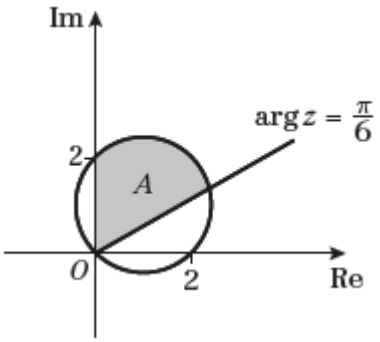
b
$$r^2 = \operatorname{cosec} 2\theta$$
$$\Rightarrow \quad r^2 \sin 2\theta = 1$$
$$\Rightarrow 2r\sin\theta\, r\cos\theta = 1$$
$$\Rightarrow \quad 2xy = 1$$
$$y = \frac{1}{2x}$$

11 a $|z - 1 - i| = \sqrt{2}$ is a circle centred at $(1,1)$ with radius $\sqrt{2}$.



b The Cartesian equation of a circle centred at $(1,1)$ with radius $\sqrt{2}$ is $(x-1)^2 + (y-1)^2 = 2$. Converting this to polar coordinates gives $(r\cos\theta - 1)^2 + (r\sin\theta - 1)^2 = 2$ which simplifies to $r = 2\cos\theta + 2\sin\theta$ when $r \neq 0$.

c The set of points $A = \left\{z : \frac{\pi}{6} \leqslant \arg z \leqslant \frac{\pi}{2}\right\} \cap \left\{z : |z - 1 - i| \leqslant \sqrt{2}\right\}$ is the green sector of the circle. It represents the intersection of all possible $\arg z$ such that $\frac{\pi}{6} \leqslant \arg z \leqslant \frac{\pi}{2}$ and the red circle which represents all z such that $|z - 1 - i| \leqslant \sqrt{2}$.



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11 d In order to find the area of the region bounded between the lines $\theta = \frac{\pi}{6}$, $\theta = \frac{\pi}{2}$ and the arc A, we calculate

$$Area = \frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} (2\cos\theta + 2\sin\theta)^2 \,\mathrm{d}\theta$$

$$= 2\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} (\cos\theta + \sin\theta)^2 \,\mathrm{d}\theta$$

$$= 2\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} (1 + 2\cos\theta\sin\theta)\,\mathrm{d}\theta$$

$$= 2\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} (1 + \sin 2\theta)\mathrm{d}\theta$$

$$= 2\left[\theta - \frac{\cos 2\theta}{2}\right]_{\frac{\pi}{6}}^{\frac{\pi}{2}}$$

$$= 2\left(\frac{\pi}{3} + \frac{3}{4}\right)$$

$$\approx 3.59 (3 \text{ s.f.})$$

12 In order to find the area of the shaded region, we must find the area of the sector bounded by the curve and the line OA, then subtract the area of the triangle OAB. The value of θ at the point A can be found by solving $r = 4\cos 2\theta = 2$, leading to $\theta = \frac{\pi}{6}$.

We now find the area of the sector bounded by the curve and the line OA.

$$A_{sector} = \frac{1}{2}\int_{0}^{\frac{\pi}{6}} (4\cos 2\theta)^2 \,\mathrm{d}\theta$$

$$= 8\int_{0}^{\frac{\pi}{6}} (\cos^2 2\theta)\mathrm{d}\theta$$

$$= 4\int_{0}^{\frac{\pi}{6}} (\cos 4\theta + 1)\mathrm{d}\theta$$

$$= 4\left[\left(\tfrac{1}{4}\sin 4\theta + \theta\right)\right]_{0}^{\frac{\pi}{6}}$$

$$= \frac{\sqrt{3}}{2} + \frac{2\pi}{3}$$

$$\approx 2.9604$$

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12 Now we find the area of the triangle OAB by using the formula

$$Area_{OAB} = \frac{1}{2} \times Base \times Height$$

$$= \frac{1}{2} |x_A||y_A|$$

where

x_A is the x coordinate of A and

x_B is the y coordinate of A.

$$Area_{OAB} = \frac{1}{2} |x_A||y_A|$$

$$= \frac{1}{2} |r\cos\theta||r\sin\theta|$$

$$= \frac{1}{2} |4\cos 2\theta\cos\theta||4\cos 2\theta\sin\theta|$$

$$= 8|\cos\tfrac{\pi}{3}\cos\tfrac{\pi}{6}||\cos\tfrac{\pi}{3}\sin\tfrac{\pi}{6}|$$

$$= \frac{\sqrt{3}}{2}$$

$$\approx 0.86603$$

Thus, the area of the shaded region is found to be

$A = Area_{sector} - Area_{OAB}$

$\approx 2.9604 - 0.86603$

≈ 2.09 (3 s.f.)

13 First we need to find the point for which the tangent to the curve is perpendicular to the initial line. We form an expression for x and differentiate with respect to θ.

$$x = r\cos\theta$$

$$= 4\sin 2\theta\cos\theta$$

$$\frac{dx}{d\theta} = 8\cos 2\theta\cos\theta - 4\sin 2\theta\sin\theta$$

$$= 8(2\cos^2\theta - 1)\cos\theta - 8\cos\theta\sin^2\theta$$

$$= 24\cos^3\theta - 16\cos\theta$$

$$= 8\cos\theta(3\cos^2\theta - 2).$$

We now solve equal to 0 in order to find our required θ values. We choose to neglect the solutions arising from the $\cos\theta = 0$ factor, since a tangent at the origin is not what we are looking for, even though it is perpendicular to the initial line.

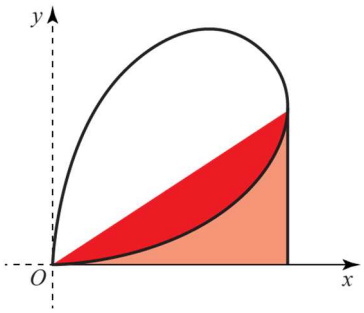
So, $3\cos^2\theta - 2 = 0$ gives $\cos\theta = \pm\sqrt{\frac{2}{3}}$ and we choose to neglect the negative solution since $0 \leqslant \theta \leqslant \frac{\pi}{2}$.

Thus our tangent perpendicular to the initial line occurs at $\theta = \theta_A = \arccos\left(\sqrt{\frac{2}{3}}\right)$.

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13 Continued

To find the area of the region, we will need to find the area of the sector that lies between $0 \leqslant \theta \leqslant \theta_A$ as shown in the diagram (red region).



$$A_{sector} = \frac{1}{2}\int_0^{\theta_A} (4\sin 2\theta)^2 \, \mathrm{d}\theta$$

$$= 8\int_0^{\theta_A} (\sin^2 2\theta)\mathrm{d}\theta$$

$$= 4\int_0^{\theta_A} (1-\cos 4\theta)\mathrm{d}\theta$$

$$= \left[4\theta - \sin 4\theta\right]_0^{\theta_A}$$

$$= 4\theta_A - \sin 4\theta_A$$

$$= 4\arccos\left(\sqrt{\frac{2}{3}}\right) - \sin\left(4\arccos\left(\sqrt{\frac{2}{3}}\right)\right)$$

$$\approx 1.8334.$$

Now we find the area of the right-angle triangle bounded by the horizontal axis, the tangent and the line $\theta = \theta_A$.

Using the formula

$$A_{tri} = \frac{1}{2} \times Base \times Height$$

$$= \frac{1}{2}|x||y|$$

$$= \frac{1}{2}r^2|\cos\theta||\sin\theta|$$

$$= 8(\sin^2 2\theta)|\cos\theta||\sin\theta|$$

and substituting in $\theta = \theta_A$, we find that $A_{tri} = \frac{64\sqrt{2}}{27}$.

So our shaded region is

$$A = A_{tri} - A_{sector}$$

$$= \frac{64\sqrt{2}}{27} - 1.8334$$

$$\approx 1.52 \text{ (2 d.p.)}$$

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Challenge

First we find expressions for x and y in terms of θ.

$x = r\cos\theta = \sqrt{2}\theta\cos\theta$

$y = r\sin\theta = \sqrt{2}\theta\sin\theta.$

Now differentiating with respect to θ we obtain

$\frac{\mathrm{d}x}{\mathrm{d}\theta} = \sqrt{2}\cos\theta - \sqrt{2}\theta\sin\theta$

$\frac{\mathrm{d}y}{\mathrm{d}\theta} = \sqrt{2}\sin\theta + \sqrt{2}\theta\cos\theta$

$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\sin\theta + \theta\cos\theta}{\cos\theta - \theta\sin\theta}.$

So at $\theta = \frac{\pi}{4}$, $\sin\theta = \cos\theta = \frac{\sqrt{2}}{2}$

$\therefore \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\sqrt{2} + \frac{\pi}{4}\sqrt{2}}{\sqrt{2} - \frac{\pi}{4}\sqrt{2}}$

$= \frac{4+\pi}{4-\pi}.$

If we use the linear formula $y = mx + c$, we can find a value for c by substituting in values for x and y at $\theta = \frac{\pi}{4}$.

At $\theta = \frac{\pi}{4}$,

$x = \sqrt{2}\theta\cos\theta = \frac{\pi}{4}$

$y = \sqrt{2}\theta\sin\theta = \frac{\pi}{4}.$

So,

$y = mx + c$

$y = \left(\frac{4+\pi}{4-\pi}\right)x + c$

$\frac{\pi}{4} = \left(\frac{4+\pi}{4-\pi}\right)\frac{\pi}{4} + c$

$c = \frac{\pi}{4}\left(1 - \frac{4+\pi}{4-\pi}\right)$

$c = \frac{\pi}{2}\left(\frac{\pi}{\pi-4}\right)$

Now we can substitute into the linear equation to obtain

$y = \left(\frac{4+\pi}{4-\pi}\right)x + \frac{\pi}{2}\left(\frac{\pi}{\pi-4}\right)$

$2(\pi-4)y + 2(\pi+4)x = \pi^2.$

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Hyperbolic Functions 6A

1 a $\sinh 4 = 27.29 \quad (2 \text{ d.p.})$

$\left(\dfrac{e^4 - e^{-4}}{2} = 27.29\right)$

Direct from calculator.

b $\cosh\left(\dfrac{1}{2}\right) = 1.13 \ (2\text{d.p.})$

$\left(\dfrac{e^{0.5} + e^{-0.5}}{2} = 1.13\right)$

Direct from calculator.

c $\tanh(-2) = -0.96 \ (2\text{d.p.})$

$\left(\dfrac{e^{-4} - 1}{e^{-4} + 1} = -0.96\right)$

Direct from calculator.

2 a $\sinh 1 = \dfrac{e^1 - e^{-1}}{2} = \dfrac{e - e^{-1}}{2}$

b $\cosh 4 = \dfrac{e^4 + e^{-4}}{2}$

c $\tanh 0.5 = \dfrac{e^1 - 1}{e^1 + 1}$

$= \dfrac{e - 1}{e + 1}$

Use $\tanh x = \dfrac{e^{2x} - 1}{e^{2x} + 1}$.

3 a $\sinh(\ln 2) = \dfrac{e^{h2} - e^{-h2}}{2}$

$= \dfrac{2 - \frac{1}{2}}{2} = \dfrac{3}{4}$

$e^{h2} = 2$, and $e^{-\ln 2} = e^{\ln 2^{-1}} = \dfrac{1}{2}$

b $\cosh(\ln 3) = \dfrac{e^{\ln 3} + e^{-\ln 3}}{2}$

$= \dfrac{3 + \frac{1}{3}}{2} = \dfrac{5}{3}$

$e^{\ln 3} = 3$, and $e^{-\ln 3} = e^{\ln 3^{-1}} = \dfrac{1}{3}$

c $\tanh(\ln 2) = \dfrac{e^{2\ln 2} - 1}{e^{2\ln 2} + 1}$

$= \dfrac{4 - 1}{4 + 1} = \dfrac{3}{5}$

$e^{2\ln 2} = e^{\ln 2^2} = 4$

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4 $\dfrac{e^x + e^{-x}}{2} = 2$

$e^x + e^{-x} = 4$ ← Multiply throughout by e^x.

$e^{2x} + 1 = 4e^x$

$e^{2x} - 4e^x + 1 = 0$

$e^x = \dfrac{4 \pm \sqrt{16-4}}{2}$ ← Solve as a quadratic in e^x.

$e^x = 3.732$ or $e^x = 0.268$

$x = \ln 3.732 = 1.32$ (2 d.p.)

$x = \ln 0.268 = -1.32$ (2 d.p.)

5 $\dfrac{e^x - e^{-x}}{2} = 1$

$e^x - e^{-x} = 2$ ← Multiply throughout by e^x.

$e^{2x} - 1 = 2e^x$

$e^{2x} - 2e^x - 1 = 0$

$e^x = \dfrac{2 \pm \sqrt{4+4}}{2}$ ← Solve as a quadratic in e^x.

$e^x = 2.414$ or $e^x = -0.414$

$e^x = 2.414$ ← e^x cannot be negative.

$x = \ln 2.414 = 0.88$ (2 d.p.)

6 $\dfrac{e^{2x} - 1}{e^{2x} + 1} = -\dfrac{1}{2}$

$2\left(e^{2x} - 1\right) = -\left(e^{2x} + 1\right)$

$2e^{2x} - 2 = -e^{2x} - 1$

$3e^{2x} = 1$

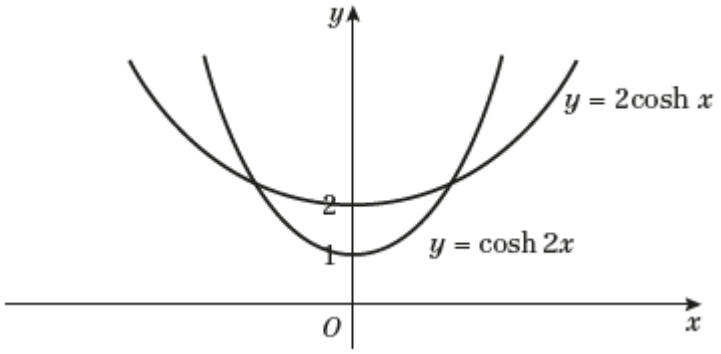
$e^{2x} = \dfrac{1}{3}$

$2x = \ln\left(\dfrac{1}{3}\right)$

$x = \dfrac{1}{2}\ln\left(\dfrac{1}{3}\right) = -0.55$ (2 d.p.)

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7



For $f(x) = \cosh x$, $f(2x) = \cosh 2x$, a horizontal sketch of scale factor $\frac{1}{2}$.

For $f(x) = \cosh x$, $2f(x) = 2\cosh x$, a vertical stretch of scale factor 2.

8 a $f(x) \in \mathbb{R}$ (All real numbers)

b $f(x) \geqslant 1$

Check the graph of each hyperbolic function to see which y values are possible.

c $-1 < f(x) < 1$

$|f(x)| < 1$

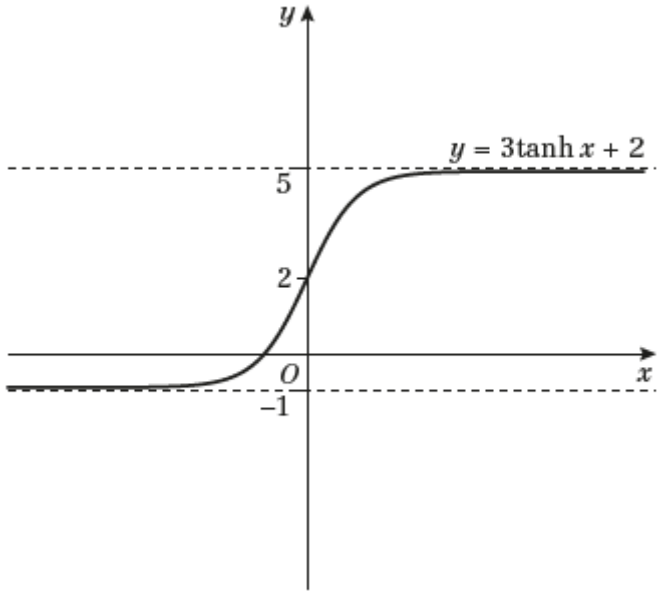
9 a $y = 3\tanh x + 2 = 3\dfrac{\sinh x}{\cosh x} + 2$

When $x = 0$, $y = 3\dfrac{\sinh 0}{\cosh 0} + 2 = 2$.

When x is large and positive, $\sinh x \approx \frac{1}{2}e^x$ and $\cosh x \approx \frac{1}{2}e^x$, so $y \approx 5$.

When x is large and negative, $\sinh x \approx -\frac{1}{2}e^{-x}$ and $\cosh x \approx \frac{1}{2}e^{-x}$, so $y \approx -1$.

Alternatively, the graph of $y = 3\tanh x + 2$ can be envisioned by scaling $y = \tanh x$ by a factor of 3 in the y direction, followed by a transition of 2 units in the positive y direction.



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9 b When x is large and positive, $\sinh x \approx \frac{1}{2}e^x$ and $\cosh \approx \frac{1}{2}e^x$.

So $y \approx 3 + 2$, thus giving $y = 5$ as an asymptote.

When x is large and negative, $\sinh x \approx -\frac{1}{2}e^{-x}$ and $\cosh \approx \frac{1}{2}e^{-x}$.

So $y \approx -3 + 2$, thus giving $y = -1$ as an asymptote.

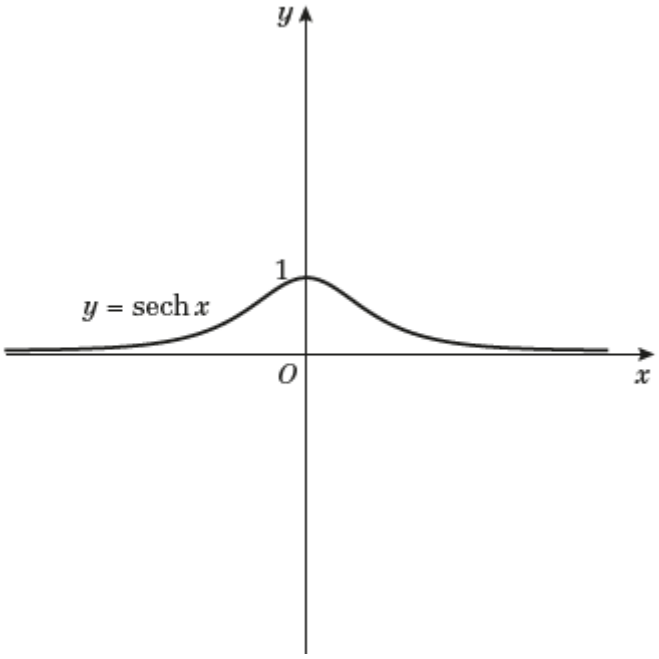
Challenge

a $y = \operatorname{sech} x = \dfrac{1}{\cosh x}$

When $x = 0$, $y = \dfrac{1}{\cosh 0} = 1$

When x is large and positive, $\cosh x \approx \frac{1}{2}e^x$, so $y \approx 0$.

When x is large and negative, $\cosh x \approx \frac{1}{2}e^{-x}$, so $y \approx 0$.



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Challenge

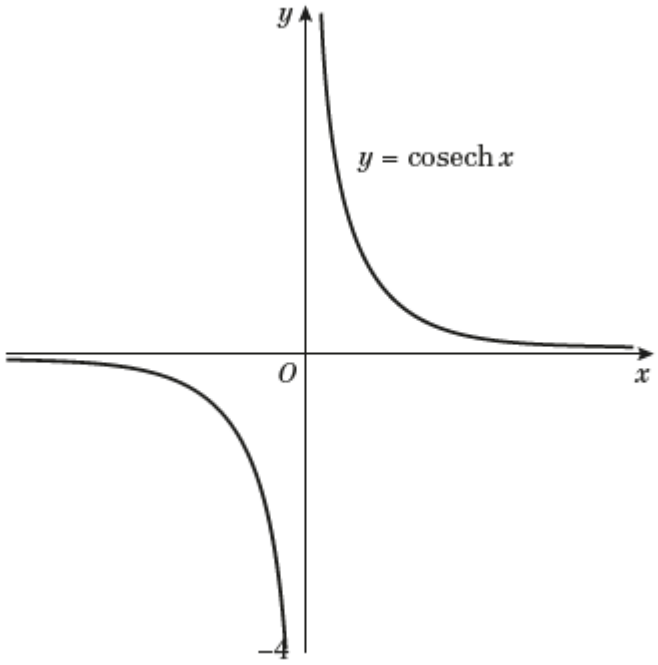
b $y = \text{cosech}\, x = \dfrac{1}{\sinh x}$ When $x = 0$, $\sinh x = 0$. So $\text{cosech}\, 0$ is not well defined.

As $x \to 0_+$ (zero from above), $\sinh x$ will be an infinitesimal positive number and so $\text{cosech}\, x \to +\infty$.

As $x \to 0_-$ (zero from below), $\sinh x$ will be an infinitesimal negative number and so $\text{cosech}\, x \to -\infty$.

When x is large and positive, $\sinh x \approx \dfrac{1}{2}e^x$, so $y \approx 0$.

When x is large and negative, $\sinh x \approx -\dfrac{1}{2}e^{-x}$, so $y \approx 0$.



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Challenge

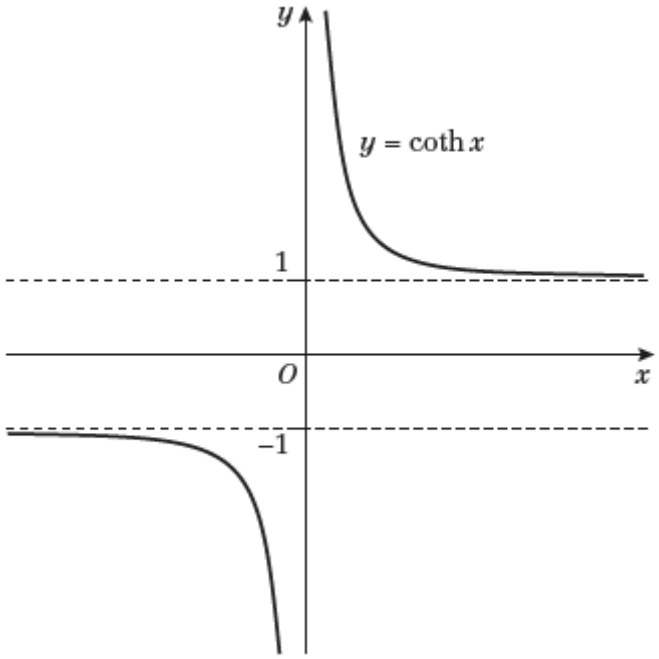
c $y = \coth x = \dfrac{\cosh x}{\sinh x}$ When $x = 0$, $\sinh x = 0$, and $\cosh x = 1$. So $\coth 0$ is not well defined.

As $x \to 0_+$ (zero from above), $\sinh x$ will be an infinitesimal positive number and $\cosh x \approx 1$, so $\operatorname{cosech} x \to +\infty$.

As $x \to 0_-$ (zero from below), $\sinh x$ will be an infinitesimal negative number and $\cosh x \approx 1$, so $\operatorname{cosech} x \to -\infty$.

When x is large and positive, $\sinh x \approx \dfrac{1}{2}e^x$ and $\cosh x \approx \dfrac{1}{2}e^x$, so $y \approx 1$.

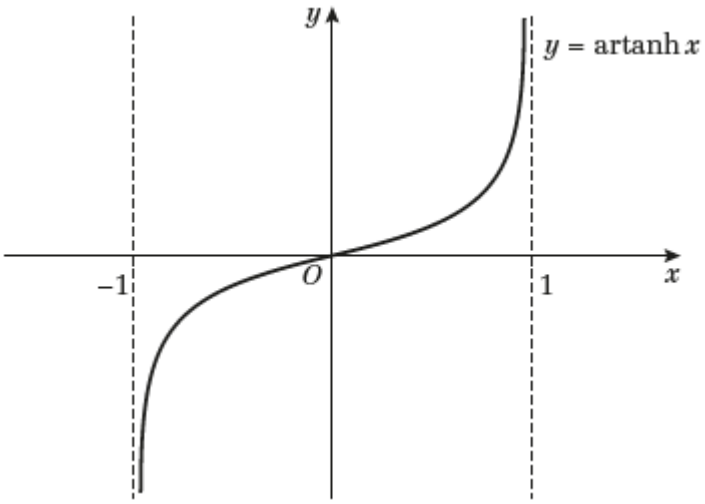
When x is large and negative, $\sinh x \approx -\dfrac{1}{2}e^{-x}$ and $\cosh x \approx \dfrac{1}{2}e^{-x}$, so $y \approx -1$.



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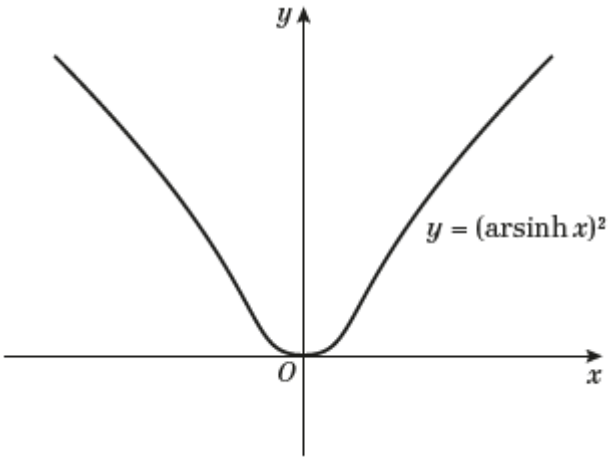
Hyperbolic Functions 6B

1



$y = \text{artanh}x,\ |x| < 1.$

2



$y = (\text{arsinh}\, x)^2$

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3 $y = \operatorname{artanh} x$

$$x = \tanh y = \frac{e^{2y}-1}{e^{2y}+1}$$
$$x\left(e^{2y}+1\right) = e^{2y}-1$$
$$1+x = e^{2y}(1-x)$$
$$e^{2y} = \frac{1+x}{1-x}$$
$$2y = \ln\left(\frac{1+x}{1-x}\right)$$
$$y = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$$
$$\operatorname{artanh} x = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right), \quad |x| < 1$$

For $|x| \geq 1$, $\ln\left(\frac{1+x}{1-x}\right)$ is not defined, since $\frac{1+x}{1-x} \leq 0$.

4 a $\operatorname{arsinh} 2 = \ln(2+\sqrt{2^2+1})$
$= \ln(2+\sqrt{5})$

b $\operatorname{arcosh} 3 = \ln(3+\sqrt{3^2-1})$
$= \ln(3+\sqrt{8})$
$= \ln(3+2\sqrt{2})$

c $\operatorname{artanh}\left(\frac{1}{2}\right) = \frac{1}{2}\ln\left(\frac{1+\frac{1}{2}}{1-\frac{1}{2}}\right)$
$= \frac{1}{2}\ln 3$

5 a $\operatorname{arsinh}\sqrt{2} = \ln(\sqrt{2}+\sqrt{2+1})$
$= \ln(\sqrt{2}+\sqrt{3})$

b $\operatorname{arcosh}\sqrt{5} = \ln(\sqrt{5}+\sqrt{5-1})$
$= \ln(2+\sqrt{5})$

c $\operatorname{artanh} 0.1 = \frac{1}{2}\ln\left(\frac{1+0.1}{1-0.1}\right)$
$= \frac{1}{2}\ln\left(\frac{11}{9}\right)$

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6 a $\operatorname{arsinh}(-3) = \ln(-3+\sqrt{(-3)^2+1})$

$= \ln(-3+\sqrt{10})$

b $\operatorname{arcosh}\left(\frac{3}{2}\right) = \ln\left(\frac{3}{2}+\sqrt{\left(\frac{3}{2}\right)^2-1}\right)$

$= \ln\left(\frac{3}{2}+\sqrt{\frac{5}{4}}\right)$

$= \ln\left(\frac{3}{2}+\frac{\sqrt{5}}{2}\right)$

$= \ln\left(\frac{3+\sqrt{5}}{2}\right)$

c $\operatorname{artanh}\left(\frac{1}{\sqrt{3}}\right) = \frac{1}{2}\ln\left(\frac{1+\frac{1}{\sqrt{3}}}{1-\frac{1}{\sqrt{3}}}\right)$

$= \frac{1}{2}\ln\left(\frac{\sqrt{3}+1}{\sqrt{3}-1}\right)$

$= \frac{1}{2}\ln\left(\frac{\sqrt{3}+1}{\sqrt{3}-1}\times\frac{\sqrt{3}+1}{\sqrt{3}+1}\right)$

$= \frac{1}{2}\ln\left(\frac{4+2\sqrt{3}}{2}\right)$

$= \frac{1}{2}\ln(2+\sqrt{3})$

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7 $\operatorname{artanh} x + \operatorname{artanh} y$

$$= \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right) + \frac{1}{2}\ln\left(\frac{1+y}{1-y}\right)$$

Use $\ln a + \ln b = \ln(ab)$.

$$= \frac{1}{2}\ln\left(\frac{1+x}{1-x} \times \frac{1+y}{1-y}\right)$$

$$= \frac{1}{2}\ln\left(\frac{1+x+y+xy}{1-x-y+xy}\right)$$

Use $\frac{1}{2}\ln a = \ln a^{\frac{1}{2}}$.

$$= \ln\sqrt{\left(\frac{1+x+y+xy}{1-x-y+xy}\right)}$$

So $\dfrac{1+x+y+xy}{1-x-y+xy} = 3$

$1+x+y+xy = 3-3x-3y+3xy$

$1+x-3+3x = -3y+3xy-y-xy$

$2xy-4y = 4x-2$

$y(x-2) = 2x-1$

$$y = \frac{2x-1}{x-2}$$

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Hyperbolic Functions 6C

1 a $\text{RHS} = 2\sinh A\cosh A$

$$= 2\left(\frac{e^{A}-e^{-A}}{2}\right)\left(\frac{e^{A}+e^{-A}}{2}\right)$$

$$= \frac{1}{2}\left(e^{2A}-1+1-e^{-2A}\right)$$

$$= \frac{e^{2A}-e^{-2A}}{2}$$

$$= \sinh 2A = \text{LHS}$$

b $\text{RHS} = \cosh A\cosh B - \sinh A\sinh B$

$$= \left(\frac{e^{A}+e^{-A}}{2}\right)\left(\frac{e^{B}+e^{-B}}{2}\right) - \left(\frac{e^{A}-e^{-A}}{2}\right)\left(\frac{e^{B}-e^{-B}}{2}\right)$$

$$= \frac{e^{A+B}+e^{-A+B}+e^{A-B}+e^{-A-B}}{4}$$

$$- \frac{e^{A+B}-e^{-A+B}-e^{A-B}+e^{-A-B}}{4}$$

$$= \frac{2\left(e^{-A+B}+e^{A-B}\right)}{4}$$

$$= \frac{e^{A-B}+e^{-(A-B)}}{2}$$

$$= \cosh(A-B) = \text{LHS}$$

c $\text{RHS} = 4\cosh^{3}A - 3\cosh A$

$$= 4\left(\frac{e^{A}+e^{-A}}{2}\right)^{3} - 3\left(\frac{e^{A}+e^{-A}}{2}\right)$$

$\left(e^{A}+e^{-A}\right)^{3} = e^{3A}+3e^{2A}e^{-A}+3e^{A}e^{-2A}+e^{-3A}$

$= e^{3A}+3e^{A}+3e^{-A}+e^{-3A}$

Use the expansion $(x+y)^{3} = x^{3}+3x^{2}y+3xy^{2}+y^{3}$.

$$\text{RHS} = \frac{e^{3A}+3e^{A}+3e^{-A}+e^{-3A}}{2} - \frac{3(e^{A}+e^{-A})}{2}$$

$$= \frac{e^{3A}+e^{-3A}}{2}$$

$$= \cosh 3A = \text{LHS}$$

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1 d $\text{RHS} = 2\sin\left(\frac{A-B}{2}\right)\cosh\left(\frac{A+B}{2}\right)$

$$= 2\left(\frac{e^{\frac{A-B}{2}} - e^{\frac{-A+B}{2}}}{2}\right)\left(\frac{e^{\frac{A+B}{2}} + e^{\frac{-A-B}{2}}}{2}\right)$$

$$= \frac{1}{2}\left(e^{\frac{A-B}{2}+\frac{A+B}{2}} - e^{\frac{-A+B}{2}+\frac{A+B}{2}} + e^{\frac{A-B}{2}+\frac{A-B}{2}} - e^{\frac{-A+B}{2}+\frac{-A-B}{2}}\right)$$

$$= \frac{1}{2}\left(e^{A} - e^{B} + e^{-B} - e^{-A}\right)$$

$$= \frac{1}{2}\left(e^{A} - e^{-A}\right) - \frac{1}{2}\left(e^{B} - e^{-B}\right)$$

$$= \sinh A - \sinh B$$

$$= \text{LHS}$$

2 a $\sin(A-B) = \sin A\cos B - \cos A\sin B$

$\sinh(A-B) = \sinh A\cosh B - \cosh A\sinh B$

Replace sin x by sinh x and cos x by cosh x.

b $\sin 3A = 3\sin A - 4\sin^3 A$

$$= 3\sin A - 4\sin A\sin^2 A$$

$\sinh 3A = 3\sinh A + 4\sinh^3 A$

Replace $\sin^2 A$, the product of two sine terms, by $-\sinh^2 A$.

c $\cos A + \cos B = 2\cos\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)$

$\cosh A + \cosh B = 2\cosh\left(\frac{A+B}{2}\right)\cosh\left(\frac{A-B}{2}\right)$

Replace cos x by cosh x.

d $\cos 2A = \frac{1-\tan^2 A}{1+\tan^2 A}$

$\cosh 2A = \frac{1+\tanh^2 A}{1-\tanh^2 A}$

$\tan^2 A = \frac{\sin^2 A}{\cos^2 A}$, so there is a product of two sines.

Replace $\tan^2 A$ by $-\tanh^2 A$.

e $\cos 2A = \cos^4 A - \sin^4 A$

$$= \cos^4 A - (\sin^2 A)(\sin^2 A)$$

$\cosh 2A = \cosh^4 A - (-\sinh^2 A)(-\sinh^2 A)$

$$= \cosh^4 A - \sinh^4 A$$

Replace $\sin^2 A$ by $-\sinh^2 A$.

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3 a Using $\cosh^2 x - \sinh^2 x = 1$

$4 - \sinh^2 x = 1$

$\sinh^2 x = 3$

$\sinh x = \pm\sqrt{3}$

← Both positive and negative values of $\sinh x$ are possible.

b Using $\tanh x = \dfrac{\sinh x}{\cosh x}$

$\tanh x = \pm\dfrac{\sqrt{3}}{2}$

c Using $\cosh 2x = 2\cosh^2 x - 1$

$\cosh 2x = (2 \times 4) - 1$

$= 7$

4 a Using $\cosh^2 x - \sinh^2 x = 1$

$\cosh^2 x - (-1)^2 = 1$

$\cosh^2 x = 2$

$\cosh x = \sqrt{2}$

← $\cosh x$ cannot be negative.

b Using $\sin 2x = 2\sinh x \cosh x$

$\sinh 2x = 2 \times (-1) \times \sqrt{2}$

$= -2\sqrt{2}$

c Using $\tanh 2x = \dfrac{2\tanh x}{1 + \tanh^2 x}$

← Alternatively use $\dfrac{\sinh 2x}{\cosh 2x} = \dfrac{\sinh 2x}{2\cosh^2 x - 1}$

$\tanh x = \dfrac{\sinh x}{\cosh x} = \dfrac{-1}{\sqrt{2}}$

$\tanh 2x = \dfrac{\left(-\frac{2}{\sqrt{2}}\right)}{1 + \left(\frac{1}{2}\right)}$

$= \dfrac{-2}{\sqrt{2}} \times \dfrac{2}{3}$

$= \dfrac{-4}{3\sqrt{2}} = -\dfrac{2\sqrt{2}}{3}$

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5 a $3\sinh x + 4\cosh x = 4$

$$\frac{3(e^x - e^{-x})}{2} + \frac{4(e^x + e^{-x})}{2} = 4$$

$$3e^x - 3e^{-x} + 4e^x + 4e^{-x} = 8$$

$$7e^x - 8 + e^{-x} = 0$$ ← Multiply throughout by e^x.

$$7e^{2x} - 8e^x + 1 = 0$$

$$(7e^x - 1)(e^x - 1) = 0$$ ← Solve as a quadratic in e^x.

$$e^x = \frac{1}{7} \text{ or } e^x = 1$$

$$x = \ln\left(\frac{1}{7}\right),\ x = 0$$ ← Note that $\ln\left(\frac{1}{7}\right) = \ln(7^{-1}) = -\ln 7$

b $7\sinh x - 5\cosh x = 1$

$$\frac{7(e^x - e^{-x})}{2} - \frac{5(e^x + e^{-x})}{2} = 1$$

$$7e^x - 7e^{-x} - 5e^x - 5e^{-x} = 2$$

$$2e^x - 2 - 12e^{-x} = 0$$ ← Multiply throughout by e^x.

$$e^x - 1 - 6e^{-x} = 0$$

$$e^{2x} - e^x - 6 = 0$$

$$(e^x - 3)(e^x + 2) = 0$$

$$e^x = 3$$ ← $e^x = -2$ is not possible for real x.

$$x = \ln 3$$

c $30\cosh x = 15 + 26\sinh x$

$$30\frac{(e^x + e^{-x})}{2} = 15 + 26\frac{(e^x - e^{-x})}{2}$$

$$15e^x + 15e^{-x} = 15 + 13e^x - 13e^{-x}$$

$$2e^x - 15 + 28e^{-x} = 0$$ ← Multiply throughout by e^x.

$$2e^{2x} - 15e^x + 28 = 0$$

$$(2e^x - 7)(e^x - 4) = 0$$ ← Solve as a quadratic in e^x.

$$e^x = \frac{7}{2},\ e^x = 4$$

$$x = \ln\left(\frac{7}{2}\right),\ x = \ln 4$$

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5 d

$$13\sinh x - 7\cosh x + 1 = 0$$

$$13\frac{(e^x - e^{-x})}{2} - 7\frac{(e^x + e^{-x})}{2} + 1 = 0$$

$$13e^x - 13e^{-x} - 7e^x - 7e^{-x} + 2 = 0$$

$$6e^x + 2 - 20e^{-x} = 0$$

$$3e^x + 1 - 10e^{-x} = 0$$ ← Multiply throughout by e^x.

$$3e^{2x} + e^x - 10 = 0$$

$$(3e^x - 5)(e^x + 2) = 0$$ ← Solve as a quadratic in e^x.

$$e^x = \frac{5}{3}$$ ← $e^x = -2$ is not possible for real x.

$$x = \ln\left(\frac{5}{3}\right)$$

e $\cosh 2x - 5\sinh x = 13$

Using $\cosh 2x = 1 + 2\sinh^2 x$,

$$1 + 2\sinh^2 x - 5\sinh x = 13$$ ← Use $\operatorname{arsinh} x = \ln(x + \sqrt{x^2 + 1})$.

$$2\sinh^2 x - 5\sinh x - 12 = 0$$

$$(2\sinh x + 3)(\sinh x - 4) = 0$$

$$\sinh x = -\frac{3}{2},\ \sinh x = 4$$

$$x = \operatorname{arsinh}\left(-\frac{3}{2}\right),\ x = \operatorname{arsinh} 4$$

$$x = \ln\left(-\frac{3}{2} + \sqrt{\frac{9}{4} + 1}\right)$$

$$= \ln\left(\frac{-3 + \sqrt{13}}{2}\right)$$

$$x = \ln(4 + \sqrt{16 + 1})$$

$$= \ln(4 + \sqrt{17})$$

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5 f $3\sinh^2 x - 13\cosh x + 7 = 0$

Using $\cosh^2 x - \sinh^2 x = 1$,

$3(\cosh^2 x - 1) - 13\cosh x + 7) = 0$

$3\cosh^2 x - 13\cosh x + 4 = 0$

$(3\cosh - 1)(\cosh x - 4) = 0$

$\cosh x = \frac{1}{3}$, $\cosh x = 4$ ← $\cosh x \geq 1$, so $\cosh x = \frac{1}{3}$ is not possible.

$\cosh x = 4$

$x = \operatorname{arcosh} 4, -\operatorname{arcosh} 4$ ← Use $\operatorname{arcosh} x = \ln(x + \sqrt{x^2 - 1})$, but remember that $\ln(x - \sqrt{x^2 - 1})$ is also a solution.

$x = \ln(4 \pm \sqrt{4^2 - 1})$

$= \ln(4 \pm \sqrt{15})$

g $\sinh 2x - 7\sinh x = 0$

$2\sinh x \cosh x - 7\sin x = 0$

$\sinh x(2\cosh x - 7) = 0$

$\sinh x = 0$, $\cosh x = \frac{7}{2}$

$x = 0$, $x = \pm \operatorname{arcosh}\left(\frac{7}{2}\right)$ ← Use $\operatorname{arcosh} x = \ln(x + \sqrt{x^2 - 1})$, but remember that $\ln(x - \sqrt{x^2 - 1})$ is also a solution.

$\operatorname{arcosh}\left(\frac{7}{2}\right) = \ln\left(\frac{7}{2} + \sqrt{\frac{49}{4} - 1}\right)$

$= \ln\left(\frac{7 + \sqrt{45}}{2}\right)$

$= \ln\left(\frac{7 + 3\sqrt{5}}{2}\right)$

$x = 0$, $x = \ln\left(\frac{7 \pm 3\sqrt{5}}{2}\right)$

h $4\cosh x + 13e^{-x} = 11$

$4\frac{(e^x + e^{-x})}{2} + 13e^{-x} = 11$

$2e^x + 2e^{-x} + 13e^{-x} = 11$ ← Multiply throughout by e^x.

$2e^x + 15e^{-x} - 11 = 0$

$2e^{2x} - 11e^x + 15 = 0$

$(2e^x - 5)(e^x - 3) = 0$ ← Solve as a quadratic in e^x.

$e^x = \frac{5}{2}$, $e^x = 3$

$x = \ln\left(\frac{5}{3}\right)$, $x = \ln 3$

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5 i

$$2\tanh x = \cosh x$$
$$\frac{2\sinh}{\cosh x} = \cosh x$$
$$2\sinh x = \cosh^2 x$$

Using $\cosh^2 x - \sinh^2 x = 1$

$$2\sinh x = 1 + \sinh^2 x$$
$$\sinh^2 x - 2\sinh x + 1 = 0$$
$$(\sinh x - 1)^2 = 0$$
$$\sinh x = 1$$
$$x = \operatorname{arsinh} 1$$

Use $\operatorname{arsinh} x = \ln(x + \sqrt{x^2 + 1})$.

$$x = \ln(1 + \sqrt{1^2 + 1})$$
$$= \ln(1 + \sqrt{2})$$

6 a

$$2\cosh^2 x - 1 = 2\left(\frac{e^x + e^{-x}}{2}\right)^2 - 1$$
$$= 2\left(\frac{e^{2x} + 2 + e^{-2x}}{4}\right) - 1$$
$$= \frac{e^{2x} + e^{-2x}}{2}$$
$$= \cosh 2x.$$

b Using the relation $\cosh 2x = 2\cosh^2 x - 1$, the equation becomes $2\cosh^2 x - 3\cosh x - 9 = 0$.
Substituting $u = \cosh x$ into this equation allows us to clearly see a quadratic equation $2u^2 - 3u - 9 = 0$ which gives solutions $u = \frac{3 \pm \sqrt{9 + 72}}{4}$, we neglect the negative solution to avoid complex numbers.
i.e.
$u = 3$.
This means that our solutions for x are the values for
$x_1 = \operatorname{ar}\cosh(3)$.

Now recalling the expression $\operatorname{ar}\cosh u = \ln\left(u \pm \sqrt{u^2 - 1}\right)$, we obtain the exact logarithm solutions

$x = \ln\left(3 \pm 2\sqrt{2}\right)$.

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7 Using the identity $\cosh^2 x - \sinh^2 x \equiv 1$, the equation in question becomes $2(\cosh^2 x - 1) - 5\cosh x = 5$.

We simplify and use the substitution $u = \cosh x$ in order to obtain $2u^2 - 5u - 7 = 0$. This equation has solutions $u = \frac{5 \pm \sqrt{25+56}}{4}$, we neglect the negative solution to avoid complex numbers, thus $u = \frac{7}{2}$.

This means that our solutions for x are the values for

$x = \text{ar}\cosh\left(\frac{7}{2}\right)$.

Now recalling the expression $\text{ar}\cosh u = \ln\left(u \pm \sqrt{u^2 - 1}\right)$, we obtain the exact logarithm solutions

$x = \ln\left(\frac{1}{2}\left(7 \pm 3\sqrt{5}\right)\right)$.

8 $\text{sech}^2 x \equiv 1 + \tanh^2 x$ is not true, the correct identity is $\text{sech}^2 x \equiv 1 - \tanh^2 x$ by Osborn's rule.
He has split the fraction via denominator on the second line. This is not valid.
In mathematics we write this as

$\frac{a}{b+c} \neq \frac{a}{b} + \frac{a}{c}$.

The final mistake he made was assuming that taking the reciprocal of all terms preserves the value.

This is incorrect. For example $3 + 1 \neq \frac{1}{3} + 1$.

The correct proof is

$$\frac{1+\tanh^2 x}{1-\tanh^2 x} \equiv \frac{2 - \text{sech}^2 x}{\text{sech}^2 x}$$
$$\equiv \frac{2}{\text{sech}^2 x} - 1$$
$$\equiv 2\cosh^2 x - 1.$$

9 a Recall the identity $R\cosh(x+a) \equiv R\cosh x \cosh a + R\sinh x \sinh a$.
From this we can set $R\cosh a = 10$ and $R\sinh a = 6$.
In order to find values for R and a:

$\frac{R\sinh a}{R\cosh a} = \tanh a = \frac{6}{10} = \frac{3}{5}$

giving

$a = \text{artanh}(0.6)$

≈ 0.693 (3 d.p.)

We can now use the expression $R\cosh a = 10$.

To find $\cosh a$: $\text{sech}^2 a = 1 - \tanh^2 a = 1 - \left(\frac{3}{5}\right)^2 = \frac{16}{25}$

$\therefore \cosh^2 a = \frac{1}{\text{sech}^2 a} = \frac{25}{16}$ and so $\cosh a = \frac{5}{4}$

So $R = \frac{10}{\cosh a} = \frac{10}{\left(\frac{5}{4}\right)} = 8$

Thus giving us $10\cosh x + 6\sinh x = 8\cosh(x + 0.693)$.

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9 b The minimum value occurs when $\cosh(x+0.693)$ is minimal. We know this occurs when $x+0.693=0$, by the graph of $\cosh x$. So the minimal value is $8\cosh 0=8$.

We could also find this answer by setting $y=8\cosh(x+0.693)$, finding the solution to $\frac{\mathrm{d}y}{\mathrm{d}x}=8\sinh(x+0.693)=0$ as $x=-0.693$ and substituting in, to get $y=8\cosh 0=8$.

The second derivative should be checked to be positive at this point in order to conclude that the stationary point is minimal, not maximal.

c $10\cosh x+6\sinh x=8\cosh(x+0.693)=11$.

We rearrange the equation to be $\cosh(x+0.693)=\frac{11}{8}$.

Set $u=x+0.693$ and now noting the symmetry about $u=0$ we obtain two solutions,

$u=\pm\operatorname{arcosh}\left(\frac{11}{8}\right)\approx\pm0.841(3\text{ d.p.})$

This means our solutions for x are

$x_1=-1.534,$

$x_2=0.148.$

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Hyperbolic Functions 6D

1 a $\frac{\mathrm{d}}{\mathrm{d}x}(\sinh 2x) = 2\cosh 2x$

b $\frac{\mathrm{d}}{\mathrm{d}x}(\cosh 5x) = 5\sinh 5x$

c $\frac{\mathrm{d}}{\mathrm{d}x}(\tanh 2x) = 2\operatorname{sech}^2 2x$

d $\frac{\mathrm{d}}{\mathrm{d}x}(\sinh 3x) = 3\cosh 3x$

e $\frac{\mathrm{d}}{\mathrm{d}x}(\coth 4x) = -4\operatorname{cosech}^2 4x$

f $\frac{\mathrm{d}}{\mathrm{d}x}(\operatorname{sech} 2x) = \frac{-1}{(\cosh 2x)^2} \times 2\sinh 2x$

$= -2\frac{\sinh 2x}{\cosh 2x} \times \frac{1}{\cosh 2x}$

$= -2\tanh 2x \operatorname{sech} 2x$

g $\frac{\mathrm{d}}{\mathrm{d}x}(\mathrm{e}^{-x}\sinh x) = -\mathrm{e}^{-x}\sinh x + \mathrm{e}^{-x}\cosh x$

$= \mathrm{e}^{-x}(\cosh x - \sinh x)$

h $\frac{\mathrm{d}}{\mathrm{d}x}(x\cosh 3x) = \cosh 3x + 3x\sinh 3x$

i $\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\sinh x}{3x}\right) = \frac{\cosh x}{3x} - \frac{\sinh x}{3x^2}$

$= \frac{x\cosh x - \sinh x}{3x^2}$

j $\frac{\mathrm{d}}{\mathrm{d}x}(x^2\cosh 3x) = 2x\cosh 3x + x^2 \times 3\sinh 3x$

$= x(2\cosh 3x + 3x\sinh 3x)$

k $\frac{\mathrm{d}}{\mathrm{d}x}(\sinh 2x\cosh 3x) = 2\cosh 2x\cosh 3x + \sinh 2x \times 3\sinh 3x$

$= 2\cosh 2x\cosh 3x + 3\sinh 2x\sinh 3x$

l $\frac{\mathrm{d}}{\mathrm{d}x}(\ln\cosh x) = \frac{1}{\cosh x} \times \sinh x$

$= \tanh x$

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1 m $\frac{\mathrm{d}}{\mathrm{d}x}(\sinh x^3) = 3x^2 \cosh x^3$

n $\frac{\mathrm{d}}{\mathrm{d}x}(\cosh^2 2x) = (2\cosh 2x)(2\sinh 2x)$

$= 4\cosh 2x \sinh 2x$

o $\frac{\mathrm{d}}{\mathrm{d}x}(\mathrm{e}^{\cosh x}) = \sinh x \mathrm{e}^{\cosh x}$

p $\frac{\mathrm{d}}{\mathrm{d}x}(\mathrm{cosech} x) = \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{\sinh x}\right) = \frac{0 - 1 \times \cosh x}{\sinh^2 x}$

$= -\coth x \,\mathrm{cosech} x$

2 $y = a\cosh nx + b\sinh nx$

Differentiate with respect to x

$\frac{\mathrm{d}y}{\mathrm{d}x} = an\sinh nx + nb\cosh nx$

$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = an^2\cosh nx + bn^2\sinh nx$

$= n^2(a\cosh nx + b\sinh nx)$

$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = n^2 y$

3 To find the stationary point of the curve $y = 12\cosh x - \sinh x$, we differentiate and set equal to 0.

$\frac{\mathrm{d}y}{\mathrm{d}x} = 12\sinh x - \cosh x = 0$

$\tanh x = \frac{1}{12}$

$x = \operatorname{artanh}\left(\frac{1}{12}\right)$.

Using the formula $\operatorname{artanh} x = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$, we find that

$x = \frac{1}{2}\ln\left(\frac{1+\frac{1}{12}}{1-\frac{1}{12}}\right)$

$= \frac{1}{2}\ln\left(\frac{13}{11}\right)$

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3 Substituting this value for x back into the given equation for y, we obtain

$$y = 12\cosh\left(\frac{1}{2}\ln\left(\frac{13}{11}\right)\right) - \sinh\left(\frac{1}{2}\ln\left(\frac{13}{11}\right)\right)$$

$$= \frac{12}{2}\left(\sqrt{\frac{13}{11}} + \sqrt{\frac{11}{13}}\right) - \frac{1}{2}\left(\sqrt{\frac{13}{11}} - \sqrt{\frac{11}{13}}\right)$$

$$= \frac{1}{2}\left(11\sqrt{\frac{13}{11}} + 13\sqrt{\frac{11}{13}}\right)$$

$$= \sqrt{143}.$$

So the coordinates of the stationary point are $\left(\frac{1}{2}\ln\left(\frac{13}{11}\right), \sqrt{143}\right)$.

4 $y = \cosh 3x \sinh x$

$$\frac{dy}{dx} = 3\sinh 3x \sinh x + \cosh 3x \cosh x$$

$$\frac{d^2y}{dx^2} = 9\cosh 3x \sinh x + 3\sinh 3x \cosh x + 3\sinh 3x \cosh x + \cosh 3x \sinh x$$

$$= 10\cosh 3x \sinh x + 6\sinh 3x \cosh x$$

$$= 2(5\cosh 3x \sinh x + 3\sinh 3x \cosh x)$$

5 a Let $y = \operatorname{arcosh} 2x$ then $\cosh y = 2x$

Differentiate with respect to x

$$\sinh y \frac{dy}{dx} = 2$$

$$\frac{dy}{dx} = \frac{2}{\sinh y}$$

$$= \frac{2}{\sqrt{\cosh^2 y - 1}} \text{ but } \cosh y = 2x$$

$$\text{so } \frac{dy}{dx} = \frac{2}{\sqrt{4x^2 - 1}}$$

b Let $y = \operatorname{arsinh}(x+1)$ then $\sinh y = x + 1$

$$\cosh y \frac{dy}{dx} = 1$$

$$\frac{dy}{dx} = \frac{1}{\cosh y}$$

$$= \frac{1}{\sqrt{\sinh^2 y + 1}} \text{ but } \sinh y = x + 1$$

$$\text{so } \frac{dy}{dx} = \frac{1}{\sqrt{(x+1)^2 + 1}}$$

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5 c Let $y = \text{artanh}3x$

$$\tanh y = 3x$$

$$\text{sech}^2 y \frac{dy}{dx} = 3$$

$$\frac{dy}{dx} = \frac{3}{\text{sech}^2 y}$$

$$\frac{dy}{dx} = \frac{3}{1-\tanh^2 y}$$

$$\frac{dy}{dx} = \frac{3}{1-9x^2}$$

d Let $y = \text{arsech}x$

$$\text{sech}y = x$$

$$\frac{1}{\cosh y} = x$$

$$1 = x\cosh y$$

Differentiate with respect to x

$$0 = \cosh y + x\sinh y \frac{dy}{dx}$$

$$x\sinh y \frac{dy}{dx} = -\cosh y$$

$$\frac{dy}{dx} = \frac{-\cosh y}{x\sinh y}$$

$$= \frac{-1}{x\tanh y}$$

$$= \frac{-1}{x(1-\text{sech}^2 y)^{\frac{1}{2}}}$$

$$= \frac{-1}{x(1-x^2)^{\frac{1}{2}}}$$

$$= -\frac{1}{x^2\sqrt{\frac{1}{x^2}-1}}$$

e Let $y = \text{arcosh}x^2$

Let $t = x^2 \quad y = \text{arcosh}t$

$$\frac{dy}{dx} = \left(\frac{dy}{dt}\right)\left(\frac{dt}{dx}\right) = \left(\frac{1}{\sqrt{t^2-1}}\right)(2x)$$

$$\frac{dy}{dx} = \frac{2x}{\sqrt{x^4-1}}$$

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5 f $y = \operatorname{arcosh} 3x$

Let $t = 3x \quad y = \operatorname{arcosh} t$

$$\frac{dy}{dx} = \left(\frac{dy}{dt}\right)\left(\frac{dt}{dx}\right) = \left(\frac{1}{\sqrt{t^2 - 1}}\right)(3)$$

$$\frac{dy}{dx} = \frac{3}{\sqrt{9x^2 - 1}}$$

g $y = x^2 \operatorname{arcosh} x$

$$\frac{dy}{dx} = 2x \operatorname{arcosh} x + \frac{x^2}{\sqrt{x^2 - 1}}$$

h $y = \operatorname{arsinh} \frac{x}{2}$

Let $t = \frac{x}{2} \quad y = \operatorname{arsinh} t$

$$\frac{dt}{dx} = \frac{1}{2} \quad \frac{dy}{dt} = \frac{1}{\sqrt{t^2 + 1}}$$

$$\frac{dy}{dx} = \left(\frac{dy}{dt}\right)\left(\frac{dt}{dx}\right) = \frac{1}{\sqrt{t^2 + 1}}\left(\frac{1}{2}\right)$$

$$= \frac{1}{2\sqrt{\left(\frac{x}{2}\right)^2 + 1}} = \frac{1}{\sqrt{x^2 + 4}}$$

i $y = e^{x^3} \operatorname{arsinh} x$

$$\frac{dy}{dx} = 3x^2 e^{x^3} \operatorname{arsinh} x + \frac{e^{x^3}}{\sqrt{x^2 + 1}}$$

j $y = \operatorname{arsinh} x \operatorname{arcosh} x$

$$\frac{dy}{dx} = \frac{1}{\sqrt{x^2 + 1}} \operatorname{arcosh} x + \frac{1}{\sqrt{x^2 - 1}} \operatorname{arsinh} x$$

k $y = \operatorname{arcosh} x \operatorname{sech} x$

$$\frac{dy}{dx} = \frac{1}{\sqrt{x^2 - 1}} \operatorname{sech} x - \operatorname{arcosh} x \tanh x \operatorname{sech} x$$

$$= \operatorname{sech} x \left(\frac{1}{\sqrt{x^2 - 1}} - \operatorname{arcosh} x \tanh x\right)$$

l $y = x \operatorname{arcosh} 3x$

$$\frac{dy}{dx} = \operatorname{arcosh} 3x + x \times \frac{3}{\sqrt{9x^2 - 1}}$$

$$\frac{dy}{dx} = \operatorname{arcosh} 3x + \frac{3x}{\sqrt{9x^2 - 1}}$$

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6 a $y = \text{arcosh}x$

$\cosh y = x$

$\sinh y \dfrac{\mathrm{d}y}{\mathrm{d}x} = 1 \Rightarrow$

$\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{1}{\sinh y} = \dfrac{1}{\sqrt{\cosh^2 y - 1}}$

but $\cosh y = x$ so

$\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{1}{\sqrt{x^2 - 1}}$

b $y = \text{artanh}x$

$\tanh y = x$

$\text{sech}^2 y \dfrac{\mathrm{d}y}{\mathrm{d}x} = 1$

$\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{1}{\text{sech}^2 y} = \dfrac{1}{1 - \tanh^2 y}$

but $\tanh y = x$ so

$\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{1}{1 - x^2}$

7 $y = \text{artanh}\dfrac{\mathrm{e}^x}{2}$

Let $t = \dfrac{\mathrm{e}^x}{2}$ $y = \text{artanh}t$

$\dfrac{\mathrm{d}t}{\mathrm{d}x} = \dfrac{\mathrm{e}^x}{2}$ $\dfrac{\mathrm{d}y}{\mathrm{d}t} = \dfrac{1}{1 - t^2}$

Then $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \left(\dfrac{\mathrm{d}y}{\mathrm{d}t}\right)\left(\dfrac{\mathrm{d}t}{\mathrm{d}x}\right) = \dfrac{1}{1 - t^2} \times \dfrac{\mathrm{e}^x}{2}$

$= \dfrac{1}{1 - \left(\frac{\mathrm{e}^x}{2}\right)^2} \times \dfrac{\mathrm{e}^x}{2}$

$= \dfrac{\frac{\mathrm{e}^x}{2}}{\frac{4 - \mathrm{e}^{2x}}{4}}$

$\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{2\mathrm{e}^x}{4 - \mathrm{e}^{2x}}$

$(4 - \mathrm{e}^{2x})\dfrac{\mathrm{d}y}{\mathrm{d}x} = 2\mathrm{e}^x$

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8 $y = \text{arsinh}x$

$$\frac{dy}{dx} = \frac{1}{\sqrt{x^2+1}} = (x^2+1)^{-\frac{1}{2}}$$

$$\frac{d^2y}{dx^2} = -\frac{1}{2}(x^2+1)^{-\frac{3}{2}}2x$$

$$= \frac{-x}{(x^2+1)^{\frac{3}{2}}}$$

$$\frac{d^3y}{dx^3} = \frac{-1(x^2+1)^{\frac{3}{2}} - \frac{3}{2}(x^2+1)^{\frac{1}{2}} \times 2x \times -x}{(x^2+1)^3}$$

$$= \frac{3x^2(x^2+1)^{\frac{1}{2}} - (x^2+1)^{\frac{3}{2}}}{(x^2+1)^3}$$

$$= \frac{3x^2}{(x^2+1)^{\frac{5}{2}}} - \frac{1}{(x^2+1)^{\frac{3}{2}}}$$

$$(x^2+1)\frac{d^3y}{dx^3} = \frac{3x^2}{(x^2+1)^{\frac{3}{2}}} - \frac{1}{(x^2+1)^{\frac{1}{2}}}$$

$$= -3x\frac{d^2y}{dx^2} - \frac{dy}{dx}$$

$$\therefore (1+x^2)\frac{d^3y}{dx^3} + 3x\frac{d^2y}{dx^2} + \frac{dy}{dx} = 0$$

9 $y = (\text{arcosh}x)^2$

$$\frac{dy}{dx} = 2\text{arcosh}x \times \frac{1}{\sqrt{x^2-1}}$$

$$= 2(x^2-1)^{-\frac{1}{2}}\text{arcosh}x$$

$$\frac{d^2y}{dx^2} = -(x^2-1)^{-\frac{3}{2}}2x\text{arcosh}x + 2(x^2-1)^{-\frac{1}{2}} \times \frac{1}{\sqrt{x^2-1}}$$

$$= \frac{-2x\text{arcosh}x}{(x^2-1)^{\frac{3}{2}}} + \frac{2}{x^2-1}$$

10 $y = \text{artanh}x \quad x = \frac{12}{13} \quad y = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right) = \frac{1}{2}\ln 25 = \ln 5$

$$\frac{dy}{dx} = \frac{1}{1-x^2} = \frac{1}{1-\left(\frac{12}{13}\right)^2} = \frac{169}{25}$$

Tangent is

$$(y - \ln 5) = \frac{169}{25}\left(x - \frac{12}{13}\right)$$

$$25y - 25\ln 5 = 169x - 156$$

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11 In order to find the normal to the curve $y = \operatorname{arcosh} 2x$ at the point $x = 2$, we first must find the value of $\dfrac{dy}{dx}$ at that point.

$\dfrac{dy}{dx} = \dfrac{2}{\sqrt{4x^2 - 1}}$, so at $x = 2$,

$\dfrac{dy}{dx} = \dfrac{2}{\sqrt{4 \times 2^2 - 1}}$

$= \dfrac{2}{\sqrt{15}}$.

The gradient of the normal at this point is therefore $-\dfrac{\sqrt{15}}{2}$

The y value at $x = 2$ is $\operatorname{arcosh} 4 = \ln\left(4 + \sqrt{15}\right)$.

So now we substitute our values for x, y, m into $y = mx + c$ in order to find c.

$y = mx + c$

$\ln\left(4 + \sqrt{15}\right) = -\dfrac{\sqrt{15}}{2} \times 2 + c$

$c = \ln\left(4 + \sqrt{15}\right) + \sqrt{15}$.

So we have $y = -\dfrac{\sqrt{15}}{2}x + \sqrt{15} + \ln\left(4 + \sqrt{15}\right)$.

12 a We differentiate and evaluate at 0 until we have 3 non-zero terms.

$f(x) = \cosh x \Rightarrow f(0) = 1$

$f'(x) = \sinh x \Rightarrow f'(0) = 0$

$f''(x) = \cosh x \Rightarrow f''(0) = 1$

$f'''(x) = \sinh x \Rightarrow f'''(0) = 0$

$f^{(4)}(x) = \cosh x \Rightarrow f^{(4)}(0) = 1$.

Now we use the standard Maclaurin series expansion and obtain

$\cosh x \approx 1 + \dfrac{x^2}{2!} + \dfrac{x^4}{4!}$

$= 1 + \dfrac{x^2}{2} + \dfrac{x^4}{24}$.

b Using the approximation, $\cosh 0.2 \approx 1 + \dfrac{0.2^2}{2!} + \dfrac{0.2^4}{4!} = 1.020067 (6 \text{ d.p.})$.

$error = \left|\dfrac{1.020066667 - \cosh 0.2}{\cosh 0.2}\right| \times 100$

$= 8.7 \times 10^{-6}\%$.

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13 a We differentiate and evaluate at 0 until we have 3 non-zero terms.

$f(x) = \sinh x \Rightarrow f(0) = 0$

$f'(x) = \cosh x \Rightarrow f'(0) = 1$

$f''(x) = \sinh x \Rightarrow f''(0) = 0$

$f'''(x) = \cosh x \Rightarrow f'''(0) = 1$

$f^{(4)}(x) = \sinh x \Rightarrow f^{(4)}(0) = 0$

$f^{(5)}(x) = \cosh x \Rightarrow f^{(5)}(0) = 1.$

Now we use the standard Maclaurin series expansion and obtain

$$\sinh x \approx x + \frac{x^3}{3!} + \frac{x^5}{5!}$$
$$= x + \frac{x^3}{6} + \frac{x^5}{120}.$$

b Since for all integers $n \geq 1$, $f^{(2n-2)}(x) = \sinh x \Rightarrow f^{(2n-2)}(0) = 0,$
$f^{(2n-1)}(x) = \cosh x \Rightarrow f^{(2n-1)}(0) = 1.$

We can conclude that only the odd derivatives will contribute to the Maclaurin series expansion, each with a denominator of $(2n-1)!$.

The first non-zero term occurs when $n = 1$ with this choice of superscript and so we can conclude that the nth non-zero term is $\frac{x^{2n-1}}{(2n-1)!}$.

14 a We differentiate and evaluate at 0 until we have 2 non-zero terms.

$f(x) = \tanh x \Rightarrow f(0) = 0$

$f'(x) = \operatorname{sech}^2 x \Rightarrow f'(0) = 1$

$f''(x) = -2\tanh x \operatorname{sech}^2 x \Rightarrow f''(0) = 0$

$f'''(x) = 4\tanh^2 x \operatorname{sech}^2 x - 2\operatorname{sech}^4 x \Rightarrow f'''(0) = -2.$

Now we use the standard Maclaurin series expansion and obtain

$$\tanh x \approx x - \frac{2x^3}{3!}$$
$$= x - \frac{x^3}{3}.$$

b Using the approximation,

$$\tanh 0.8 \approx 0.8 - \frac{0.8^3}{3}$$
$$= 0.629333 (6 \text{ d.p.})$$
$$error = \left| \frac{0.629333 - \tanh 0.8}{\tanh 0.8} \right| \times 100$$
$$= 5.23\% (3 \text{ s.f.})$$

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15 a We differentiate and evaluate at 0 until we have three non-zero terms.

$f(x) = \text{ar tanh}\, x \Rightarrow f(0) = 0$

$$f'(x) = \frac{1}{1-x^2} \Rightarrow f'(0) = 1$$

$$f''(x) = \frac{2x}{\left(1-x^2\right)^2} \Rightarrow f''(0) = 0$$

$$f'''(x) = \frac{2(3x^2+1)}{\left(1-x^2\right)^3} \Rightarrow f'''(0) = 2$$

$$f^{(4)}(x) = \frac{24x(x^2+1)}{\left(1-x^2\right)^4} \Rightarrow f^{(4)}(0) = 0$$

$$f^{(5)}(x) = \frac{24(5x^4+10x^2+1)}{\left(1-x^2\right)^5} \Rightarrow f^{(5)}(0) = 24.$$

Now we use the standard Maclaurin series expansion and obtain

$$\begin{aligned}\text{artanh}\, x &\approx x + \frac{2x^3}{3!} + \frac{24x^5}{5!} \\ &= x + \frac{x^3}{3} + \frac{x^5}{5}.\end{aligned}$$

b By observation, the n th non-zero term in the series expansion appears to be $\dfrac{x^{2n-1}}{2n-1}$.

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15 c We differentiate and evaluate at 0 until we have two non-zero terms.

$f(x) = \cosh x \operatorname{artanh} x \Rightarrow f(0) = 0$

$$f'(x) = \frac{\cosh x}{1-x^2} + \sinh x \operatorname{artanh} x \Rightarrow f'(0) = 1$$

$$f''(x) = \frac{2\sinh x}{1-x^2} + \frac{2x\cosh x}{\left(1-x^2\right)^2} + \cosh x \operatorname{artanh} x \Rightarrow f''(0) = 0$$

$$f'''(x) = \frac{6x\sinh x + 2\cosh x}{\left(1-x^2\right)^2} + \frac{8x^2\cosh x}{\left(1-x^2\right)^3}$$

$$+\frac{3\cosh x}{1-x^2} + \sinh x \operatorname{artanh} x \Rightarrow f'''(0) = 5.$$

Now we use the standard Maclaurin series expansion and obtain

$$\cosh x \operatorname{artanh} x \approx x + \frac{5x^3}{3!}$$

$$= x + \frac{5x^3}{6}$$

A less tedious way of doing this would be to take the expansions of $\cosh x$ and $\operatorname{artanh} x$ then multiply together, omitting higher order terms.

$$\cosh x \approx 1 + \frac{x^2}{2} + \frac{x^4}{24}$$

$$\operatorname{artanh} x \approx x + \frac{x^3}{3} + \frac{x^5}{5}$$

$$\cosh x \operatorname{artanh} x \approx x + \frac{x^3}{2} + \frac{x^3}{3} + \ldots$$

$$= x + \frac{5x^3}{6}.$$

16 We differentiate and evaluate at 0 until we have three non-zero terms.

$f(x) = \sinh x \cosh 2x \Rightarrow f(0) = 0$

$f'(x) = \cosh x \cosh 2x + 2\sinh x \sinh 2x \Rightarrow f'(0) = 1$

$f''(x) = 5\sinh x \cosh 2x + 4\cosh x \sinh 2x \Rightarrow f''(0) = 0$

$f'''(x) = 13\cosh x \cosh 2x + 14\sinh x \sinh 2x \Rightarrow f'''(0) = 13$

$f^{(4)}(x) = 41\sinh x \cosh 2x + 40\cosh x \sinh 2x \Rightarrow f^{(4)}(0) = 0$

$f^{(5)}(x) = 121\cosh x \cosh 2x + 122\sinh x \sinh 2x \Rightarrow f^{(5)}(0) = 121.$

Now we use the standard Maclaurin series expansion and obtain

$$\sinh x \cosh 2x \approx x + \frac{13x^3}{3!} + \frac{121x^5}{5!}$$

$$= x + \frac{13x^3}{6} + \frac{121x^5}{120}.$$

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17 a We differentiate $y = \cos x \cosh x$ with respect to x four times.

$y = \cos x \cosh x$

$\frac{dy}{dx} = \cos x \sinh x - \sin x \cosh x$

$\frac{d^2y}{dx^2} = -2\sin x \sinh x$

$\frac{d^3y}{dx^3} = -2(\cos x \sinh x + \sin x \cosh x)$

$\frac{d^4y}{dx^4} = -4\cos x \cosh x = -4y.$

b We evaluate the differentials at 0 until we have three non-zero terms.

$f(x) = \cos x \cosh x \Rightarrow f(0) = 1$

$f'(x) = \cos x \sinh x - \sin x \cosh x \Rightarrow f'(0) = 0$

$f''(x) = -2\sin x \sinh x \Rightarrow f''(0) = 0$

$f'''(x) = -2(\cos x \sinh x + \sin x \cosh x) \Rightarrow f'''(0) = 0$

$f^{(4)}(x) = -4\cos x \cosh x = -4f(x) \Rightarrow f^{(4)}(0) = -4.$

Since we have the relation $\frac{d^4y}{dx^4} = -4y,$ we can conclude that all differentials that are not of the form $\frac{d^{4n}y}{dx^{4n}},$ (where n is an integer) is 0 when evaluated at 0. So our third non-zero term is

$$\frac{d^8y}{dx^8} = \frac{d^4}{dx^4}\left(\frac{d^4y}{dx^4}\right)$$

$$= \frac{d^4}{dx^4}(-4y)$$

$$= -4\frac{d^4y}{dx^4}$$

$$= 16y$$

$$= 16\cos x \cosh x \Rightarrow f^{(8)}(0) = 16.$$

Now we use the standard Maclaurin series expansion and obtain

$$\cos x \cosh x \approx 1 - \frac{4x^4}{4!} + \frac{16x^8}{8!}$$

$$= 1 - \frac{x^4}{6} + \frac{x^8}{2520}$$

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17 c From the previous question, we know that the $\frac{\mathrm{d}^{4n}y}{\mathrm{d}x^{4n}}$ terms are the only non-zero contributions.

Recalling that the Maclaurin series expansion is $\mathrm{f}(x)=\sum_{k=0}^{\infty}\frac{\mathrm{f}^{(k)}(0)x^k}{k!}$ and considering

$$\begin{aligned}\frac{\mathrm{d}^{4n}y}{\mathrm{d}x^{4n}}&=\frac{\mathrm{d}^{4n-4}}{\mathrm{d}x^{4n-4}}\left(\frac{\mathrm{d}^4y}{\mathrm{d}x^4}\right)\\&=\frac{\mathrm{d}^{4n-4}}{\mathrm{d}x^{4n-4}}(-4y)\\&=-4\frac{\mathrm{d}^{4n-8}}{\mathrm{d}x^{4n-8}}\left(\frac{\mathrm{d}^4y}{\mathrm{d}x^4}\right)\\&=-4\frac{\mathrm{d}^{4n-8}}{\mathrm{d}x^{4n-8}}(-4y)\\&\vdots\\&=(-4)^n y,\end{aligned}$$

along with all differentials that are not of the form $\frac{\mathrm{d}^{4n}y}{\mathrm{d}x^{4n}}$ (where n is an integer) is 0 when evaluated at 0, we can write an expression.

$$\mathrm{f}(x)=\sum_{k=0}^{\infty}\frac{\mathrm{f}^{(k)}(0)x^k}{k!}$$

$$\begin{aligned}\cos x\cosh x&=\sum_{r=0}^{\infty}\frac{\frac{\mathrm{d}^{4r}y}{\mathrm{d}x^{4r}}(0)\,x^{4r}}{(4r)!}\\&=\sum_{r=0}^{\infty}\frac{(-4)^r y(0)x^{4r}}{(4r)!}\\&=\sum_{r=0}^{\infty}\frac{(-4)^r x^{4r}}{(4r)!}\end{aligned}$$

Challenge

We differentiate and evaluate at 0 until we have three non-zero terms.

$\mathrm{f}(x)=\operatorname{sech}x\Rightarrow\mathrm{f}(0)=1$

$\mathrm{f}'(x)=-\tanh x\operatorname{sech}x\Rightarrow\mathrm{f}'(0)=0$

$\mathrm{f}''(x)=\tanh^2 x\operatorname{sech}x-\operatorname{sech}^3 x\Rightarrow\mathrm{f}''(0)=-1$

$\mathrm{f}'''(x)=2\tanh x\operatorname{sech}^3 x-\tanh^3 x\operatorname{sech}x+3\tanh x\operatorname{sech}^3 x$

$=5\tanh x\operatorname{sech}^3 x-\tanh^3 x\operatorname{sech}x\Rightarrow\mathrm{f}'''(0)=0$

$\mathrm{f}^{(4)}(x)=5\operatorname{sech}^5 x-15\tanh^2 x\operatorname{sech}^3 x$

$-3\tanh^2 x\operatorname{sech}^3 x+\tanh^4 x\operatorname{sech}x\Rightarrow\mathrm{f}^{(4)}(0)=5.$

Now we use the standard Maclaurin series expansion and obtain

$$\begin{aligned}\sinh x\cosh 2x&\approx 1-\frac{x^2}{2!}+\frac{5x^4}{4!}\\&=1-\frac{x^2}{2}+\frac{5x^4}{24}\end{aligned}$$

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Hyperbolic Functions 6E

1 a $\int(\sinh x + 3\cosh x)\,\mathrm{d}x = \cosh x + 3\sinh x + C$

b $\int\left(\cosh x - \frac{1}{\cosh^2 x}\right)\mathrm{d}x = \int(\cosh x - \operatorname{sech}^2 x)\,\mathrm{d}x = \sinh x - \tanh x + C$

c $\int \frac{\sinh x}{\cosh^2 x}\,\mathrm{d}x = \int \frac{1}{\cosh x}\cdot\frac{\sinh x}{\cosh x}\,\mathrm{d}x = \int \operatorname{sech}x \tanh x\ \mathrm{d}x = -\operatorname{sech}x + C$

2 a $\int \sinh 2x\,\mathrm{d}x = \frac{1}{2}\cosh 2x + C$

b $\int \cosh\left(\frac{x}{3}\right)\mathrm{d}x = \frac{1}{\left(\frac{1}{3}\right)}\sinh\left(\frac{x}{3}\right) + C = 3\sinh\left(\frac{x}{3}\right) + C$

3 a $\int \frac{1+x}{\sqrt{(x^2-1)}}\,\mathrm{d}x = \int \frac{1}{\sqrt{(x^2-1)}}\,\mathrm{d}x + \int \frac{x}{\sqrt{(x^2-1)}}\,\mathrm{d}x$

$= \int \frac{1}{\sqrt{(x^2-1)}}\,\mathrm{d}x + \int x(x^2-1)^{-\frac{1}{2}}\,\mathrm{d}x$

$= \operatorname{arcosh}x + \sqrt{(x^2-1)} + C$

b $\int \frac{x-3}{\sqrt{(1+x^2)}}\,\mathrm{d}x = \int \frac{x}{\sqrt{(1+x^2)}}\,\mathrm{d}x - \int \frac{3}{\sqrt{(1+x^2)}}\,\mathrm{d}x$

$= \int x(1+x^2)^{-\frac{1}{2}}\,\mathrm{d}x - \int \frac{3}{\sqrt{(1+x^2)}}\,\mathrm{d}x$

$= \sqrt{(1+x^2)} - 3\operatorname{arsinh}x + C$

4 a $\int \sinh^3 x \cosh x\,\mathrm{d}x = \int (\sinh x)^3 \cosh x\,\mathrm{d}x = \frac{1}{4}\sinh^4 x + C$

b $\int \tanh 4x\,\mathrm{d}x = \int \frac{\sinh 4x}{\cosh 4x}\,\mathrm{d}x = \frac{1}{4}\ln\cosh 4x + C$

c $\int \sqrt{\cosh 2x}\sinh 2x\,\mathrm{d}x = \frac{1}{2}\int (\cosh 2x)^{\frac{1}{2}}(2\sinh 2x)\,\mathrm{d}x$

$= \frac{1}{2}\left\{\frac{(\cosh 2x)^{\frac{3}{2}}}{\left(\frac{3}{2}\right)}\right\} + C$

$= \frac{1}{3}(\cosh 2x)^{\frac{3}{2}} + C$

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5 a $\displaystyle\int \frac{\sinh x}{2+3\cosh x}\,\mathrm{d}x = \frac{1}{3}\int \frac{3\sinh x}{2+3\cosh x}\,\mathrm{d}x$

$$= \frac{1}{3}\ln(2+3\cosh x) + C$$

b $\displaystyle\int \frac{1+\tanh x}{\cosh^2 x}\,\mathrm{d}x = \int (1+\tanh x)\operatorname{sech}^2 x\,\mathrm{d}x$

$$= \int (\operatorname{sech}^2 x + \tanh x \operatorname{sech}^2 x)\mathrm{d}x$$

$$= \tanh x + \frac{1}{2}\tanh^2 x + C \quad \text{or} \quad \tanh x - \frac{1}{2}\operatorname{sech}^2 x + C$$

c $\displaystyle\int \frac{5\cosh x + 2\sinh x}{\cosh x}\,\mathrm{d}x = \int (5 + 2\tanh x)\mathrm{d}x$

$$= 5x + 2\ln\cosh x + C$$

6 $\displaystyle\int x\sinh 3x\,\mathrm{d}x = \frac{1}{3}x\cosh 3x - \int \frac{1}{3}\cosh 3x\,\mathrm{d}x$

$$= \frac{1}{3}x\cosh 3x - \frac{1}{9}\sinh 3x + C$$

Using $\displaystyle\int u\frac{\mathrm{d}v}{\mathrm{d}x}\mathrm{d}x = uv - \int v\frac{\mathrm{d}u}{\mathrm{d}x}\mathrm{d}x$ with $u = x$ and $\displaystyle\frac{\mathrm{d}v}{\mathrm{d}x} = \sinh 3x$

7 a $\displaystyle\int \mathrm{e}^x \cosh x\,\mathrm{d}x = \int \mathrm{e}^x\left(\frac{\mathrm{e}^x + \mathrm{e}^{-x}}{2}\right)\mathrm{d}x$

$$= \frac{1}{2}\int (\mathrm{e}^{2x} + 1)\,\mathrm{d}x$$

$$= \frac{1}{4}\mathrm{e}^{2x} + \frac{1}{2}x + C$$

You cannot use integration by parts.

b $\displaystyle\int \mathrm{e}^{-2x}\sinh 3x\,\mathrm{d}x = \int \mathrm{e}^{-2x}\left(\frac{\mathrm{e}^{3x} - \mathrm{e}^{-3x}}{2}\right)\mathrm{d}x$

$$= \frac{1}{2}\int (\mathrm{e}^{x} - \mathrm{e}^{-5x})\,\mathrm{d}x$$

$$= \frac{1}{2}\mathrm{e}^{x} + \frac{1}{10}\mathrm{e}^{-5x} + C$$

You could use integration by parts twice.

c $\displaystyle\int \cosh x\cosh 3x\,\mathrm{d}x = \int\left(\frac{\mathrm{e}^x + \mathrm{e}^{-x}}{2}\right)\left(\frac{\mathrm{e}^{3x} + \mathrm{e}^{-3x}}{2}\right)\mathrm{d}x$

or write as $\displaystyle\frac{1}{2}(\cosh 4x + \cosh 2x)$

$$= \frac{1}{4}\int (\mathrm{e}^{4x} + \mathrm{e}^{-4x} + \mathrm{e}^{2x} + \mathrm{e}^{-2x})\,\mathrm{d}x$$

$$= \frac{1}{16}\mathrm{e}^{4x} - \frac{1}{16}\mathrm{e}^{-4x} + \frac{1}{8}\mathrm{e}^{2x} - \frac{1}{8}\mathrm{e}^{-2x} + C \quad \text{or} \quad \frac{1}{8}\sinh 4x + \frac{1}{4}\sinh 2x + C$$

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8 $\sinh x + \cosh x = \frac{1}{2}(e^x - e^{-x}) + \frac{1}{2}(e^x + e^{-x}) = e^x$

So $\int_0^1 \left(\frac{1}{\sinh x + \cosh x}\right) dx = \int_0^1 e^{-x} dx = \left[-e^{-x}\right]_0^1 = 1 - \frac{1}{e}$

9 **a** $\int \sinh^2 x \, dx = \frac{1}{2}\int (\cosh 2x - 1) \, dx = \frac{1}{4}\sinh 2x - \frac{1}{2}x + C$

b $\int \sinh^2 x \cosh^2 x \, dx = \int \left(\frac{1}{2}\sinh 2x\right)^2 dx$

$= \frac{1}{4}\int \sinh^2 2x \, dx$

$= \frac{1}{4}\int \left(\frac{\cosh 4x - 1}{2}\right) dx$ ← Using $\cosh 2u = 1 + 2\sinh^2 u$

$= -\frac{1}{8}x + \frac{1}{32}\sinh 4x + C$

c $\int \cosh^5 x \, dx = \int \cosh^4 x \cosh x \, dx$

$= \int (1 + \sinh^2 x)^2 \cosh x \, dx$

$= \int (1 + 2\sinh^2 x + \sinh^4 x)\cosh x \, dx$

$= \int (\cosh x + 2\sinh^2 x \cosh x + \sinh^4 x \cosh x) \, dx$

$= \sinh x + \frac{2}{3}\sinh^3 x + \frac{1}{5}\sinh^5 x + C$

10 $\int_0^{\ln 2} \cosh^2\left(\frac{x}{2}\right) dx = \int_0^{\ln 2} \left(\frac{1 + \cosh x}{2}\right) dx$

$= \frac{1}{2}\left[x + \sinh x\right]_0^{\ln 2}$

$= \frac{1}{2}\left[\ln 2 + \left(\frac{e^{\ln 2} - e^{-\ln 2}}{2}\right)\right]$ ← $e^{\ln 2} = 2$, $e^{-\ln 2} = e^{\ln \frac{1}{2}} = \frac{1}{2}$

$= \frac{1}{2}\left[\ln 2 + \frac{3}{4}\right]$

$= \frac{1}{8}\left[3 + 4\ln 2\right]$

$= \frac{1}{8}(3 + \ln 16)$

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11 a Let $x = 3\cosh u$, so $\mathrm{d}x = 3\sinh u\,\mathrm{d}u$

$$\int \frac{1}{\sqrt{x^2-9}}\mathrm{d}x = \int \frac{1}{\sqrt{9\cosh^2 u - 9}} 3\sinh u\ \mathrm{d}u$$

$$= \int \frac{1}{3\sqrt{\cosh^2 u - 1}} 3\sinh u\ \mathrm{d}u$$

$$= \int \frac{3\sinh u}{3\sinh u}\mathrm{d}u$$

$$= \int 1\,\mathrm{d}u$$

$$= u + C$$

$$= \operatorname{arcosh}\left(\frac{x}{3}\right) + C$$

b You need $4x^2 = 25\sinh^2 u$, or $2x = 5\sinh u$, then $\mathrm{d}x = \frac{5}{2}\cosh u\mathrm{d}u$

$$\int \frac{1}{\sqrt{4x^2+25}}\mathrm{d}x = \int \frac{1}{\sqrt{25\sinh^2 u + 25}}\left(\frac{5}{2}\cosh u\right)\mathrm{d}u$$

$$= \frac{5}{2}\int \frac{\cosh u}{5\sqrt{\sinh^2 u + 1}}\mathrm{d}u$$

$$= \frac{1}{2}\int \frac{\cosh u}{\cosh u}\mathrm{d}u$$

$$= \frac{1}{2}\int 1\,\mathrm{d}u$$

$$= \frac{1}{2}u + C$$

$$= \frac{1}{2}\operatorname{arsinh}\left(\frac{2x}{5}\right) + C$$

12 a $\displaystyle\int \frac{3}{\sqrt{x^2+9}}\mathrm{d}x = 3\operatorname{arsinh}\left(\frac{x}{3}\right) + C$ ← Using $\displaystyle\int \frac{1}{\sqrt{x^2+a^2}}\mathrm{d}x = \operatorname{arsinh}\left(\frac{x}{a}\right) + C.$

b $\displaystyle\int \frac{1}{\sqrt{x^2-2}}\mathrm{d}x = \operatorname{arcosh}\left(\frac{x}{\sqrt{2}}\right) + C$ ← Using $\displaystyle\int \frac{1}{\sqrt{x^2-a^2}}\mathrm{d}x = \operatorname{arcosh}\left(\frac{x}{a}\right) + C$

13 a $$\int \frac{1}{\sqrt{4x^2-12}}\mathrm{d}x = \int \frac{1}{\sqrt{4(x^2-3)}}\mathrm{d}x$$

$$= \frac{1}{2}\int \frac{1}{\sqrt{x^2-3}}\mathrm{d}x$$

$$= \frac{1}{2}\operatorname{arcosh}\left(\frac{x}{\sqrt{3}}\right) + C$$

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13 b $\displaystyle\int \frac{1}{\sqrt{9x^2+16}}\,\mathrm{d}x = \int \frac{1}{\sqrt{9\left\{x^2+\left(\frac{16}{9}\right)\right\}}}\,\mathrm{d}x$

$$= \frac{1}{3}\int \frac{1}{\sqrt{\left\{x^2+\left(\frac{16}{9}\right)\right\}}}\,\mathrm{d}x$$

$$= \frac{1}{3}\operatorname{arsinh}\left(\frac{x}{\left(\frac{4}{3}\right)}\right)+C$$

$$= \frac{1}{3}\operatorname{arsinh}\left(\frac{3x}{4}\right)+C$$

14 $\displaystyle\int_1^2 \frac{3}{\sqrt{1+4x^2}}\,\mathrm{d}x = 3\int_1^2 \frac{1}{2\sqrt{\frac{1}{4}+x^2}}\,\mathrm{d}x$

$$= \frac{3}{2}\left[\operatorname{arsinh}\frac{x}{\left(\frac{1}{2}\right)}\right]_1^2$$

$$= \frac{3}{2}\left[\operatorname{arsinh}(2x)\right]_1^2$$

$$= \frac{3}{2}\left[\operatorname{arsinh}4 - \operatorname{arsinh}2\right]$$

$$= 0.977 \quad (3\text{ s.f.})$$

15 Reminder: The logarithmic form of an inverse hyperbolic function is in the Edexcel formulae booklet.

a $\displaystyle\int_0^4 \frac{1}{\sqrt{x^2+16}}\,\mathrm{d}x = \left[\operatorname{arsinh}\left(\frac{x}{4}\right)\right]_0^4$

$$= \operatorname{arsinh}1 - \operatorname{arsinh}0$$

$$= \ln\left(1+\sqrt{2}\right)$$

← Using $\operatorname{arsinh}x = \ln\left\{x+\sqrt{x^2+1}\right\}$

b $\displaystyle\int_{13}^{15} \frac{1}{\sqrt{x^2-144}}\,\mathrm{d}x = \left[\operatorname{arcosh}\left(\frac{x}{12}\right)\right]_{13}^{15}$

$$= \operatorname{arcosh}\left(\frac{5}{4}\right) - \operatorname{arcosh}\left(\frac{13}{12}\right)$$

← Using $\operatorname{arcosh}x = \ln\left\{x+\sqrt{x^2-1}\right\}$

$$= \ln\left\{\frac{5}{4}+\sqrt{\frac{25}{16}-1}\right\} - \ln\left\{\frac{13}{12}+\sqrt{\frac{169}{144}-1}\right\}$$

$$= \ln\left\{\frac{5}{4}+\sqrt{\frac{9}{16}}\right\} - \ln\left\{\frac{13}{12}+\sqrt{\frac{25}{144}}\right\}$$

$$= \ln 2 - \ln\left(\frac{3}{2}\right)$$

$$= \ln\left(\frac{4}{3}\right)$$

← Using $\ln a - \ln b = \ln\left(\frac{a}{b}\right)$

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16 With $x = \sinh^2 u$ and $\mathrm{d}x = 2\sin u \cosh u \,\mathrm{d}u$,

and $\dfrac{x}{x+1} = \dfrac{\sinh^2 u}{\sinh^2 u + 1} = \dfrac{\sinh^2 u}{\cosh^2 u}$

$$\int \sqrt{\frac{x}{x+1}}\,\mathrm{d}x = \int \frac{\sinh u}{\cosh u} 2\sinh u \cosh u \,\mathrm{d}u$$
$$= \int 2\sinh^2 u \,\mathrm{d}u$$
$$= \int (\cosh 2u - 1)\,\mathrm{d}u$$
$$= \frac{\sinh 2u}{2} - u + C$$
$$= \sinh u \cosh u - \operatorname{arsinh}(\sqrt{x}) + C$$
$$= \sqrt{x}\sqrt{1+x} - \operatorname{arsinh}(\sqrt{x}) + C$$

$\sinh u = \sqrt{x}$ and $\cosh u = \sqrt{1 + \sinh^2 u}$

17 With $u = x^2$ and $\mathrm{d}u = 2x\,\mathrm{d}x$,

$$\int_2^3 \frac{2x}{\sqrt{x^4 - 1}}\mathrm{d}x = \int_4^9 \frac{\mathrm{d}u}{\sqrt{u^2 - 1}}$$
$$= \left[\operatorname{arcosh} u\right]_4^9$$
$$= \operatorname{arcosh} 9 - \operatorname{arcosh} 4$$
$$= 0.824 \quad (3\text{ s.f.})$$

18 Using $x = 2\cosh u$, $\mathrm{d}x = 2\sinh u \,\mathrm{d}u$

$$\int \sqrt{x^2 - 4}\,\mathrm{d}x = \int 2\sqrt{\cosh^2 u - 1} \times 2\sinh u \,\mathrm{d}u$$
$$= 4\int \sinh^2 u \,\mathrm{d}u$$
$$= 2\int (\cosh 2u - 1)\,\mathrm{d}u$$
$$= 2\left\{\frac{\sinh 2u}{2} - u\right\} + C$$
$$= 2\sinh u \cosh u - 2u + C$$
$$= 2\left(\sqrt{\left(\frac{x}{2}\right)^2 - 1}\right)\left(\frac{x}{2}\right) - 2\operatorname{arcosh}\left(\frac{x}{2}\right) + C$$
$$= 2\left(\frac{\sqrt{x^2 - 4}}{2}\right)\left(\frac{x}{2}\right) - 2\operatorname{arcosh}\left(\frac{x}{2}\right) + C$$
$$= \frac{1}{2}x\sqrt{x^2 - 4} - 2\operatorname{arcosh}\left(\frac{x}{2}\right) + C$$

$\cosh u = \dfrac{x}{2}$ and $\sinh u = \sqrt{\cosh^2 u - 1}$

19 a $2\cosh x - \sinh x = 2\left(\dfrac{\mathrm{e}^x + \mathrm{e}^{-x}}{2}\right) - \left(\dfrac{\mathrm{e}^x - \mathrm{e}^{-x}}{2}\right) = \dfrac{\mathrm{e}^x + 3\mathrm{e}^{-x}}{2}$

So $\displaystyle\int \frac{1}{2\cosh x - \sinh x}\mathrm{d}x = \int \frac{2}{\mathrm{e}^x + 3\mathrm{e}^{-x}}\mathrm{d}x$

$$= \int \frac{2\mathrm{e}^x}{\mathrm{e}^{2x} + 3}\mathrm{d}x$$

Multiplying numerator and denominator by e^x.

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19 b Using the substitution $u = e^x$, $du = e^x\,dx$ and

$$\int \frac{2e^x}{e^{2x}+3}dx = 2\int \frac{du}{u^2+3}$$

$$= \frac{2}{\sqrt{3}}\arctan\left(\frac{u}{\sqrt{3}}\right)+C$$

$$= \frac{2}{\sqrt{3}}\arctan\left(\frac{e^x}{\sqrt{3}}\right)+C$$

20 With $u = \frac{2}{3}\sinh x$, $du = \frac{2}{3}\cosh x dx$ or $\cosh x dx = \frac{3}{2}du$

$$4\sinh^2 x + 9 = 4\left(\frac{3u}{2}\right)^2 + 9 = 9u^2 + 9 = 9(u^2+1)$$

$$\text{So} \int_0^1 \frac{\cosh x}{\sqrt{4\sinh^2 x + 9}}dx = \int_0^{\frac{2}{3}\sinh 1} \frac{1}{3\sqrt{u^2+1}} \times \frac{3}{2}du$$

$$= \left[\frac{1}{2}\operatorname{arsinh}(u)\right]_0^{\frac{2}{3}\sinh 1}$$

$$= \frac{1}{2}\operatorname{arsinh}\left(\frac{2}{3}\sinh 1\right)$$

$$= 0.360 \quad (3 \text{ s.f.})$$

21 a $x^2 - 4x - 12 = \{(x-2)^2 - 16\}$

$$\text{So} \int \frac{1}{\sqrt{x^2-4x-12}}dx = \int \frac{1}{\sqrt{(x-2)^2-16}}dx$$

Let $u = (x-2)$, so $du = dx$.

$$\text{Then} \int \frac{1}{\sqrt{x^2-4x-12}}dx = \int \frac{1}{\sqrt{u^2-16}}dx$$

$$= \operatorname{arcosh}\left(\frac{u}{4}\right)+C$$

$$= \operatorname{arcosh}\left(\frac{x-2}{4}\right)+C$$

b $x^2 + 6x + 10 = \{(x+3)^2 + 1\}$

$$\text{So} \int \frac{1}{\sqrt{x^2+6x+10}}dx = \int \frac{1}{\sqrt{(x+3)^2+1}}dx$$

Let $u = (x+3)$, so $du = dx$.

$$\text{Then} \int \frac{1}{\sqrt{x^2+6x+10}}dx = \int \frac{1}{\sqrt{u^2+1}}du$$

$$= \operatorname{arsinh}(u)+C$$

$$= \operatorname{arsinh}(x+3)+C$$

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21 c $2x^2+4x+7=2\left(x^2+2x+\frac{7}{2}\right)=2\left\{(x+1)^2+\frac{5}{2}\right\}$

Let $u=(x+1)$, so $\mathrm{d}u=\mathrm{d}x$.

Then $\int\frac{1}{2x^2+4x+7}\mathrm{d}x=\frac{1}{2}\int\frac{1}{u^2+\left(\frac{\sqrt{5}}{\sqrt{2}}\right)^2}\mathrm{d}u$

$$=\frac{1}{2}\left\{\frac{\sqrt{2}}{\sqrt{5}}\arctan\left(\frac{\sqrt{2}u}{\sqrt{5}}\right)\right\}+C$$

$$=\frac{\sqrt{10}}{10}\arctan\left(\frac{\sqrt{2}(x+1)}{\sqrt{5}}\right)+C$$

d $9x^2-8x+1=9\left(x^2-\frac{8}{9}x+\frac{1}{9}\right)=9\left\{\left(x-\frac{4}{9}\right)^2-\frac{7}{81}\right\}$

So $\int\frac{1}{\sqrt{9x^2-8x+1}}\mathrm{d}x=\frac{1}{3}\int\frac{1}{\sqrt{\left(x-\frac{4}{9}\right)^2-\left(\frac{\sqrt{7}}{9}\right)^2}}\mathrm{d}x$

Let $u=\left(x-\frac{4}{9}\right)$, so $\mathrm{d}u=\mathrm{d}x$.

Then $\int\frac{1}{\sqrt{9x^2-8x+1}}\mathrm{d}x=\frac{1}{3}\int\frac{1}{\sqrt{u^2-\left(\frac{\sqrt{7}}{9}\right)^2}}\mathrm{d}u$

$$=\frac{1}{3}\operatorname{arcosh}\left(\frac{9u}{\sqrt{7}}\right)+C$$

$$=\frac{1}{3}\operatorname{arcosh}\left(\frac{9x-4}{\sqrt{7}}\right)+C$$

22 a $4x^2-12x+10=4\left(x^2-3x+\frac{5}{2}\right)=4\left\{\left(x-\frac{3}{2}\right)^2+\frac{1}{4}\right\}$

So $\int\frac{1}{\sqrt{4x^2-12x+10}}\mathrm{d}x=\frac{1}{2}\int\frac{1}{\sqrt{\left(x-\frac{3}{2}\right)^2+\left(\frac{1}{2}\right)^2}}\mathrm{d}x$

Let $u=\left(x-\frac{3}{2}\right)$, so $\mathrm{d}u=\mathrm{d}x$.

Then $\int\frac{1}{\sqrt{4x^2-12x+10}}\mathrm{d}x=\frac{1}{2}\int\frac{1}{\sqrt{u^2+\left(\frac{1}{2}\right)^2}}\mathrm{d}u$

$$=\frac{1}{2}\operatorname{arsinh}(2u)+C$$

$$=\frac{1}{2}\operatorname{arsinh}(2x-3)+C$$

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22 b $4x^2-12x+4=4(x^2-3x+1)=4\left\{\left(x-\frac{3}{2}\right)^2-\frac{5}{4}\right\}$

So $\int\frac{1}{\sqrt{4x^2-12x+4}}\,\mathrm{d}x=\frac{1}{2}\int\frac{1}{\sqrt{\left(x-\frac{3}{2}\right)^2-\left(\frac{\sqrt{5}}{2}\right)^2}}\,\mathrm{d}x$

Let $u=\left(x-\frac{3}{2}\right)$, so $\mathrm{d}u=\mathrm{d}x$.

Then $\int\frac{1}{\sqrt{4x^2-12x+4}}\,\mathrm{d}x=\frac{1}{2}\int\frac{1}{\sqrt{u^2-\left(\frac{\sqrt{5}}{2}\right)^2}}\,\mathrm{d}u$

$$=\frac{1}{2}\operatorname{arcosh}\left(\frac{2u}{\sqrt{5}}\right)+C$$

$$=\frac{1}{2}\operatorname{arcosh}\left(\frac{2x-3}{\sqrt{5}}\right)+C$$

23 $x^2+2x+5=(x+1)^2+4$

So $\int_0^1\frac{1}{\sqrt{x^2+2x+5}}\,\mathrm{d}x=\int_0^1\frac{1}{\sqrt{(x+1)^2+4}}\,\mathrm{d}x$

Let $u=(x+1)$, so $\mathrm{d}u=\mathrm{d}x$.

Then $\int_0^1\frac{1}{\sqrt{x^2+2x+5}}\,\mathrm{d}x=\int_1^2\frac{1}{\sqrt{u^2+2^2}}\,\mathrm{d}u$

$$=\left[\operatorname{arsinh}\left(\frac{u}{2}\right)\right]_1^2$$

$$=\left[\operatorname{arsinh}1-\operatorname{arsinh}\left(\frac{1}{2}\right)\right]$$

$$=0.400\quad(3\text{ s.f.})$$

24 $x^2-2x+2=(x-1)^2+1$

So $\int_1^3\frac{1}{\sqrt{x^2-2x+2}}\,\mathrm{d}x=\int_1^3\frac{1}{\sqrt{(x-1)^2+1}}\,\mathrm{d}x$

$$=\left[\operatorname{arsinh}(x-1)\right]_1^3$$

$$=\operatorname{arsinh}2$$

$$=\ln\{2+\sqrt{5}\}$$

$$\operatorname{arsinh}x=\ln\{x+\sqrt{x^2+1}\}$$

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25 $3x^2-6x+7=3\left(x^2-2x+\frac{7}{3}\right)=3\left\{(x-1)^2+\frac{4}{3}\right\}$

So $\int\frac{1}{\sqrt{3x^2-6x+7}}\,dx=\frac{1}{\sqrt{3}}\int\frac{1}{\sqrt{(x-1)^2+\left(\frac{2}{\sqrt{3}}\right)^2}}\,dx$

Let $u=(x-1)$, so $du=dx$.

Then $\int_1^3\frac{1}{\sqrt{3x^2-6x+7}}=\frac{1}{\sqrt{3}}\int_0^2\frac{1}{\sqrt{u^2+\left(\frac{2}{\sqrt{3}}\right)^2}}\,du$

$$=\frac{1}{\sqrt{3}}\left[\operatorname{arsinh}\left(\frac{\sqrt{3}u}{2}\right)\right]_0^2$$

$$=\frac{1}{\sqrt{3}}\operatorname{arsinh}\sqrt{3}$$

$$=\frac{1}{\sqrt{3}}\ln\{\sqrt{3}+\sqrt{3+1}\}\qquad \operatorname{arsinh}x=\ln\{x+\sqrt{x^2+1}\}$$

$$=\frac{1}{\sqrt{3}}\ln\{2+\sqrt{3}\}$$

26 a In order to find the intersection point of the two curves, we solve

$5\cosh x=7-\sinh x$

$\frac{5}{2}\left(e^x+e^{-x}\right)=7-\frac{1}{2}\left(e^x-e^{-x}\right)$

$\frac{5}{2}\left(e^{2x}+1\right)=7e^x-\frac{1}{2}\left(e^{2x}-1\right)$

$3e^{2x}-7e^x+2=0.$

Set $u=e^x$ in order to clearly see the quadratic equation $3u^2-7u+2=0$ with solutions

$u=\frac{7\pm\sqrt{49-24}}{6}$

$u=\frac{7\pm5}{6}$

i.e.

$u_1=\frac{1}{3}$

$u_2=2.$

This gives us the x values for the intersection points as

$x_1=\ln\frac{1}{3}=-\ln 3$

$x_2=\ln 2$

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26 b We find the area of the region R by calculating

$$\begin{aligned}Area_R &= \int_{x_1}^{x_2}(7-\sinh x)\,\mathrm{d}x - \int_{x_1}^{x_2}(5\cosh x)\,\mathrm{d}x\\ &= \int_{x_1}^{x_2}(7-\sinh x-5\cosh x)\,\mathrm{d}x\\ &= \left[7x-\cosh x-5\sinh x\right]_{-\ln 3}^{\ln 2}\\ &= (7\ln 2-5)-(-7\ln 3+5)\\ &= 7\ln 6-10\\ &= \ln 279936-10.\end{aligned}$$

27 $\text{Volume} = \pi\int_0^1 \sinh^2 x\,\mathrm{d}x = \frac{\pi}{2}\int_0^1(\cosh 2x-1)\mathrm{d}x$

$$\begin{aligned}&= \frac{\pi}{2}\left[\frac{1}{2}\sinh 2x - x\right]_0^1\\ &= \frac{\pi}{2}\left[\frac{1}{2}\sinh 2-1\right]\\ &= \frac{\pi}{2}\left[\frac{1}{4}(\mathrm{e}^2-\mathrm{e}^{-2})-1\right]\\ &= \frac{\pi}{8}\left[\mathrm{e}^2-4-\mathrm{e}^{-2}\right]\\ &= \frac{\pi}{8\mathrm{e}^2}(\mathrm{e}^4-4\mathrm{e}^2-1).\end{aligned}$$

Challenge

1 Using the substitution $x = 1+\sinh\theta$, $\mathrm{d}x = \cosh\theta\ \mathrm{d}\theta$

$x^2-2x+2 = (\sinh^2\theta+2\sinh\theta+1)-2(\sinh\theta+1)+2 = \sinh^2\theta+1 = \cosh^2\theta$

$$\text{So}\int\frac{1}{(x^2-2x+2)^{\frac{3}{2}}}\mathrm{d}x = \int\frac{1}{\cosh^3\theta}\cdot\cosh\theta\,\mathrm{d}\theta$$

$$\begin{aligned}&= \int\operatorname{sech}^2\theta\,\mathrm{d}\theta\\ &= \tanh\theta + C\\ &= \frac{x-1}{\sqrt{x^2-2x+2}} + C\end{aligned}$$

$$\sinh\theta = x-1$$

$$\cosh\theta = \sqrt{1+\sinh^2\theta} = \sqrt{2-2x+x^2}$$

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Challenge

2 Using the substitution $u = x^2$, $\mathrm{d}u = 2x\mathrm{d}x$,

a So $\int x\cosh^2(x^2)\,\mathrm{d}x = \frac{1}{2}\int \cosh^2 \mathrm{u}\,\mathrm{d}u$

$$= \frac{1}{4}\int (\cosh 2u + 1)\,\mathrm{d}u$$

$$= \frac{1}{8}\sinh 2u + \frac{u}{4} + C$$

$$= \frac{1}{8}\sinh(2x^2) + \frac{x^2}{4} + C$$

b So $\int \frac{x}{\cosh^2(x^2)}\mathrm{d}x = \int x\,\mathrm{sech}^2(x^2)\,\mathrm{d}x$

$$= \frac{1}{2}\int \mathrm{sech}^2 u\ \mathrm{d}u$$

$$= \frac{1}{2}\tanh u + C$$

$$= \frac{1}{2}\tanh(x^2) + C$$

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Hyperbolic Functions Mixed Exercise 6

1 a $\sinh(\ln 3) = \dfrac{e^{\ln 3} - e^{-\ln 3}}{2}$

$= \dfrac{3 - \frac{1}{3}}{2} = \dfrac{4}{3}$

$e^{\ln 3} = 3$, and $e^{-\ln 3} = e^{\ln 3^{-1}} = \dfrac{1}{3}$

b $\cosh(\ln 5) = \dfrac{e^{\ln 5} + e^{-\ln 5}}{2}$

$= \dfrac{5 + \frac{1}{5}}{2} = \dfrac{13}{5}$

$e^{\ln 5} = 5$, and $e^{-\ln 5} = e^{\ln 5^{-1}} = \dfrac{1}{5}$

c $\tanh\left(\ln\dfrac{1}{4}\right) = \dfrac{e^{2\ln\frac{1}{4}} - 1}{e^{2\ln\frac{1}{4}} + 1}$

$= \dfrac{\left(\frac{1}{16} - 1\right)}{\left(\frac{1}{16} + 1\right)}$

$= -\dfrac{15}{17}$

$e^{2\ln\frac{1}{4}} = e^{\ln\left(\frac{1}{4}\right)^2} = \dfrac{1}{16}$

2 $\operatorname{artanh} x - \operatorname{artanh} y$

$= \dfrac{1}{2}\ln\left(\dfrac{1+x}{1-x}\right) - \dfrac{1}{2}\ln\left(\dfrac{1+y}{1-y}\right)$

Use $\ln a - \ln b = \ln\left(\dfrac{a}{b}\right)$

$= \dfrac{1}{2}\ln\left(\dfrac{1+x}{1-x} \times \dfrac{1-y}{1+y}\right)$

$= \dfrac{1}{2}\ln\left(\dfrac{1+x-y-xy}{1-x+y-xy}\right)$

Use $\dfrac{1}{2}\ln a = \ln a^{\frac{1}{2}}$

$= \ln\sqrt{\left(\dfrac{1+x-y-xy}{1-x+y-xy}\right)}$

So $\sqrt{\left(\dfrac{1+x-y-xy}{1-x+y-xy}\right)} = 5$

$\dfrac{1+x-y-xy}{1-x+y-xy} = 25$

$1 + x - y - xy = 25 - 25x + 25y - 25xy$

$24xy - 26y = 24 - 26x$

$y(12x - 13) = 12 - 13x$

$y = \dfrac{12 - 13x}{12x - 13}$

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3 $\text{RHS} = \sinh A\cosh B - \cosh A\sinh B$

$$= \left(\frac{e^{A} - e^{-A}}{2}\right)\left(\frac{e^{B} + e^{-B}}{2}\right) - \left(\frac{e^{A} + e^{-A}}{2}\right)\left(\frac{e^{B} - e^{-B}}{2}\right)$$
$$= \frac{e^{A+B} - e^{-A+B} + e^{A-B} - e^{-A-B}}{4} - \frac{e^{A+B} + e^{-A+B} - e^{A-B} - e^{-A-B}}{4}$$
$$= \frac{2(e^{A-B} - e^{-A+B})}{4}$$
$$= \frac{e^{A-B} - e^{-(A-B)}}{2}$$
$$= \sinh(A-B) = \text{LHS}$$

4 $\text{RHS} = \dfrac{2\tanh\frac{1}{2}x}{1-\tanh^2\frac{1}{2}x}$

$$2\tanh\frac{1}{2}x = \frac{2(e^{x}-1)}{e^{x}+1}$$

$$1-\tanh^2\frac{1}{2}x = 1-\left(\frac{e^{x}-1}{e^{x}+1}\right)^2$$
$$= \frac{(e^{x}+1)^2 - (e^{x}-1)^2}{(e^{x}+1)^2}$$
$$= \frac{4e^{x}}{(e^{x}+1)^2}$$

$$\text{So RHS} = \frac{2(e^{x}-1)}{e^{x}+1} \times \frac{(e^{x}+1)^2}{4e^{x}}$$
$$= \frac{(e^{x}-1)(e^{x}+1)}{2e^{x}}$$
$$= \frac{e^{2x}-1}{2e^{x}}$$
$$= \frac{e^{x}-e^{-x}}{2}$$
$$= \sinh x = \text{LHS}$$

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5 $9\cosh x - 5\sinh x = 15$

$$9\frac{\left(e^{x}+e^{-x}\right)}{2} - 5\frac{\left(e^{x}-e^{-x}\right)}{2} = 15$$

$$9e^{x} + 9e^{-x} - 5e^{x} + 5e^{-x} = 30$$

$$4e^{x} - 30 + 14e^{-x} = 0$$

$$2e^{x} - 15 + 7e^{-x} = 0$$ ← Multiply throughout by e^x.

$$2e^{2x} - 15e^{x} + 7 = 0$$

$$(2e^{x} - 1)(e^{x} - 7) = 0$$ ← Solve as a quadratic in e^x.

$$e^{x} = \frac{1}{2},\ e^{x} = 7$$

$$x = \ln\left(\frac{1}{2}\right),\ x = \ln 7$$

6 $23\sinh x - 17\cosh x + 7 = 0$

$$23\frac{(e^{x}-e^{-x})}{2} - 17\frac{(e^{x}+e^{-x})}{2} + 7 = 0$$

$$23e^{x} - 23e^{-x} - 17e^{x} - 17e^{-x} + 14 = 0$$

$$6e^{x} + 14 - 40e^{-x} = 0$$

$$3e^{x} + 7 - 20e^{-x} = 0$$

$$3e^{2x} + 7e^{x} - 20 = 0$$ ← Multiply throughout by e^x.

$$(3e^{x} - 5)(e^{x} + 4) = 0$$

$$e^{x} = \frac{5}{3}$$ ← $e^x = -4$ is not possible for real x.

$$x = \ln\left(\frac{5}{3}\right)$$

7 $3\cosh^2 x + 11\sinh x = 17$

Using $\cosh^2 x - \sinh^2 x = 1$

$$3(1 + \sinh^2 x) + 11\sinh x = 17$$

$$3\sinh^2 x + 11\sinh x - 14 = 0$$

$$(3\sinh x + 14)(\sinh x - 1) = 0$$

$$\sinh x = -\frac{14}{3},\ \sinh x = 1$$

$$x = \operatorname{arsinh}\left(-\frac{14}{3}\right), x = \operatorname{arsinh}1$$ ← Use $\operatorname{arsinh}x = \ln(x + \sqrt{x^2+1})$.

$$x = \ln\left(-\frac{14}{3} + \sqrt{\frac{196}{9} + 1}\right)$$

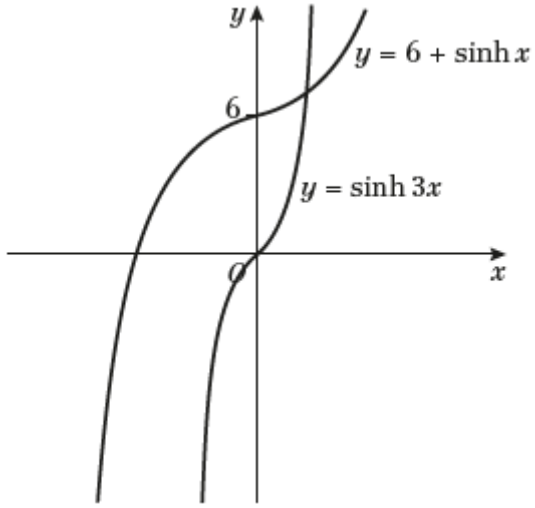
$$= \ln\left(\frac{-14 + \sqrt{205}}{3}\right)$$

$$x = \ln(1 + \sqrt{1+1})$$

$$= \ln(1 + \sqrt{2})$$

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8 a



b At the interaction,

$$6 + \sinh x = \sinh 3x$$

$$6 + \sinh x = 3\sinh x + 4\sinh^3 x$$

$$4\sinh^3 x + 2\sinh x - 6 = 0$$

$$2\sinh^3 x + \sinh x - 3 = 0$$

You can see, by inspection that $\sinh x = 1$ satisfies this equation.

$$(\sinh x - 1)(2\sinh^2 x + 2\sinh x + 3) = 0$$

The equation $2\sinh^2 x + 2\sinh x + 3 = 0$ has no real roots, because

$b^2 - 4ac = 4 - 24 < 0.$

The only intersection is where $\sinh x = 1$

For $\sinh x = 1$,

$x = \text{arsinh}1$

$= \ln(1 + \sqrt{1^2 + 1})$

$= \ln(1 + \sqrt{2})$

Using $y = 6 + \sinh x$

$y = 7$

Coordinates of the point of intersection are $(\ln(1 + \sqrt{2}),\ 7)$

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9 a $13\cosh x + 5\sinh x = R\cosh x\cosh\alpha + R\sinh x\sinh\alpha$

So $R\cosh\alpha = 13$

$R\sinh\alpha = 5$

$R^2\cosh^2\alpha - R^2\sinh^2\alpha = 13^2 - 5^2$ ← Use the identity $\cosh^2 A - \sinh^2 A = 1$.

$R^2(\cosh^2\alpha - \sinh^2\alpha) = 144$

$R^2 = 144$

$R = 12$

$\dfrac{R\sinh\alpha}{R\cosh\alpha} = \dfrac{5}{13}$

$\tanh\alpha = \dfrac{5}{13}$

$\alpha = 0.405$ ← Direct from calculator.

b $13\cosh x + 5\sinh x = 12\cosh(x + 0.405)$ ← For any value A, $\cosh A \geqslant 1$.

The minimum value of $13\cosh x + 5\sinh x$ is 12.

10 a $3\cosh x + 5\sinh x = R\sinh x\cosh\alpha + R\cosh x\sinh\alpha$

So $R\cosh\alpha = 5$

$R\sinh\alpha = 3$

$R^2\cosh^2\alpha - R^2\sinh^2\alpha = 5^2 - 3^2$ ← Use the identity $\cosh^2 A - \sinh^2 A = 1$.

$R^2(\cosh^2\alpha - \sinh^2\alpha) = 16$

$R^2 = 16$

$R = 4$

$\dfrac{R\sinh\alpha}{R\cosh\alpha} = \dfrac{3}{5}$

$\tanh\alpha = \dfrac{3}{5}$

$\alpha = 0.693$ ← Direct from calculator.

$3\cosh x + 5\sinh x = 4\sinh(x + 0.693)$

b $4\sinh(x + 0.693) = 8$

$\sinh(x + 0.693) = 2$

$x + 0.693 = \text{arsinh}2$

$= 1.44$ (3 s.f.) ← Direct from calculator.

$x = 0.75$ (2 d.p.)

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10 c $3\cosh x + 5\sinh = 8$

$$3\frac{(e^x + e^{-x})}{2} + 5\frac{(e^x - e^{-x})}{2} = 8$$

$$3e^x + 3e^{-x} + 5e^x - 5e^{-x} = 16$$

$$8e^x - 16 - 2e^{-x} = 0$$

$$4e^x - 8 - e^{-x} = 0$$

Multiply throughout by e^x.

$$4e^{2x} - 8e^x - 1 = 0$$

Solve as a quadratic in e^x.

$$e^x = \frac{8 \pm \sqrt{64+16}}{8}$$

$$e^x = 1 \pm \frac{\sqrt{80}}{8} = 1 \pm \frac{\sqrt{5}}{2}$$

$e^x = 1 - \frac{\sqrt{5}}{2}$ is negative, so not possible for real x.

$$e^x = 1 + \frac{\sqrt{5}}{2}$$

$$x = \ln\left(1 + \frac{\sqrt{5}}{2}\right)$$

$$= 0.75 \quad (2 \text{ d.p.})$$

11 $y = \cosh 2x$

$$\frac{dy}{dx} = 2\sinh 2x$$

12 a $y = \text{arsinh}3x$

Let $t = 3x$ $y = \text{arsinh}t$

$$\frac{dt}{dx} = 3 \quad \frac{dy}{dt} = \frac{1}{\sqrt{t^2+1}}$$

$$\frac{dy}{dx} = \frac{1}{\sqrt{t^2+1}} \times 3$$

$$= \frac{3}{\sqrt{9x^2+1}}$$

b $y = \text{arsinh}x^2$

Let $t = x^2$ $y = \text{arsinh}t$

$$\frac{dt}{dx} = 2x \quad \frac{dy}{dt} = \frac{1}{\sqrt{t^2+1}}$$

$$\frac{dy}{dt} = \frac{1}{\sqrt{t^2+1}} \times 2x$$

$$= \frac{2x}{\sqrt{x^4+1}}$$

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12 c $y = \text{arcosh}\dfrac{x}{2}$

Let $t = \dfrac{x}{2}$ $y = \text{arcosh}t$

$\dfrac{dt}{dx} = \dfrac{1}{2}$ $\dfrac{dy}{dt} = \dfrac{1}{\sqrt{t^2 - 1}}$

$\dfrac{dy}{dx} = \dfrac{1}{\sqrt{t^2 - 1}} \times \dfrac{1}{2}$

$= \dfrac{1}{2\sqrt{\frac{x^2}{4} - 1}} = \dfrac{1}{\sqrt{x^2 - 4}}$

d $y = x^2 \text{arcosh}2x$

$\dfrac{dy}{dx} = 2x\text{arcosh}2x + x^2 \times \dfrac{2}{\sqrt{4x^2 - 1}}$

$= 2x\left(\text{arcosh}2x + \dfrac{x}{\sqrt{4x^2 - 1}}\right)$

13 $y = (\text{arsinh}x)^2$

$\dfrac{dy}{dx} = \dfrac{2(\text{arsinh}x)^1}{\sqrt{x^2 + 1}}$

$\dfrac{d^2y}{dx^2} = \dfrac{\frac{2}{\sqrt{x^2+1}} \times \sqrt{x^2 + 1} - \frac{1}{2}(x^2 + 1)^{-\frac{1}{2}} \times 2x \times 2\text{arsinh}x}{\left(\sqrt{x^2 + 1}\right)^2}$

$(x^2 + 1)\dfrac{d^2y}{dx^2} = 2 - 2x(x^2 + 1)^{-\frac{1}{2}}\text{arsinh}x$

$= 2 - x\dfrac{dy}{dx}$

$(x^2 + 1)\dfrac{d^2y}{dx^2} + x\dfrac{dy}{dx} - 2 = 0$

14 a Differentiating with respect to x gives us $f'(x) = 5\sinh x - 3\cosh x$.

b $y = f(x)$ has a turning point when $f'(x) = 0$.

$f'(x) = 5\sinh x - 3\cosh x = 0$

$5\sinh x = 3\cosh x$

$5\tanh x = 3$

$\tanh x = \dfrac{3}{5}$

$x = \text{ar}\tanh\left(\dfrac{3}{5}\right)$

$x = \ln 2$

Thus, $x = \ln 2$, and $y = 5\cosh(\ln 2) - 3\sinh(\ln 2)$

Therefore $(\ln 2, 4)$ are the coordinates of the turning point.

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15 $y = \text{arcosh}(\sinh 2x)$

Let $t = \sinh 2x \quad y = \text{arcosh}t$

$$\frac{dt}{dx} = 2\cosh 2x \quad \frac{dy}{dt} = \frac{1}{\sqrt{t^2 - 1}}$$

$$\frac{dy}{dx} = \frac{1}{\sqrt{t^2 - 1}} \times 2\cosh 2x$$

$$= \frac{2\cosh 2x}{\sqrt{\sinh^2 2x - 1}}$$

16 a Differentiating y with respect to x four times and applying the product rule where necessary gives

$$\frac{dy}{dx} = \cos x \cosh x + \sin x \sinh x,$$

$$\frac{d^2y}{dx^2} = 2\cos x \sinh x,$$

$$\frac{d^3y}{dx^3} = -2\sin x \sinh x + 2\cos x \cosh x,$$

$$\frac{d^4y}{dx^4} = -4\sin x \cosh x.$$

So, $\frac{d^4y}{dx^4} = -4y$

b Evaluating the expressions we found in part a at $x = 0$ gives

$$y(0) = 0,$$

$$\frac{dy}{dx}(0) = 1,$$

$$\frac{d^2y}{dx^2}(0) = 0,$$

$$\frac{d^3y}{dx^3}(0) = 2,$$

$$\frac{d^4y}{dx^4}(0) = 0.$$

Since we know that $\frac{d^4y}{dx^4} = -4y$, we conclude $\frac{d^5y}{dx^5} = -4$.

This gives us the first three terms in the Maclaurin expansion as

$$y \approx 0 + 1x + \frac{0x^2}{2!} + \frac{2x^3}{3!} + \frac{0x^4}{4!} + \frac{-4x^5}{5!}$$

$$y \approx x + \frac{x^3}{3} - \frac{x^5}{30}$$

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17 Differentiating y with respect to x five times and applying the product rule where necessary give

$$\frac{dy}{dx} = 2\cosh 2x \cosh x + \sinh 2x \sinh x,$$

$$\frac{d^2y}{dx^2} = 5\sinh 2x \cosh x + 4\cosh 2x \sinh x,$$

$$\frac{d^3y}{dx^3} = 14\cosh 2x \cosh x + 13\sinh 2x \sinh x,$$

$$\frac{d^4y}{dx^4} = 41\sinh 2x \cosh x + 40\cosh 2x \sinh x,$$

$$\frac{d^5y}{dx^5} = 122\cosh 2x \cosh x + 121\sinh 2x \sinh x.$$

Evaluating these expressions at 0 gives

$y(0) = 0$

$\frac{dy}{dx}(0) = 2$

$\frac{d^2y}{dx^2}(0) = 0$

$\frac{d^3y}{dx^3}(0) = 14$

$\frac{d^4y}{dx^4}(0) = 0$

$\frac{d^5y}{dx^5}(0) = 122$

So $y \approx 2x + \frac{7x^3}{3} + \frac{61x^5}{60}$.

18 $4x^2 + 4x + 17 \equiv (ax+b)^2 + c, \quad a > 0$

a $4x^2 + 4x + 17 \equiv (2x+b)^2 + c \quad a = 2$

$\equiv 4x^2 + 4bx + b^2 + c$

Comparing coefficient of x: $b = 1$

Comparing constant term: $17 = 1 + c \Rightarrow c = 16$

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18 b Using **a**, $\int \frac{1}{4x^2+4x+17}\,\mathrm{d}x = \int \frac{1}{(2x+1)^2+16}\,\mathrm{d}x$

Let $2x+1 = 4\tan\theta$, then $2\mathrm{d}x = 4\sec^2\theta\,\mathrm{d}\theta$

and $\int \frac{1}{(2x+1)^2+16}\,\mathrm{d}x = \int \frac{2\sec^2\theta}{16\tan^2\theta+16}\,\mathrm{d}\theta$

$$= \int \frac{2\sec^2\theta}{16\sec^2\theta}\,\mathrm{d}\theta$$

$$= \frac{1}{8}\theta$$

$$= \frac{1}{8}\arctan\left(\frac{2x+1}{4}\right)(+C)$$

So $\int_{-0.5}^{1.5} \frac{1}{4x^2+4x+17}\,\mathrm{d}x = \frac{1}{8}[\arctan 1 - \arctan 0]$

$$= \frac{\pi}{32}$$

19 Using the definitions of $\sinh 4x$ and $\cosh 6x$

$$\int \sinh 4x \cosh 6x\,\mathrm{d}x = \int \left(\frac{\mathrm{e}^{4x}-\mathrm{e}^{-4x}}{2}\right)\left(\frac{\mathrm{e}^{6x}+\mathrm{e}^{-6x}}{2}\right)\mathrm{d}x$$

$$= \frac{1}{4}\int \left(\mathrm{e}^{10x}+\mathrm{e}^{-2x}-\mathrm{e}^{2x}-\mathrm{e}^{-10x}\right)\mathrm{d}x$$

$$= \frac{1}{4}\left\{\frac{\mathrm{e}^{10x}}{10}+\frac{\mathrm{e}^{-2x}}{-2}-\frac{\mathrm{e}^{2x}}{2}-\frac{\mathrm{e}^{-10x}}{-10}\right\}+C$$

$$= \frac{1}{4}\left\{\frac{\mathrm{e}^{10x}}{10}+\frac{\mathrm{e}^{-10x}}{10}-\frac{\mathrm{e}^{2x}}{2}-\frac{\mathrm{e}^{-2x}}{2}\right\}+C$$

$$= \frac{1}{20}\cosh 10x - \frac{1}{4}\cosh 2x + C$$

You could use hyperbolic identities to split up into a difference of two $\sinh s$.

as $\cosh ax = \frac{\mathrm{e}^{ax}+\mathrm{e}^{-ax}}{2}$

b You cannot use by parts for $\int \mathrm{e}^x \sinh x\,\mathrm{d}x$

Using the definition of $\sinh x$

$$\int \mathrm{e}^x \sinh x\,\mathrm{d}x = \int \mathrm{e}^x\left(\frac{\mathrm{e}^x-\mathrm{e}^{-x}}{2}\right)\mathrm{d}x$$

$$= \frac{1}{2}\int \left(\mathrm{e}^{2x}-1\right)\mathrm{d}x$$

$$= \frac{1}{2}\left(\frac{1}{2}\mathrm{e}^{2x}-x\right)+C$$

$$= \frac{1}{4}\mathrm{e}^{2x}-\frac{1}{2}x+C$$

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20 Area under curve $= \int_0^5 y\,\mathrm{d}x = \int_0^5 \frac{10}{\sqrt{4x^2+9}}\,\mathrm{d}x$

$$= 5\int_0^5 \frac{1}{\sqrt{x^2+\frac{9}{4}}}\,\mathrm{d}x$$

$$= 5\left[\operatorname{arsinh}\left(\frac{2x}{3}\right)\right]_0^5$$

$$= 5\operatorname{arsinh}\left(\frac{10}{3}\right)$$

'Real' area $= 5\operatorname{arsinh}\left(\frac{10}{3}\right) \times 100 = 960$ m^2.

Using $\int \frac{1}{\sqrt{x^2+a^2}}\,\mathrm{d}x = \operatorname{arsinh}\left(\frac{x}{a}\right)$

21 $x^2 - 2x + 10 = (x-1)^2 + 9$

So $\int \frac{\mathrm{d}x}{\sqrt{x^2-2x+10}} = \int \frac{\mathrm{d}x}{\sqrt{(x-1)^2+9}}$

Let $x - 1 = 3\sinh u$, then $\mathrm{d}x = 3\cosh u\,\mathrm{d}u$

so $\int \frac{\mathrm{d}x}{\sqrt{x^2-2x+10}} = \int \frac{3\cosh u}{3\cosh u}\,\mathrm{d}u$

$$= u + C$$

$$= \operatorname{arsinh}\left(\frac{x-1}{3}\right) + C$$

22 a Using the exponential forms

$$\int \frac{1}{\sinh x + 2\cosh x}\,\mathrm{d}x = \int \frac{1}{\left(\frac{\mathrm{e}^x - \mathrm{e}^{-x}}{2}\right) + 2\left(\frac{\mathrm{e}^x + \mathrm{e}^{-x}}{2}\right)}\,\mathrm{d}x$$

$$= \int \frac{2}{3\mathrm{e}^x + \mathrm{e}^{-x}}\,\mathrm{d}x$$

$$= \int \frac{2\mathrm{e}^x}{3\mathrm{e}^{2x}+1}\,\mathrm{d}x$$

Using the substitution $u = \mathrm{e}^x$, then $\frac{\mathrm{d}u}{\mathrm{d}x} = \mathrm{e}^x = u$

So $\int \frac{2\mathrm{e}^x}{3\mathrm{e}^{2x}+1}\,\mathrm{d}x = \int \frac{2u}{3u^2+1}\left(\frac{\mathrm{d}u}{u}\right) = \int \frac{2}{3u^2+1}\,\mathrm{d}u$

$$= \frac{2}{3}\int \frac{1}{u^2+\frac{1}{3}}\,\mathrm{d}u$$

$$= \frac{2}{3}\left(\sqrt{3}\right)\arctan\left(\sqrt{3}u\right) + C$$

$$= \frac{2}{\sqrt{3}}\arctan\left(\sqrt{3}\mathrm{e}^x\right) + C$$

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22 b $x^2-2x+10=(x-1)^2+9$

So let $x-1=3\sinh u$, then $\mathrm{d}x=3\cosh u\ \mathrm{d}u$

$$\text{and}\int\frac{3x-1}{\sqrt{x^2-2x+10}}\mathrm{d}x=\int\frac{9\sinh u+2}{\sqrt{9\sinh^2 u+9}}3\cosh u\,\mathrm{d}u$$

$$=\int\frac{9\sinh u+2}{3\cosh u}3\cosh u\,\mathrm{d}u$$

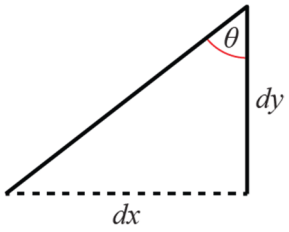
$$=9\cosh u+2u+C$$

$$=9\sqrt{1+\left(\frac{x-1}{3}\right)^2}+2\text{arsinh}\left(\frac{x-1}{3}\right)+C$$

$$\text{So}\int_1^4\frac{3x-1}{\sqrt{x^2-2x+10}}=\left[9\sqrt{2}+2\text{arsinh}1\right]-[9]$$

$$=9\left(\sqrt{2}-1\right)+2\text{arsinh}1$$

23 a An infinitesimal region close to the point of intersection can be seen in this diagram.



So $\tan\theta=\dfrac{\mathrm{d}x}{\mathrm{d}y}$.

We differentiate with respect to x and obtain

$$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{1}{3}\sinh\left(\frac{x}{3}\right)$$

That is, $\dfrac{\mathrm{d}x}{\mathrm{d}y}=\dfrac{3}{\sinh\left(\frac{x}{3}\right)}$.

So at $x=4$,

$$\theta=\arctan\left(\frac{3}{\sinh\left(\frac{4}{3}\right)}\right)\approx 59.5°\ \ (3\text{s.f.})$$

b $\displaystyle\int_{-3}^{4}\cosh\left(\frac{x}{3}\right)\mathrm{d}x=\left[3\sinh\left(\frac{x}{3}\right)\right]_{-3}^{4}=8.82\text{m}^2.$

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24 a $3\cosh 2x = \frac{3}{2}(e^{2x} + e^{-2x})$, $8 + \sinh 2x = 8 + \frac{1}{2}(e^{2x} - e^{-2x})$,

we find the intersection points by setting these expressions equal to each other and solving for x.

$\frac{3}{2}(e^{2x} + e^{-2x}) = 8 + \frac{1}{2}(e^{2x} - e^{-2x})$

$e^{2x} - 8 + 2e^{-2x} = 0.$

Multiplying through by e^{2x} and using the substitution $z = e^{2x}$, we obtain the quadratic equation $z^2 - 8z + 2 = 0$, with solutions $z_1 = 4 - \sqrt{14}$, $z_2 = 4 + \sqrt{14}$.

Therefore, the solutions are $x_1 = \frac{1}{2}\ln\left(4 - \sqrt{14}\right)$ and $x_2 = \frac{1}{2}\ln\left(4 + \sqrt{14}\right)$.

b $\int_{\frac{1}{2}\ln\left(4-\sqrt{14}\right)}^{\frac{1}{2}\ln\left(4+\sqrt{14}\right)} (8 + \sinh 2x - 3\cosh 2x)\,\mathrm{d}x$

$= \left[8x + \frac{1}{2}\cosh 2x - \frac{3}{2}\sinh 2x\right]_{\frac{1}{2}\ln\left(4-\sqrt{14}\right)}^{\frac{1}{2}\ln\left(4+\sqrt{14}\right)}$

≈ 6.12 (3s.f.)

25 First we find the cross-sectional area of the loaf. Note that since a unit is 5cm, we will have a number which we will multiply by $(5\text{cm})^2 = 25\text{cm}^2$ as opposed to the usual cm^2. Let A denote cross-sectional area.

$A = \int_{-2}^{2} \frac{5}{\sqrt{x^2 + 4}}\,\mathrm{d}x$

$= \frac{5}{2}\int_{-2}^{2} \frac{1}{\sqrt{(\frac{x}{2})^2 + 1}}\,\mathrm{d}x$

Let $\frac{x}{2} = \sinh u$, so that the integral becomes

$A = \frac{5}{2}\int_{\text{arsinh}(-1)}^{\text{arsinh}(1)} \frac{1}{\sqrt{(\sinh u)^2 + 1}} \times 2\cosh u\,\mathrm{d}u$

$= 5\int_{\text{arsinh}(-1)}^{\text{arsinh}(1)} 1\,\mathrm{d}u$

$= 5\left[u\right]_{\text{arsinh}(-1)}^{\text{arsinh}(1)}$

$= 10\,\text{arsinh}(1)$ units^2

$= 10\,\text{arsinh}(1)\ (5\text{cm})^2$

$= 250\,\text{arsinh}(1)\ \text{cm}^2.$

Then the volume, V, is calculated by multiplying the cross-sectional area in centimetres by the length in centimetres.

$V = 250\,\text{arsinh}(1) \times 30$

$= 7500\,\text{arsinh}(1)$

$\approx 6610\,\text{cm}^3$

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26 a $y = 3 - \sinh x$ crosses the $x-$axis when $x = \operatorname{arsinh}(3)$

$$= \ln\left(3 + \sqrt{3^2 + 1}\right)$$

$$= \ln\left(3 + \sqrt{10}\right).$$

So the coordinates are $\left(\ln\left(3 + \sqrt{10}\right), 0\right)$.

b The volume of revolution, V, is found by

$$V = \int_0^{\ln\left(3+\sqrt{10}\right)} \pi(3 - \sinh x)^2 \, \mathrm{d}x$$

$$V = \int_0^{\ln\left(3+\sqrt{10}\right)} \pi\left(3 - \frac{1}{2}(e^x - e^{-x})\right)^2 \, \mathrm{d}x$$

$$= \int_0^{\ln\left(3+\sqrt{10}\right)} \pi\left(9 - 3(e^x - e^{-x}) + \frac{1}{4}(e^{2x} - 2 + e^{-2x})\right) \mathrm{d}x$$

$$= \pi\left[9x - 3(e^x + e^{-x}) + \frac{1}{4}\left(\frac{1}{2}e^{2x} - 2x - \frac{1}{2}e^{-2x}\right)\right]_0^{\ln\left(3+\sqrt{10}\right)}$$

$$\approx 22.7 \quad (3\text{s.f.})$$

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Challenge

First we find the area between two arbitrarily large values equidistant from the origin; we will call these values $-R$ and R.

So area $= \int_{-R}^{R} \operatorname{sech} x \, \mathrm{d}x$

We convert to exponential form and use the substitution $u = e^x$, noting that $\frac{\mathrm{d}u}{\mathrm{d}x} = e^x = u$.

$$\begin{aligned}
\int_{-R}^{R} \operatorname{sech} x \, \mathrm{d}x &= \int_{-R}^{R} \frac{2}{e^x + e^{-x}} \, \mathrm{d}x \\
&= \int_{-R}^{R} \frac{2e^x}{e^{2x} + 1} \, \mathrm{d}x \\
&= \int_{e^{-R}}^{e^R} \frac{2u}{u^2 + 1} \, \frac{\mathrm{d}u}{u} \\
&= 2\int_{e^{-R}}^{e^R} \frac{1}{u^2 + 1} \, \mathrm{d}u \\
&= 2\left[\arctan(u)\right]_{e^{-R}}^{e^R} \\
&= 2\left[\arctan\left(e^x\right)\right]_{-R}^{R}
\end{aligned}$$

Now that we have this expression for arbitrarily large R, we take the limit as $R \to \infty$.

$$\begin{aligned}
A &= 2\lim_{R\to\infty}\left[\arctan\left(e^x\right)\right]_{-R}^{R} \\
&= 2\lim_{R\to\infty}\left(\arctan\left(e^R\right) - \arctan\left(e^{-R}\right)\right) \\
&= 2\left(\frac{\pi}{2} - 0\right) \\
&= \pi
\end{aligned}$$

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Methods in differential equations 7A

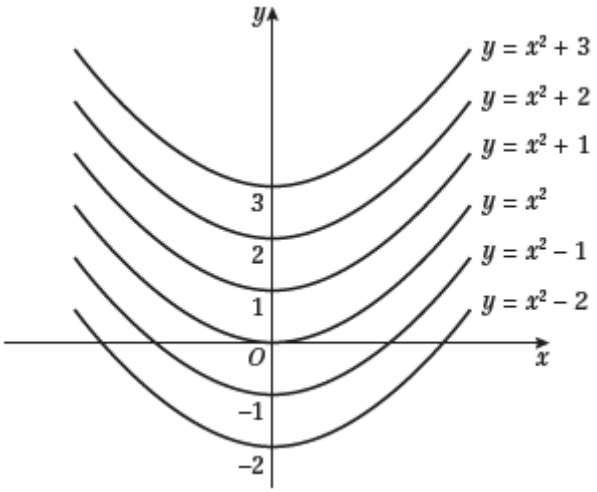
1 a Integrating both sides of the equation and including a constant:

$$\frac{dy}{dx} = 2x$$

$$\Rightarrow y = \int 2x\,dx$$

$$\Rightarrow y = x^2 + c \quad \text{where } c \text{ is a constant}$$

The family of solution curves are parabola.
Sketching the solution curves for $c = -2, -1, 0, 1, 2$ and 3 gives:



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1 b Separating the variables and integrating:

$$\frac{dy}{dx} = y$$

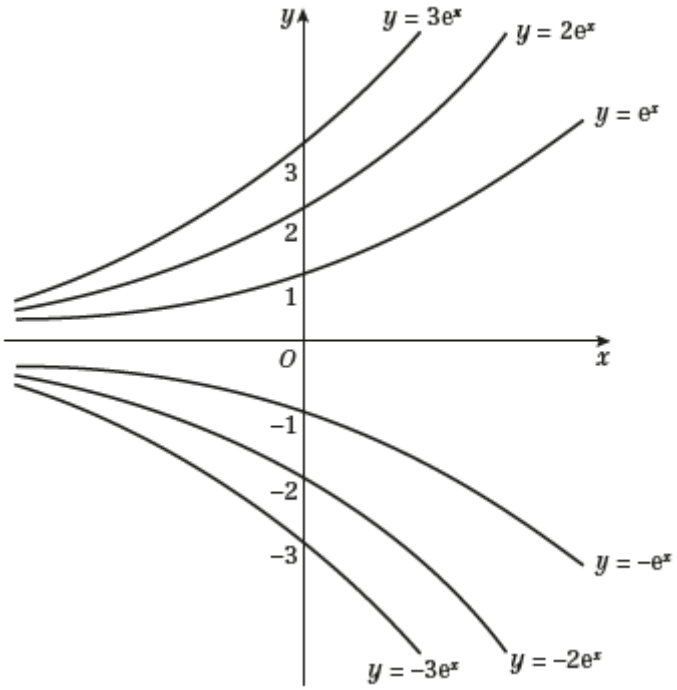
$$\Rightarrow \int \frac{1}{y} dy = \int 1 dx$$

$$\Rightarrow \ln y = x + c \quad \text{where } c \text{ is a constant}$$

$$\Rightarrow y = e^{x+c} = e^c \times e^x$$

$$\Rightarrow y = Ae^x \quad \text{where } A \text{ is a constant } (A = e^c)$$

The family of solution curves are exponential curves.
Sketching the solution curves for $A = -3, -2, -1, 1, 2$ and 3 gives:



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1 c Separating the variables and integrating:

$$\frac{dy}{dx} = \frac{2y}{x}$$

$$\Rightarrow \int \frac{1}{y} dy = \int \frac{2}{x} dx$$

$$\Rightarrow \ln y = 2 \ln x + c$$

Expressing the constant as $\ln A$ and simplifying using the laws of logarithms:

$$\ln y = 2 \ln x + \ln A$$

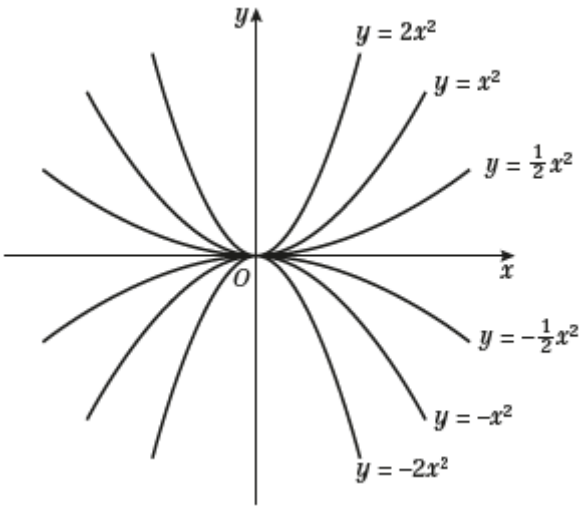
$$\Rightarrow \ln y = \ln x^2 + \ln A \quad \text{using the power law}$$

$$\Rightarrow \ln y = \ln Ax^2 \quad \text{using the multiplication law}$$

$$\Rightarrow y = Ax^2$$

The family of solution curves are parabola.

Sketching the solution curves for $A = -2, -1, -\frac{1}{2}, \frac{1}{2}, 1, 2$ gives:



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1 d Separating the variables and integrating:

$$\frac{dy}{dx} = \frac{x}{y}$$

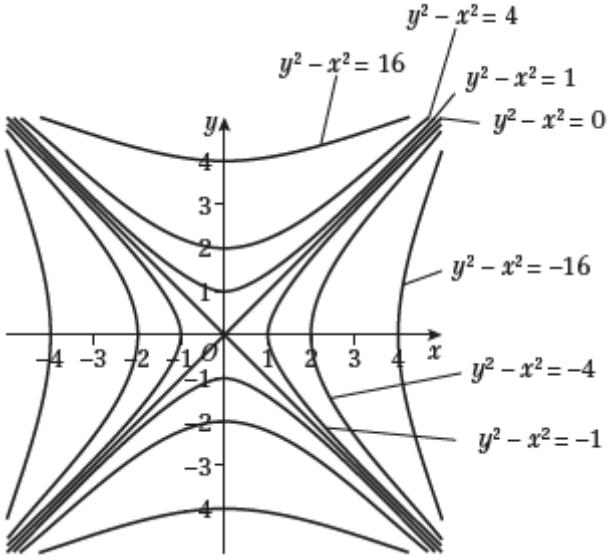
$$\Rightarrow \int y\,dy = \int x\,dx$$

$$\Rightarrow \frac{y^2}{2} = \frac{x^2}{2} + c \quad \text{or} \quad y^2 - x^2 = 2c$$

$y^2 - x^2 = 0 \Rightarrow (y - x)(y + x) = 0$, and the graph of this equation are straight lines $y = x$ and $y = -x$

$y^2 - x^2 = 2c$ for $c \neq 0$ is a hyperbola with asymptotes $y = x$ and $y = -x$

Sketching some of the solution curves gives:



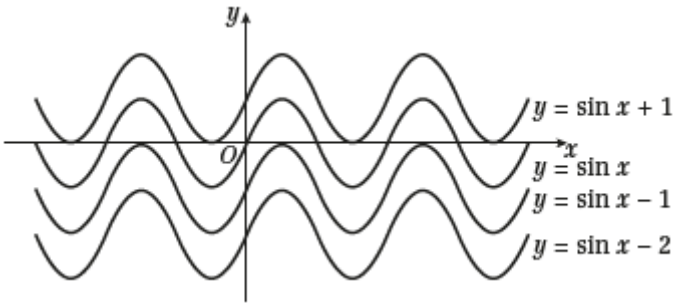
e $\frac{dy}{dx} = \cos x$

$$\Rightarrow y = \sin x + c$$

The family of solution curves are sin curves.

The graph of $y = \sin x + c$ is a translation of $y = \sin x$ by the vector $\begin{pmatrix} 0 \\ c \end{pmatrix}$

Sketching some of the solution curves gives:



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1 f Separating the variables and integrating:

$$\frac{dy}{dx} = y\cot x \quad 0 < x < \pi$$

$$\Rightarrow \int \frac{1}{y}\,dy = \int \frac{\cos x}{\sin x}\,dx$$

$$\Rightarrow \ln|y| = \ln|\sin x| + c \qquad \text{integrating } \frac{\cos x}{\sin x} \text{ using the reverse chain rule}$$

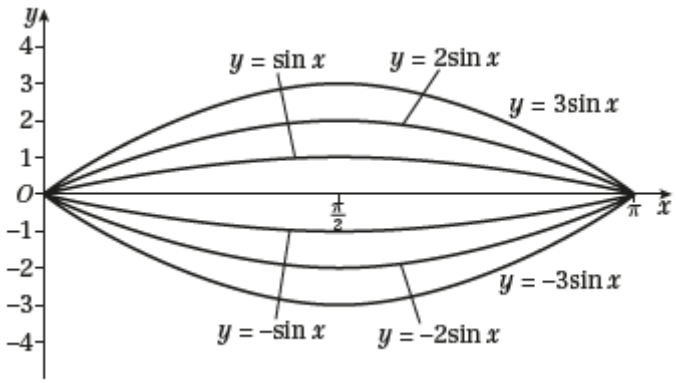
Expressing the constant as $\ln|A|$ and simplifying using the laws of logarithms:

$$\ln|y| = \ln|\sin x| + \ln|A|$$

$$\Rightarrow \ln|y| = \ln|A\sin x|$$

$$\Rightarrow y = A\sin x$$

The family of solution curves are sin curves for $0 < x < \pi$ with varying amplitudes.
Sketching some of the solution curves gives:



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2 a Separating the variables and integrating:

$$\frac{dy}{dx} = \frac{-xy}{9-x^2}$$

$$\Rightarrow \int \frac{1}{y}\,dy = -\int \frac{x}{9-x^2}\,dx$$

$$\Rightarrow \ln y = \frac{1}{2}\ln(9-x^2) + \ln A$$

$$\Rightarrow 2\ln y = \ln A^2(9-x^2)$$

$$\Rightarrow \ln y^2 = \ln A^2(9-x^2)$$

$$\Rightarrow y^2 = 9A^2 - A^2x^2$$

Let $A^2 = k$, so be definition k is a positive constant

Then $y^2 + kx^2 = 9k$

b If the solution passes through (2, 5) then

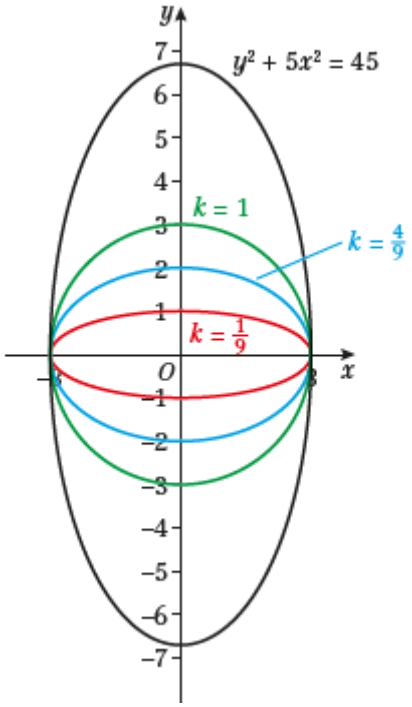
$25 + 4k = 9k$

$25 = 5k \Rightarrow k = 5$

So the equation is $y^2 + 5x^2 = 45$

c The solution curves are all ellipses, except when $k = 1$ when the curve is circle.

When $y = 0$, $x = \pm 3$, when $x = 0$, $y = \pm\sqrt{9k}$



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3 a $x\dfrac{dy}{dx}+y=\cos x$

$\Rightarrow \dfrac{d}{dx}(xy)=\cos x$ as $\dfrac{d}{dx}(xy)=x\dfrac{dy}{dx}+y$ from the product rule

$\Rightarrow xy=\int \cos dx=\sin x+c$

So $y=\dfrac{1}{x}\sin x+\dfrac{c}{x}$

b $e^{-x}\dfrac{dy}{dx}-e^{-x}y=xe^{x}$

$\Rightarrow \dfrac{d}{dx}(e^{-x}y)=xe^{x}$ as $\dfrac{d}{dx}(e^{-x}y)=e^{-x}\dfrac{dy}{dx}-e^{-x}y$ from the product rule

$\Rightarrow e^{-x}y=\int xe^{x}dx=xe^{x}-\int e^{x}dx$ using integration by parts formula (with $u=x, v=e^{x}$)

$=xe^{x}-e^{x}+c$

So $y=xe^{2x}-e^{2x}+ce^{x}$ multiplying both sides by e^{x}

c $\sin x\dfrac{dy}{dx}+y\cos x=3$

$\Rightarrow \dfrac{d}{dx}(y\sin x)=3$ as $\dfrac{d}{dx}(y\sin x)=\sin x\dfrac{dy}{dx}+y\cos x$ from the product rule

$\Rightarrow y\sin x=\int 3\ dx$

$\Rightarrow y\sin x=3x+c$

So $y=\dfrac{3x}{\sin x}+\dfrac{c}{\sin x}=3x\operatorname{cosec}x+c\operatorname{cosec}x$

d $\dfrac{1}{x}\dfrac{dy}{dx}-\dfrac{1}{x^{2}}y=e^{x}$

$\Rightarrow \dfrac{d}{dx}\left(\dfrac{1}{x}y\right)=e^{x}$ as $\dfrac{d}{dx}\left(\dfrac{1}{x}y\right)=\dfrac{1}{x}\dfrac{dy}{dx}-\dfrac{1}{x^{2}}y$ from the product rule

$\Rightarrow \dfrac{1}{x}y=\int e^{x}\ dx=e^{x}+c$

So $y=xe^{x}+cx$

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3 e Simplify the left-hand side by noting that from the product and chain rules

$$\frac{\mathrm{d}}{\mathrm{d}x}(x^2\mathrm{e}^y) = x^2\frac{\mathrm{d}(\mathrm{e}^y)}{\mathrm{d}x} + 2x\mathrm{e}^y \qquad \text{the product rule}$$

$$= x^2\frac{\mathrm{d}(\mathrm{e}^y)}{\mathrm{d}y}\frac{\mathrm{d}y}{\mathrm{d}x} + 2x\mathrm{e}^y \qquad \text{the chain rule}$$

$$= x^2\mathrm{e}^y\frac{\mathrm{d}y}{\mathrm{d}x} + 2x\mathrm{e}^y$$

So $x^2\mathrm{e}^y\dfrac{\mathrm{d}y}{\mathrm{d}x} + 2x\mathrm{e}^y = x \Rightarrow \dfrac{\mathrm{d}}{\mathrm{d}x}(x^2\mathrm{e}^y) = x$

$$\Rightarrow x^2\mathrm{e}^y = \int x\,\mathrm{d}x = \frac{x^2}{2} + c$$

$$\Rightarrow \mathrm{e}^y = \frac{1}{2} + \frac{c}{x^2}$$

So $y = \ln\left(\dfrac{1}{2} + \dfrac{c}{x^2}\right)$

f $4xy\dfrac{\mathrm{d}y}{\mathrm{d}x} + 2y^2 = x^2$

$\Rightarrow \dfrac{\mathrm{d}}{\mathrm{d}x}(2xy^2) = x^2$ using the product and chain rules

$$\Rightarrow 2xy^2 = \int x^2\,\mathrm{d}x = \frac{1}{3}x^3 + c$$

$$\Rightarrow y^2 = \frac{1}{6}x^2 + \frac{c}{2x}$$

So $y = \pm\sqrt{\dfrac{1}{6}x^2 + \dfrac{c}{2x}}$

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4 a The equation is in the form $\frac{dy}{dx} + P(x)y = Q(x)$, so the integrating factor is $e^{\int P(x)dx} = e^{\int 2x dx} = e^{x^2}$

Multiplying the equation by this factor gives:

$e^{x^2}\frac{dy}{dx} + e^{x^2}2xy = 1$

$\Rightarrow \frac{d}{dx}(ye^{x^2}) = 1$

$\Rightarrow ye^{x^2} = \int 1 dx = x + c$

So $y = \frac{x + c}{e^{x^2}} = xe^{-x^2} + ce^{-x^2}$

b As $x \to \infty$, e^{x^2} becomes much larger than x; therefore, $y \to 0$.

5 a $x^2\frac{dy}{dx} + 2xy = 2x + 1$

$\Rightarrow \frac{d}{dx}(x^2y) = 2x + 1$

$\Rightarrow x^2y = \int (2x + 1) dx$

$\Rightarrow x^2y = x^2 + x + c$

So $y = 1 + \frac{1}{x} + \frac{c}{x^2}$

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5 b When $x=-\frac{1}{2}$, $y=0$, $1-2+4c=0\Rightarrow 4c=1\Rightarrow c=\frac{1}{4}$

So $y=1+\frac{1}{x}+\frac{1}{4x^2}$

When $x=-\frac{1}{2}$, $y=3$, $1-2+4c=3\Rightarrow 4c=4\Rightarrow c=1$

So $y=1+\frac{1}{x}+\frac{1}{x^2}$

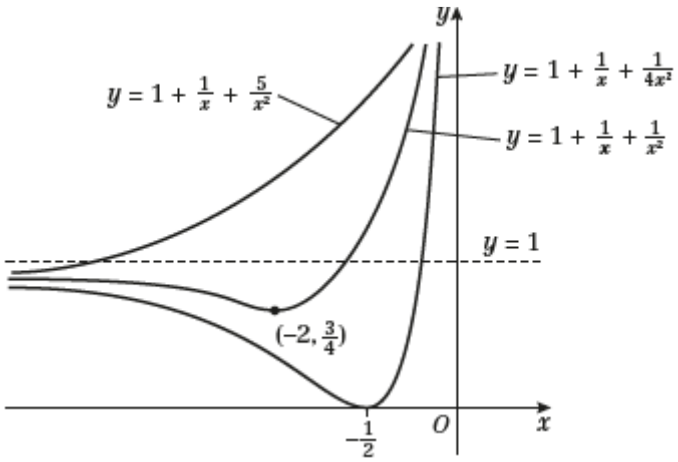
When $x=-\frac{1}{2}$, $y=19$, $1-2+4c=19\Rightarrow 4c=20\Rightarrow c=5$

So $y=1+\frac{1}{x}+\frac{5}{x^2}$

The curves have a horizontal asymptote at $y=1$ and a vertical asymptote at $x=0$

When $y=1, \frac{1}{x}+\frac{c}{x^2}=0\Rightarrow x=-c$. When $y=0$, $x^2+x+c=0$. There are no real roots for c $>\frac{1}{4}$.

So a sketch of the three curves for $x<0$ is



6 a $\ln x\frac{\mathrm{d}y}{\mathrm{d}x}+\frac{y}{x}=\frac{1}{(x+1)(x+2)}$

$\Rightarrow \frac{\mathrm{d}}{\mathrm{d}x}(y\ln x)=\frac{1}{(x+1)(x+2)}$ as $\frac{\mathrm{d}}{\mathrm{d}x}(y\ln x)=\ln x\frac{\mathrm{d}y}{\mathrm{d}x}+\frac{y}{x}$ using the product rule

$\Rightarrow y\ln x=\int\frac{1}{(x+1)(x+2)}\mathrm{d}x$

$=\int\left(\frac{(x+2)-(x+1)}{(x+1)(x+2)}\right)\mathrm{d}x=\int\left(\frac{1}{x+1}-\frac{1}{x+2}\right)\mathrm{d}x$

$=\ln(x+1)-\ln(x+2)+\ln A$

So $y=\frac{\ln(x+1)-\ln(x+2)+\ln A}{\ln x}=\frac{\ln\frac{A(x+1)}{(x+2)}}{\ln x}$

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6 b When $x = 2,\ y = 2,\ 2 = \dfrac{\ln \frac{3}{4}A}{\ln 2}$

So $\ln \dfrac{3}{4}A = 2\ln 2 = \ln 4$

$\Rightarrow \dfrac{3}{4}A = 4 \Rightarrow A = \dfrac{16}{3}$

So the solution is $y = \dfrac{\ln \dfrac{16(x+1)}{3(x+2)}}{\ln x}$

7 a The integrating factor is $e^{\int 2\,dx} = e^{2x}$

Multiplying the equation by this factor gives:

$e^{2x}\dfrac{dy}{dx} + 2e^{2x}y = e^{2x}e^{x}$

$\Rightarrow \dfrac{d}{dx}(e^{2x}y) = e^{3x}$

$\Rightarrow e^{2x}y = \int e^{3x}\,dx = \dfrac{1}{3}e^{3x} + c$

So $y = \dfrac{1}{3}e^{x} + ce^{-2x}$

b The integrating factor is $e^{\int \cot x\,dx} = e^{\ln \sin x} = \sin x$

Multiplying the equation by this factor gives:

$\sin x\dfrac{dy}{dx} + y\cos x = \sin x$

$\Rightarrow \dfrac{d}{dx}(y\sin x) = \sin x$

$\Rightarrow y\sin x = \int \sin x\,dx = -\cos x + c$

So $y = -\cot x + c\operatorname{cosec} x$

c The integrating factor is $e^{\int \sin x\,dx} = e^{-\cos x}$

Multiplying the equation by this factor gives:

$e^{-\cos x}\dfrac{dy}{dx} + y\sin x e^{-\cos x} = e^{-\cos x}e^{\cos x}$

$\Rightarrow \dfrac{d}{dx}(ye^{-\cos x}) = 1$

$\Rightarrow ye^{-\cos x} = x + c$

So $y = xe^{\cos x} + ce^{\cos x}$

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7 d The integrating factor is $e^{\int -1\,dx} = e^{-x}$

Note that $P(x) = -1$ and the minus sign is important.

Multiplying the equation by this factor gives:

$$e^{-x}\frac{dy}{dx} - ye^{-x} = e^{2x}e^{-x}$$

$$\Rightarrow \frac{d}{dx}(ye^{-x}) = e^{x}$$

$$\Rightarrow ye^{-x} = \int e^{x}\,dx = e^{x} + c$$

So $y = e^{2x} + ce^{x}$

e The integrating factor is $e^{\int \tan x\,dx} = e^{\ln \sec x} = \sec x$

Multiplying the equation by this factor gives:

$$\sec x\frac{dy}{dx} + y\sec x\tan x = x\cos x\sec x$$

$$\Rightarrow \frac{d}{dx}(y\sec x) = x$$

$$\Rightarrow y\sec x = \int x\,dx = \frac{1}{2}x^{2} + c$$

So $y = \left(\frac{1}{2}x^{2} + c\right)\cos x$

f The integrating factor is $e^{\int \frac{1}{x}dx} = e^{\ln x} = x$

Multiplying the equation by this factor gives:

$$x\frac{dy}{dx} + y = \frac{x}{x^{2}}$$

$$\Rightarrow \frac{d}{dx}(xy) = \frac{1}{x}$$

$$\Rightarrow xy = \int \frac{1}{x}dx = \ln x + c$$

So $y = \frac{1}{x}\ln x + \frac{c}{x}$

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7 g Divide both sides by x^2 to get an equation in the form $\frac{dy}{dx}+P(x)y=Q(x)$. This gives:

$$\frac{dy}{dx}-\frac{1}{x}y=\frac{x}{x+2} \qquad \textbf{(1)}$$

The integrating factor is $e^{\int -\frac{1}{x}dx}=e^{-\ln x}=e^{\ln\frac{1}{x}}=\frac{1}{x}$

Multiplying equation **(1)** by this factor gives:

$$\frac{1}{x}\frac{dy}{dx}-\frac{1}{x^2}y=\frac{1}{x+2}$$

$$\Rightarrow \frac{d}{dx}\left(\frac{1}{x}y\right)=\frac{1}{x+2}$$

$$\Rightarrow \frac{1}{x}y=\int\frac{1}{x+2}dx=\ln(x+2)+c$$

So $y=x\ln(x+2)+cx$

h Divide both sides by x^2 to get an equation in the form $\frac{dy}{dx}+P(x)y=Q(x)$. This gives:

$$\frac{dy}{dx}+\frac{1}{3x}y=\frac{1}{3} \qquad \textbf{(1)}$$

The integrating factor is $e^{\int\frac{1}{3x}dx}=e^{\frac{1}{3}\ln x}=e^{\ln x^{\frac{1}{3}}}=x^{\frac{1}{3}}$

Multiplying equation **(1)** by $x^{\frac{1}{3}}$ gives:

$$x^{\frac{1}{3}}\frac{dy}{dx}+\frac{1}{3}x^{-\frac{2}{3}}y=\frac{1}{3}x^{\frac{1}{3}}$$

$$\Rightarrow \frac{d}{dx}\left(x^{\frac{1}{3}}y\right)=\frac{1}{3}x^{\frac{1}{3}}$$

$$\Rightarrow x^{\frac{1}{3}}y=\int\frac{1}{3}x^{\frac{1}{3}}dx=\frac{1}{4}x^{\frac{4}{3}}+c$$

So $y=\frac{1}{4}x+cx^{-\frac{1}{3}}$

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7 i Dividing both sides by $(x+2)$ gives:

$$\frac{dy}{dx}-\frac{1}{(x+2)}y=1 \qquad \textbf{(1)}$$

The integrating factor is $e^{\int\frac{-1}{(x+2)}dx}=e^{-\ln(x+2)}=e^{\ln\frac{1}{x+2}}=\frac{1}{x+2}$

Multiplying equation **(1)** by the integrating factor:

$$\frac{1}{(x+2)}\frac{dy}{dx}-\frac{1}{(x+2)^2}y=\frac{1}{(x+2)}$$

$$\Rightarrow\frac{d}{dx}\left[\frac{1}{(x+2)}y\right]=\frac{1}{x+2}$$

$$\Rightarrow\frac{1}{(x+2)}y=\int\frac{1}{x+2}\,dx=\ln(x+2)+c$$

So $y=(x+2)\ln(x+2)+c(x+2)$

j Dividing both sides by x gives:

$$\frac{dy}{dx}+\frac{4}{x}y=\frac{e^x}{x^3} \qquad \textbf{(1)}$$

The integrating factor is $e^{\int\frac{4}{x}dx}=e^{4\ln x}=e^{\ln x^4}=x^4$

Multiplying equation **(1)** by the integrating factor:

$$x^4\frac{dy}{dx}+4x^3y=xe^x$$

$$\Rightarrow\frac{d}{dx}(x^4y)=xe^x$$

$$\Rightarrow x^4y=\int xe^x dx=xe^x-\int e^x dx=xe^x-e^x+c$$

Integrating xe^x using integration by parts

So $y=\frac{1}{x^3}e^x-\frac{1}{x^4}e^x+\frac{c}{x^4}$

8 Dividing both sides by x gives:

$$\frac{dy}{dx}+\frac{2}{x}y=\frac{1}{x}e^x \qquad \textbf{(1)}$$

The integrating factor is $e^{\int\frac{2}{x}dx}=e^{2\ln x}=e^{\ln x^2}=x^2$

Multiplying equation **(1)** by x^2

$$x^2\frac{dy}{dx}+2xy=xe^x$$

$$\Rightarrow\frac{d}{dx}(x^2y)=xe^x$$

$$\Rightarrow x^2y=\int xe^x dx=xe^x-\int e^x dx=xe^x-e^x+c$$

So $y=\frac{1}{x}e^x-\frac{1}{x^2}e^x+\frac{c}{x^2}$

Given that $y=1$ when $x=1$, then $1=e-e+c\Rightarrow c=1$

So the required equation is $y=\frac{1}{x}e^x-\frac{1}{x^2}e^x+\frac{1}{x^2}$

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9 Dividing both sides by x^3 gives:

$$\frac{dy}{dx} - \frac{1}{x}y = \frac{1}{x^3} \qquad \textbf{(1)}$$

The integrating factor is $e^{-\int \frac{1}{x} dx} = e^{-\ln x} = e^{\ln \frac{1}{x}} = \frac{1}{x}$

Multiplying equation **(1)** by $\frac{1}{x}$

$$\frac{1}{x}\frac{dy}{dx} - \frac{1}{x^2}y = \frac{1}{x^4}$$

$$\Rightarrow \frac{d}{dx}\left(\frac{1}{x}y\right) = \frac{1}{x^4}$$

$$\frac{1}{x}y = \int \frac{1}{x^4} dx = \int x^{-4} dx = -\frac{1}{3}x^{-3} + c$$

So $y = -\frac{1}{3}x^{-2} + cx = -\frac{1}{3x^2} + cx$

But $y = 1$ when $x = 1$, so $1 = -\frac{1}{3} + c \Rightarrow c = \frac{4}{3}$

So the required equation is $y = -\frac{1}{3x^2} + \frac{4x}{3}$

10 a Dividing both sides by $\left(x + \frac{1}{x}\right)$ gives:

$$\frac{dy}{dx} + \frac{2}{\left(x + \frac{1}{x}\right)}y = \frac{2(x^2+1)^2}{\left(x + \frac{1}{x}\right)}, \text{ which simplifies to}$$

$$\frac{dy}{dx} + \frac{2x}{x^2+1} \times y = 2x(x^2+1) \qquad \textbf{(1)}$$

The integrating factor is $e^{\int \frac{2x}{x^2+1} dx} = e^{\ln(x^2+1)} = (x^2+1)$

Multiplying equation **(1)** by (x^2+1)

$$(x^2+1)\frac{dy}{dx} + 2xy = 2x(x^2+1)^2$$

$$\Rightarrow \frac{d}{dx}\left((x^2+1)y\right) = 2x(x^2+1)^2$$

$$y(x^2+1) = \int 2x(x^2+1)^2 dx = \frac{1}{3}(x^2+1)^3 + c$$

So $y = \frac{1}{3}(x^2+1)^2 + \frac{c}{(x^2+1)}$

b Given that $y = 1$ when $x = 1$, then $1 = \frac{1}{3} \times 4 + \frac{1}{2}c \Rightarrow c = -\frac{2}{3}$

So the required equation is $y = \frac{1}{3}(x^2+1)^2 - \frac{2}{3(x^2+1)}$

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11 a Dividing both sides by $\cos x$ gives:

$$\frac{dy}{dx} + y\sec x = \sec x \qquad (1)$$

Using the standard result $\int \sec x\,dx = \ln(\sec x + \tan x)$, (you will not be expected to prove this result)the integrating factor is

$$e^{\int \sec x\,dx} = e^{\ln(\sec x + \tan x)} = \sec x + \tan x$$

Multiplying equation (**1**) by this factor gives:

$$(\sec x + \tan x)\frac{dy}{dx} + (\sec^2 x + \sec x \tan x)y = \sec^2 x + \sec x \tan x$$

$$\Rightarrow \frac{d}{dx}\left((\sec x + \tan x)y\right) = \sec^2 x + \sec x \tan x$$

$$\Rightarrow (\sec x + \tan x)y = \int \sec^2 x + \sec x \tan x\,dx = \tan x + \sec x + c$$

So $y = 1 + \dfrac{c}{\sec x + \tan x}$

b Given that $y = 2$ when $x = 0$, then $2 = 1 + \dfrac{c}{1+0} \Rightarrow c = 1$

So $y = 1 + \dfrac{1}{\sec x + \tan x}$

Dividing top and bottom by $\cos x$ gives $y = 1 + \dfrac{\cos x}{1 + \sin x}$

12 Rewriting the equation as $\dfrac{1}{\sqrt{y^2 - 4}}dy = x\,dx$

Integrate the left-hand side using the substitution $y = 2\cosh u$

$\dfrac{dy}{du} = 2\sinh u$, so dy can be replaced by $2\sinh u\,du$

$$\int \frac{1}{\sqrt{y^2 - 4}}dy = \int \frac{1}{\sqrt{4\cosh^2 u - 4}} 2\sinh u\,du = \int \frac{1}{2\sinh u} 2\sinh u\,du = \int 1\,du$$

$$= u + c = \operatorname{arcosh}\left(\frac{y}{2}\right) + c$$

So integrating both sides of the rewritten equation gives:

$$\operatorname{arcosh}\left(\frac{y}{2}\right) = \frac{x^2}{2} + c \Rightarrow \frac{y}{2} = \cosh\left(\frac{x^2}{2} + c\right)$$

So the general solution is $y = 2\cosh\left(\dfrac{x^2}{2} + c\right)$

13 a Rewriting the equation as $\dfrac{1}{y}dy = \cosh x\,dx$

Integrating both sides gives

$\ln y = \sinh x + c$ and therefore the general solution is $y = e^{\sinh x + c}$

b Given that $y = e$ when $x = 0$, then $e = e^{\sinh 0 + c} = e^c \Rightarrow c = 1$. So the particular solution is $y = e^{\sinh x + 1}$

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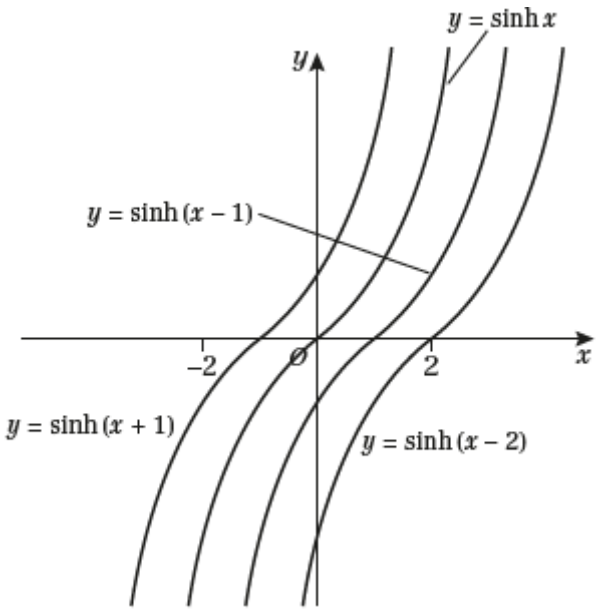
14 a Rewriting the equation as $\frac{1}{\sqrt{1+y^2}}\mathrm{d}y = \mathrm{d}x$

The left-hand side is the derivative of $\operatorname{arsinh} y$, so integrating both sides gives:

$\operatorname{arsinh} y = x + c \Rightarrow y = \sinh(x+c)$

b The graph of $y = \sinh(x+c)$ is a translation of $y = \sinh x$ by the vector $\begin{pmatrix} -c \\ 0 \end{pmatrix}$

This is a sketch of some solution curves.



15 a Dividing both sides by $\cos x$ gives

$\frac{\mathrm{d}y}{\mathrm{d}x} + y\tan x = \sec x$ **(1)**

The integrating factor is $e^{\int \tan x\ \mathrm{d}x} = e^{\ln|\sec x|} = \sec x$

Multiply both sides of equation **(1)** by sec x

$\sec x\frac{\mathrm{d}y}{\mathrm{d}x} + y\tan x = \sec^2 x$

$\frac{\mathrm{d}}{\mathrm{d}x}(y\sec x) = \sec^2 x$

$\Rightarrow y\sec x = \int \sec^2 x\ \mathrm{d}x$

$\Rightarrow y\sec x = \tan x + c$

$y = \sin x + c\cos x$

b Given that $y = 3$ when $x = \pi$, then $3 = \sin\pi + c\cos\pi = 0 - c \Rightarrow c = -3$.

So the particular solution is $y = \sin x - 3\cos x$

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15 c If $x = \frac{\pi}{2}$, then $y = \sin\frac{\pi}{2} + c\cos\frac{\pi}{2} = 1 + c \times 0 = 1$ for any value of c

Similarly if $x = \frac{3\pi}{2}$, then $y = \sin\frac{3\pi}{2} + c\cos\frac{3\pi}{2} = -1 + c \times 0 = -1$ for any value of c

So $\left(\frac{\pi}{2}, 1\right)$ and $\left(\frac{3\pi}{2}, -1\right)$ lie on all solution curves.

16 Dividing by a gives $\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{b}{a}y = 0$

The integrating factor is $\mathrm{e}^{\int \frac{b}{a}\mathrm{d}x} = \mathrm{e}^{\frac{bx}{a}}$

Multiplying by this factor gives

$$\mathrm{e}^{\frac{bx}{a}}\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{b}{a}\mathrm{e}^{\frac{bx}{a}}y = 0$$

$$\Rightarrow \frac{\mathrm{d}}{\mathrm{d}x}\left(\mathrm{e}^{\frac{bx}{a}}y\right) = 0$$

$$\Rightarrow \mathrm{e}^{\frac{bx}{a}}y = c$$

So the general solution is $y = c\mathrm{e}^{-\frac{bx}{a}}$

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Methods in differential equations 7B

1 a The auxiliary equation is

$$m^2 + 5m + 6 = 0$$
$$(m+3)(m+2) = 0$$
$$m = -3 \text{ or } -2$$

So the general solution is $y = Ae^{-3x} + Be^{-2x}$

> The auxiliary equation of $a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0$ is $am^2 + bm^2 + c = 0$. If α and β are real roots of this quadratic then $y = Ae^{\alpha x} + Be^{\beta x}$ is the general solution of the differential equation.

b The auxiliary equation is

$$m^2 - 8m + 12 = 0$$
$$(m-6)(m-2) = 0$$
$$m = 2 \text{ or } 6$$

So the general solution is $y = Ae^{2x} + Be^{6x}$

c The auxiliary equation is

$$m^2 + 2m - 15 = 0$$
$$(m+5)(m-3) = 0$$
$$m = -5 \text{ or } 3$$

So the general solution is $y = Ae^{-5x} + Be^{3x}$

d The auxiliary equation is

$$m^2 - 3m - 28 = 0$$
$$(m-7)(m+4) = 0$$
$$m = 7 \text{ or } -4$$

So the general solution is $y = Ae^{7x} + Be^{-4x}$

e The auxiliary equation is

$$m^2 + 5m = 0$$
$$m(m+5) = 0$$
$$m = 0 \text{ or } -5$$

The auxiliary equation has two real roots, but one of them is zero. As $e^{0x} = 1$, the general solution is $y = Ae^{0x} + Be^{-5x} = A + Be^{-5x}$

f The auxiliary equation is

$$3m^2 + 7m + 2 = 0$$
$$(3m+1)(m+2) = 0$$
$$m = -\frac{1}{3} \text{ or } -2$$

So the general solution is $y = Ae^{-\frac{1}{3}x} + Be^{-2x}$

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1 g The auxiliary equation is

$$4m^2 - 7m - 2 = 0$$
$$(4m+1)(m-2) = 0$$
$$m = -\frac{1}{4} \text{ or } 2$$

So the general solution is $y = Ae^{-\frac{1}{4}x} + Be^{2x}$

h The auxiliary equation is

$$15m^2 - 7m - 2 = 0$$
$$(5m+1)(3m-2) = 0$$
$$m = -\frac{1}{5} \text{ or } \frac{2}{3}$$

So the general solution is $y = Ae^{-\frac{1}{5}x} + Be^{\frac{2}{3}x}$

2 a The auxiliary equation is

$$m^2 + 10m + 25 = 0$$
$$(m+5)(m+5) = 0$$
$$m = -5$$

So the general solution is $y = (A + Bx)e^{-5x}$

> The auxiliary equation of $a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0$ is $am^2 + bm^2 + c = 0$. If this equation has one repeated real root α then the general solution of the differential equation is $y = (A + Bx)e^{\alpha x}$

b The auxiliary equation is

$$m^2 - 18m + 81 = 0$$
$$(m-9)(m-9) = 0$$
$$m = 9$$

So the general solution is $y = (A + Bx)e^{9x}$

c The auxiliary equation is

$$m^2 + 2m + 1 = 0$$
$$(m+1)(m+1) = 0$$
$$m = -1$$

So the general solution is $y = (A + Bx)e^{-x}$

d The auxiliary equation is

$$m^2 - 8m + 16 = 0$$
$$(m-4)(m-4) = 0$$
$$m = 4$$

So the general solution is $y = (A + Bx)e^{4x}$

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2 e The auxiliary equation is

$$16m^2 + 8m + 1 = 0$$
$$(4m+1)(4m+1) = 0$$
$$m = -\frac{1}{4}$$

So the general solution is $y = (A + Bx)\mathrm{e}^{-\frac{1}{4}x}$

f The auxiliary equation is

$$4m^2 - 4m + 1 = 0$$
$$(2m-1)(2m-1) = 0$$
$$m = \frac{1}{2}$$

So the general solution is $y = (A + Bx)\mathrm{e}^{\frac{1}{2}x}$

g The auxiliary equation is

$$4m^2 + 20m + 25 = 0$$
$$(2m+5)(2m+5) = 0$$
$$m = -\frac{5}{2}$$

So the general solution is $y = (A + Bx)\mathrm{e}^{-\frac{5}{2}x}$

h The auxiliary equation is

$$m^2 + 2\sqrt{3}m + 3 = 0$$
$$(m+\sqrt{3})(m+\sqrt{3}) = 0$$
$$m = -\sqrt{3}$$

So the general solution is $y = (A + Bx)\mathrm{e}^{-\sqrt{3}x}$

3 a The auxiliary equation is

$$m^2 + 25 = 0$$
$$\Rightarrow m = \pm 5\mathrm{i}$$

The general solution is $y = A\cos 5x + B\sin 5x$

b The auxiliary equation is

$$m^2 + 81 = 0$$
$$\Rightarrow m = \pm 9\mathrm{i}$$

The general solution is $y = A\cos 9x + B\sin 9x.$

If the auxiliary equation has purely imaginary roots, the general solution has the form $y = A\cos\omega x + B\sin\omega x$, where A and B are constants and $\mathrm{i}\omega$ is the solution of the auxiliary equation.

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3 c The auxiliary equation is

$m^2 + 1 = 0$

$\Rightarrow m = \pm \mathrm{i}$

The general solution is $y = A\cos x + B\sin x$

d The auxiliary equation is

$9m^2 + 16 = 0$

$m^2 = -\frac{16}{9}$

$\Rightarrow m = \pm\frac{4}{3}\mathrm{i}$

The general solution is $y = A\cos\frac{4}{3}x + B\sin\frac{4}{3}x$

e The auxiliary equation is

$m^2 + 8m + 17 = 0$

$m = \frac{-8 \pm \sqrt{64 - 4\times 17}}{2} = -4 \pm \frac{1}{2}\sqrt{-4} = -4 \pm \mathrm{i}$ using the quadratic formula

The general solution is $y = \mathrm{e}^{-4x}(A\cos x + B\sin x)$

If the auxiliary equation has complex roots, the general solution has the form $y = \mathrm{e}^{px}(A\cos qx + B\sin qx)$, where A and B are constants and $p \pm \mathrm{i}q$ are solutions of the auxiliary equation.

f The auxiliary equation is

$m^2 - 4m + 5 = 0$

$m = \frac{4 \pm \sqrt{16 - 20}}{2} = 2 \pm \frac{1}{2}\sqrt{-4} = 2 \pm \mathrm{i}$

The general solution is $y = \mathrm{e}^{2x}(A\cos x + B\sin x)$

g The auxiliary equation is

$m^2 + 20m + 109 = 0$

$m = \frac{-20 \pm \sqrt{400 - 436}}{2} = \frac{-20 \pm \sqrt{-36}}{2} = -10 \pm 3\mathrm{i}$

The general solution is $y = \mathrm{e}^{-10x}(A\cos 3x + B\sin 3x)$

h The auxiliary equation is

$m^2 + \sqrt{3}m + 3 = 0$

$m = \frac{-\sqrt{3} \pm \sqrt{3 - 12}}{2} = \frac{-\sqrt{3} \pm \sqrt{-9}}{2} = \frac{-\sqrt{3} \pm 3\mathrm{i}}{2}$

The general solution is $y = \mathrm{e}^{-\frac{\sqrt{3}}{2}x}\left(A\cos\frac{3}{2}x + B\sin\frac{3}{2}x\right)$

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4 a The auxiliary equation is

$m^2 + 14m + 49 = 0$

$(m+7)(m+7) = 0$

$m = -7$

So the general solution is $y = (A + Bx)\mathrm{e}^{-7x}$

b The auxiliary equation is

$m^2 + m - 12 = 0$

$(m+4)(m-3) = 0$

$m = -4$ or 3

So the general solution is $y = A\mathrm{e}^{-4x} + B\mathrm{e}^{3x}$

c The auxiliary equation is

$m^2 + 4m + 13 = 0$

$$m = \frac{-4 \pm \sqrt{16 - 52}}{2} = \frac{-4 \pm \sqrt{-36}}{2} = -2 \pm 3\mathrm{i}$$

The general solution is $y = \mathrm{e}^{-2x}(A\cos 3x + B\sin 3x)$

d The auxiliary equation is

$16m^2 - 24m + 9 = 0$

$(4m-3)(4m-3) = 0$

$$m = \frac{3}{4}$$

So the general solution is $y = (A + Bx)\mathrm{e}^{\frac{3}{4}x}$

e The auxiliary equation is

$9m^2 - 6m + 5 = 0$

$$m = \frac{6 \pm \sqrt{36 - 4 \times 9 \times 5}}{2 \times 9} = \frac{6 \pm \sqrt{36 - 180}}{18} = \frac{6 \pm \sqrt{-144}}{18} = \frac{6 \pm 12\mathrm{i}}{18} = \frac{1 \pm 2\mathrm{i}}{3}$$

So the general solution is $y = \mathrm{e}^{\frac{1}{3}x}\left(A\cos\frac{2}{3}x + B\sin\frac{2}{3}x\right)$

f The auxiliary equation is

$6m^2 - m - 2 = 0$

$(3m-2)(2m+1) = 0$

$$m = \frac{2}{3} \text{ or } -\frac{1}{2}$$

So the general solution is $y = A\mathrm{e}^{\frac{2}{3}x} + B\mathrm{e}^{-\frac{1}{2}x}$

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5 a The auxiliary equation is

$m^2 + 2km + 9 = 0$

$\Rightarrow m = \dfrac{-2k \pm \sqrt{4k^2 - 36}}{2} = -k \pm \sqrt{k^2 - 9}$ using the quadratic formula

i If $|k| > 3$, the auxiliary equation has two real solutions

So the differential equation has the general solution $x = Ae^{(-k+\sqrt{k^2-9})t} + Be^{(-k-\sqrt{k^2-9})t}$

ii If $|k| < 3$, then the auxiliary equation has two complex conjugate roots, $-k \pm i\sqrt{9-k^2}$

So the general solution is $x = e^{-kt}\left(A\cos\left(\left(\sqrt{9-k^2}\right)t\right) + B\sin\left(\left(\sqrt{9-k^2}\right)t\right)\right)$

iii If $|k| = 3$, then the solution of the auxiliary equation is one repeated root –k,

So the differential equation has general solution $x = (A + Bt)e^{-kt}$;

b i The general solution can be found simply substituting k = 2 in the equation found in part **a ii**

So the general solution is $x = e^{-2t}\left(A\cos\sqrt{5}t + B\sin\sqrt{5}t\right)$

ii When $t \to \infty$, $e^{-2t} \to 0$, while the trigonometric functions take values that are bounded, so $x \to 0$.

6 If α is the only root of the equation, then:

(**1**) $a\alpha^2 + b\alpha + c = 0$ by definition

(**2**) $\alpha = -\dfrac{b}{2a}$ from the quadratic formula, as equal roots $\Rightarrow b^2 = 4ac$

Let $y = (A + Bx)e^{\alpha x}$

Then $\dfrac{dy}{dx} = Be^{\alpha x} + (A + Bx)\alpha e^{\alpha x}$

And $\dfrac{d^2y}{dx^2} = B\alpha e^{\alpha x} + B\alpha e^{\alpha x} + (A + Bx)\alpha^2 e^{\alpha x} = 2B\alpha e^{\alpha x} + (A + Bx)\alpha^2 e^{\alpha x}$

Substituting these results into the differential equation gives:

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + c = a(2B\alpha e^{\alpha x} + (A + Bx)\alpha^2 e^{\alpha x}) + b(Be^{\alpha x} + (A + Bx)\alpha e^{\alpha x}) + c(A + Bx)e^{\alpha x}$$

$$= Be^{\alpha x}(2a\alpha + b) + (A + Bx)e^{\alpha x}(a\alpha^2 + b\alpha + c)$$

But from equation (**1**) $a\alpha^2 + b\alpha + c = 0$ and from equation (**2**) $\alpha = -\dfrac{b}{2a} \Rightarrow 2a\alpha + b = 0$

Hence $a\dfrac{d^2y}{dx^2} + b\dfrac{dy}{dx} + c = 0$, and so $y = (A + Bx)e^{\alpha x}$ is a solution to this equation.

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7 Let $y = A\mathrm{f}(x) + B\mathrm{g}(x)$

Then $\dfrac{\mathrm{d}y}{\mathrm{d}x} = A\dfrac{\mathrm{d}\,\mathrm{f}(x)}{\mathrm{d}x} + B\dfrac{\mathrm{d}\,\mathrm{g}(x)}{\mathrm{d}x}$

And $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2} = A\dfrac{\mathrm{d}^2\,\mathrm{f}(x)}{\mathrm{d}x^2} + B\dfrac{\mathrm{d}^2\,\mathrm{g}(x)}{\mathrm{d}x^2}$

Then substituting these results into the differential equation gives:

$$a\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + b\frac{\mathrm{d}y}{\mathrm{d}x} + c = a\left(A\frac{\mathrm{d}^2\,\mathrm{f}(x)}{\mathrm{d}x^2} + B\frac{\mathrm{d}^2\,\mathrm{g}(x)}{\mathrm{d}x^2}\right) + b\left(A\frac{\mathrm{d}\,\mathrm{f}(x)}{\mathrm{d}x} + B\frac{\mathrm{d}\,\mathrm{g}(x)}{\mathrm{d}x}\right) + c\left(A\mathrm{f}(x) + B\mathrm{g}(x)\right)$$

$$= A\left(a\frac{\mathrm{d}^2\,\mathrm{f}(x)}{\mathrm{d}x^2} + b\frac{\mathrm{d}\,\mathrm{f}(x)}{\mathrm{d}x} + c\,\mathrm{f}(x)\right) + B\left(a\frac{\mathrm{d}^2\,\mathrm{g}(x)}{\mathrm{d}x^2} + b\frac{\mathrm{d}\,\mathrm{g}(x)}{\mathrm{d}x} + c\,\mathrm{g}(x)\right)$$

As $y = \mathrm{f}(x)$ and $y = \mathrm{g}(x)$ are solutions of the differential equation, it follows that

$$a\frac{\mathrm{d}^2\,\mathrm{f}(x)}{\mathrm{d}x^2} + b\frac{\mathrm{d}\,\mathrm{f}(x)}{\mathrm{d}x} + c\,\mathrm{f}(x) = 0 \quad \text{and} \quad a\frac{\mathrm{d}^2\,\mathrm{g}(x)}{\mathrm{d}x^2} + b\frac{\mathrm{d}\,\mathrm{g}(x)}{\mathrm{d}x} + c\,\mathrm{g}(x) = 0$$

Therefore $A\left(a\dfrac{\mathrm{d}^2\,\mathrm{f}(x)}{\mathrm{d}x^2} + b\dfrac{\mathrm{d}\,\mathrm{f}(x)}{\mathrm{d}x} + c\,\mathrm{f}(x)\right) + B\left(a\dfrac{\mathrm{d}^2\,\mathrm{g}(x)}{\mathrm{d}x^2} + b\dfrac{\mathrm{d}\,\mathrm{g}(x)}{\mathrm{d}x} + c\,\mathrm{g}(x)\right) = 0$

So $y = A\mathrm{f}(x) + B\mathrm{g}(x)$ is a solution.

Challenge

If a real-valued quadratic equation has complex roots $p \pm q\mathrm{i}$ then by Euler's formula,

$$\mathrm{e}^{p+\mathrm{i}q} = \mathrm{e}^p\mathrm{e}^{\mathrm{i}q} = \mathrm{e}^p(\cos q + \mathrm{i}\sin q)$$

$$\mathrm{e}^{p-\mathrm{i}q} = \mathrm{e}^p\mathrm{e}^{\mathrm{i}q} = \mathrm{e}^p(\cos(-q) + \mathrm{i}\sin(-q)) = \mathrm{e}^p(\cos q - \mathrm{i}\sin q)$$

So substituting for $\alpha = p + \mathrm{i}q$ and $\beta = p - \mathrm{i}q$ into $Ae^{\alpha x} + Be^{\beta x}$ gives:

$$Ae^{\alpha x} + Be^{\beta x} = A\mathrm{e}^{px}(\cos qx + \mathrm{i}\sin qx) + B\mathrm{e}^{px}(\cos qx - \mathrm{i}\sin qx)$$

$$= \mathrm{e}^{px}((A + B)\cos qx + (A - B)\mathrm{i}\sin qx)$$

Choose A and B such that they are complex conjugates, i.e. there are real numbers λ and μ where $A = \lambda + \mu\mathrm{i}$ and $B = \lambda - \mu\mathrm{i}$

So $A + B = 2\lambda$ and $(A - B)\mathrm{i} = (2u\mathrm{i})\mathrm{i} = 2u\mathrm{i}^2 = -2u$

Hence $Ae^{\alpha x} + Be^{\beta x} = \mathrm{e}^{px}(2\lambda\cos qx - 2\mu\sin qx) = \mathrm{e}^{px}(C\cos qx + D\sin qx)$,
where C and D are real constants, $C = 2\lambda$ and $D = -2\mu$

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Methods in differential equations 7C

1 a First consider the corresponding homogeneous equation $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+6\frac{\mathrm{d}y}{\mathrm{d}x}+5y=0$

The auxiliary equation is

$m^2+6m+5=0$

$(m+5)(m+1)=0$

$m=-5$ or -1

So the complementary function is $y=A\mathrm{e}^{-x}+B\mathrm{e}^{-5x}$

The form of the particular integral is $y=\lambda$, so $\frac{\mathrm{d}y}{\mathrm{d}x}=0$ and $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=0$

Substituting into $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+6\frac{\mathrm{d}y}{\mathrm{d}x}+5y=10$ gives:

$5\lambda=10\Rightarrow\lambda=2$

So the general solution is $y=A\mathrm{e}^{-x}+B\mathrm{e}^{-5x}+2$

b First consider the corresponding homogeneous equation $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}-8\frac{\mathrm{d}y}{\mathrm{d}x}+12y=0$

The auxiliary equation is

$m^2-8m+12=0$

$(m-6)(m-2)=0$

$m=6$ or 2

So the complementary function is $y=A\mathrm{e}^{6x}+B\mathrm{e}^{2x}$

The form of the particular integral is $y=\lambda+\mu x$, so $\frac{\mathrm{d}y}{\mathrm{d}x}=\mu$, $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=0$

Substituting into $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}-8\frac{\mathrm{d}y}{\mathrm{d}x}+12y=36x$ gives:

$-8\mu+12\lambda+12\mu x=36x$

Comparing coefficients of x: $12\mu=36\Rightarrow\mu=3$

Comparing constants: $-8\mu+12\lambda=0\Rightarrow3\lambda=2\mu$

Substituting for μ: $3\lambda=6\Rightarrow\lambda=2$

So a particular integral is $2+3x$

The general solution is $y=A\mathrm{e}^{6x}+B\mathrm{e}^{2x}+2+3x$

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1 c Solving the corresponding homogeneous equation $\frac{d^2y}{dx^2}+\frac{dy}{dx}-12y=0$

The auxiliary equation is

$m^2+m-12=0$

$(m+4)(m-3)=0$

$m=-4$ or 3

So the complementary function is $y=Ae^{-4x}+Be^{3x}$

The form of the particular integral is $y=\lambda e^{2x}$, so $\frac{dy}{dx}=2\lambda e^{2x}$ and $\frac{d^2y}{dx^2}=4\lambda e^{2x}$

Substituting into $\frac{d^2y}{dx^2}+\frac{dy}{dx}-12y=12e^{2x}$ gives:

$4\lambda e^{2x}+2\lambda e^{2x}-12\lambda e^{2x}=12e^{2x}$

$\Rightarrow -6\lambda e^{2x}=12e^{2x}\Rightarrow \lambda=-2$

So a particular integral is $-2e^{2x}$

The general solution is $y=Ae^{-4x}+Be^{3x}-2e^{2x}$

d Solving the corresponding homogeneous equation $\frac{d^2y}{dx^2}+2\frac{dy}{dx}-15y=0$

The auxiliary equation is

$m^2+2m-15=0$

$(m+5)(m-3)=0$

$m=-5$ or 3

So the complementary function is $y=Ae^{-5x}+Be^{3x}$

The form of the particular integral is $y=\lambda$, so $\frac{dy}{dx}=0$ and $\frac{d^2y}{dx^2}=0$

Substituting into $\frac{d^2y}{dx^2}+2\frac{dy}{dx}-15y=5$ gives:

$-15\lambda=5\Rightarrow \lambda=-\frac{1}{3}$

So a particular integral is $-\frac{1}{3}$

The general solution is $y=Ae^{-5x}+Be^{3x}-\frac{1}{3}$

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1 e Solving the corresponding homogeneous equation $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}-8\frac{\mathrm{d}y}{\mathrm{d}x}+16y=0$

The auxiliary equation is

$m^2-8m+16=0$

$(m-4)(m-4)=0$

$m=4$

The auxiliary equation has a repeated root so the complementary function is of the form $(A+Bx)\mathrm{e}^{\alpha x}$

So the complementary function is $y=(A+Bx)\mathrm{e}^{4x}$

The particular integral is $y=\lambda+\mu x$, so $\frac{\mathrm{d}y}{\mathrm{d}x}=\mu$ and $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=0$

Substituting in $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}-8\frac{\mathrm{d}y}{\mathrm{d}x}+16y=8x+12$ gives:

$-8\mu+16\lambda+16\mu x=8x+12$

Comparing coefficients of x: $16\mu=8\Rightarrow\mu=\frac{1}{2}$

Comparing constants: $-8\mu+16\lambda=12\Rightarrow 4\lambda=3+2\mu$

Substituting for μ: $4\lambda=3+1\Rightarrow\lambda=1$

So a particular integral is $1+\frac{1}{2}x$

The general solution is $y=(A+Bx)\mathrm{e}^{4x}+1+\frac{1}{2}x$

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1 f Solving the corresponding homogeneous equation $\frac{d^2y}{dx^2}+2\frac{dy}{dx}+y=0$

The auxiliary equation is

$m^2+2m+1=0$

$(m+1)(m+1)=0$

$m=-1$

So the complementary function is $y=(A+Bx)e^{-x}$

The particular integral is $y=\lambda\cos 2x+\mu\sin 2x$, so

$\frac{dy}{dx}=-2\lambda\sin 2x+2\mu\cos 2x$

$\frac{d^2y}{dx^2}=-4\lambda\cos 2x-4\mu\sin 2x$

Substituting in $\frac{d^2y}{dx^2}+2\frac{dy}{dx}+y=25\cos 2x$ gives:

$-4\lambda\cos 2x-4\mu\sin 2x+2(-2\lambda\sin 2x+2\mu\cos 2x)+\lambda\cos 2x+\mu\sin 2x=25\cos 2x$

$\Rightarrow(-4\lambda+4\mu+\lambda)\cos 2x+(-4\mu-4\lambda+\mu)\sin 2x=25\cos 2x$

$\Rightarrow(4\mu-3\lambda)\cos 2x-(3\mu+4\lambda)\sin 2x=25\cos 2x$

Equating the coefficients of cos 2x: $4\mu-3\lambda=25$ **(1)**

Equating the coefficients of sin 2x: $3\mu+4\lambda=0$ **(2)**

Adding 4 × equation **(1)** to 3 × equation **(2)** gives:

$16\mu-12\lambda+9\mu+12\lambda=100\Rightarrow 25\mu=100\Rightarrow\mu=4$

Substituting for in equation **(2)** gives: $3\times 4+4\lambda=0\Rightarrow\lambda=-3$

So a particular integral is $y=4\sin 2x-3\cos 2x$

The general solution is $y=(A+Bx)e^{-x}+4\sin 2x-3\cos 2x.$

g Solving the corresponding homogeneous equation $\frac{d^2y}{dx^2}+81y=0$

> If the auxiliary equation has imaginary roots, the complementary function is of the form $A\cos\omega x+B\sin\omega x$

The auxiliary equation is

$m^2+81=0$

$m=\pm 9i$

The complementary function is $y=A\cos 9x+B\sin 9x$

The particular integral is $y=\lambda e^{3x}$, so $\frac{dy}{dx}=3\lambda e^{3x}$ and $\frac{d^2y}{dx^2}=9\lambda e^{3x}$

Substituting into $\frac{d^2y}{dx^2}+81y=15e^{3x}$ gives:

$9\lambda e^{3x}+81\lambda e^{3x}=15e^{3x}\Rightarrow 90\lambda e^{3x}=15e^{3x}\Rightarrow\lambda=\frac{15}{90}=\frac{1}{6}$

So a particular integral is $\frac{1}{6}e^{3x}$

The general solution is $y=A\cos 9x+B\sin 9x+\frac{1}{6}e^{3x}$

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1 h Solving the corresponding homogeneous equation $\frac{d^2y}{dx^2}+4y=0$

The auxiliary equation is

$m^2+4=0$

$m=\pm 2i$

The complementary function is $y=A\cos 2x+B\sin 2x$

The particular integral is $y=\lambda\cos x+\mu\sin x$

$\frac{dy}{dx}=-\lambda\sin x+\mu\cos x$

$\frac{d^2y}{dx^2}=-\lambda\cos x-\mu\sin x$

Substituting into $\frac{d^2y}{dx^2}+4y=\sin x$ gives:

Then $-\lambda\cos x-\mu\sin x+4(\lambda\cos x+\mu\sin x)=\sin x$

Equating the coefficients of $\cos x$: $3\lambda=0\Rightarrow\lambda=0$

Equating the coefficients of $\sin x$: $3\mu=1\Rightarrow\mu=\frac{1}{3}$

So a particular integral is $\frac{1}{3}\sin x$

The general solution is $y=A\cos 2x+B\sin 2x+\frac{1}{3}\sin x$

i Solving the corresponding homogeneous equation $\frac{d^2y}{dx^2}-4\frac{dy}{dx}+5y=0$

The auxiliary equation is

$m^2-4m+5=0$

$m=\frac{4\pm\sqrt{16-20}}{2}=2\pm i$

> If the auxiliary equation has complex roots, the complementary function is of the form $e^{px}(A\cos ax+B\sin ax)$

The complementary function is $y=e^{2x}(A\cos x+B\sin x)$

The particular integral is $y=\lambda+\mu x+\upsilon x^2$, so $\frac{dy}{dx}=\mu+2\upsilon x$ and $\frac{d^2y}{dx^2}=2\upsilon$

Substituting into $\frac{d^2y}{dx^2}-4\frac{dy}{dx}+5y=25x^2-7$ gives:

$2\upsilon-4(\mu+2\upsilon x)+5(\lambda+\mu x+\upsilon x^2)=25x^2-7$

$\Rightarrow 5\upsilon x^2+(5\mu-8\upsilon)x+2\upsilon-4\mu+5\lambda=25x^2-7$

Equating the coefficients of x^2: $5\upsilon=25\Rightarrow\upsilon=5$

Equating the coefficients of x: $5\mu-8\upsilon=0\Rightarrow 5\mu=8\upsilon=40\Rightarrow\mu=8$

Equating constant terms: $2\upsilon-4\mu+5\lambda=-7\Rightarrow 10-32+5\lambda=-7\Rightarrow 5\lambda=15\Rightarrow\lambda=3$

So the particular integral is $3+8x+5x^2$

The general solution is $y=e^{2x}(A\cos x+B\sin x)+3+8x+5x^2$

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1 j Solving the corresponding homogeneous equation $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}-2\frac{\mathrm{d}y}{\mathrm{d}x}+26y=0$

The auxiliary equation is

$m^2-2m+26=0$

$m=\frac{2\pm\sqrt{4-4\times26}}{2}=\frac{2\pm\sqrt{-100}}{2}=1\pm5\mathrm{i}$

The complementary function is $y=\mathrm{e}^x(A\cos5x+B\sin5x)$

The particular integral is $y=\lambda\mathrm{e}^x$, so $\frac{\mathrm{d}y}{\mathrm{d}x}=\lambda\mathrm{e}^x$ and $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=\lambda\mathrm{e}^x$

Substitute into $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}-2\frac{\mathrm{d}y}{\mathrm{d}x}+26y=\mathrm{e}^x$

$\lambda\mathrm{e}^x-2\lambda\mathrm{e}^x+26\lambda\mathrm{e}^x=1$

$\Rightarrow25\lambda=1\Rightarrow\lambda=\frac{1}{25}$

So the particular integral is $\frac{1}{25}\mathrm{e}^x$

The general solution is $y=\mathrm{e}^x(A\cos5x+B\sin5x)+\frac{1}{25}\mathrm{e}^x$

2 a Consider a particular integral of the form $y=\upsilon x^2+\mu x+\lambda$, so $\frac{\mathrm{d}y}{\mathrm{d}x}=\mu+2\upsilon x$ and $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=2\upsilon$

Substituting into $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}-5\frac{\mathrm{d}y}{\mathrm{d}x}+4y=x^2-3x+2$ gives:

$2\upsilon-5(\mu+2\upsilon x)+4(\lambda+\mu x+\upsilon x^2)=x^2-3x+2$

$\Rightarrow4\upsilon x^2+(4\mu-10\upsilon)x+2\upsilon-5\mu+4\lambda=x^2-3x+2$

Equating the coefficients of x^2: $4\upsilon=1\Rightarrow\upsilon=\frac{1}{4}$

Equating the coefficients of x: $4\mu-10\upsilon=-3\Rightarrow4\mu=10\upsilon-3\Rightarrow\mu=-\frac{1}{8}$

Equating constant terms: $2\upsilon-5\mu+4\lambda=2\Rightarrow\frac{1}{2}+\frac{5}{8}+4\lambda=2\Rightarrow4\lambda=\frac{7}{8}\Rightarrow\lambda=\frac{7}{32}$

So the particular integral is $\frac{1}{4}x^2-\frac{1}{8}x+\frac{7}{32}$

b Solving the corresponding homogeneous equation $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}-2\frac{\mathrm{d}y}{\mathrm{d}x}+26y=0$

The auxiliary equation is

$m^2-5m+4=0$

$(m-4)(m-1)=0$

$m=1$ or 4

So the complementary function is $y=A\mathrm{e}^{4x}+B\mathrm{e}^x$

The general solution of the given equation is $y=A\mathrm{e}^{4x}+B\mathrm{e}^x+\frac{1}{4}x^2-\frac{1}{8}x+\frac{7}{32}$

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3 a The complementary function is the general solution of the equation $\frac{d^2y}{dx^2} - 6\frac{dy}{dx} = 0$

The auxiliary equation is

$m^2 - 6m = 0$

$m(m-6) = 0$

$m = 0$ or 6

So the complementary function is $y = Ae^{6x} + Be^{0x} = Ae^{6x} + B$

b As the complementary function includes a constant term, multiply the 'expected' particular integral by x, so consider a particular integral of the form $y = \upsilon x^3 + \mu x^2 + \lambda x$

$\frac{dy}{dx} = 3\upsilon x^2 + 2\mu x + \lambda$ and $\frac{d^2y}{dx^2} = 6\upsilon x + 2\mu$

Substituting into $\frac{d^2y}{dx^2} - 6\frac{dy}{dx} = 2x^2 - x + 1$ gives:

$6\upsilon x + 2\mu - 6(3\upsilon x^2 + 2\mu x + \lambda) = 2x^2 - x + 1$

$\Rightarrow -18\upsilon x^2 + (6\upsilon - 12\mu)x + 2\mu - 6\lambda = 2x^2 - x + 1$

Equating the coefficients of x^2: $-18\upsilon = 2 \Rightarrow \upsilon = -\frac{1}{9}$

Equating the coefficients of x: $6\upsilon - 12\mu = -1 \Rightarrow 12\mu = 6\upsilon + 1 \Rightarrow 12\mu = \frac{1}{3} \Rightarrow \mu = \frac{1}{36}$

Equating constant terms: $2\mu - 6\lambda = 1 \Rightarrow 6\lambda = -\frac{34}{36} \Rightarrow \lambda = \frac{17}{108}$

So the particular integral is $-\frac{1}{9}x^3 + \frac{1}{36}x^2 + \frac{17}{108}x$

The general solution is $y = Ae^{6x} + B - \frac{1}{9}x^3 + \frac{1}{36}x^2 + \frac{17}{108}x$

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4 The complementary function is the general solution of the equation $\frac{d^2y}{dx^2}+4\frac{dy}{dx}=0$

The auxiliary equation is

$m^2+4m=0$

$m(m+4)=0$

$m=0$ or -4

So the complementary function is $y=A+Be^{-4x}$

As the complementary function includes a constant term, multiply the 'expected' particular integral by x, so consider a particular integral of the form $y=\upsilon x^3+\mu x^2+\lambda x$

$\frac{dy}{dx}=3\upsilon x^2+2\mu x+\lambda$ and $\frac{d^2y}{dx^2}=6\upsilon x+2\mu$

Substituting into $\frac{d^2y}{dx^2}+4\frac{dy}{dx}=24x^2$ gives:

$6\upsilon x+2\mu+4(3\upsilon x^2+2\mu x+\lambda)=24x^2$

$\Rightarrow 12\upsilon x^2+(6\upsilon+8\mu)x+2\mu+4\lambda=24x^2$

Equating the coefficients of x^2: $12\upsilon=24\Rightarrow\upsilon=2$

Equating the coefficients of x: $6\upsilon+8\mu=0\Rightarrow 8\mu=-12\Rightarrow\mu=-\frac{3}{2}$

Equating constant terms: $2\mu+4\lambda=0\Rightarrow 4\lambda=3\Rightarrow\lambda=\frac{3}{4}$

So the particular integral is $2x^3-\frac{3}{2}x^2+\frac{3}{4}x$

The general solution is $y=A+Be^{-4x}+2x^3-\frac{3}{2}x^2+\frac{3}{4}x$

5 a Solving the corresponding homogeneous equation $\frac{d^2y}{dx^2}-2\frac{dy}{dx}+y=0$

The auxiliary equation is

$m^2-2m+1=0$

$(m-1)(m-1)=0$

$m=1$

So the complementary function is $y=(A+Bx)e^x$

The complementary function contains an xe^x and so λxe^x is not a suitable form for the particular integral of this equation.

Note that if $y=\lambda xe^x$, then $\frac{dy}{dx}=\lambda e^x+\lambda xe^x$ and $\frac{d^2y}{dx^2}=2\lambda e^x+\lambda xe^x$

Substituting into $\frac{d^2y}{dx^2}-2\frac{dy}{dx}+y=e^x$ gives:

$2\lambda e^x+\lambda xe^x-2(\lambda e^x+\lambda xe^x)+\lambda xe^x=e^x$

$\Rightarrow 0=e^x$, which is impossible as $e^x>0$

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5 b if $y = \lambda x^2 e^x$, then $\frac{dy}{dx} = 2\lambda x e^x + \lambda x^2 e^x$ and $\frac{d^2y}{dx^2} = 2\lambda e^x + 4\lambda x e^x + \lambda x^2 e^x$

Substituting into $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = e^x$ gives:

$2\lambda e^x + 4\lambda x e^x + \lambda x^2 e^x - 2(2\lambda x e^x + \lambda x^2 e^x) + \lambda x^2 e^x = e^x$

$\Rightarrow 2\lambda e^x = e^x \Rightarrow \lambda = \frac{1}{2}$

c The general solution is the complementary function (from part **a**) plus the particular integral (from part **b**), so it is

$$y = (A + Bx)e^x + \frac{1}{2}x^2 e^x = \left(A + Bx + \frac{1}{2}x^2\right)e^x$$

6 a Solving the corresponding homogeneous equation $\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 3y = 0$

The auxiliary equation is

$m^2 + 4m + 3 = 0$

$(m+3)(m+1) = 0$

$m = -1$ or -3

So the complementary function is $y = Ae^{-t} + Be^{-3t}$

The form of the particular integral is $y = \lambda + \mu t$, so $\frac{dy}{dt} = \mu$, $\frac{d^2y}{dt^2} = 0$

Substituting into $\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 3y = kt + 5$ gives:

$4\mu + 3\lambda + 3\mu t = kt + 5$

Equating the coefficients of t: $3\mu = k \Rightarrow \mu = \frac{k}{3}$

Equating constant terms: $4\mu + 3\lambda = 5 \Rightarrow 3\lambda = 5 - \frac{4k}{3} \Rightarrow \lambda = \frac{5}{3} - \frac{4k}{9}$

So the particular integral is $\frac{5}{3} - \frac{4k}{9} + \frac{k}{3}t$

The general solution is $y = Ae^{-t} + Be^{-3t} + \frac{5}{3} - \frac{4k}{9} + \frac{kt}{3}$

b If $k = 6$, then the general solution is $y = Ae^{-t} + Be^{-3t} + 2t - 1$.

As $t \to \infty$, $e^{-t} \to 0$, so for large values of t the general solution may be approximated by $y = 2t - 1$

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Challenge

Solving the corresponding homogeneous equation $\frac{\mathrm{d}^2y}{\mathrm{d}t^2}+y=0$

The auxiliary equation is $m^2+1=0\Rightarrow m=\pm\mathrm{i}$

So the complementary function is $y=A\cos x+B\sin x$

To find a particular integral, consider functions of the form $y=\lambda x\mathrm{e}^{2x}+\mu\mathrm{e}^{2x}$

$\frac{\mathrm{d}y}{\mathrm{d}x}=\lambda\mathrm{e}^{2x}+2\lambda x\mathrm{e}^{2x}+2\mu\mathrm{e}^{2x}$

$\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=2\lambda\mathrm{e}^{2x}+2\lambda\mathrm{e}^{x}+4\lambda x\mathrm{e}^{2x}+4\mu\mathrm{e}^{2x}=4\lambda\mathrm{e}^{2x}+4\lambda x\mathrm{e}^{2x}+4\mu\mathrm{e}^{2x}$

Substituting into $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+y=5x\mathrm{e}^{2x}$ gives:

$4\lambda\mathrm{e}^{2x}+4\lambda x\mathrm{e}^{2x}+4\mu\mathrm{e}^{2x}+\lambda x\mathrm{e}^{2x}+\mu\mathrm{e}^{2x}=5x\mathrm{e}^{2x}$

$\Rightarrow(4\lambda+5\mu)\mathrm{e}^{2x}+5\lambda x\mathrm{e}^{2x}=5x\mathrm{e}^{2x}$

Equating the coefficients of $x\mathrm{e}^{2x}$: $5\lambda=5\Rightarrow\lambda=1$

Equating the coefficients of e^{2x}: $4\lambda+5\mu=0\Rightarrow\mu=-\frac{4}{5}$

So the particular integral is $x\mathrm{e}^{2x}-\frac{4}{5}\mathrm{e}^{2x}$

The general solution is $y=A\cos x+B\sin x+x\mathrm{e}^{2x}-\frac{4}{5}\mathrm{e}^{2x}$

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Methods in differential equations 7D

1 a The auxiliary equation for the corresponding homogeneous equation is

$m^2 + 5m + 6 = 0$

$(m+2)(m+3) = 0$

$m = -2$ or -3

So the complementary function is $y = Ae^{-2x} + Be^{-3x}$

A particular integral for the equation is of the form $y = \lambda e^x$, so $\frac{dy}{dx} = \lambda e^x$ and $\frac{d^2y}{dx^2} = \lambda e^x$

Substituting into $\frac{d^2y}{dx^2} + 5\frac{dy}{dx} + 6y = 12e^x$ gives $12\lambda e^x = 12e^x \Rightarrow \lambda = 1$

So a particular integral is e^x

The general solution is $y = Ae^{-2x} + Be^{-3x} + e^x$

b When $x = 0$, $y = 1$ so $A + B + 1 = 1 \Rightarrow A = -B$ **(1)**

$\frac{dy}{dx} = -3Ae^{-3x} - 2Be^{-2x} + e^x$

When $x = 0$, $\frac{dy}{dx} = 0$ so $-3A - 2B + 1 = 0$

$\Rightarrow 3A + 2B = 1$ **(2)**

Substituting the value for A from equation **(1)** into equation **(2)**

$-3B + 2B = 1 \Rightarrow B = -1$

So $A = -B = 1$

Substituting values for A and B into the general solution from part **a** gives the particular solution

$y = e^{-3x} - e^{-2x} + e^x$

2 a Solving the corresponding homogeneous equation $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} = 0$

The auxiliary equation is

$m^2 + 2m = 0$

$m(m+2) = 0$

$m = 0$ or -2

So the complementary function is $y = Ae^{0x} + Be^{-2x} = A + Be^{-2x}$

The form of the particular integral is $y = \lambda e^{2x}$, so $\frac{dy}{dx} = 2\lambda e^{2x}$ and $\frac{d^2y}{dx^2} = 4\lambda e^{2x}$

Substituting into $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} = 12e^{2x}$ gives: $4\lambda e^{2x} + 4\lambda e^{2x} = 12e^{2x} \Rightarrow \lambda = \frac{3}{2}$

So a particular integral is $\frac{3}{2}e^{2x}$

The general solution is $y = A + Be^{-2x} + \frac{3}{2}e^{2x}$

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2 b When $x = 0$, $y = 2$ so $A + B + \frac{3}{2} = 2 \Rightarrow A + B = \frac{1}{2}$ **(1)**

$\frac{dy}{dx} = -2Be^{-2x} + 3e^{2x}$

When $x = 0$, $\frac{dy}{dx} = 6$ so $-2B + 3 = 6 \Rightarrow B = -\frac{3}{2}$

Substituting into equation **(1)** $A - \frac{3}{2} = \frac{1}{2} \Rightarrow A = 2$

Substituting values for A and B into the general solution from part **a** gives the particular solution

$y = 2 - \frac{3}{2}e^{-2x} + \frac{3}{2}e^{2x}$

3 Solving the corresponding homogeneous equation $\frac{d^2y}{dx^2} - \frac{dy}{dx} - 42y = 0$

The auxiliary equation is

$m^2 - m - 42 = 0$

$(m - 7)(m + 6) = 0$

$m = -6$ or 7

So the complementary function is $y = Ae^{-6x} + Be^{7x}$

The form of the particular integral is $y = \lambda$, so $\frac{dy}{dx} = 0$ and $\frac{d^2y}{dx^2} = 0$

Substituting in $\frac{d^2y}{dx^2} - \frac{dy}{dx} - 42y = 14$ gives: $-42\lambda = 14 \Rightarrow \lambda = -\frac{1}{3}$

So the general solution is $y = Ae^{-6x} + Be^{7x} - \frac{1}{3}$

Now applying the boundary conditions

When $x = 0$, $y = 0$ so $A + B - \frac{1}{3} = 0 \Rightarrow 3A + 3B = 1$ **(1)**

$\frac{dy}{dx} = -6Ae^{-6x} + 7Be^{7x}$

When $x = 0$, $\frac{dy}{dx} = \frac{1}{6}$ so $-6A + 7B = \frac{1}{6}$ **(2)**

Adding 2 × equation **(1)** to equation **(2)** gives:

$6A + 6B - 6A + 7B = 2 + \frac{1}{6}$

$\Rightarrow 13B = \frac{13}{6} \Rightarrow B = \frac{1}{6}$

Substituting into equation **(1)** $A + \frac{1}{6} = \frac{1}{3} \Rightarrow A = \frac{1}{6}$

Substituting values for A and B into the general solution gives the particular solution

$y = \frac{1}{6}e^{-6x} + \frac{1}{6}e^{7x} - \frac{1}{3}$

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4 a Solving the corresponding homogeneous equation $\frac{d^2y}{dx^2}+9y=0$

The auxiliary equation is $m^2+9=0 \Rightarrow m=\pm 3i$

So the complementary function is $y=A\cos 3x+B\sin 3x$

To find a particular integral consider functions of the form $y=\lambda\cos x+\mu\sin x$

$\frac{dy}{dx}=-\lambda\sin x+\mu\cos x$ and $\frac{d^2y}{dx^2}=-\lambda\cos x-\mu\sin x$

Substituting in $\frac{d^2y}{dx^2}+9y=16\sin x$ gives:

$-\lambda\cos x-\mu\sin x+9(\lambda\cos x+\mu\sin x)=16\sin x$

$\Rightarrow 8\lambda\cos x+8\mu\sin x=16\sin x$

$\Rightarrow \lambda=0,\ \mu=2$

So a particular integral is $2\sin x$

The general solution of the equation is $y=A\cos 3x+B\sin 3x+2\sin x$

b When $x=0,\ y=1$ so $A\cos 0+B\sin 0+2\sin 0=1 \Rightarrow A=1$

$\frac{dy}{dx}=-3A\sin 3x+3B\cos 3x+2\cos x$

When $x=0, \frac{dy}{dx}=8$ so $3B+2=8 \Rightarrow B=2$

Substituting values for A and B into the general solution from part **a** gives the particular solution

$y=\cos 3x+2\sin 3x+2\sin x$

5 a Solving the corresponding homogeneous equation $4\frac{d^2y}{dx^2}+4\frac{dy}{dx}+5y=0$

The auxiliary equation is $4m^2+4m+5=0$

$m=\frac{-4\pm\sqrt{16-80}}{8}=\frac{-4\pm\sqrt{-64}}{8}=\frac{-4\pm 8i}{8}=-\frac{1}{2}\pm i$ using the quadratic formula

So the complementary function is $y=e^{-\frac{1}{2}x}(A\cos x+B\sin x)$

The form of the particular integral is $y=\lambda\cos x+\mu\sin x$

$\frac{dy}{dx}=-\lambda\sin x+\mu\cos x$ and $\frac{d^2y}{dx^2}=-\lambda\cos x-\mu\sin x$

Substituting into $4\frac{d^2y}{dx^2}+4\frac{dy}{dx}+5y=\sin x+4\cos x$ gives:

$-4\lambda\cos x-4\mu\sin x-4\lambda\sin x+4\mu\cos x+5\lambda\cos x+5\mu\sin x=\sin x+4\cos x$

Equating the coefficients of $\cos x$: $\lambda+4\mu=4$ **(1)**

Equating the coefficients of $\sin x$: $-4\lambda+\mu=1$ **(2)**

Adding 4 × equation **(1)** to equation **(2)** gives: $17\mu=17 \Rightarrow \mu=1$

And hence from equation **(1)** $\lambda=0$, so a particular integral is $\sin x$

The general solution is $y=e^{-\frac{1}{2}x}(A\cos x+B\sin x)+\sin x$

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5 b When $x = 0, y = 0$ so $e^0 A = 0 \Rightarrow A = 0$

So the particular solution is $y = Be^{-\frac{1}{2}x} \sin x + \sin x$

$\frac{dy}{dx} = Be^{-\frac{1}{2}x} \cos x - \frac{1}{2} Be^{-\frac{1}{2}x} \sin x + \cos x$

When $x = 0, \frac{dy}{dx} = 0$ so $B + 1 = 0 \Rightarrow B = -1$

Substituting values for A and B into the general solution from part **a** gives the particular solution

$y = -e^{-\frac{1}{2}x} \sin x + \sin x = \sin x(1 - e^{-\frac{1}{2}x})$

6 a Solving the corresponding homogeneous equation $\frac{d^2x}{dt^2} - 3\frac{dx}{dt} + 2x = 0$

The auxiliary equation is

$m^2 - 3m + 2 = 0$

$(m - 2)(m - 1) = 0$

$m = 1$ or 2

In this question t is the independent variable, and x the dependent variable. The method of solution is the same as in questions connecting x and y.

So the complementary function is $x = Ae^t + Be^{2t}$

The form of the particular integral is $x = \lambda + ut$, so $\frac{dx}{dt} = \mu$ and $\frac{d^2x}{dt^2} = 0$

Substituting into $\frac{d^2x}{dt^2} - 3\frac{dx}{dt} + 2x = 2t - 3$ gives:

$-3\mu + 2\lambda + 2\mu t = 2t - 3$

Equate coefficients of t: $2\mu = 2 \Rightarrow \mu = 1$

Equate constant terms: $2\lambda - 3\mu = -3 \Rightarrow \lambda = 0$

So a particular integral is t.

The general solution is $x = Ae^t + Be^{2t} + t$

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6 b When $t = 0$, $x = 1$ so $A + B = 1$
When $t = 1$, $x = 2$ so $A\mathrm{e} + B\mathrm{e}^2 = 1$

$$A\mathrm{e} + (1 - A)\mathrm{e}^2 = 1$$

$$A\mathrm{e} + \mathrm{e}^2 - \mathrm{e}^2 A = 1$$

$$A\mathrm{e}(1 - \mathrm{e}) = 1 - \mathrm{e}^2$$

$$A = \frac{1 - \mathrm{e}^2}{\mathrm{e}(1 - \mathrm{e})}$$

$$A = \frac{1 + \mathrm{e}}{\mathrm{e}}$$

$$B = 1 - \frac{1 + \mathrm{e}}{\mathrm{e}}$$

$$B = -\frac{1}{\mathrm{e}}$$

So the particular solution is

$$x = \left(\frac{1 + \mathrm{e}}{\mathrm{e}}\right)\mathrm{e}^t + \left(-\frac{1}{\mathrm{e}}\right)\mathrm{e}^{2t} + t$$

$$= \frac{\mathrm{e}^t + \mathrm{e}^{t+1}}{\mathrm{e}} - \frac{\mathrm{e}^{2t}}{\mathrm{e}} + t$$

$$= \mathrm{e}^{t-1} + \mathrm{e}^t - \mathrm{e}^{2t-1} + t$$

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7 Solving the corresponding homogeneous equation $\frac{d^2x}{dt^2} - 9x = 0$

The auxiliary equation is $m^2 - 9 = 0 \Rightarrow m = \pm 3$

So the complementary function is $x = Ae^{3t} + Be^{-3t}$

The form of the particular integral is $x = \lambda\cos t + \mu\sin t$

The particular integral is of the form $\lambda\cos t + \mu\sin t$.

$\frac{dx}{dt} = -\lambda\sin t + \mu\cos t$ and $\frac{d^2x}{dt^2} = -\lambda\cos t - \mu\sin t$

Substituting into $\frac{d^2x}{dt^2} - 9x = 10\sin t$ gives:

$-\lambda\cos t - \mu\sin t - 9\lambda\cos t - 9\mu\sin t = 10\sin t$

Equating the coefficients of cos t: $-10\lambda = 0 \Rightarrow \lambda = 0$

Equating the coefficients of sin t: $-10\mu = 10 \Rightarrow \mu = -1$

So a particular integral is $-\sin t$

The general solution is $x = Ae^{3t} + Be^{-3t} - \sin t$

When $t = 0, x = 2$ so $A + B = 2$ **(1)**

$\frac{dx}{dt} = 3Ae^{3t} - 3Be^{-3t} - \cos t$

When $t = 0, \frac{dx}{dt} = -1$ so $3A - 3B - 1 = -1 \Rightarrow A = B$

Substituting $A = B$ in equation **(1)** gives $2B = 2 \Rightarrow A = B = 1$

Substituting values for A and B into the general solution gives the particular solution

$x = e^{3t} + e^{-3t} - \sin t$

8 a i $x = \lambda t^3 e^{2t}$, so $\frac{dx}{dt} = 3\lambda t^2 e^{2t} + 2\lambda t^3 e^{2t}$

$\frac{d^2x}{dt^2} = 6\lambda t e^{2t} + 6\lambda t^2 e^{2t} + 6\lambda t^2 e^{2t} + 4\lambda t^3 e^{2t} = 6\lambda t e^{2t} + 12\lambda t^2 e^{2t} + 4\lambda t^3 e^{2t}$

Substituting into $\frac{d^2x}{dt^2} - 4\frac{dx}{dt} + 4x = 3te^{2t}$ gives:

$6\lambda t e^{2t} + 12\lambda t^2 e^{2t} + 4\lambda t^3 e^{2t} - 4(3\lambda t^2 e^{2t} + 2\lambda t^3 e^{2t}) + 4\lambda t^3 e^{2t} = 3te^{2t}$

$\Rightarrow 6\lambda t e^{2t} = 3te^{2t}$

$\Rightarrow \lambda = \frac{1}{2}$

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8 a ii Solving the corresponding homogeneous equation $\frac{\mathrm{d}^2x}{\mathrm{d}t^2} - 4\frac{\mathrm{d}x}{\mathrm{d}t} + 4x = 0$

The auxiliary equation is

$m^2 - 4m + 4 = 0$

$(m-2)(m-2) = 0$

$m = 2$

So the complementary function is $x = (A + Bt)\mathrm{e}^{2t}$

Therefore using the particular integral from part **ai** the general solution is

$$x = (A + Bt)\mathrm{e}^{2t} + \frac{1}{2}t^3\mathrm{e}^{2t}$$

This equation can also be written as

$$x = A\mathrm{e}^{2t} + Bt\mathrm{e}^{2t} + \frac{1}{2}t^3\mathrm{e}^{2t} \quad \text{or} \quad x = \left(A + Bt + \frac{1}{2}t^3\right)\mathrm{e}^{2t}$$

b When $t = 0, x = 0$ so $A = 0$

$$\frac{\mathrm{d}x}{\mathrm{d}t} = 2\left(A + Bt + \frac{1}{2}t^3\right)\mathrm{e}^{2t} + \left(B + \frac{3}{2}t^2\right)\mathrm{e}^{2t}$$

When $t = 0, \frac{\mathrm{d}x}{\mathrm{d}t} = 1$ so $2A + B = 1$

As $A = 0 \Rightarrow B = 1$

Substituting values for A and B into the general solution gives the particular solution

$$x = t\mathrm{e}^{2t} + \frac{1}{2}t^3\mathrm{e}^{2t} = \left(t + \frac{1}{2}t^3\right)\mathrm{e}^{2t} = \left(1 + \frac{1}{2}t^2\right)t\mathrm{e}^{2t}$$

9 Solving the corresponding homogeneous equation $25\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + 36x = 0$

The auxiliary equation is

$25m^2 + 36 = 0$

$$m^2 = -\frac{36}{25}$$

$$m = \pm\frac{6}{5}\mathrm{i}$$

So the complementary function is $x = A\cos\frac{6}{5}t + B\sin\frac{6}{5}t$

The form of the particular integral is $x = \lambda$, so $\frac{\mathrm{d}x}{\mathrm{d}t} = 0$ and $\frac{\mathrm{d}^2x}{\mathrm{d}t^2} = 0$

Substituting into $25\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + 36x = 18$ gives:

$36\lambda = 18 \Rightarrow \lambda = \frac{18}{36} = \frac{1}{2}$

So the general solution is $x = A\cos\frac{6}{5}t + B\sin\frac{6}{5}t + \frac{1}{2}$

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9 When $t=0, x=1$ so $A+\frac{1}{2}=1 \Rightarrow A=\frac{1}{2}$

$\frac{\mathrm{d}x}{\mathrm{d}t}=-\frac{6}{5}A\sin\frac{6}{5}t+\frac{6}{5}B\cos\frac{6}{5}t$

When $t=0, \frac{\mathrm{d}x}{\mathrm{d}t}=0.6$ so $1.2B=0.6 \Rightarrow B=\frac{1}{2}$

Substituting values for A and B into the general solution gives the particular solution

$x=\frac{1}{2}\left(\cos\frac{6}{5}t+\sin\frac{6}{5}t+1\right)$

10 a Solving the corresponding homogeneous equation $\frac{\mathrm{d}^2x}{\mathrm{d}t^2}-2\frac{\mathrm{d}x}{\mathrm{d}t}+2x=0$

The auxiliary equation is

$m^2-2m+2=0$

$m=\frac{2\pm\sqrt{4-8}}{2}=1\pm\mathrm{i}$

So the complementary function is $x=\mathrm{e}^t(A\cos t+B\sin t)$

The form of the particular integral is $x=\upsilon t^2+\mu t+\lambda$, so $\frac{\mathrm{d}x}{\mathrm{d}t}=\mu+2\upsilon t$ and $\frac{\mathrm{d}^2x}{\mathrm{d}t^2}=2\upsilon$

Substituting into $\frac{\mathrm{d}^2x}{\mathrm{d}t^2}-2\frac{\mathrm{d}x}{\mathrm{d}t}+2x=2t^2$ gives:

$2\upsilon-2(\mu+2\upsilon t)+2(\lambda+\mu t+\upsilon t^2)=2t^2$

Equating the coefficients of t^2: $2\upsilon=2 \Rightarrow \upsilon=1$

Equating the coefficients of t: $2\mu-4\upsilon=0 \Rightarrow 2\mu=4\upsilon \Rightarrow \mu=2$

Equating constant terms: $2\upsilon-2\mu+2\lambda=0 \Rightarrow 2-4+2\lambda=0 \Rightarrow 2\lambda=2 \Rightarrow \lambda=1$

So a particular integral is t^2+2t+1

The general solution is $x=\mathrm{e}^t(A\cos t+B\sin t)+t^2+2t+1$

b When $t=0, x=1$ so $A+1=1 \Rightarrow A=0$

$\frac{\mathrm{d}x}{\mathrm{d}t}=\mathrm{e}^t(A\cos t+B\sin t)+\mathrm{e}^t(-A\sin t+B\cos t)+2t+2$

When $t=0, \frac{\mathrm{d}x}{\mathrm{d}t}=3$ and given $A=0$ so $B+2=3 \Rightarrow B=1$

Substituting values for A and B into the general solution gives the particular solution

$x=\mathrm{e}^t\sin t+t^2+2t+1$ which can be rewitten as $x=\mathrm{e}^t\sin t+(1+t)^2$

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11 a Solving the corresponding homogeneous equation $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = 0$

The auxiliary equation is

$m^2 - 3m + 2 = 0$

$(m-2)(m-1) = 0$

$m = 1$ or 2

So the complementary function is $y = Ae^x + Be^{2x}$

As the complementary function has an e^{2x} term, try a particular integral in the form $\lambda x e^{2x}$

So $\frac{dy}{dx} = \lambda e^{2x} + 2\lambda x e^{2x}$ and $\frac{d^2y}{dx^2} = 4\lambda e^{2x} + 4\lambda x e^{2x}$

Substituting into $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = 3e^{2x}$ gives:

$4\lambda e^{2x} + 4\lambda x e^{2x} - 3(\lambda e^{2x} + 2\lambda x e^{2x}) + 2\lambda x e^{2x} = 3e^{2x}$

$\Rightarrow \lambda e^{2x} = 3e^{2x} \Rightarrow \lambda = 3$

So a particular integral is $3xe^{2x}$

The general solution is $y = Ae^x + Be^{2x} + 3xe^{2x}$

b When $x = 0, y = 0$ so $A + B = 0 \Rightarrow A = -B$

$\frac{dy}{dx} = Ae^x + 2Be^{2x} + 3e^{2x} + 6xe^{2x}$

When $x = 0, \frac{dy}{dx} = 0$ and given $A = -B$ so $-B + 2B + 3 = 0 \Rightarrow B = -3$ and $A = 3$

Substituting values for A and B into the general solution from part **a** gives the particular solution

$y = 3e^x - 3e^{2x} + 3xe^{2x}$

12 Solving the corresponding homogeneous equation $\frac{d^2y}{dx^2} + 9y = 0$

The auxiliary equation is $m^2 + 9 = 0 \Rightarrow m = \pm 3i$

So the complementary function is $y = A\cos 3x + B\sin 3x$

As the complementary function has a $\sin 3x$ term, try a particular integral $y = \lambda x \cos 3x + \mu x \sin 3x$

$\frac{dy}{dx} = -3\lambda x \sin 3x + \lambda \cos 3x + 3\mu x \cos 3x + \mu \sin 3x$

$\frac{d^2y}{dx^2} = -9\lambda x \cos 3x - 3\lambda \sin 3x - 3\lambda \sin 3x - 9\mu x \sin 3x + 3\mu \cos 3x - 3\mu \cos 3x$

$= -9\lambda x \cos 3x - 6\lambda \sin 3x - 9\mu x \sin 3x$

Substituting into $\frac{d^2y}{dx^2} + 9y = \sin 3x$ gives:

$-9\lambda x \cos 3x - 6\lambda \sin 3x - 9\mu x \sin 3x + 9\lambda x \cos 3x + 9\mu x \sin 3x = \sin 3x$

$\Rightarrow -6\lambda \sin 3x = \sin 3x \Rightarrow \lambda = -\frac{1}{6}$

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12 The equation is satisfied for any μ, so choose $\mu = 0$ and a particular integral is $-\frac{1}{6}x\cos 3x$

The general solution is $y = A\cos 3x + B\sin 3x - \frac{1}{6}x\cos 3x$

When $x = 0, y = 0$ so $A = 0$

$\frac{dy}{dx} = -3A\sin 3x + 3B\cos 3x - \frac{1}{6}\cos 3x - \frac{1}{2}x\sin 3x$

When $x = 0, \frac{dy}{dx} = 0$ so $3B - \frac{1}{6} = 0 \Rightarrow B = \frac{1}{18}$

Substituting values for A and B into the general solution gives the particular solution

$y = \frac{1}{18}\sin 3x - \frac{1}{6}x\cos 3x$

13 a Solving the corresponding homogeneous equation $\frac{d^2x}{dt^2} + 5\frac{dx}{dt} + 6x = 0$

The auxiliary equation is

$m^2 + 5m + 6 = 0$

$(m+3)(m+2) = 0$

$m = -3$ or -2

So the complementary function is $x = Ae^{-2t} + Be^{-3t}$

The form of the particular integral is $x = \lambda e^{-t}$, so $\frac{dx}{dt} = -\lambda e^{-t}$ and $\frac{d^2x}{dt^2} = \lambda e^{-t}$

Substituting into $\frac{d^2x}{dt^2} + 5\frac{dx}{dt} + 6x = 2e^{-t}$ gives:

$\lambda e^{-t} - 5\lambda e^{-t} + 6\lambda e^{-t} = 2e^{-t}$

$\Rightarrow 2\lambda e^{-t} = 2e^{-t} \Rightarrow \lambda = 1$

So a particular integral is e^{-t}

The general solution is $x = Ae^{-2t} + Be^{-3t} + e^{-t}$

When $t = 0, x = 0$ so $A + B + 1 = 0 \Rightarrow A = -B - 1$

$\frac{dx}{dt} = -2Ae^{-2t} - 3Be^{-3t} - e^{-t}$

When $t = 0, \frac{dx}{dt} = 2$ so $-2A - 3B - 1 = 2$

$\Rightarrow -2(-B-1) - 3B - 1 = 2$

$\Rightarrow B = -1$ and hence $A = 0$

Substituting values for A and B into the general solution gives the particular solution

$x = -e^{-3t} + e^{-t}$

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13 b $\dfrac{\mathrm{d}x}{\mathrm{d}t} = 3\mathrm{e}^{-3t} - \mathrm{e}^{-t}$

When $\dfrac{\mathrm{d}x}{\mathrm{d}t} = 0, 3\mathrm{e}^{-3t} = \mathrm{e}^{-t}$ and taking logarithms of both sides gives $\ln 3\mathrm{e}^{-3t} = \ln^{-t}$

$\Rightarrow \ln 3 + \ln \mathrm{e}^{-3t} = \ln \mathrm{e}^{-t} \Rightarrow \ln 3 - 3t = -t$

$\Rightarrow t = \dfrac{1}{2}\ln 3$

$\dfrac{\mathrm{d}^2x}{\mathrm{d}t^2} = -9\mathrm{e}^{-3t} + \mathrm{e}^{-t}$, so when $t = \dfrac{1}{2}\ln 3$

$\dfrac{\mathrm{d}^2x}{\mathrm{d}t^2} = -9\mathrm{e}^{-\frac{3}{2}\ln 3} + \mathrm{e}^{-\frac{1}{2}\ln 3} = -9\mathrm{e}^{-\ln 3^{\frac{3}{2}}} + \mathrm{e}^{\ln 3^{-\frac{1}{2}}} = -\dfrac{9}{3\sqrt{3}} + \dfrac{1}{\sqrt{3}} = -\dfrac{2\sqrt{3}}{3}$

So when $t = \dfrac{1}{2}\ln 3$, $\dfrac{\mathrm{d}x}{\mathrm{d}t} = 0$ and $\dfrac{\mathrm{d}^2x}{\mathrm{d}t^2} < 0$, there is a maximum, and as the function is continuous this is its maximum value. Substituting in the equation gives

Maximum value $= -\mathrm{e}^{-\frac{3}{2}\ln 3} + \mathrm{e}^{-\frac{1}{2}\ln 3} = -\mathrm{e}^{\ln 3^{-\frac{3}{2}}} + \mathrm{e}^{\ln 3^{-\frac{1}{2}}} = -3^{-\frac{3}{2}} + 3^{-\frac{1}{2}} = \dfrac{1}{\sqrt{3}} - \dfrac{1}{3\sqrt{3}} = \dfrac{2}{3\sqrt{3}} = \dfrac{2\sqrt{3}}{9}$

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Methods in differential equations – mixed exercise 7

1 The integrating factor is $e^{\int \tan x \, dx} = e^{\ln \sec x} = \sec x$

Multiplying the equation by this factor gives:

$$\sec x \frac{dy}{dx} + y \sec x \tan x = 2\sec^2 x$$

$$\Rightarrow \frac{d}{dx}(y \sec x) = 2\sec^2 x$$

$$\Rightarrow y \sec x = \int 2\sec^2 x \, dx = 2\tan x + c$$

$$\Rightarrow y = 2\sin x + c\cos x$$

2 Rewrite the equation as $\frac{dy}{dx} + \frac{x}{1-x^2} y = \frac{5x}{1-x^2}$

The integrating factor is $e^{\int \frac{x}{1-x^2} dx} = e^{-\frac{1}{2}\ln(1-x^2)} = e^{\ln(1-x^2)^{-\frac{1}{2}}} = (1-x^2)^{-\frac{1}{2}}$

Multiplying the equation by this factor gives:

$$\frac{1}{(1-x^2)^{\frac{1}{2}}} \frac{dy}{dx} + \frac{x}{(1-x^2)^{\frac{3}{2}}} y = \frac{5x}{(1-x^2)^{\frac{3}{2}}}$$

$$\Rightarrow \frac{dy}{dx} \frac{y}{(1-x^2)^{\frac{1}{2}}} = \frac{5x}{(1-x^2)^{\frac{3}{2}}}$$

$$\Rightarrow \frac{y}{(1-x^2)^{\frac{1}{2}}} = \int \frac{5x}{(1-x^2)^{\frac{3}{2}}} dx$$

$$\Rightarrow \frac{y}{(1-x^2)^{\frac{1}{2}}} = \frac{5}{(1-x^2)^{\frac{1}{2}}} + c$$

So $y = 5 + c(1-x^2)^{\frac{1}{2}}$

3 Rewrite the equation as $x\frac{dy}{dx} + y = -x$

The left-hand side is the derivative of the product xy, so the equation is equivalent to

$$\frac{d}{dx}(xy) = -x$$

$$\Rightarrow xy = -\int x dx = -\frac{x^2}{2} + c$$

So $y = -\frac{x}{2} + \frac{c}{x}$

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4 The integrating factor is $e^{\int \frac{1}{x} dx} = e^{\ln x} = x$

Multiplying the equation by this factor gives:

$$x\frac{dy}{dx} + y = x\sqrt{x}$$

$$\Rightarrow \frac{d}{dx}(xy) = x\sqrt{x}$$

$$\Rightarrow xy = \int x\sqrt{x}\,dx = \frac{2}{5}x^{\frac{5}{2}} + c$$

So $y = \frac{2}{5}x^{\frac{3}{2}} + \frac{c}{x}$

5 The integrating factor is $e^{\int 2x dx} = e^{x^2}$

Multiplying the equation by this factor gives:

$$e^{x^2}\frac{dy}{dx} + 2xe^{x^2}y = xe^{x^2}$$

$$\Rightarrow \frac{d}{dx}(e^{x^2}y) = xe^{x^2}$$

$$\Rightarrow e^{x^2}y = \int xe^{x^2}\,dx = \frac{1}{2}e^{x^2} + c$$

So $y = \frac{1}{2} + ce^{-x^2}$

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6 Rewrite the equation in the form $\frac{\mathrm{d}y}{\mathrm{d}x}+\frac{2x^2-1}{x(1-x^2)}y=\frac{2x^2}{(1-x^2)}$

The integrating factor is $\mathrm{e}^{\int\frac{2x^2-1}{x(1-x^2)}\mathrm{d}x}$ and this simplifies as follows:

$$\int\frac{2x^2-1}{x(1-x^2)}\mathrm{d}x=\int\frac{3x^2-1-x^2}{x-x^3}\mathrm{d}x=\int\frac{3x^2-1}{x-x^3}-\frac{x}{1-x^2}\mathrm{d}x$$

$$=-\ln(x-x^3)+\frac{1}{2}\ln(1-x^2)=\ln(1-x^2)^{\frac{1}{2}}-\ln(x-x^3)$$

$$=\ln\left(\frac{(1-x^2)^{\frac{1}{2}}}{(x-x^3)}\right)=\ln\left(\frac{1}{x(1-x^2)^{\frac{1}{2}}}\right)$$

So the integrating factor simplifies to $\frac{1}{x(1-x^2)^{\frac{1}{2}}}$

Multiplying the equation by this factor gives:

$$\frac{1}{x(1-x^2)^{\frac{1}{2}}}\frac{\mathrm{d}y}{\mathrm{d}x}+\frac{2x^2-1}{x^2(1-x^2)^{\frac{3}{2}}}y=\frac{2x}{(1-x^2)^{\frac{3}{2}}}$$

$$\Rightarrow\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{y}{x(1-x^2)^{\frac{1}{2}}}\right)=\frac{2x}{(1-x^2)^{\frac{3}{2}}}$$

$$\Rightarrow\frac{y}{x(1-x^2)^{\frac{1}{2}}}=\int\frac{2x}{(1-x^2)^{\frac{3}{2}}}\mathrm{d}x=\frac{2}{(1-x^2)^{\frac{1}{2}}}+c$$

So $y=2x+cx(1-x^2)^{\frac{1}{2}}=2x+cx\sqrt{1-x^2}$

7 a Given that $\frac{\mathrm{d}y}{\mathrm{d}x}-ay=k\mathrm{e}^{\lambda x}$, the integrating factor is $\mathrm{e}^{\int-a\,\mathrm{d}x}=\mathrm{e}^{-ax}$

Multiplying the equation by this factor gives:

$$\mathrm{e}^{-ax}\frac{\mathrm{d}y}{\mathrm{d}x}-a\mathrm{e}^{-ax}y=k\mathrm{e}^{\lambda x}\mathrm{e}^{-ax}$$

$$\Rightarrow\frac{\mathrm{d}}{\mathrm{d}x}(y\mathrm{e}^{-ax})=k\mathrm{e}^{\lambda x}\mathrm{e}^{-ax}$$

$$\Rightarrow y\mathrm{e}^{-ax}=\int k\mathrm{e}^{(\lambda-a)x}\,\mathrm{d}x=\frac{k\mathrm{e}^{(\lambda-a)x}}{\lambda-a}\mathrm{e}^{(\lambda-a)x}+c$$

So $y=\frac{k\mathrm{e}^{(\lambda-a)x}\mathrm{e}^{ax}}{\lambda-a}+c\mathrm{e}^{ax}=\frac{k\mathrm{e}^{\lambda x}}{\lambda-a}+c\mathrm{e}^{ax}$ for $\lambda\neq a$

b When $Q(x)=kx^n\mathrm{e}^{ax}$

$$\frac{\mathrm{d}}{\mathrm{d}x}(y\mathrm{e}^{-ax})=k\mathrm{e}^{\lambda x}\mathrm{e}^{ax}\mathrm{e}^{-ax}=k\mathrm{e}^{\lambda x}$$

$$\Rightarrow y\mathrm{e}^{-ax}=\int kx^n\mathrm{d}x=\frac{kx^{n+1}}{n+1}+c$$

So $y=\frac{kx^{n+1}}{n+1}\mathrm{e}^{ax}+c\mathrm{e}^{ax}$

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8 Rewrite the equation in the form $\frac{\mathrm{d}y}{\mathrm{d}x}+\frac{y}{\tan x}=2\cos x$

The integrating factor is $\mathrm{e}^{\int\frac{1}{\tan x}\mathrm{d}x}=\mathrm{e}^{\int\cot x\,\mathrm{d}x}=\mathrm{e}^{\ln\sin x}=\sin x$

Multiplying both sides by this factor gives:

$\sin x\frac{\mathrm{d}y}{\mathrm{d}x}+y\cos x=2\cos x\sin x=\sin 2x$ using a trigonometric identity

$\Rightarrow\frac{\mathrm{d}}{\mathrm{d}x}(y\sin x)=\sin 2x$

$\Rightarrow y\sin x=\int\sin 2x\mathrm{d}x=-\frac{1}{2}\cos 2x+c=-\frac{1}{2}+\sin^2 x+c$ using $\cos 2x=1-\sin^2 x$

So $y=\sin x+A\operatorname{cosec} x$ defining $A=c-\frac{1}{2}$

Note that without making the simplification using $\cos 2x=1-\sin^2 x$, an alternative but correct answer is obtained.

$y\sin x=\int\sin 2x\mathrm{d}x=-\frac{1}{2}\cos 2x+c$

$\Rightarrow y=-\frac{1}{2}\cos 2x\operatorname{cosec} x+c\operatorname{cosec} x$

9 a The integrating factor is $\mathrm{e}^{\int\tan x\,\mathrm{d}x}=\mathrm{e}^{\ln\sec x}=\frac{1}{\cos x}$

Multiplying both sides by this factor gives:

$\left(\frac{1}{\cos x}\right)\frac{\mathrm{d}y}{\mathrm{d}x}+y\frac{\sin x}{\cos^2 x}=\mathrm{e}^x$

$\Rightarrow\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{y}{\cos x}\right)=\int\mathrm{e}^x\,\mathrm{d}x$

$\Rightarrow\frac{y}{\cos x}=\mathrm{e}^x+c$

So $y=\mathrm{e}^x\cos x+c\cos x$

b When $x=\pi, y=1$ so $-\mathrm{e}^\pi-c=1\Rightarrow c=-\mathrm{e}^\pi-1$

So $y=\mathrm{e}^x\cos x-(1+\mathrm{e}^\pi)\cos x$

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10 The integrating factor is $e^{\int -3dx} = e^{-3x}$

Multiplying the equation by this factor gives:

$$e^{-3x}\frac{dy}{dx} - 3e^{-3x}y = e^{-3x}\sin x$$

$$\Rightarrow \frac{d}{dx}(e^{-3x}y) = e^{-3x}\sin x$$

$$\Rightarrow e^{-3x}y = \int e^{-3x}\sin x dx \qquad (\mathbf{1})$$

The expression $e^{-3x}\sin x$ can be integrated by using integration by parts twice

$$\int e^{-3x}\sin x\,dx = -\frac{1}{3}e^{-3x}\sin x + \int \frac{1}{3}e^{-3x}\cos x\,dx \qquad \text{using integration by parts with } u = \sin x, v = -\frac{1}{3}e^{x}$$

$$= -\frac{1}{3}e^{-3x}\sin x - \frac{1}{9}e^{-3x}\cos x - \int \frac{1}{9}e^{-3x}\sin x\,dx \qquad \text{this time with } u = \cos x, v = -\frac{1}{9}e^{x}$$

Simplifying this equation gives:

$$\frac{10}{9}\int e^{-3x}\sin x\,dx = -\frac{1}{3}e^{-3x}\sin x - \frac{1}{9}e^{-3x}\cos x$$

$$\Rightarrow \int e^{-3x}\sin x\,dx = -\frac{3}{10}e^{-3x}\sin x - \frac{1}{10}e^{-3x}\cos x$$

Using this expression in equation (**1**) and adding in a constant gives:

$$e^{-3x}y = \int e^{-3x}\sin x\,dx = -\frac{3}{10}e^{-3x}\sin x - \frac{1}{10}e^{-3x}\cos x + c$$

So $y = -\frac{3}{10}\sin x - \frac{1}{10}\cos x + ce^{3x}$

Applying the condition that when $x = 0, y = 0$ so $-\frac{1}{10} + c = 0 \Rightarrow c = \frac{1}{10}$

So the solution is $y = -\frac{3}{10}\sin x - \frac{1}{10}\cos x + \frac{1}{10}e^{3x}$

11 a Separating the variables and integrating:

$$\frac{dy}{dx} = y\sinh x$$

$$\Rightarrow \int \frac{1}{y}dy = \int \sinh x\,dx$$

$\Rightarrow \ln y = \cosh x + c$ where c is a constant

$\Rightarrow y = e^{\cosh x + c} = e^{c} \times e^{\cosh x}$

$\Rightarrow y = Ae^{\cosh x}$ where A is a constant ($A = e^{c}$)

b When $x = 0, y = 1$ so $Ae = 1 \Rightarrow A = e^{-1}$

So the solution is $y = e^{-1}e^{\cosh x} = e^{\cosh x - 1}$

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12 Separating the variables and integrating:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = x(4 - y^2)$$

$$\Rightarrow \int \frac{1}{4 - y^2}\,\mathrm{d}y = \int x\,\mathrm{d}x$$

$$\Rightarrow \int \frac{1}{(2 - y)(2 + y)}\,\mathrm{d}y = \int x\,\mathrm{d}x$$ rearranging the left-hand side so that it is easier to integrate

$$\Rightarrow \frac{1}{4}\int \frac{1}{(2 - y)} + \frac{1}{(2 + y)}\,\mathrm{d}y = \int x\,\mathrm{d}x$$

$$\Rightarrow -\frac{1}{4}\ln(2 - y) + \frac{1}{4}\ln(2 + y) = \frac{1}{2}x^2 + c$$

$$\Rightarrow \ln\left(\frac{2 + y}{2 - y}\right)^{\frac{1}{4}} = \frac{1}{2}x^2 + c$$ simplifying using the rules of logarithms

$$\Rightarrow \left(\frac{2 + y}{2 - y}\right)^{\frac{1}{4}} = \mathrm{e}^{\frac{1}{2}x^2 + c} = A\mathrm{e}^{\frac{x^2}{2}}$$

$$\Rightarrow \frac{2 + y}{2 - y} = C\mathrm{e}^{2x^2}$$ taking the power of 4 of each side

$$\Rightarrow 2 + y = 2C\mathrm{e}^{2x^2} - yC\mathrm{e}^{2x^2}$$ rearranging to find an expression in y

$$\Rightarrow y = \frac{2C\mathrm{e}^{2x^2} - 2}{C\mathrm{e}^{2x^2} + 1}$$

When $x = 0, y = 1$ so $\frac{2C - 2}{C + 1} = 1 \Rightarrow 2C - 2 = C + 1 \Rightarrow C = 3$

So the solution is $y = \frac{6\mathrm{e}^{2x^2} - 2}{3\mathrm{e}^{2x^2} + 1} = \frac{2(3\mathrm{e}^{2x^2} - 1)}{3\mathrm{e}^{2x^2} + 1}$

13 The auxiliary equation is

$$m^2 + m + 1 = 0$$

$$m = \frac{-1 \pm \sqrt{1 - 4}}{2} = -\frac{1}{2} \pm \frac{\sqrt{3}}{2}\mathrm{i}$$

So the general solution is $y = \mathrm{e}^{-\frac{1}{2}x}\left(A\cos\frac{\sqrt{3}}{2}x + B\sin\frac{\sqrt{3}}{2}x\right)$

If the auxiliary equation has complex roots, the solution is of the form
$y = \mathrm{e}^{px}(A\cos qx + B\sin qx)$

14 The auxiliary equation is

$$m^2 - 12m + 36 = 0$$

$$(m - 6)(m - 6) = 0$$

$$m = 6$$

So the general solution is $y = (A + Bx)\mathrm{e}^{6x}$

If the auxiliary equation has a repeated solution, the solution is of the form
$y = (A + Bx)\mathrm{e}^{\alpha x}$

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15 The auxiliary equation is

$m^2 - 4m = 0$

$m(m-4) = 0$

$m = 0$ or 4

So the general solution is $y = Ae^{0x} + Be^{4x} = A + Be^{4x}$

16 The auxiliary equation is $m^2 + k^2 = 0 \Rightarrow m = \pm ik$

The general solution is $y = A\cos kx + B\sin kx$

When $x = 0, y = 1$ so $A = 1$

$\frac{dy}{dx} = -kA\sin kx + kB\cos kx$

When $x = 0, \frac{dy}{dx} = 1$ so $kB = 1 \Rightarrow B = \frac{1}{k}$

Substituting values for A and B into the general solution gives the particular solution

$y = \cos kx + \frac{1}{k}\sin kx$

If the auxiliary equation has imaginary solutions, and the general solution has the form $y = A\cos \omega x + B\sin \omega x$.

17 The auxiliary equation is

$m^2 - 2m + 10 = 0$

$m = \frac{2 \pm \sqrt{4-40}}{2} = \frac{2 \pm \sqrt{-36}}{2} = 1 \pm 3i$

The general solution is $y = e^x(A\cos 3x + B\sin 3x)$

When $x = 0, y = 0$ so $A = 0$

$\frac{dy}{dx} = e^x B\sin 3x + 3e^x B\cos 3x$

When $x = 0, \frac{dy}{dx} = 3$ so $3B = 3 \Rightarrow B = 1$

Substituting values for A and B into the general solution gives the particular solution

$y = e^x \sin 3x$

18 a $y = ke^{2x}$ so $\frac{dy}{dx} = ke^{2x} + 2kxe^{2x}$ and $\frac{d^2y}{dx^2} = 4ke^{2x} + 4kxe^{2x}$

Substituting into $\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 13y = e^{2x}$ gives:

$4ke^{2x} + 4kxe^{2x} - 4(ke^{2x} + 2kxe^{2x}) + 13kxe^{2x} = e^{2x}$

$\Rightarrow 9ke^{2x} = e^{2x} \Rightarrow k = \frac{1}{9}$

So a particular integral is $\frac{1}{9}e^{2x}$

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18 b Solving the corresponding homogeneous equation $\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 13y = 0$

The auxiliary equation is

$m^2 - 4m + 13 = 0$

$m = \frac{4 \pm \sqrt{16 - 52}}{2} = \frac{4 \pm \sqrt{-36}}{2} = 2 \pm 3i$

So the complementary function is $y = e^{2x}(A\cos 3x + B\sin 3x)$

Using the particular integral from part **a**, the general solution is $y = e^{2x}(A\cos 3x + B\sin 3x) + \frac{1}{9}e^{2x}$

19 Solving the corresponding homogeneous equation $\frac{d^2y}{dx^2} - y = 0$

The auxiliary equation is

$m^2 - 1 = 0$

$(m+1)(m-1) = 0$

$m = 1$ or -1

So the complementary function is $y = Ae^x + Be^{-x}$

As the complementary function has an e^x term, try a particular integral in the form $\lambda x e^x$

So $\frac{dy}{dx} = \lambda e^x + \lambda x e^x$ and $\frac{d^2y}{dx^2} = 2\lambda e^x + \lambda x e^x$

Substituting into $\frac{d^2y}{dx^2} - y = 4e^x$ gives:

$2\lambda e^x + \lambda x e^x - \lambda x e^x = 4e^x$

$\Rightarrow 2\lambda e^x = 4e^x \Rightarrow \lambda = 2$

So a particular integral is $2xe^x$

The general solution is $y = Ae^x + Be^{-x} + 2xe^x$

20 a Solving the corresponding homogeneous equation $\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 4y = 0$

The auxiliary equation is

$m^2 - 4m + 4 = 0$

$(m-2)(m-2) = 0$

$m = 2$

So the complementary function is $y = (A + Bx)e^{2x}$

b As the complementary function has e^{2x} and xe^{2x} terms, these cannot be terms in the particular integral.

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20 c $y = kx^2e^{2x}$, so $\frac{dy}{dx} = 2kx^2e^{2x} + 2kxe^{2x}$

$\frac{d^2y}{dx^2} = 4kx^2e^{2x} + 4kxe^{2x} + 4kxe^{2x} + 2ke^{2x} = 4kx^2e^{2x} + 8kxe^{2x} + 2ke^{2x}$

Substituting into $\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 4y = 4e^{2x}$ gives:

$(4kx^2 + 8kx + 2k - 8kx^2 - 8kx + 4kx^2)e^{2x} = 4e^{2x}$

$\Rightarrow 2ke^{2x} = 4e^{2x} \Rightarrow k = 2$

So a particular interval is $2x^2e^{2x}$

Using the complementary function from part **a**, the general solution is $y = (A + Bx + 2x^2)e^{2x}$

21 Solving the corresponding homogeneous equation $\frac{d^2y}{dt^2} + 4y = 0$

The auxiliary equation is $m^2 + 4 = 0 \Rightarrow m = \pm 2i$

So the complementary function is $y = A\cos 2t + B\sin 2t$

The form of the particular integral is $y = \lambda\cos 3t + \mu\sin 3t$

$\frac{dy}{dt} = -3\lambda\sin 3t + 3\mu\cos 3t$ and $\frac{d^2y}{dt^2} = -9\lambda\cos 3t - 9\mu\sin 3t$

Substituting into $\frac{d^2y}{dt^2} + 4y = 5\cos 3t$ gives:

$-9\lambda\cos 3t - 9\mu\sin 3t + 4\lambda\cos 3t + 4\mu\sin t = 5\cos 3t$

$\Rightarrow -5\lambda\cos 3t - 5\mu\sin 3t = 5\cos 3t$

$\Rightarrow \lambda = -1$ and $\mu = 0$

So a particular integral is $-\cos 3t$

The general solution is $y = A\cos 2t + B\sin 2t - \cos 3t$

When $t = 0$, $y = 1$ so $A - 1 = 1 \Rightarrow A = 2$

$\frac{dy}{dt} = -2A\sin 2t + 2B\cos 2t + 3\sin 3t$

When $t = 0$, $\frac{dy}{dt} = 2$ so $2B = 2 \Rightarrow B = 1$

Substituting values for A and B into the general solution gives the particular solution

$y = 2\cos 2t + \sin 2t - \cos 3t$

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22 a Let $y = \lambda + \mu x + kx\mathrm{e}^{2x}$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \mu + 2kx\mathrm{e}^{2x} + k\mathrm{e}^{2x}$$

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 4kx\mathrm{e}^{2x} + 2k\mathrm{e}^{2x} + 2k\mathrm{e}^{2x} = 4kx\mathrm{e}^{2x} + 4k\mathrm{e}^{2x}$$

Substituting into $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - 3\frac{\mathrm{d}y}{\mathrm{d}x} + 2y = 4x + \mathrm{e}^{2x}$ gives:

$$4kx\mathrm{e}^{2x} + 4k\mathrm{e}^{2x} - 3\mu - 6kx\mathrm{e}^{2x} - 3k\mathrm{e}^{2x} + 2\lambda + 2\mu x + 2kx\mathrm{e}^{2x} = 4x + \mathrm{e}^{2x}$$

$$\Rightarrow k\mathrm{e}^{2x} + 2\mu x + 2\lambda - 3\mu = 4x + \mathrm{e}^{2x}$$

Equating the coefficients of e^{2x}: $k = 1$
Equating the coefficients of x: $2\mu = 4 \Rightarrow \mu = 2$
Equating constants: $2\lambda - 3\mu = 0 \Rightarrow \lambda = 3$

So a particular integral is $3 + 2x + x\mathrm{e}^{2x}$

b Solving the corresponding homogeneous equation $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - 3\frac{\mathrm{d}y}{\mathrm{d}x} + 2y = 0$

The auxiliary equation is

$$m^2 - 3m + 2 = 0$$

$$(m-2)(m-1) = 0$$

$$m = 1 \text{ or } 2$$

So the complementary function is $y = A\mathrm{e}^{x} + B\mathrm{e}^{2x}$
Using the particular integral from part **a**, the general solution is

$$y = A\mathrm{e}^{x} + B\mathrm{e}^{2x} + 3 + 2x + x\mathrm{e}^{2x} = A\mathrm{e}^{x} + (B + x)\mathrm{e}^{2x} + 2x + 3$$

23 a Solving the corresponding homogeneous equation $16\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 8\frac{\mathrm{d}y}{\mathrm{d}x} + 5y = 0$

The auxiliary equation is

$$16m^2 + 8m + 5 = 0$$

$$m = \frac{-8 \pm \sqrt{64 - 320}}{32} = -\frac{1}{4} \pm \frac{\sqrt{-256}}{32} = -\frac{1}{4} \pm \frac{1}{2}\mathrm{i}$$

So the complementary function is $y = \mathrm{e}^{-\frac{1}{4}x}\left(A\cos\frac{1}{2}x + B\sin\frac{1}{2}x\right)$

The particular integral has the form $y = \lambda x + \mu$, so $\frac{\mathrm{d}y}{\mathrm{d}x} = \lambda$ and $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 0$

Substituting into $16\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 8\frac{\mathrm{d}y}{\mathrm{d}x} + 5y = 5x + 23$ gives:

$$8\lambda + 5\lambda x + 5\mu = 5x + 23$$

Equating the coefficients of x: $5\lambda = 5 \Rightarrow \lambda = 1$
Equating constant terms: $8\lambda + 5\mu = 23 \Rightarrow \mu = 3$

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23 a So a particular integral is $x+3$

The general solution is $y = \mathrm{e}^{-\frac{1}{4}x}\left(A\cos\frac{1}{2}x + B\sin\frac{1}{2}x\right) + x + 3$

When $x = 0$, $y = 3$ so $A + 3 = 3 \Rightarrow A = 0$

$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{1}{4}\mathrm{e}^{-\frac{1}{4}x}\left(A\cos\frac{1}{2}x + B\sin\frac{1}{2}x\right) - \frac{1}{2}\mathrm{e}^{-\frac{1}{4}x}\left(A\sin\frac{1}{2}x + B\cos\frac{1}{2}x\right) + 1$

When x = 0, $\frac{\mathrm{d}y}{\mathrm{d}x} = 3$ so $-\frac{1}{4}A + \frac{1}{2}B + 1 = 3$

As $A = 0 \Rightarrow B = 4$

Substituting values for A and B into the general solution gives the particular solution

$y = 4\mathrm{e}^{-\frac{1}{4}x}B\sin\frac{1}{2}x + x + 3$

b As $x \to \infty$, $\mathrm{e}^{-\frac{1}{4}x} \to 0$ so $y \to x + 3$

Therefore $y \approx x + 3$ for large values of x.

24 Solving the corresponding homogeneous equation $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - \frac{\mathrm{d}y}{\mathrm{d}x} - 6y = 0$

The auxiliary equation is

$m^2 - m - 6 = 0$

$(m-3)(m+2) = 0$

$m = 3$ or -2

So the complementary function is $y = A\mathrm{e}^{3x} + B\mathrm{e}^{-2x}$

The particular integral is of the form $y = \lambda\sin 3x + \mu\cos 3x$

$\frac{\mathrm{d}y}{\mathrm{d}x} = 3\lambda\cos 3x - 3\mu\sin 3x$

$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = -9\lambda\sin 3x - 9\mu\cos 3x$

Substituting into $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - \frac{\mathrm{d}y}{\mathrm{d}x} - 6y = 3\sin 3x - 2\cos 3x$ gives:

$-9\lambda\sin 3x - 9\mu\cos 3x - 3\lambda\cos 3x + 3\mu\sin 3x - 6\lambda\sin 3x - 6\mu\cos 3x = 3\sin 3x - 2\cos 3x$

$\Rightarrow (3\mu - 15\lambda)\sin 3x - (15\mu + 3\lambda)\cos 3x = 3\sin 3x - 2\cos 3x$

Equating the coefficients of sin 3x: $3\mu - 15\lambda = 3$ **(1)**

Equating the coefficients of sin 3x: $15\mu + 3\lambda = 2$ **(2)**

Adding 5 × equation **(2)** to equation **(1)** gives: $78\mu = 13 \Rightarrow \mu = \frac{1}{6}$

Substituting in equation **(1)** gives: $\frac{1}{2} - 15\lambda = 3 \Rightarrow 15\lambda = -\frac{5}{2} = -\frac{1}{6}$

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24 So a particular integral is $\frac{1}{6}(\cos 3x - \sin 3x)$

The general solution is $y = Ae^{3x} + Be^{-2x} + \frac{1}{6}(\cos 3x - \sin 3x)$

When $x = 0, y = 1$ so $A + B + \frac{1}{6} = 1 \Rightarrow A + B = \frac{5}{6}$

However as y remains finite for large values of x, $A = 0 \Rightarrow B = \frac{5}{6}$

Substituting values for A and B into the general solution gives the particular solution

$y = \frac{5}{6}e^{-2x} + \frac{1}{6}(\cos 3x - \sin 3x)$

25 a Solving the corresponding homogeneous equation $\frac{d^2x}{dt^2} + 8\frac{dx}{dt} + 16x = 0$

The auxiliary equation is

$m^2 + 8m + 16 = 0$

$(m+4)(m+4) = 0$

$m = -4$

So the complementary function is $y = (A + Bt)e^{-4t}$

The particular integral is of the form $x = \lambda \sin 4t + \mu \cos 4t$

$\frac{dx}{dt} = 4\lambda \cos 4t - 4\mu \sin 4t$

$\frac{d^2x}{dt^2} = -16\lambda \sin 4t - 16\mu \cos 4t$

Substituting into $\frac{d^2x}{dt^2} + 8\frac{dx}{dt} + 16x = \cos 4t$ gives:

$-16\lambda \sin 4t - 16\mu \cos 4t + 32\lambda \cos 4t - 32\mu \sin 4t + 16\lambda \sin 4t + 16\mu \cos 4t = \cos 4t$

$\Rightarrow -32\mu \sin 4t + 32\lambda \cos 4t = \cos 4t$

$\Rightarrow \lambda = \frac{1}{32}$ and $\mu = 0$

So a particular integral is $\frac{1}{32}\sin 4t$

The general solution is $x = (A + Bt)e^{-4t} + \frac{1}{32}\sin 4t$

b When $t = 0, x = \frac{1}{2}$ so $A = \frac{1}{2}$

$\frac{dx}{dt} = -4(A + Bt)e^{-4t} + Be^{-4t} + \frac{1}{8}\cos 4t$

When $t = 0$, $\frac{dx}{dt} = 0$ so $-4A + B + \frac{1}{8} = 0$

$\Rightarrow B = 4A - \frac{1}{8} = 2 - \frac{1}{8} = \frac{15}{8}$

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25 b Substituting values for A and B into the general solution from part **a** gives the particular solution

$$x = \frac{1}{2}e^{-4t} + \frac{15}{8}te^{-4t} + \frac{1}{32}\sin 4t$$

c As $t \to \infty$, $e^{-4t} \to 0$ and $te^{-4t} \to 0$, as e^{-4t} decreases faster than the linear factor.

So for large values of t, the function behaves like $x = \frac{1}{32}\sin 4t$.

It will oscillate like a sine wave with amplitude $\frac{1}{32}$ and period $\frac{\pi}{2}$.

Challenge

1 a Given that $z = y^2$, then $y = z^{\frac{1}{2}}$ and $\frac{dy}{dx} = \frac{1}{2}z^{-\frac{1}{2}}\frac{dz}{dx}$

The equation $2(1+x^2)\frac{dy}{dx} + 2xy = \frac{1}{y}$ becomes

$$2(1+x^2)\frac{1}{2}z^{-\frac{1}{2}}\frac{dz}{dx} + 2xz^{\frac{1}{2}} = z^{-\frac{1}{2}}$$

Multiply the equation by $\frac{z^{\frac{1}{2}}}{1+x^2}$ gives

$$\frac{dz}{dx} + \frac{2x}{1+x^2}z = \frac{1}{1+x^2}$$

The integrating factor is $e^{\int \frac{2x}{1+x^2}dx} = e^{\ln(1+x^2)} = 1+x^2$

$$(1+x^2)\frac{dz}{dx} + 2xz = 1$$

$$\Rightarrow \frac{d}{dx}\left((1+x^2)z\right) = 1$$

$$\Rightarrow (1+x^2)z = \int 1\,dx = x + c$$

$$\Rightarrow z = \frac{x+c}{1+x^2}$$

As $y = z^{\frac{1}{2}}$, $y = \sqrt{\frac{x+c}{1+x^2}}$

b When $x = 0$, $y = 2$ so $\sqrt{c} = 2 \Rightarrow c = 4$

So the particular solution is $y = \sqrt{\frac{x+4}{1+x^2}}$

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Challenge

2 a Let $x = \mathrm{e}^u$, then $\dfrac{\mathrm{d}x}{\mathrm{d}u} = \mathrm{e}^u$

$$\frac{\mathrm{d}y}{\mathrm{d}u} = \frac{\mathrm{d}y}{\mathrm{d}x} \times \frac{\mathrm{d}x}{\mathrm{d}u} = \mathrm{e}^u \frac{\mathrm{d}y}{\mathrm{d}x} = x\frac{\mathrm{d}y}{\mathrm{d}x}$$

$$\frac{\mathrm{d}^2y}{\mathrm{d}u^2} = \frac{\mathrm{d}x}{\mathrm{d}u} \times \frac{\mathrm{d}y}{\mathrm{d}x} + x\frac{\mathrm{d}^2y}{\mathrm{d}x^2} \times \frac{\mathrm{d}x}{\mathrm{d}u}$$

$$= x\frac{\mathrm{d}y}{\mathrm{d}x} + x^2\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$$

Find $\dfrac{\mathrm{d}y}{\mathrm{d}u}$ in terms of x and $\dfrac{\mathrm{d}y}{\mathrm{d}x}$, and show that $\dfrac{\mathrm{d}^2y}{\mathrm{d}u^2} = x\dfrac{\mathrm{d}y}{\mathrm{d}x} + x^2\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}$ then substitute into the differential equation.

So $x^2\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2} + 4x\dfrac{\mathrm{d}y}{\mathrm{d}x} + 2y = \ln x$ transforms to $\dfrac{\mathrm{d}^2y}{\mathrm{d}u^2} + 3\dfrac{\mathrm{d}y}{\mathrm{d}u} + 2y = \ln x = u$

The auxiliary equation is

$$m^2 + 3m + 2 = 0$$

$$(m+2)(m+1) = 0$$

$$m = -1 \text{ or } -2$$

So the complementary function is $y = A\mathrm{e}^{-u} + B\mathrm{e}^{-2u}$

Let the particular integral be of the form $y = \lambda u + \mu$, so $\dfrac{\mathrm{d}y}{\mathrm{d}u} = \lambda$ and $\dfrac{\mathrm{d}^2y}{\mathrm{d}u^2} = 0$

Substituting into $\dfrac{\mathrm{d}^2y}{\mathrm{d}u^2} + 3\dfrac{\mathrm{d}y}{\mathrm{d}u} + 2y = u$ gives:

$$3\lambda + 2\lambda u + 2\mu = u$$

Equating the coefficients of u: $2\lambda = 1 \Rightarrow \lambda = \dfrac{1}{2}$

Equating constants: $3\lambda + 2\mu = 0 \Rightarrow 2\mu = -\dfrac{3}{2} \Rightarrow \mu = -\dfrac{3}{4}$

So a particular integral is $\dfrac{1}{2}u - \dfrac{3}{4}$

The general solution is $y = A\mathrm{e}^{-u} + B\mathrm{e}^{-2u} + \dfrac{1}{2}u - \dfrac{3}{4}$

But $x = \mathrm{e}^u \Rightarrow u = \ln x$ and $\mathrm{e}^{-u} = x^{-1} = \dfrac{1}{x}$, $\mathrm{e}^{-2u} = x^{-2} = \dfrac{1}{x^2}$

So the general solution of the original equation is $y = \dfrac{A}{x} + \dfrac{B}{x^2} + \dfrac{1}{2}\ln x - \dfrac{3}{4}$

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Challenge

2 b When $x=1, y=1$ so $A+B-\frac{3}{4}=1 \Rightarrow A+B=\frac{7}{4}$ **(1)**

$\frac{dy}{dx}=-\frac{A}{x^2}-\frac{2B}{x^3}+\frac{1}{2x}$

When $x=1, \frac{dy}{dx}=1$ so $-A-2B+\frac{1}{2}=1 \Rightarrow A+2B=-\frac{1}{2}$ **(2)**

Subtracting equation (**1**) from equation (**2**) gives $B=-\frac{9}{4}$ and so from equation (**1**) $A=\frac{9}{4}+\frac{7}{4}=4$

Substituting values for A and B into the general solution from part **a** gives the particular solution

$y=\frac{4}{x}-\frac{9}{4x^2}+\frac{1}{2}\ln x-\frac{3}{4}$

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Modelling with differential equations 8A

1 Let s be the displacement along the x-axis from O.

So $\frac{\mathrm{d}s}{\mathrm{d}t} = v$, and therefore $s = \int v\,\mathrm{d}t = \int t\sin t\,\mathrm{d}t$

Using integration by parts with $u = t$ and $v = -\cos t$

$s = -t\cos t + \int \cos t\,\mathrm{d}t = -t\cos t + \sin t + c$

When $t = 0, s = 0$, so $0 + 0 + c = 0 \Rightarrow c = 0$

So the equation for displacement is $s = -t\cos t + \sin t$

When $t = \frac{\pi}{2}$, $s = -\frac{\pi}{2}\cos\frac{\pi}{2} + \sin\frac{\pi}{2} = 1$

Hence P is 1 metre from O, as required.

2 $a = \frac{\mathrm{d}v}{\mathrm{d}t} = \frac{6t}{(2+t^2)^2}$ so $v = \int a\,\mathrm{d}t = \int \frac{6t}{(2+t^2)^2}\mathrm{d}t$

Using integration by substitution, let $u = 2 + t^2$ and so $\frac{\mathrm{d}u}{\mathrm{d}t} = 2t$

$v = \int \frac{6t}{(2+t^2)^2}\mathrm{d}t = \int \frac{3}{(2+t^2)^2} \times 2t\,\mathrm{d}t$

$= \int \frac{3}{u^2}\mathrm{d}u = \int 3u^{-2}\,\mathrm{d}u = \frac{3u^{-1}}{-1} + c = c - \frac{3}{u}$

$= c - \frac{3}{2+t^2}$

When $t = 0, v = 0$ so $c - \frac{3}{2} = 0 \Rightarrow c = \frac{3}{2}$

So the solution is $v = \frac{3}{2} - \frac{3}{2+t^2}$

3 a $v = \int a\,\mathrm{d}t = \int -4\mathrm{e}^{0.2t}\,\mathrm{d}t = -20\mathrm{e}^{0.2t} + c$

When $t = 0, v = 20$ so $-20 + c = 20 \Rightarrow c = 40$

The solution is $v = 40 - 20\mathrm{e}^{0.2t}$

b $x = \int v\,\mathrm{d}t = \int (40 - 20\mathrm{e}^{0.2t})\,\mathrm{d}t = 40t - 100\mathrm{e}^{0.2t} + c$

When $t = 0, x = 0$ so $0 - 100 + c = 0 \Rightarrow c = 100$

$x = 40t - 100\mathrm{e}^{0.2t} + 100$

The maximum value of x occurs when

$\frac{\mathrm{d}x}{\mathrm{d}t} = v = 40 - 20\mathrm{e}^{0.2t} = 0 \Rightarrow \mathrm{e}^{0.2t} = 2$

Taking logarithms of both sides gives $0.2t = \ln 2 \Rightarrow t = 5\ln 2$

The maximum value of x is given by

$x = 40 \times 5\ln 2 - 100e^{0.2\times 5\ln 2} + 100 = 40 \times 5\ln 2 - 100 \times 2 + 100 = 200\ln 2 - 100$

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4 Velocity is the derivative of the displacement with respect to time, hence $\frac{dx}{dt} = e^{-\frac{x}{2}}$

So $\frac{dt}{dx} = e^{\frac{x}{2}}$

$t = \int e^{\frac{x}{2}} dx = 2e^{\frac{x}{2}} + c$

When $t = 0$ the particle is at the origin, so $x = 0$: $2e^{\frac{0}{2}} + c = 2 + c = 0 \Rightarrow c = -2$

So an equation for t in terms of x is $t = 2e^{\frac{x}{2}} - 2$
Solving for x:

$e^{\frac{x}{2}} = \frac{t+2}{2}$

$\frac{x}{2} = \ln\left(\frac{t+2}{2}\right)$

$x = 2\ln\left(\frac{t}{2} + 1\right)$

5 a To find an expression for v integrate $\frac{dv}{dt} - 2vt = t$. The integrating factor is $e^{-\int 2t dt} = e^{-t^2}$

Multiplying by this factor, the original equations becomes

$e^{-t^2}\frac{dv}{dt} - e^{-t^2}2vt = e^{-t^2}t$

$\Rightarrow \frac{d}{dt}(e^{-t^2}v) = e^{-t^2}t$

$\Rightarrow e^{-t^2}v = \int e^{-t^2}t dt$

Using integration by substitution, let $u = -t^2$ and so $\frac{du}{dt} = -2t$

$e^{-t^2}v = \int e^{-t^2}t dt = -\frac{1}{2}\int e^u du = -\frac{1}{2}e^u + c = -\frac{1}{2}e^{-t^2} + c$

At $t = 0$, $v = 1\,\text{m s}^{-1}$, so $e^0 \times 1 = -\frac{1}{2}e^0 + c \Rightarrow c = \frac{3}{2}$

This gives

$e^{-t^2}v = -\frac{1}{2}e^{-t^2} + \frac{3}{2}$

$\Rightarrow v = \frac{3}{2}e^{t^2} - \frac{1}{2} = \frac{1}{2}(3e^{t^2} - 1)$ as required

b After 2 seconds, the velocity of the car will be equal to

$v = \frac{1}{2}(3e^{2^2} - 1) = \frac{1}{2}(3e^4 - 1) = 81.4\,\text{m s}^{-1}$ (3 s.f.)

c For $t = 4$, $\frac{1}{2}(3e^{4^2} - 1) = \frac{1}{2}(3e^{16} - 1) = 1.34 \times 10^7\,\text{m s}^{-1}$

No car can travel at 13 million m s^{-1} (nearly 30 million mph), so the model is not appropriate.

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6 a $(t+4)\dfrac{\mathrm{d}v}{\mathrm{d}t}+4v=9.8(t+4)$

As t is positive, rewrite as $\dfrac{\mathrm{d}v}{\mathrm{d}t}+4v(t+4)^{-1}=9.8$

The integrating factor is $\mathrm{e}^{\int\frac{4}{t+4}\mathrm{d}t}=\mathrm{e}^{4\ln(t+4)}=(t+4)^4$

So the equation becomes

$(t+4)^4\dfrac{\mathrm{d}v}{\mathrm{d}t}+4v(t+4)^3=9.8(t+4)^4$

$\Rightarrow\dfrac{\mathrm{d}}{\mathrm{d}t}\left(v(t+4)^4\right)=9.8(t+4)^4$

$\Rightarrow v(t+4)^4=9.8\int(t+4)^4\,\mathrm{d}t$

$\Rightarrow v(t+4)^4=\dfrac{9.8}{5}(t+4)^5+C$

$\Rightarrow v=\dfrac{49(t+4)^5}{25(t+4)^4}+\dfrac{C}{(t+4)^4}=\dfrac{49(t+4)^5-c}{25(t+4)^4}$ $\left(\text{rearranging by } C=\dfrac{-c}{25}\right)$

At time $t=0$, the drop is at rest so $v=0$, hence

$\dfrac{49\times4^5-c}{25\times4^4}=0\Rightarrow c=49\times4^5=50176$

b So after 5 seconds, $t=5$ and

$v=\dfrac{49\times9^5-50176}{25\times9^4}=17.3\,\mathrm{m\,s^{-1}}$ (3 s.f.)

c The function is increasing in t. This means that as $t\to\infty, v\to\infty$, i.e. the velocity will keep increasing without any limit. This is an unrealistic scenario.

7 a The gas mixture added to the tank in an hour contains $0.05\times50=2.5\mathrm{cm}^3$ of oxygen.

The total volume of the gas mixture in the tank is

$V=500+50t-20t=500+30t$

There is $x\ \mathrm{cm}^3$ of oxygen in the tank at time t and so the proportion of oxygen in the tank is

$\dfrac{x}{500+30t}$

Since the oxygen is distributed uniformly throughout the tank, the proportion of oxygen in the $20\ \mathrm{cm}^3$ gas leaking every hour is

$20\times\dfrac{x}{500+30t}$

Thus the rate of change of the amount of oxygen in the tank can be expressed as

$\dfrac{\mathrm{d}x}{\mathrm{d}t}=2.5-20\times\dfrac{x}{500+30t}=2.5-\dfrac{2x}{50+3t}$ as required.

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7 b To find x solve $\dfrac{\mathrm{d}x}{\mathrm{d}t}+\dfrac{2x}{50+3t}=2.5$

The integrating factor is $\mathrm{e}^{\int\frac{2}{50+3t}\mathrm{d}t}=\mathrm{e}^{\frac{2}{3}\ln(3t+50)}=(3t+50)^{\frac{2}{3}}$

Multiplying by this factor gives:

$$(3t+50)(3t+50)^{\frac{2}{3}}\frac{\mathrm{d}x}{\mathrm{d}t}+\frac{2x}{(3t+50)^{\frac{1}{3}}}=2.5(3t+50)^{\frac{2}{3}}$$

$$\Rightarrow\frac{\mathrm{d}}{\mathrm{d}t}\left(x(3t+50)^{\frac{2}{3}}\right)=2.5(3t+50)^{\frac{2}{3}}$$

$$\Rightarrow x(3t+50)^{\frac{2}{3}}=\int 2.5(3t+50)^{\frac{2}{3}}\mathrm{d}t=\frac{1}{2}(3t+50)^{\frac{5}{3}}+c$$

$$\Rightarrow x=\frac{1}{2}(3t+50)+\frac{c}{(3t+50)^{\frac{2}{3}}}$$

We know that at time t = 0 there is no oxygen in the tank, so

$$\frac{50}{2}+\frac{c}{50^{\frac{2}{3}}}=0\Rightarrow c=-25\times50^{\frac{2}{3}}=339.3\text{ (4 s.f.)}$$

Thus, after 4 hours

$$x=\frac{1}{2}(3\times4+50)-\frac{339.3}{(3\times4+50)^{\frac{2}{3}}}=31-\frac{339.3}{62^{\frac{2}{3}}}=9.34\,\text{cm}^3\text{ (3 s.f.)}$$

c The model as it is currently formulated assumes that the oxygen mixes in to the tank immediately and uniformly. In reality, this is unlikely and the model should account for that.

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Modelling with differential equations 8B

1 a The particle moves with simple harmonic motion.

b Rewriting $\frac{\mathrm{d}^2x}{\mathrm{d}^2t} = -9x$ as $\frac{\mathrm{d}^2x}{\mathrm{d}^2t} + 9x = 0$

The auxiliary equation is

$m^2 + 9 = 0$

$\Rightarrow m = \pm 3\mathrm{i}$

So $x = A\cos 3t + B\sin 3t$

Using the initial conditions, when $t = 0$, $x = 2$ so $A\cos 0 + B\sin 0 = 2 \Rightarrow A = 2$

When $t = 0$, $v = \frac{\mathrm{d}x}{\mathrm{d}t} = 3$

$\frac{\mathrm{d}x}{\mathrm{d}t} = -3A\sin 3t + 3B\cos 3t$

So $-3A\sin 0 + 3B\cos 0 = 3 \Rightarrow B = 1$

Substituting for A and B gives the solution:

$x = 2\cos 3t + \sin 3t$

c Writing the solution to part **b** in the form $R\sin(\theta + \alpha)$

$R\sin(3t + \alpha) = R\sin\alpha\cos 3t + R\cos\alpha\sin 3t$

So $R\sin\alpha = 2$ and $R\cos\alpha = 1$

$R^2\sin^2\alpha + R^2\cos^2\alpha = R^2(\sin^2\alpha + \cos^2\alpha) = R^2 = 5 \Rightarrow R = \sqrt{5}$

$\tan\alpha = 2 \Rightarrow \alpha = 1.12$ (3 s.f.)

So the equation can be written as $x = \sqrt{5}\sin(3t + 1.12)$

Hence $x_{\max} = \sqrt{5}$

2 a Rewriting $\ddot{x} = -16x$ as $\ddot{x} + 16x = 0$ and solving the auxiliary equation

$m^2 + 16 = 0$

$\Rightarrow m = \pm 4\mathrm{i}$

Hence $x = A\cos 4t + B\sin 4t$

Using the initial conditions, when $t = 0$, $x = 2$ so $A\cos 0 + B\sin 0 = 5 \Rightarrow A = 5$

When $t = 0$, $\dot{x} = 2$

$\dot{x} = -4A\sin 4t + 4B\cos 4t$

So $-4A\sin 0 + 4B\cos 0 = 2 \Rightarrow B = \frac{1}{2}$

Substituting for A and B gives the solution:

$x = 5\cos 4t + \frac{1}{2}\sin 4t$

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2 b The period of motion can be found by solving $4t = 2\pi$

So the period of motion is $\frac{\pi}{2}$

To find the maximum displacement, write x in the form $R\sin(\theta + \alpha)$:

$R\sin(4t + \alpha) = R\sin\alpha\cos 4t + R\cos\alpha\sin 4t$

So $R\sin\alpha = 5$ and $R\cos\alpha = \frac{1}{2}$

$R^2 = 25 + \frac{1}{4} \Rightarrow R = \sqrt{\frac{101}{4}} = \frac{\sqrt{101}}{2}$

$\tan\alpha = 10 \Rightarrow \alpha = 1.47$ (3 s.f.)

So the equation can be written as $x = \frac{\sqrt{101}}{2}\sin(4t + 1.47)$

Hence $x_{\max} = \frac{\sqrt{101}}{2}$

3 a The particle moves with simple harmonic motion.

b The acceleration is proportional to the distance from the origin, so $\ddot{x} = ax$ for some constant of proportionality, a.

When $x = 1$, $\ddot{x} = -5$, this gives $-5 = a$

So $\ddot{x} = -5x$

c Rewriting $\ddot{x} = -5x$ as $\ddot{x} + 5x = 0$ and solving the auxiliary equation

$m^2 + 5 = 0$

$\Rightarrow m = \pm\sqrt{5}\mathrm{i}$

Hence $x = A\cos\sqrt{5}t + B\sin\sqrt{5}t$

Using the initial conditions, when $t = 0$ and $x = 5$ so $A\cos 0 + B\sin 0 = 5 \Rightarrow A = 5$

When $t = 0$, $\dot{x} = 6$

$\dot{x} = -\sqrt{5}A\sin\sqrt{5}t + \sqrt{5}B\cos\sqrt{5}t$

So $-\sqrt{5}A\sin 0 + \sqrt{5}B\cos 0 = 6 \Rightarrow B = \frac{6}{\sqrt{5}} = \frac{6\sqrt{5}}{5}$

Substituting for A and B gives the solution:

$x = 5\cos\sqrt{5}t + \frac{6\sqrt{5}}{5}\sin\sqrt{5}t$

d Rewriting x in the form $R\sin(\theta + \alpha)$

$R^2 = 5^2 + \left(\frac{6\sqrt{5}}{5}\right)^2 = 25 + \frac{180}{25} = \frac{625}{25} + \frac{180}{25} = \frac{805}{25} \Rightarrow R = \frac{\sqrt{805}}{5}$

$\tan\alpha = \frac{5}{\frac{6\sqrt{5}}{5}} = \frac{25}{6\sqrt{5}} \Rightarrow \alpha = 1.08$ (3 s.f.)

So the equation can be written as $x = \frac{\sqrt{805}}{5}\sin(\sqrt{5}t + 1.08)$, hence $x_{\max} = \frac{\sqrt{805}}{5}$

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4 a $\dfrac{\mathrm{d}^2x}{\mathrm{d}t^2} = -kx$

As $\dfrac{\mathrm{d}^2x}{\mathrm{d}^2t} = -7$ when $x = 2$, this gives $-7 = -2k \Rightarrow k = \dfrac{7}{2}$

b Rewriting $\dfrac{\mathrm{d}^2x}{\mathrm{d}^2t} = -kx = -\dfrac{7}{2}x$ as $\dfrac{\mathrm{d}^2x}{\mathrm{d}^2t} + \dfrac{7}{2}x = 0$

The auxiliary equation is

$m^2 + \dfrac{7}{2} = 0$

$\Rightarrow m = \pm\sqrt{\dfrac{7}{2}}\mathrm{i}$

So $x = A\cos\sqrt{\dfrac{7}{2}}t + B\sin\sqrt{\dfrac{7}{2}}t$

Using the initial conditions, when $t = 0$, $x = 6$ so $A\cos 0 + B\sin 0 = 6 \Rightarrow A = 6$

When $t = 0$, $\dot{x} = \dfrac{\mathrm{d}x}{\mathrm{d}t} = 1$

$\dfrac{\mathrm{d}x}{\mathrm{d}t} = -\sqrt{\dfrac{7}{2}}A\sin\sqrt{\dfrac{7}{2}}t + \sqrt{\dfrac{7}{2}}B\cos\sqrt{\dfrac{7}{2}}t$

So $-\sqrt{\dfrac{7}{2}}A\sin 0 + \sqrt{\dfrac{7}{2}}B\cos 0 = 1 \Rightarrow B = \sqrt{\dfrac{2}{7}}$

Substituting for A and B gives the solution:

$x = 6\cos\sqrt{\dfrac{7}{2}}t + \sqrt{\dfrac{2}{7}}\sin\sqrt{\dfrac{7}{2}}t$

c The period of motion can be found by solving $\sqrt{\dfrac{7}{2}}t = 2\pi$

So the period of motion is $\dfrac{2\sqrt{2}\pi}{\sqrt{7}} = 3.36$ seconds (2 d.p.)

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5 a Rewriting the equation as $\frac{d^2x}{dt^2} + 2.25x = 0$ and solving the auxiliary equation:

$m^2 + 2.25 = 0$

$\Rightarrow m = \pm 1.5\text{i}$

So $x = A\sin 1.5t + B\cos 1.5t$

At $t = 2$, $x = 1.3$, the maximum displacement so

$A\sin 3 + B\cos 3 = 1.3$ **(1)**

At maximum displacement, the velocity of the boat is 0, so at $t = 2$, $v = 0$, so

$\dot{x} = 1.5A\cos 1.5t - 1.5B\sin 1.5t$

So $1.5A\cos 3 - 1.5B\sin 3 = 0$

$\Rightarrow A\cos 3 = B\sin 3 \Rightarrow A = B\tan 3$

Substituting for A into equation (1) gives:

$B\tan 3\sin 3 + B\cos 3 = 1.3$

$B(-0.1425 \times .1411 + -.9900) = 1.3$

$B = -1.287$ (3 d.p.)

So $A = B\tan 3 = -1.287 \times \tan 3 = 0.183$ (3 d.p.)

Substituting for A and B gives the solution:

$x = 0.183\sin 1.5t - 1.287\cos 1.5t$

b The time elapsed between the boat being at its highest and its lowest point is half of the full period. The period of motion is found by solving:

$1.5t = 2\pi \Rightarrow t = \frac{4\pi}{3}$

Hence the time elapsed between the boat being at its highest and lowest point is $\frac{2\pi}{3}$ seconds

c The model assumes that the amplitude of the boat's motion is fixed and that it will float to a fixed highest and fixed lowest point. This does not account for any changes in the motion of the sea over time due to tides, weather, etc.

6 a The particle moves with a simple harmonic motion.

b Rewriting the equation as $\ddot{x} + 200x = 0$ and solving the auxiliary equation

$m^2 + 200 = 0$

$\Rightarrow m = \pm\text{i}\sqrt{200} = \pm 10\text{i}\sqrt{2}$

So $x = A\cos 10\sqrt{2}t + B\sin 10\sqrt{2}t$

Using the initial conditions, when $t = 0$, $x = 0.3$ so $A\cos 0 + B\sin 0 = 0.3 \Rightarrow A = 0.3$

When $t = 0$, $\dot{x} = 0$

$\dot{x} = -10\sqrt{2}A\sin 10\sqrt{2}t + 10\sqrt{2}B\cos 10\sqrt{2}t$

So $-10\sqrt{2}A\sin 0 + 10\sqrt{2}B\cos 0 = 0 \Rightarrow B = 0$

Substituting for A and B gives the solution:

$x = 0.3\cos 10\sqrt{2}t$

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6 c The period of motion can be found by solving $10\sqrt{2}t = 2\pi$

So the period of motion is $\frac{2\pi}{10\sqrt{2}} = \frac{\sqrt{2}\pi}{10}$ seconds

The largest displacement occurs when $\cos 10\sqrt{2}t = 1$ and hence $x = 0.3$, so the amplitude is 0.3 m.

d The velocity is given by $\dot{x} = -3\sqrt{2}\sin 10\sqrt{2}t$

The maximum velocity occurs when $\sin 10\sqrt{2}t = -1$ and so $v_{\max} = 3\sqrt{2}\,\text{m s}^{-1}$

7 a Rewriting the equation as $\ddot{x} + \frac{100}{0.64}x = 0$ and solving the auxiliary equation

$m^2 + \frac{100}{0.84} = 0$

$\Rightarrow m = \pm\frac{10}{0.8}\text{i} = \pm 12.5\text{i}$

So $x = A\cos 12.5t + B\sin 12.5t$

Using the initial conditions, when $t = 0$, $x = 1$ so $A\cos 0 + B\sin 0 = 1 \Rightarrow A = 1$

When $t = 0$, $\dot{x} = 0$

$\dot{x} = -12.5A\sin 12.5t + 12.5B\cos 12.5t$

So $-12.5A\sin 0 + 12.5B\cos 0 = 0 \Rightarrow B = 0$

Substituting for A and B gives the solution:

$x = \cos 12.5t$

b The period of motion can be found by solving $12.5t = 2\pi$

So the period of motion is $\frac{2\pi}{12.5} = \frac{4\pi}{25}$ seconds

8 a Rewriting the equation as $\ddot{x} + 320x = 0$ and solving the auxiliary equation

$m^2 + 320 = 0$

$\Rightarrow m = \pm\sqrt{320}\text{i} = \pm\sqrt{64 \times 5}\text{i} = \pm 8\sqrt{5}\text{i}$

So $x = A\cos 8\sqrt{5}t + B\sin 8\sqrt{5}t$

Using the initial conditions, when $t = 0$, $x = 8$ so $A\cos 0 + B\sin 0 = 8 \Rightarrow A = 8$

When $t = 0$, $\dot{x} = 0$

$v = \dot{x} = -8\sqrt{5}A\sin 8\sqrt{5}t + 8\sqrt{5}B\cos 8\sqrt{5}t$

So $-8\sqrt{5}A\sin 0 + 8\sqrt{5}B\cos 0 = 0 \Rightarrow B = 0$

Substituting for A and B gives the solution:

$x = 8\cos 8\sqrt{5}t$

b The period of the resulting oscillations can be found by solving $8\sqrt{5}t = 2\pi$

So the period of oscillation is $\frac{2\pi}{8\sqrt{5}} = \frac{2\sqrt{5}\pi}{40} = 0.351$ seconds (3 d.p.)

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9 a Rewriting the equation as $\ddot{x}+\frac{250}{3}x=0$ and solving the auxiliary equation

$m^2+\frac{250}{3}=0$

$\Rightarrow m=\pm\frac{5\sqrt{10}}{\sqrt{3}}\mathrm{i}$

So $x=A\cos\frac{5\sqrt{10}}{\sqrt{3}}t+B\sin\frac{5\sqrt{10}}{\sqrt{3}}t$

Using the initial conditions, when $t=0$, $x=15$ so $A\cos 0+B\sin 0=15\Rightarrow A=15$

When $t=0$, $\dot{x}=0$

$v=\dot{x}=-\frac{5\sqrt{10}}{\sqrt{3}}A\sin\frac{5\sqrt{10}}{\sqrt{3}}t+\frac{5\sqrt{10}}{\sqrt{3}}B\cos\frac{5\sqrt{10}}{\sqrt{3}}t$

So $-\frac{5\sqrt{10}}{\sqrt{3}}A\sin 0+\frac{5\sqrt{10}}{\sqrt{3}}B\cos 0=0\Rightarrow B=0$

Substituting for A and B gives the solution:

$x=15\cos\frac{5\sqrt{10}}{\sqrt{3}}t$

b The period of motion can be found by solving $\frac{5\sqrt{10}}{\sqrt{3}}t=2\pi$

So the period of motion is $\frac{2\sqrt{3}\pi}{5\sqrt{10}}=\frac{2\sqrt{30}\pi}{50}=\frac{\sqrt{30}\pi}{25}=0.688$ seconds (3 d.p.)

The maximum displacement occurs when $\cos\frac{5\sqrt{10}}{\sqrt{3}}t=1$, so $x_{\max}=15$

Hence the amplitude is 15 cm

c The model does not account for air resistance, which will cause the pendulum to stop swinging eventually. A refinement to the model would be to incorporate a damping effect so that as $t\to\infty, x\to 0$.

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Modelling with differential equations 8C

1 a $\dfrac{\mathrm{d}^2x}{\mathrm{d}t^2}+4\dfrac{\mathrm{d}x}{\mathrm{d}t}+8x=0$

The auxiliary equation is

$m^2+4m+8=0$

$m=\dfrac{-4\pm\sqrt{16-32}}{2}=\dfrac{-4\pm\sqrt{-16}}{2}=-2\pm 2\mathrm{i}$

So the general solution is $x=\mathrm{e}^{-2t}(A\cos 2t+B\sin 2t)$

Using the initial conditions, when $t=0$, $x=2$ so $A\cos 0+B\sin 0=2\Rightarrow A=2$

$\dot{x}=-2\mathrm{e}^{-2t}(A\cos 2t+B\sin 2t)+\mathrm{e}^{-2t}(-2A\sin 2t+2B\cos 2t)$

When $t=0$, $\dot{x}=0\Rightarrow -2A+2B=0$

$\Rightarrow B=A=2$

Substituting for A and B gives the solution:

$x=\mathrm{e}^{-2t}(2\cos 2t+2\sin 2t)=2\mathrm{e}^{-2t}(\cos 2t+\sin 2t)$

b $x=2\mathrm{e}^{-\frac{2\pi}{3}}\left(\cos\dfrac{2\pi}{3}+\sin\dfrac{2\pi}{3}\right)=2\times 0.1231\left(-\dfrac{1}{2}+\dfrac{\sqrt{3}}{2}\right)=0.0901$ (3 s.f.)

c As the auxiliary equation has complex roots, the motion is lightly damped.

2 $\dfrac{\mathrm{d}^2x}{\mathrm{d}t^2}+8\dfrac{\mathrm{d}x}{\mathrm{d}t}+12x=0$

The auxiliary equation is

$m^2+8m+12=0$

$(m+6)(m+2)=0$

$m=-6$ or -2

So the general solution is $x=A\mathrm{e}^{-6t}+B\mathrm{e}^{-2t}$

When $t=0$, $x=4\Rightarrow A+B=4$ **(1)**

$\dot{x}=-6A\mathrm{e}^{-6t}-2B\mathrm{e}^{-2t}$

$t=0$, $\dot{x}=0$ So $-6A-2B=0\Rightarrow 3A+B=0$ **(2)**

Subtracting equation **(1)** from equation **(2)** gives:

$2A=-4\Rightarrow A=-2$

Substituting in equation **(1)** gives $-2+B=4\Rightarrow B=6$

Substituting for A and B gives the solution:

$x=6\mathrm{e}^{-2t}-2\mathrm{e}^{-6t}$

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3 a $\dfrac{\mathrm{d}^2x}{\mathrm{d}t^2}+2\dfrac{\mathrm{d}x}{\mathrm{d}t}+6x=0$

The auxiliary equation is

$m^2+2m+6=0$

$m=\dfrac{-2\pm\sqrt{4-24}}{2}=\dfrac{-2\pm\sqrt{-20}}{2}=-1\pm\mathrm{i}\sqrt{5}$

So the general solution is $x=\mathrm{e}^{-t}(A\cos\sqrt{5}t+B\sin\sqrt{5}t)$

When $t=0, x=1$ so $A\cos 0+B\sin 0=1\Rightarrow A=1$

$\dot{x}=-\mathrm{e}^{-t}(A\cos\sqrt{5}t+B\sin\sqrt{5}t)+\mathrm{e}^{-t}(-\sqrt{5}A\sin\sqrt{5}t+\sqrt{5}B\cos\sqrt{5}t)$

When $t=0,\ \dot{x}=0\Rightarrow -A+\sqrt{5}B=0$

$\Rightarrow B=\dfrac{A}{\sqrt{5}}=\dfrac{1}{\sqrt{5}}$

Substituting for A and B gives the solution:

$x=\mathrm{e}^{-t}\left(\cos\sqrt{5}t+\dfrac{1}{\sqrt{5}}\sin\sqrt{5}t\right)$

b $\dot{x}=0,\ \ t=T$

So $-\mathrm{e}^{-T}\left(\cos\sqrt{5}T+\dfrac{1}{\sqrt{5}}\sin\sqrt{5}T\right)+\mathrm{e}^{-T}\left(-\sqrt{5}\sin\sqrt{5}T+\dfrac{1}{\sqrt{5}}\sqrt{5}\cos\sqrt{5}T\right)=0 \qquad \mathrm{e}^{-T}\neq 1$

$\Rightarrow -\cos\sqrt{5}T-\dfrac{1}{\sqrt{5}}\sin\sqrt{5}T-\sqrt{5}\sin\sqrt{5}T+\cos\sqrt{5}T=0$

$\Rightarrow \dfrac{5-1}{\sqrt{5}}\sin\sqrt{5}T=0\Rightarrow\sin\sqrt{5}T=0$

$\Rightarrow \sqrt{5}T=\pi, 2\pi, 3\pi,\ldots \qquad$ as $T>0$

So the smallest value of T for which P is at rest is $\dfrac{\pi}{\sqrt{5}}=1.40$ (3 s.f.)

4 a $\dfrac{\mathrm{d}^2x}{\mathrm{d}t^2}+4k\dfrac{\mathrm{d}x}{\mathrm{d}t}+4k^2x=0$

The auxiliary equation is

$m^2+4km+4k^2=0$

$(m+2k)(m+2k)=0$

$m=-2k$

So $x=(A+Bt)\mathrm{e}^{-2kt}$

When $t=0, x=0$ so $A\mathrm{e}^0=0\Rightarrow A=0$

$\dot{x}=-2k(A+Bt)\mathrm{e}^{-2kt}+B\mathrm{e}^{-2kt}$

When $t=0,\ \dot{x}=u\Rightarrow -2kA+B=u$

As $A=0\Rightarrow B=u$

Substituting for A and B gives the solution:

$x=ut\mathrm{e}^{-2kt}$

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4 b The particle comes to stop when $v = 0$, i.e. when

$$\frac{\mathrm{d}x}{\mathrm{d}t} = u\mathrm{e}^{-2kt} - 2kut\mathrm{e}^{-2kt} = 0$$

$$\Rightarrow u\mathrm{e}^{-2kt}(1-2kt) = 0$$

Assuming $u \neq 0$ this means $1-2kt = 0 \Rightarrow t = \frac{1}{2k}$

So in this model the particle will stop after $\frac{1}{2k}$ s

5 a The force of $6x$ N and the resistance force are acting in the same direction
So using $F = ma$ gives $-(6x + 2v) = 2\ddot{x}$

Rearranging this equation gives

$$2\ddot{x} + 2v + 6x = 0$$

$$\Rightarrow \ddot{x} + \dot{x} + 3x = 0 \quad \text{as required (as } v = \dot{x})$$

b Solving the auxiliary equation:

$$m^2 + m + 3 = 0$$

$$m = \frac{-1 \pm \sqrt{1-12}}{2} = \frac{-1 \pm \mathrm{i}\sqrt{11}}{2}$$

So $x = \mathrm{e}^{-\frac{1}{2}t}\left(A\cos\frac{\sqrt{11}}{2}t + B\sin\frac{\sqrt{11}}{2}t\right)$

When $t = 0$, $x = 1$ so $\mathrm{e}^0(A\cos 0 + B\sin 0) = 1 \Rightarrow A = 1$. Hence

$$\dot{x} = -\frac{1}{2}\mathrm{e}^{-\frac{1}{2}t}\left(A\cos\frac{\sqrt{11}}{2}t + B\sin\frac{\sqrt{11}}{2}t\right) + \mathrm{e}^{-\frac{1}{2}t}\left(-\frac{\sqrt{11}}{2}A\sin\frac{\sqrt{11}}{2}t + \frac{\sqrt{11}}{2}B\cos\frac{\sqrt{11}}{2}t\right)$$

When $t = 0$, $\dot{x} = 2$ so $-\frac{1}{2}A + \frac{\sqrt{11}}{2}B = 2$

As $A = 1 \Rightarrow \frac{\sqrt{11}}{2}B = 2 + \frac{1}{2} = \frac{5}{2}$

$$\Rightarrow B = \frac{5}{\sqrt{11}} = \frac{5\sqrt{11}}{11}$$

Substituting for A and B gives the solution:

$$x = \mathrm{e}^{-\frac{1}{2}t}\left(\cos\frac{\sqrt{11}}{2}t + \frac{5\sqrt{11}}{11}\sin\frac{\sqrt{11}}{2}t\right)$$

c When $t = 2$ we have:

$$x = \mathrm{e}^{-\frac{1}{2}2}\left(\cos\frac{\sqrt{11}}{2}2 + \frac{5\sqrt{11}}{11}\sin\frac{\sqrt{11}}{2}2\right) = \mathrm{e}^{-1}\left(\cos\sqrt{11} + \frac{5\sqrt{11}}{11}\sin\sqrt{11}\right) = -0.459\,\text{m (3 s.f.)}$$

d As $\left(\cos\frac{\sqrt{11}}{2}t + \frac{5\sqrt{11}}{11}\sin\frac{\sqrt{11}}{2}t\right)$ is bounded, the long-term behaviour of P is determined by $\mathrm{e}^{-\frac{1}{2}t}$, which will decay to 0. So the maximum displacement will decrease exponentially.

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6 Solving the corresponding homogeneous equation $\frac{\mathrm{d}^2x}{\mathrm{d}t^2}+9x=0$

The auxiliary equation is

$m^2+9=0 \Rightarrow m=\pm 3\mathrm{i}$

So the complementary function is $x=A\cos 3t+B\sin 3t$

The form of the particular integral is $x=\lambda\cos t+\mu\sin t$

$\frac{\mathrm{d}x}{\mathrm{d}t}=-\lambda\sin t+\mu\cos t$ and $\frac{\mathrm{d}^2x}{\mathrm{d}t^2}=-\lambda\cos t-\mu\sin t$

Substituting into $\frac{\mathrm{d}^2x}{\mathrm{d}t^2}+9x=k\cos t$ gives:

$-\lambda\cos t-\mu\sin t+9\lambda\cos t+9\mu\sin t=k\cos t$

Equating coefficients of $\cos t$: $8\lambda=k \Rightarrow \lambda=\frac{k}{8}$

Equating coefficients of $\sin t$: $8\mu=0 \Rightarrow \mu=0$

So a particular interval is $\frac{k}{8}\cos t$

The general solution is $x=A\cos 3t+B\sin 3t+\frac{k}{8}\cos t$

When $t=0$, $x=0$ so $A\cos 0+B\sin 0+\frac{k}{8}\cos 0=0 \Rightarrow A=-\frac{k}{8}$. Hence

$\dot{x}=-3A\sin 3t+3B\cos 3t-\frac{k}{8}\sin t$

When $t=0$, $\dot{x}=\frac{k}{5}$ so $-3A\sin 0+3B\cos 0-\frac{k}{8}\sin 0=\frac{k}{5} \Rightarrow B=\frac{k}{15}$

Substituting for A and B gives the solution:

$$x=-\frac{k}{8}\cos 3t+\frac{k}{15}\sin 3t+\frac{k}{8}\cos t$$

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7 Solving the corresponding homogeneous equation $\frac{\mathrm{d}^2x}{\mathrm{d}t^2}+5k\frac{\mathrm{d}x}{\mathrm{d}t}+6k^2x=0$

The auxiliary equation is

$m^2+5km+6k^2=0$

$(m+2k)(m+3k)=0$

$m=-2k$ or $-3k$

So the complementary function is $x=A\mathrm{e}^{-2kt}+B\mathrm{e}^{-3kt}$

Try a constant for the particular integral, $x=\lambda$, so $\dot{x}=\ddot{x}=0$

Substituting into $\frac{\mathrm{d}^2x}{\mathrm{d}t^2}+5k\frac{\mathrm{d}x}{\mathrm{d}t}+6k^2x=5kU$ gives:

$6k^2\lambda=5kU\Rightarrow\lambda=\frac{5U}{6k}$

So the general solution is $x=A\mathrm{e}^{-2kt}+B\mathrm{e}^{-3kt}+\frac{5U}{6k}$

Now use the initial conditions to determine A and B

When $t=0$, $x=0$ so $A+B+\frac{5U}{6k}=0\Rightarrow A=-\frac{5U}{6k}-B$ **(1)**

$\dot{x}=-2kA\mathrm{e}^{-2kt}-3kB\mathrm{e}^{-3kt}$

When $t=0$, $\dot{x}=U$ so $-2kA-3kB=U\Rightarrow 2kA=-3kB-U$ **(2)**

Substituting from equation **(1)** into equation **(2)** gives:

$2kA=-3kB-U$

$\Rightarrow-\frac{5U}{3}-2kB=-3kB-U$

$\Rightarrow B=\frac{2U}{3k}$

So from equation **(1)** $A=-\frac{5U}{6k}-\frac{2U}{3k}=-\frac{9U}{6k}=-\frac{3U}{2k}$

Substituting for A and B gives the solution:

$x=\frac{2U}{3k}\mathrm{e}^{-3kt}-\frac{3U}{2k}\mathrm{e}^{-2kt}+\frac{5U}{6k}$

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8 a Try a particular integral of the form $x = \lambda\cos t + \mu\sin t$

$\frac{\mathrm{d}x}{\mathrm{d}t} = -\lambda\sin t + \mu\cos t$ and $\frac{\mathrm{d}^2x}{\mathrm{d}t^2} = -\lambda\cos t - \mu\sin t$

Substituting into $2\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + 3\frac{\mathrm{d}x}{\mathrm{d}t} + x = 100\cos t$ gives:

$-2\lambda\cos t - 2\mu\sin t - 3\lambda\sin t + 3\mu\cos t + \lambda\cos t + \mu\sin t = 100\cos t$

$\Rightarrow (3\mu - \lambda)\cos t - (3\lambda + \mu)\sin t = 100\cos t$

Equating coefficients of $\cos t$: $3\mu - \lambda = 100 \Rightarrow \lambda = 3\mu - 100$ **(1)**

Equating coefficients of $\sin t$: $3\lambda + \mu = 0 \Rightarrow \mu = -3\lambda$ **(2)**

Substituting equation **(2)** in equation **(1)** gives:

$\lambda = -9\lambda - 100 \Rightarrow \lambda = -10$

So $\mu = -3\lambda = 30$

So the particular solution is $x = 30\sin t - 10\cos t$ as required.

b To find the general solution solving the corresponding homogeneous equation $2\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + 3\frac{\mathrm{d}x}{\mathrm{d}t} + x = 0$

$2m^2 + 3m + 1 = 0$

$(2m+1)(m+1) = 0$

$m = -\frac{1}{2}$ or -1

Hence the general solution is $x = Ae^{-t} + Be^{-\frac{1}{2}t} + 30\sin t - 10\cos t$

c To find the value of x at 5 seconds, use initial conditions to find the values of A and B.

When $t = 0$, $x = 0$ so $Ae^0 + Be^0 + 30\sin 0 - 10\cos 0 = 0 \Rightarrow A + B = 10$ **(1)**

$\dot{x} = -Ae^{-t} - \frac{1}{2}Be^{-\frac{1}{2}t} + 30\cos t + 10\sin t$

When $t = 0$, $\dot{x} = 0$ so $-Ae^0 - \frac{1}{2}Be^{0t} + 30\cos 0 + 10\sin 0 = 0 \Rightarrow 2A + B = 60$ **(2)**

Subtracting equation **(1)** from equation **(2)** gives

$A = 60 - 10 = 50$, so from equation **(1)** $B = -40$

Substituting for A and B gives the solution:

$x = 50e^{-t} - 40e^{-\frac{1}{2}t} + 30\sin t - 10\cos t$

At $t = 5$

$x = 50e^{-5} - 40e^{-2.5} + 30\sin 5 - 10\cos 5 = -34.55$ m (2 d.p.)

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Modelling with differential equations 8D

1 The system of equations is

$$\frac{dx}{dt} = x + y \qquad (\mathbf{1})$$

$$\frac{dy}{dt} = x - y \qquad (\mathbf{2})$$

Rearranging equation (**1**) and differentiating with respect to t gives:

$$y = \frac{dx}{dt} - x \qquad (\mathbf{3})$$

$$\frac{dy}{dt} = \frac{d^2x}{dt^2} - \frac{dx}{dt}$$

Substituting into equation (**2**) gives:

$$\frac{d^2x}{dt^2} - \frac{dx}{dt} = x - \frac{dx}{dt} + x$$

$$\frac{d^2x}{dt^2} - 2x = 0$$

Solving the auxiliary equation

$$m^2 - 2 = 0 \Rightarrow m = \pm\sqrt{2}$$

So $x = Ae^{\sqrt{2}t} + Be^{-\sqrt{2}t}$

Then differentiating with respect to t and substituting in equation (**3**) gives:

$$y = \frac{dx}{dt} - x = \sqrt{2}Ae^{\sqrt{2}t} - \sqrt{2}Be^{-\sqrt{2}t} - Ae^{\sqrt{2}t} - Be^{-\sqrt{2}t}$$

$$y = A(\sqrt{2}-1)e^{\sqrt{2}t} - B(\sqrt{2}+1)e^{-\sqrt{2}t}$$

Using the initial conditions at $t = 0$, $x = 1$ and $y = 2$ gives

$$A + B = 1 \Rightarrow A = 1 - B \qquad (\mathbf{4})$$

$$A(\sqrt{2}-1) - B(\sqrt{2}+1) = 2 \qquad (\mathbf{5})$$

Substituting equation (**4**) into equation (**5**)

$$(1-B)(\sqrt{2}-1) - B(\sqrt{2}+1) = 2$$

$$\Rightarrow \sqrt{2} - 1 - \sqrt{2}B + B - \sqrt{2}B - B = 2$$

$$\Rightarrow B = \frac{\sqrt{2}-3}{2\sqrt{2}} = \frac{2-3\sqrt{2}}{4}$$

$$\text{So } A = 1 - \frac{2-3\sqrt{2}}{4} = \frac{2+3\sqrt{2}}{4}$$

Substituting for A and B gives the particular solutions

$$x = \frac{2+3\sqrt{2}}{4}e^{\sqrt{2}t} + \frac{2-3\sqrt{2}}{4}e^{-\sqrt{2}t} = \frac{1}{4}(2+3\sqrt{2})e^{\sqrt{2}t} + \frac{1}{4}(2-3\sqrt{2})e^{-\sqrt{2}t}$$

$$y = \frac{2+3\sqrt{2}}{4}(\sqrt{2}-1)e^{\sqrt{2}t} - \frac{2-3\sqrt{2}}{4}(\sqrt{2}+1)e^{-\sqrt{2}t}$$

$$= \frac{4-\sqrt{2}}{4}e^{\sqrt{2}t} + \frac{4+\sqrt{2}}{4}e^{-\sqrt{2}t} = \frac{1}{4}(4-\sqrt{2})e^{\sqrt{2}t} + \frac{1}{4}(4+\sqrt{2})e^{-\sqrt{2}t}$$

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2 a The system of equations is

$$\frac{dx}{dt} = x + 5y \qquad (\mathbf{1})$$

$$\frac{dy}{dt} = -3y - x \qquad (\mathbf{2})$$

Rearranging equation (**2**) and differentiating with respect to t gives:

$$x = -3y - \frac{dy}{dt}$$

$$\frac{dx}{dt} = -3\frac{dy}{dt} - \frac{d^2y}{dt^2}$$

Substituting into equation (**1**) gives:

$$-3\frac{dy}{dt} - \frac{d^2y}{dt^2} = -3y - \frac{dy}{dt} + 5y$$

$$\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + 2y = 0$$

Solving the auxiliary equation

$$m^2 + 2m + 2 = 0$$

$$m = \frac{-2 \pm \sqrt{4-8}}{2} = -1 \pm \mathrm{i}$$

So $y = \mathrm{e}^{-t}(A\cos t + B\sin t)$

Then differentiating with respect to t

$$\frac{dy}{dt} = -\mathrm{e}^{-t}(A\cos t + B\sin t) + \mathrm{e}^{-t}(-A\sin t + B\cos t)$$

Substituting in equation (**2**) gives:

$$-\mathrm{e}^{-t}(A\cos t + B\sin t) + \mathrm{e}^{-t}(-A\sin t + B\cos t) = -3(\mathrm{e}^{-t}(A\cos t + B\sin t)) - x$$

$$\Rightarrow x = \mathrm{e}^{-t}(A\cos t + B\sin t + A\sin t - B\cos t - 3A\cos t - 3B\sin t)$$

$$\Rightarrow x = -\mathrm{e}^{-t}((2A + B)\cos t + (2B - A)\sin t)$$

So the general solutions are

$$x = -\mathrm{e}^{-t}((2A + B)\cos t + (2B - A)\sin t) \text{ and } y = \mathrm{e}^{-t}(A\cos t + B\sin t)$$

Note that the problem can also be solved by rearranging equation (**1**) to obtain $5y = -\frac{dx}{dt} + x$

Then differentiating with respect to t and substituting in equation (**2**) and following a similar methodology gives the mathematically equivalent results:

$$x = \mathrm{e}^{-t}(A\cos t + B\sin t) \text{ and } y = \frac{1}{5}\mathrm{e}^{-t}((B - 2A)\cos t - (A + 2B)\sin t)$$

b At $t = 0$, $x = 1$ and $y = 2$ so

From the equation for x: $-(2A + B) = 1 \Rightarrow B = -1 - 2A$

From the equation for y: $A = 2 \Rightarrow B = -5$

Substituting for A and B gives the particular solutions

$$x = \mathrm{e}^{-t}(\cos t + 12\sin t)$$

$$y = \mathrm{e}^{-t}(2\cos t - 5\sin t)$$

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3 The system of equations is

$$\frac{dx}{dt} = 2x - 3y - 2 \qquad (\mathbf{1})$$

$$\frac{dy}{dt} = x + y - 1 \qquad (\mathbf{2})$$

Rearranging equation (**2**) and differentiating with respect to t gives:

$$x = \frac{dy}{dt} - y + 1$$

$$\frac{dx}{dt} = \frac{d^2y}{dt^2} - \frac{dy}{dt}$$

Substituting into equation (**1**) gives:

$$\frac{d^2y}{dt^2} - \frac{dy}{dt} = 2\frac{dy}{dt} - 2y + 2 - 3y - 2$$

$$\frac{d^2y}{dt^2} - 3\frac{dy}{dt} + 5y = 0$$

Solving the auxiliary equation

$$m^2 - 3m + 5 = 0 \Rightarrow m = \frac{3 \pm \sqrt{9 - 20}}{2} = \frac{3}{2} \pm \frac{\sqrt{11}}{2}$$

$$\text{So } y = e^{\frac{3t}{2}}\left(A\cos\frac{\sqrt{11}}{2}t + B\sin\frac{\sqrt{11}}{2}t\right)$$

Then differentiating with respect to t

$$\frac{dy}{dt} = \frac{3}{2}e^{\frac{3t}{2}}\left(A\cos\frac{\sqrt{11}}{2}t + B\sin\frac{\sqrt{11}}{2}t\right) + e^{\frac{3t}{2}}\left(-\frac{\sqrt{11}}{2}A\sin\frac{\sqrt{11}}{2}t + \frac{\sqrt{11}}{2}B\cos\frac{\sqrt{11}}{2}t\right)$$

$$= e^{\frac{3t}{2}}\left(\left(\frac{3}{2}A + \frac{\sqrt{11}}{2}B\right)\cos\frac{\sqrt{11}}{2}t + \left(\frac{3}{2}B - \frac{\sqrt{11}}{2}A\right)\sin\frac{\sqrt{11}}{2}t\right)$$

Substituting into equation (**2**) gives

$$e^{\frac{3t}{2}}\left(\left(\frac{3}{2}A + \frac{\sqrt{11}}{2}B\right)\cos\frac{\sqrt{11}}{2}t + \left(\frac{3}{2}B - \frac{\sqrt{11}}{2}A\right)\sin\frac{\sqrt{11}}{2}t\right) = x + e^{\frac{3t}{2}}\left(A\cos\frac{\sqrt{11}}{2}t + B\sin\frac{\sqrt{11}}{2}t\right) - 1$$

$$\Rightarrow x = e^{\frac{3t}{2}}\left(\left(\frac{1}{2}A + \frac{\sqrt{11}}{2}B\right)\cos\frac{\sqrt{11}}{2}t + \left(\frac{1}{2}B - \frac{\sqrt{11}}{2}A\right)\sin\frac{\sqrt{11}}{2}t\right) + 1$$

Using the initial conditions at $t = 0$, $x = 0$ and $y = 1$ gives:

$$\frac{1}{2}A + \frac{\sqrt{11}}{2}B + 1 = 0 \Rightarrow A = -\sqrt{11}B - 2 \quad \text{and } A = 1, \text{ so } B = -\frac{3}{\sqrt{11}}$$

Substituting for A and B gives the particular solutions

$$x = e^{\frac{3t}{2}}\left(\left(\frac{1}{2} - \frac{3\sqrt{11}}{2\sqrt{11}}\right)\cos\frac{\sqrt{11}}{2}t + \left(-\frac{3}{2\sqrt{11}} - \frac{\sqrt{11}}{2}\right)\sin\frac{\sqrt{11}}{2}t\right) + 1 = e^{\frac{3t}{2}}\left(-\cos\frac{\sqrt{11}}{2}t - \frac{7}{\sqrt{11}}\sin\frac{\sqrt{11}}{2}t\right) + 1$$

$$y = e^{\frac{3t}{2}}\left(\cos\frac{\sqrt{11}}{2}t - \frac{3}{\sqrt{11}}\sin\frac{\sqrt{11}}{2}t\right)$$

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4 a The system of equations is

$$\frac{dx}{dt} = 0.2x + 0.2y \qquad (\mathbf{1})$$

$$\frac{dy}{dt} = -0.5x + 0.4y \qquad (\mathbf{2})$$

Rearranging equation (**1**) and differentiating with respect to t gives:

$$y = 5\frac{dx}{dt} - x \qquad (\mathbf{3})$$

$$\frac{dy}{dt} = 5\frac{d^2x}{dt^2} - \frac{dx}{dt}$$

Substituting into equation (**2**) gives:

$$5\frac{d^2x}{dt^2} - \frac{dx}{dt} = -0.5x + 2\frac{dx}{dt} - 0.4x$$

$$5\frac{d^2x}{dt^2} - 3\frac{dx}{dt} + 0.9x = 0$$

Dividing through by 5 gives:

$$\frac{d^2x}{dt^2} - 0.6\frac{dx}{dt} + 0.18x = 0$$

b To find x, first solve the auxiliary equation

$$m^2 - 0.6m + 0.18 = 0$$

$$m = \frac{0.6 \pm \sqrt{0.36 - 0.72}}{2} = \frac{0.6 \pm \sqrt{-0.36}}{2} = 0.3 \pm 0.3\text{i}$$

So $x = e^{0.3t}(A\cos 0.3t + B\sin 0.3t)$

c To find y, first differentiate the equation for x from part **b**

$$\frac{dx}{dt} = e^{0.3t}(-0.3A\sin 0.3t + 0.3B\cos 0.3t) + 0.3e^{0.3t}(A\cos 0.3t + B\sin 0.3t)$$

$$= 0.3e^{0.3t}(B - A)\sin 0.3t + 0.3e^{0.3t}(A + B)\cos 0.3t$$

Then substituting into equation (**3**)

$$y = 5\frac{dx}{dt} - x = 5\left(0.3e^{0.3t}(B - A)\sin 0.3t + 0.3e^{0.3t}(A + B)\cos 0.3t\right) - e^{0.3t}(A\cos 0.3t + B\sin 0.3t)$$

$$= e^{0.3t}\left((1.5A + 1.5B - A)\cos 0.3t + (1.5B - 1.5A - B)\sin 0.3t\right)$$

$$= 0.1e^{0.3t}\left((5A + 15B)\cos 0.3t + (5B - 15A)\sin 0.3t\right)$$

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4 d At $t = 0$, there were 3 sand foxes and 111 meerkats on the island, i.e. $x = 3$ and $y = 111$. So:

$e^0 A\cos 0 + B\sin 0 = 3 \Rightarrow A = 3$

$0.1e^0(5A + 15B)\cos 0 = 111 \Rightarrow 0.1(15 + 15B) = 111$

$\Rightarrow 111 = 1.5 + 1.5B \Rightarrow B = 73$

So need to find t for which $y = 0$

$$y = 0.1e^{0.3t}(1110\cos(0.3t) + 320\sin(0.3t))$$
$$0 = 0.1e^{0.3t}(1110\cos(0.3t) + 320\sin(0.3t))$$
$$0 = 1110\cos(0.3t) + 320\sin(0.3t)$$
$$1110\cos(0.3t) = -320\sin(0.3t)$$
$$-\frac{1110}{320} = \tan(0.3t)$$
$$0.3t = 1.8515...$$
$$t = 6.17 \text{ years (3 s.f.)}$$

So, since the first measurement was taken in 2012, the meerkats will die out some time in 2018.

e In the year when the meerkats die out, $t = 6$, so

$x = e^{0.3t}(3\cos(0.3t) + 73\sin(0.3t))$

$= e^{0.3\times 6.17}(3\cos(0.3\times 6.17) + 73\sin(0.3\times 6.17))$

$= 441$ (3 s.f.)

The number of foxes has to be a natural number, so round to 441 foxes.

f The model seems reasonable for the first few years, where both the number of meerkats and the foxes are sensible. When after 6 years (see part **d**) the meerkats die out, the model becomes unsuitable. Considering larger values of t shows that the number of meerkats becomes negative. For instance, consider $t = 10$, we have

$y = 0.1e^3(1110\cos 3 + 320\sin 3) \approx -2096.5$

which is not a feasible number of animals. Similarly, for the number of foxes, if we consider $t = 15$, we get:

$x = e^{0.3\times 15}\left(3\cos(0.3\times 15) + 73\sin(0.3\times 15)\right) \approx -6480.521$

This behaviour can also be seen from the equations for x and y – as the coefficient of cos and sin are very large, they will lead to large negative expressions inside the brackets. Since the exponential on front of the bracket is always positive, the expressions will periodically become negative, making this model unsuitable for modelling the number of animals in the long term.

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5 a The system of equations is

$$\frac{dx}{dt} = -3x + 2y \qquad (\mathbf{1})$$

$$\frac{dy}{dt} = -2x + y \qquad (\mathbf{2})$$

Rearranging equation (**1**) and differentiating with respect to t gives:

$$y = 1.5x + 0.5\frac{dx}{dt} \qquad (\mathbf{3})$$

$$\frac{dy}{dt} = 1.5\frac{dx}{dt} + 0.5\frac{d^2x}{dt^2}$$

Substituting into equation (**2**)

$$1.5\frac{dx}{dt} + 0.5\frac{d^2x}{dt^2} = -2x + 1.5x + 0.5\frac{dx}{dt}$$

$$\frac{d^2x}{dt^2} + 2\frac{dx}{dt} + x = 0$$

Solving the auxiliary equation

$$m^2 + 2m + 1 = 0$$

$$(m+1)(m+1) = 0$$

$$m = -1$$

So $x = (A + Bt)e^{-t}$

Then differentiating with respect to t

$$\frac{dx}{dt} = Be^{-t} - (A + Bt)e^{-t} = e^{-t}(B - A - Bt)$$

Substituting into equation (**3**) gives:

$$y = 1.5(A + Bt)e^{-t} + 0.5e^{-t}(B - A - Bt)$$

$$= (A + 0.5B + Bt)e^{-t}$$

Using the initial conditions, at $t = 0$, $x = 1$ and $y = 2$. Hence:

$$(A + B \times 0)e^0 = 1 \Rightarrow A = 1$$

$$(A + 0.5B + B \times 0)e^0 = 2 \Rightarrow B = 2$$

Substituting for A and B gives the particular solutions

$x = (1 + 2t)e^{-t} = Pe^{-t}$ where $P(t) = 1 + 2t$

$y = (2 + 2t)e^{-t} = Qe^{-t}$ where $Q(t) = 2 + 2t$

b At $t = 2$, $x = 5e^{-2} = 0.677$ litres (3 d.p.) and $y = 6e^{-2} = 0.812$ litres (3 d.p.)

c As t gets large, e^t becomes much larger than t, so $te^{-t} \to 0$. So the amounts of both chemicals in the tank tend to zero.

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6 a The system of equations is

$$\frac{dx}{dt} = -4y \qquad (1)$$

$$\frac{dy}{dt} = 4x \qquad (2)$$

Rearranging equation (**2**) and differentiating with respect to t gives:

$$x = 0.25\frac{dy}{dt} \qquad (3)$$

$$\frac{dx}{dt} = 0.25\frac{d^2y}{dt^2}$$

Substituting into equation (**1**) gives:

$$0.25\frac{d^2y}{dt^2} = -4y$$

$$\frac{d^2y}{dt^2} = -16y \qquad (4)$$

This equation describes simple harmonic motion for y.

b Solving the auxiliary equation for equation (**3**)

$m^2 + 16 = 0 \Rightarrow m = \pm 4\text{i}$

So $y = A\cos 4t + B\sin 4t$

Then differentiating with respect to t

$$\frac{dy}{dt} = -4A\sin 4t + 4B\cos 4t$$

Then substituting into equation (**3**)

$x = B\cos 4t - A\sin 4t$

Using the initial conditions, at $t = 0$, $x = 4$ and $y = 5$. So:

$B\cos 0 - A\sin 0 = 4 \Rightarrow B = 4$

$A\cos 0 + B\sin 0 = 5 \Rightarrow A = 5$

Substituting for A and B gives the particular solutions

$x = 4\cos 4t - 5\sin 4t$

$y = 5\cos 4t + 4\sin 4t$

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7 a The system of equations is

$$\frac{dx}{dt} = -0.03x + 0.01y + 50 \qquad (\mathbf{1})$$

$$\frac{dy}{dt} = 0.01x - 0.03y \qquad (\mathbf{2})$$

Rearranging equation (**1**) and differentiating with respect to t gives:

$$y = 100\frac{dx}{dt} + 3x - 5000 \qquad (\mathbf{3})$$

$$\frac{dy}{dt} = 100\frac{d^2x}{dt^2} + 3\frac{dx}{dt}$$

Substituting into equation (**2**) gives:

$$100\frac{d^2x}{dt^2} + 3\frac{dx}{dt} = 0.01x - 3\frac{dx}{dt} - 0.09x + 150$$

$$\frac{d^2x}{dt^2} + 0.06\frac{dx}{dt} + 0.0008x = 1.5$$

b Solving the auxiliary equation

$$m^2 + 0.06m + 0.0008 = 0$$

$$(m + 0.04)(m + 0.02) = 0$$

$$m = -0.04 \text{ or } -0.02$$

So the complementary function is $x = Ae^{-0.04t} + Be^{-0.02t}$

Try a constant for the particular integral, $x = \lambda$, so $\dot{x} = \ddot{x} = 0$

Substituting into $\frac{d^2x}{dt^2} + 0.006\frac{dx}{dt} + 0.0008x = 1.5$ gives:

$0.0008\lambda = 1.5 \Rightarrow \lambda = 1875$

So $x = Ae^{-0.04t} + Be^{-0.02t} + 1875$

Differentiating with respect to t

$$\frac{dx}{dt} = -0.04Ae^{-0.04t} - 0.02Be^{-0.02t}$$

Then substituting into equation (**3**)

$$y = 100(-0.04Ae^{-0.04t} - 0.02Be^{-0.02t}) + 3(Ae^{-0.04t} + Be^{-0.02t} + 1875) - 5000$$

$$= Be^{-0.02t} - Ae^{-0.04t} + 625$$

So the general solutions are

$$x = Ae^{-0.04t} + Be^{-0.02t} + 1875$$

$$y = Be^{-0.02t} - Ae^{-0.04t} + 625$$

c Since the exponential terms tend to zero in the expressions for both x and y, the total amount of toxin in the blood will tend to $x_{\lim} = 1875\,\text{mg}$ and in the organs to $y_{\lim} = 625\,\text{mg}$.

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8 a The system of equations is

$$\frac{dx}{dt} = -2x + y + 1 \qquad (\mathbf{1})$$

$$\frac{dy}{dt} = -4x + y + 2 \qquad (\mathbf{2})$$

Rearranging equation (**1**) and differentiating with respect to t gives:

$$y = \frac{dx}{dt} + 2x - 1 \qquad (\mathbf{3})$$

$$\frac{dy}{dt} = \frac{d^2x}{dt^2} + 2\frac{dx}{dt}$$

Substitute into equation (**2**) gives:

$$\frac{d^2x}{dt^2} + 2\frac{dx}{dt} = 4x + \frac{dx}{dt} + 2x - 1 + 2$$

$$\frac{d^2x}{dt^2} + \frac{dx}{dt} - 6x = 1$$

b Solving the auxiliary equation

$$m^2 + m - 6 = 0$$

$$(m-2)(m+3) = 0$$

$$m = 2 \text{ or } -3$$

So the complementary function is $x = Ae^{2t} + Be^{-3t}$

Try a constant for the particular integral, $x = \lambda$, so $\dot{x} = \ddot{x} = 0$

Substituting into $\frac{d^2x}{dt^2} + \frac{dx}{dt} - 6x = 1$ gives:

$$-6\lambda = 1 \Rightarrow \lambda = -\frac{1}{6}$$

So $x = Ae^{2t} + Be^{-3t} - \frac{1}{6}$

Differentiating with respect to t

$$\frac{dx}{dt} = 2Ae^{2t} - 3Be^{-3t}$$

Then substituting into equation (**3**)

$$y = 2Ae^{2t} - 3Be^{-3t} + 2\left(Ae^{2t} + Be^{-3t} - \frac{1}{6}\right) - 1$$

$$= 4Ae^{2t} - Be^{-3t} - \frac{4}{3}$$

c Note that both x, y are dominated by the positive exponential term, so as t gets large, both quantities will grow to infinity. Thus the model does not seem to be well suited to describe the amounts of nutrients.

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Challenge

$$\frac{dy}{dt} = 2y - \frac{y^2}{6000} - 60x$$

$$\frac{dx}{dt} = 0.02y - x$$

The populations are stable if $\frac{dx}{dt} = \frac{dy}{dt} = 0$. So:

$$2y - \frac{y^2}{6000} - 60x = 0 \qquad (\mathbf{1})$$

$$0.02y - x = 0 \Rightarrow x = 0.02y \qquad (\mathbf{2})$$

Substituting for x from equation (**2**) into equation (**1**) gives

$$2y - \frac{y^2}{6000} - 60 \times 0.02y = 0$$

$$\Rightarrow \frac{y^2}{6000} - 0.8y = 0$$

$$\Rightarrow y^2 - 4800y = 0$$

$$\Rightarrow y(y - 4800) = 0$$

So $y = 0$ or 4800

When $y = 0$, $x = 0$; when $y = 4800$, $x = 0.02(4800) = 96$
So ignoring the trivial case when both populations are 0, the solution is 96 owls and 4800 field mice.

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Modelling with differential equations – mixed exercise 8

1 a $a = \dfrac{\mathrm{d}v}{\mathrm{d}t} = 20t\ \mathrm{e}^{-t^2}$

$v = \int 20t\ \mathrm{e}^{-t^2}\mathrm{d}t = -10\mathrm{e}^{-t^2} + c$

When $t = 0,\ v = 8$ so $-10 + c = 8 \Rightarrow c = 18$

So $v = 18 - 10\mathrm{e}^{-t^2}$

b As $t \to \infty,\ \mathrm{e}^{-t} \to 0$ and $v \to 18$

The limiting velocity of P is $18\,\mathrm{ms}^{-1}$

2 $a = \dfrac{\mathrm{d}v}{\mathrm{d}t} = \dfrac{18}{(2t+3)^3} = 18(2t+3)^{-3}$

$v = \int 18(2t+3)^{-3}\mathrm{d}t = \dfrac{18}{-2\times 2}(2t+3)^{-2} + c$

$= c - \dfrac{9}{2(2t+3)^2}$

When $t = 0,\ v = 0$ so $c - \dfrac{9}{2\times 3^2} = 0 \Rightarrow c = \dfrac{1}{2}$

So formula for velocity is $v = \dfrac{1}{2} - \dfrac{9}{2(2t+3)^2}$

When $v = 0.48$

$v = \dfrac{1}{2} - \dfrac{9}{2(2t+3)^2} = 0.48 \Rightarrow \dfrac{9}{2(2t+3)^2} = 0.02$

$(2t+3)^2 = \dfrac{9}{2\times 0.02} = 225$

As $t \geqslant 0,\ 2t + 3 = +\sqrt{225} = 15$

So $t = \dfrac{15-3}{2} = 6$

3 a $a = \dfrac{\mathrm{d}v}{\mathrm{d}t} = \dfrac{100}{(2t+5)^2} = 100(2t+5)^{-2}$

$v = \int 100(2t+5)^{-2}\,\mathrm{d}t = \dfrac{100}{2\times -1}(2t+5)^{-1} + c$

$= c - \dfrac{50}{2t+5}$

When $t = 0,\ v = 0$ so $c - \dfrac{50}{5} = 0 \Rightarrow c = 10$

So $v = 10 - \dfrac{50}{2t+5}$

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3 b $x = \int\left(10 - \frac{50}{2t+5}\right)\mathrm{d}t = 10t - 25\ln(2t+5) + c$

When $t = 0, x = 0$ so $-25\ln 5 + c = 0 \Rightarrow c = 25\ln 5$

So formula for distance travelled is $x = 10t - 25\ln(2t+5) + 25\ln 5$

When $t = 10$

$x = 100 - 25\ln 25 + 25\ln 5 = 100 - 25\ln\frac{25}{5}$

$= 100 - 25\ln 5 = 59.8\,\text{m}$ (1 d.p.)

The distance moved by the car in the first 10 seconds of its motion is 59.8 metres.

4 a $a = \frac{\mathrm{d}v}{\mathrm{d}t} = \cos^2 t = \frac{1}{2} + \frac{1}{2}\cos 2t$ using the trigonometric identity $\cos 2t = 2\cos^2 t - 1$

$v = \int\left(\frac{1}{2} + \frac{1}{2}\cos 2t\right)\mathrm{d}t = \frac{1}{2}t + \frac{1}{4}\sin 2t + c$

When $t = 0, v = 0 \Rightarrow A = 0$

So formula for velocity is $v = \frac{1}{2}t + \frac{1}{4}\sin 2t$

When $t = \pi$ $v = \frac{\pi}{2} + \frac{1}{4}\sin 2\pi = \frac{\pi}{2} + 0 = \frac{\pi}{2}$

The speed of P when $t = \pi$ is $\frac{\pi}{2}\,\text{m s}^{-1}$

b The distance of P from O when $t = \frac{\pi}{4}$ is given by

$x = \int_0^{\frac{\pi}{4}}\left(\frac{1}{2}t + \frac{1}{4}\sin 2t\right)\mathrm{d}t = \left[\frac{1}{4}t^2 - \frac{1}{8}\cos 2t\right]_0^{\frac{\pi}{4}}$

$= \left(\frac{\pi^2}{64} - \frac{1}{8}\cos\frac{\pi}{2}\right) - \left(0 - \frac{1}{8}\right)$

$= \frac{\pi^2}{64} + \frac{1}{8} = \frac{1}{64}(\pi^2 + 8)$

The distance of P from O when $t = \frac{\pi}{4}$ is $\frac{1}{64}(\pi^2 + 8)\,\text{m}$, as required.

5 a $a = \frac{\mathrm{d}v}{\mathrm{d}t} = \frac{2t+3}{t+1} = \frac{2(t+1)+1}{t+1} = 2 + \frac{1}{t+1}$

$v = 2t + \ln(t+1) + c$

When $t = 0, v = 0$ so $0 + c = 0 \Rightarrow c = 0$

So $v = 2t + \ln(t+1)$

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5 b $x = \int 2t + \ln(t+1)\,\mathrm{d}t$

Using integration by parts

$$\int \ln(t+1)\,\mathrm{d}t = \int 1\ln(t+1)\,\mathrm{d}t = t\ln(t+1) - \int \frac{t}{t+1}\,\mathrm{d}t$$

$$= t\ln(t+1) - \int\left(1 - \frac{1}{t+1}\right)\mathrm{d}t = t\ln(t+1) - t + \ln(t+1)$$

$$= (t+1)\ln(t+1) - t$$

So $x = \int 2t + \ln(t+1)\,\mathrm{d}t = t^2 + (t+1)\ln(t+1) - t + c$

The distance of P from O when $t = 2$ is given by $x = \int_0^2 (2t + \ln(t+1))\,\mathrm{d}t$

Hence $x = \left[t^2 - t + (t+1)\ln(t+1)\right]_0^2 = 4 - 2 + 3\ln 3 = 2 + 3\ln 3$

The distance of P from O when $t = 2$ is $(2 + 3\ln 3)\,\mathrm{m}$.

6 a $\dfrac{\mathrm{d}V}{\mathrm{d}t} = 2t + 3V + 5$ can be rearranged as $\dfrac{\mathrm{d}V}{\mathrm{d}t} - 3V = 2t + 5$

The integrating factor is $\mathrm{e}^{\int -3\mathrm{d}t} = \mathrm{e}^{-3t}$

Multiplying the equation by the integrating factor

$$\mathrm{e}^{-3t}\frac{\mathrm{d}V}{\mathrm{d}t} - 3\mathrm{e}^{-3t}V = 2t\mathrm{e}^{-3t} + 5\mathrm{e}^{-3t}$$

$$\Rightarrow \frac{\mathrm{d}}{\mathrm{d}t}(\mathrm{e}^{-3t}V) = 2t\mathrm{e}^{-3t} + 5\mathrm{e}^{-3t}$$

$\Rightarrow \mathrm{e}^{-3t}V = 2\int t\mathrm{e}^{-3t}\,\mathrm{d}t + 5\int \mathrm{e}^{-3t}\,\mathrm{d}t$ use integration by parts for the first integral

$$\Rightarrow \mathrm{e}^{-3t}V = -\frac{2}{3}t\mathrm{e}^{-3t} - \frac{2}{9}\mathrm{e}^{-3t} - \frac{5}{3}\mathrm{e}^{-3t} + c$$

$$\Rightarrow \mathrm{e}^{-3t}V = -\frac{1}{3}\mathrm{e}^{-3t}\left(2t + \frac{17}{3}\right) + c$$

So $V = -\dfrac{2}{3}t - \dfrac{17}{9} + c\mathrm{e}^{3t}$

When $t = 0, V = 1$ so $-\dfrac{17}{9} + c\mathrm{e}^{0} = 1 \Rightarrow c = \dfrac{26}{9}$

So solution is $V = -\dfrac{2}{3}t - \dfrac{17}{9} + \dfrac{26}{9}\mathrm{e}^{3t}$

Which is of the required form with $A = -\dfrac{2}{3}$, $B = -\dfrac{17}{9}$, $C = \dfrac{26}{9}$

b When $t = 2$, $V = -\dfrac{2}{3}\times 2 - \dfrac{17}{9} + \dfrac{26}{9}\mathrm{e}^{3\times 2} = 1162.2\,\mathrm{cm}^3$ (1 d.p.)

c Because of the exponential term, this model predicts that the bacteria will reproduce infinitely many times and the colony will grow without limit. A decay factor could be added to make the model more realistic.

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7 a The rate at which the contaminant enters the reservoir is constant, 4g in every litre of fluid, which flows at the rate of 300 l/day. So $4\times300=1200\,\text{g}$ of contaminant enters the reservoir every day.

Let x be the grams of contaminant in the reservoir

So the proportion of contaminant in the reservoir is $\dfrac{x}{10000+100t}$

The denominator is the total amount of liquid in the reservoir on day t: the initial 10 000 litres, minus the 200 litres which leaks every day, plus the 300 litres which enters the container each day.

So the amount of contaminant that leaks from the reservoir each day is $\dfrac{x}{10000+100t}\times200$

Thus $\dfrac{\mathrm{d}x}{\mathrm{d}t}=1200-\dfrac{200x}{10000+100t}=1200-\dfrac{2x}{100+t}$

b Rearrange the differential equation as $\dfrac{\mathrm{d}x}{\mathrm{d}t}+\dfrac{2x}{100+t}=1200$

The integrating factor is $\mathrm{e}^{\int\frac{2}{100+t}\mathrm{d}t}=\mathrm{e}^{2\ln(100+t)}=(t+100)^2$

Multiplying the equation by the integrating factor

$$(t+100)^2\frac{\mathrm{d}x}{\mathrm{d}t}+x=(t+100)^2\times1200$$

$$\Rightarrow\frac{\mathrm{d}x}{\mathrm{d}t}\left((t+100)^2x\right)=1200t^2+240\,000t+12\,000\,000$$

$$\Rightarrow x(t+100)^2=400t^3+120\,000t^2+12\,000\,000t+c$$

$$\Rightarrow x=400\left(\frac{t^3+300t^2+30\,000t+c'}{(t+100)^2}\right)$$

When $t=0, x=0$ (the reservoir initially contains fresh water so $\dfrac{400c'}{100}=0\Rightarrow c'=0$

So $x=400\left(\dfrac{t^3+300t^2+30\,000t}{(t+100)^2}\right)$

When $t=7$, $x=400\left(\dfrac{343+300\times49+30\,000\times7}{107^2}\right)=7860\,\text{g}$ (3 s.f.)

c The model assumes that the contaminant mixes uniformly and instantaneously, which is not a realistic representation of this situation.

8 a The particle moves with simple harmonic motion.

b Given $\ddot{x}=-49x$, solve the auxiliary equation

$m^2+49=0\Rightarrow m=\pm7\mathrm{i}$

So the general solution is $x=A\cos7t+B\sin7t$

$\dot{x}=-7A\sin7t+7B\cos7t$

At time $t=0$ the particle is at rest, $\dot{x}=0$ so $-7A\sin0+7B\cos0=0\Rightarrow B=0$

At time $t=0$ the particle is at point B. Suppose the distance from B to O is equal to d.

Then $d=A\cos0\Rightarrow A=d$, so the solution is $x=d\cos7t$

To find the period of motion, solve $7t=2\pi\Rightarrow$ the period of motion is $\dfrac{2\pi}{7}$ seconds

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9 a The particle moves with simple harmonic motion.

b Given $\ddot{x} = -\frac{50}{3}x$, solve the auxiliary equation

$m^2 + \frac{50}{3} = 0 \Rightarrow m = \pm\sqrt{-\frac{50}{3}} = \pm\frac{5\mathrm{i}\sqrt{2}}{\sqrt{3}} = \pm\frac{5\mathrm{i}\sqrt{6}}{3}$

So $x = A\cos\frac{5\sqrt{6}}{3}t + B\sin\frac{5\sqrt{6}}{3}t$

At time $t = 0$, $x = 4$ so $A\cos\frac{5\sqrt{6}}{3}0 + B\sin\frac{5\sqrt{6}}{3}0 = 4 \Rightarrow A = 4$

$\dot{x} = -\frac{5\sqrt{6}}{3}A\sin\frac{5\sqrt{6}}{3}t + \frac{5\sqrt{6}}{3}B\cos\frac{5\sqrt{6}}{3}t$

At time $t = 0$ the particle is at rest, so $\dot{x} = 0$ and $-\frac{5\sqrt{6}}{3}A\sin\frac{5\sqrt{6}}{3}0 + \frac{5\sqrt{6}}{3}B\cos\frac{5\sqrt{6}}{3}0 = 0 \Rightarrow B = 0$

Substituting for A and B gives the solution:

$x = 4\cos\frac{5\sqrt{6}}{3}t$ or $x = 4\cos 5\sqrt{\frac{2}{3}}t$ as $\frac{\sqrt{6}}{3} = \frac{\sqrt{2}\sqrt{3}}{3} = \frac{\sqrt{2}}{\sqrt{3}}$

To find the period of motion, solve $\frac{5\sqrt{6}}{3}t = 2\pi$

So the period of motion is $\frac{6\pi}{5\sqrt{6}} = \frac{\sqrt{6}\pi}{5} = 1.54\,\text{s}$ (3 s.f.)

The amplitude of motion will be described by the oscillation of the particle around the natural length of the spring, so $a = 4 - 2.5 = 1.5$ m

10 a $\frac{\mathrm{d}^2x}{\mathrm{d}t^2} = -0.25x$, so solving the auxiliary equation

$m^2 + 0.25 = 0 \Rightarrow m = \pm 0.5\mathrm{i}$

So $x = A\cos 0.5t + B\sin 0.5t$

Writing x in the form $x = R\sin(0.5t + \alpha) = R\sin\alpha\cos 0.5t + R\cos\alpha\sin 0.5t$

So $R\sin\alpha = A$ and $R\cos\alpha = B$

And hence $R^2 = A^2 + B^2$, $\tan\alpha = \frac{A}{B}$

Maximum displacement occurs when $\sin(0.5t + \alpha) = \pm 1 \Rightarrow R = 4$

It occurs when $t = 2$, so $1 + \alpha = 0.5\pi + k\pi \Rightarrow \alpha = 0.5\pi + k\pi - 1$

So $\tan\alpha = \tan(0.5\pi + k\pi - 1) = \tan(k\pi + (0.5\pi - 1)) = \tan(0.5\pi - 1) = \cot 1$

So $\frac{A}{B} = \cot 1 \Rightarrow A = B\cot 1$ and $R^2 = A^2 + B^2 = 4^2 = 16$

Thus $B^2\cot^2 1 + B^2 = 16 \Rightarrow B^2 = \frac{16}{\cot^2 1 + 1} = 11.3292 \Rightarrow B = 3.366$ (3 d.p.)

And $A = B\cot 1 = 2.161$ (3 d.p.)

So $x = 2.161\cos 0.5t + 3.366\sin 0.5t$

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10 b The float travels between its highest and lowest point in half of the time it takes it to complete the oscillation (the period of the motion).
To find the period of motion, solve $0.5t = 2\pi$, so the period of motion is 4π
So it takes the boat 2π seconds.

c This model does not account for any changes in the height of the waves, i.e. the amplitude of the motion over time.

11 a $\dfrac{\mathrm{d}^2x}{\mathrm{d}t^2} + 2k\dfrac{\mathrm{d}x}{\mathrm{d}t} + n^2x = 0$

The auxiliary equation is $m^2 + 2km + n^2 = 0$

$m^2 + 2km + n^2 = 0$

$m = \dfrac{-2k \pm \sqrt{4k^2 - 4n^2}}{2} = -k \pm \sqrt{k^2 - n^2}$

As $0 < k < n \Rightarrow k^2 - n^2 < 0$ so $m = -k \pm \mathrm{i}\sqrt{n^2 - k^2}$

The general solution is $x = \mathrm{e}^{-kt}\left(A\cos t\sqrt{n^2 - k^2} + B\sin t\sqrt{n^2 - k^2}\right)$

The general solution can also be written in the form $x = A'\mathrm{e}^{-kt}\cos(\omega t + \varepsilon)$, where $\omega = \sqrt{(n^2 - k^2)}$

b To find the period of motion, solve $t\sqrt{n^2 - k^2} = 2\pi$

So the period of motion is $\dfrac{2\pi}{\sqrt{n^2 - k^2}}$

12 a The auxiliary equation is

$m^2 + 2km + 2k^2 = 0$

$m = \dfrac{-2k \pm \sqrt{(4k^2 - 8k^2)}}{2} = \dfrac{-2k \pm \sqrt{(-4k^2)}}{2} = -k \pm k\mathrm{i}$

So $x = \mathrm{e}^{-kt}(A\cos kt + B\sin kt)$

Using the initial conditions, when $t = 0$, $x = 0 \Rightarrow 0 = A$

As $A = 0$, $\dot{x} = -k\mathrm{e}^{-kt}B\sin kt + B\mathrm{e}^{-kt}k\cos kt$

$t = 0$, $\dot{x} = U$ so $Bk = U \Rightarrow B = \dfrac{U}{k}$

So the general solution is $x = \mathrm{e}^{-kt}\dfrac{U}{k}\sin kt$

$\dot{x} = -k\mathrm{e}^{-kt}\dfrac{U}{k}\sin kt + \dfrac{U}{k}\mathrm{e}^{-kt}k\cos kt$

When $\dot{x} = 0$, $U\mathrm{e}^{-kt}(\sin kt - \cos kt) = 0$

$\Rightarrow \sin kt = \cos kt$

$\Rightarrow \tan kt = 1$

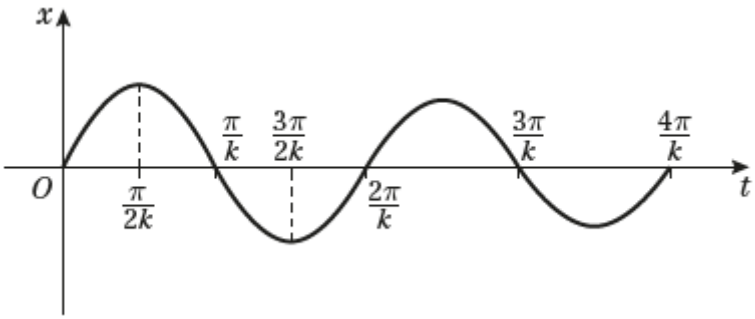
$\Rightarrow kt = \dfrac{\pi}{4} + n\pi$

So $kt = \left(n + \dfrac{1}{4}\right)\pi$, $n \in \mathbb{N}$ as required

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12 b From c, the maxima and minima occur when $kt = \left(n + \frac{1}{4}\right)\pi$

Multiplying $\sin kt$ by e^{-kt} causes the amplitude to decrease as t increases.



13 a The auxiliary equation is

$m^2 + n^2 = 0 \Rightarrow m = \pm in$

So the complementary function is $x = A\cos nt + B\sin nt$

The form of the particular integral is $x = \mu t + \lambda$, so $\dot{x} = \mu$ and $\ddot{x} = 0$

Substituting into $\frac{d^2x}{dt^2} + n^2x = n^2Ut$ gives:

$n^2(\mu t + \lambda) = n^2Ut$

$\Rightarrow \mu = U$ and $\lambda = 0$

The general solution is $x = A\cos nt + B\sin nt + Ut$

Using the initial conditions when $t = 0, x = 0 \Rightarrow A = 0$

$\dot{x} = Bn\cos nt + U$

When $t = 0$, $\dot{x} = 0$ so $Bn + U = 0 \Rightarrow B = -\frac{U}{n}$

Substituting for A and B gives the solution:

$$x = Ut - \frac{U}{n}\sin nt$$

b Using the result in part **a**, $\dot{x} = U - U\cos nt$

When $\dot{x} = 0$, $1 - \cos nt = 0$

$\Rightarrow \cos nt = 1$

$\Rightarrow nt = 0, 2\pi, 4\pi, \ldots$

So smallest positive value of T is $\frac{2\pi}{n}$

c When $t = \frac{2\pi}{n}$, $x = U\frac{2\pi}{n} - \frac{U}{n}\sin 2\pi = \frac{2U\pi}{n}$

P has moved a distance $\frac{2U\pi}{n}$ when it first comes to rest.

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14 a $\dfrac{\mathrm{d}^2x}{\mathrm{d}t^2}+4\dfrac{\mathrm{d}x}{\mathrm{d}t}+3x=150\cos t$

Try a form of the particular integral as $x=\lambda\cos t+\mu\sin t$

Then $\dfrac{\mathrm{d}x}{\mathrm{d}t}=-\lambda\sin t+\mu\cos t$ and $\dfrac{\mathrm{d}^2x}{\mathrm{d}t^2}=-\lambda\cos t-\mu\sin t$

Substituting into the equation gives

$-\lambda\cos t-\mu\sin t-4\lambda\sin t+4\mu\cos t+3\lambda\cos t+3\mu\sin t=150\cos t$

$\Rightarrow(-\lambda+4\mu+3\lambda)\cos t+(3\mu-\mu-4\lambda)\sin t=150\cos t$

$\Rightarrow(2\lambda+4\mu)\cos t+(2\mu-4\lambda)\sin t=150\cos t$

Equating coefficients of $\cos t$: $2\lambda+4\mu=150\Rightarrow\lambda=75-2\mu$

Equating coefficients of $\sin t$: $2\mu-4\lambda=0\Rightarrow\ \mu=2\lambda$

So $5\lambda=75\Rightarrow\lambda=15$

And $\mu=30$

So a particular solution is $x=15\cos t+30\sin t$

b To find the general solution, solve the auxiliary equation

$m^2+4m+3=0$

$(m+3)(m+1)=0$

$m=-3$ or -1

So the general solution is

$x=Ae^{-3t}+Be^{-t}+15\cos t+30\sin t$

c Using the initial conditions to find the particular solution.

At time $t=0$, $x=0$ so $A+B+15=0\Rightarrow A=-B-15$ **(1)**

$\dfrac{\mathrm{d}x}{\mathrm{d}t}=-3Ae^{-3t}-Be^{-t}-15\sin t+30\cos t$

At time $t=0$, $\dfrac{\mathrm{d}x}{\mathrm{d}t}=0$ so $-3A-B+30=0\Rightarrow B=30-3A$ **(2)**

Substituting equation **(2)** in equation **(1)** gives

$A=-30+3A-15\Rightarrow 2A=45\Rightarrow A=22.5$

$B=30-3A=30-67.5=-37.5$

So $x=22.5e^{-3t}-37.5e^{-t}+15\cos t+30\sin t$

When $t=10$ $x=22.5e^{-30}-37.5e^{-10}+15\cos 10+30\sin 10=-29$ m (2 s.f.)

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15 a First solving the corresponding homogeneous equation $\frac{d^2x}{dt^2}+2\frac{dx}{dt}+10x=0$

The auxiliary equation is

$m^2+2m+10=0$

$m=\frac{-2\pm\sqrt{4-40}}{2}=-1\pm 3i$

So the complementary function is $x=e^{-t}(A\cos 3t+B\sin 3t)$

Try a particular integral of the form $x=\lambda\cos t+\mu\sin t$

$\frac{dx}{dt}=-\lambda\sin t+\mu\cos t$ and $\frac{d^2x}{dt^2}=-\lambda\cos t-\mu\sin t$

Substituting into $\frac{d^2x}{dt^2}+2\frac{dx}{dt}+10x=27\cos t-6\sin t$ gives:

$-\lambda\cos t-\mu\sin t-2\lambda\sin t+2\mu\cos t+10\lambda\cos t+10\mu\sin t=27\cos t-6\sin t$

$\Rightarrow(2\mu+9\lambda)\cos t+(9\mu-2\lambda)\sin t=27\cos t-6\sin t$

Equating coefficients of $\cos t$: $2\mu+9\lambda=27\Rightarrow 18\lambda=54-4\mu$ **(1)**

Equating coefficients of $\sin t$: $9\mu-2\lambda=-6\Rightarrow 18\lambda=81\mu+54$ **(2)**

Substituting equation **(2)** in equation **(1)** gives:

$85\mu=0\Rightarrow\mu=0$

So $18\lambda=54\Rightarrow\lambda=3$

So the particular integral is $3\cos t$

The general solution is $x=e^{-t}(A\cos 3t+B\sin 3t)+3\cos t$

b When $t=0, x=3$ so $A+3=0\Rightarrow A=0$

So $x=3\cos t+Be^{-t}\sin 3t$

$\frac{dx}{dt}=-3\sin t+3Be^{-t}\cos 3t-Be^{-t}\sin 3t$

When $t=0, \frac{dx}{dt}=6$ so $3B=6\Rightarrow B=2$

Substituting for A and B gives the solution:

$x=2e^{-t}\sin 3t+3\cos t$

c After a week $t\approx 7$ days, $e^{-t}\approx 0$, so $x\approx 3\cos t$

Therefore, the distance between highest and lowest water level is $3-(-3)=6$m

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16 a The system of equations is

$$\frac{dx}{dt} = 2x + y \qquad (1)$$

$$\frac{dy}{dt} = -2x + 4y \qquad (2)$$

Rearranging equation (**1**) and differentiating with respect to t gives:

$$y = \frac{dx}{dt} - 2x \qquad (3)$$

$$\frac{dy}{dt} = \frac{d^2x}{dt^2} - 2\frac{dx}{dt}$$

Substituting into equation (**2**) gives:

$$\frac{d^2x}{dt^2} - 2\frac{dx}{dt} = -2x + 4\frac{dx}{dt} - 8x$$

$$\frac{d^2x}{dt^2} - 6\frac{dx}{dt} + 10x = 0$$

b Solving the auxiliary equation

$$m^2 - 6x + 10 = 0$$

$$m = \frac{6 \pm \sqrt{36 - 40}}{2} = \frac{6 \pm \sqrt{-4}}{2} = 3 \pm \mathrm{i}$$

So $x = \mathrm{e}^{3t}(A\cos t + B\sin t)$, which is of the required form with $\alpha = 3,\ \beta = 1$

c $$\frac{dx}{dt} = 3\mathrm{e}^{3t}(A\cos t + B\sin t) + \mathrm{e}^{3t}(-A\sin t + B\cos t)$$

So substituting in $y = \frac{dx}{dt} - 2x$, i.e. equation (**3**) from part **a**

$$y = 3\mathrm{e}^{3t}(A\cos t + B\sin t) + \mathrm{e}^{3t}(-A\sin t + B\cos t) - 2\mathrm{e}^{3t}(A\cos t + B\sin t)$$

$$= \mathrm{e}^{3t}\left((3A + B - 2A)\cos t + (3B - A - 2B)\sin t\right)$$

$$= \mathrm{e}^{3t}\left((A + B)\cos t + (B - A)\sin t\right)$$

d At $t = 0$, $x = 10$ and $y = 20$. Thus

$$\mathrm{e}^0 A\cos 0 + B\sin 0 = 10 \Rightarrow A = 10$$

$$\mathrm{e}^0\left((A + B)\cos 0 + (B - A)\sin 0\right) = 20 \Rightarrow A + B = 20 \Rightarrow B = 10$$

Substituting for A and B gives:

$x = 10\mathrm{e}^{3t}(\cos t + \sin t)$ and $y = 20\mathrm{e}^{3t}\cos t$

So find t for which $y = 0$, $20\mathrm{e}^{3t}\cos t = 0 \Rightarrow t = \frac{\pi}{2} \approx 1.5$

So, since the study began at the start of 2008, the slugs die out some time in 2009.

e At $t = \frac{\pi}{2}$, $x = 10\mathrm{e}^{\frac{3\pi}{2}}\left(\cos\frac{\pi}{2} + \sin\frac{\pi}{2}\right) = 10\mathrm{e}^{\frac{3\pi}{2}} = 1113$ to nearest whole number

f The model predicts a very large number of hedgehogs when the slugs die out, and a very large increase in only 1.5 years, so perhaps it is not a very good model.

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17 a The system of equations is

$$\frac{dx}{dt} = -4x + 3y \qquad (1)$$

$$\frac{dy}{dt} = -3x + 2y \qquad (2)$$

Rearranging equation (**1**) and differentiating with respect to t gives:

$$y = \frac{1}{3}\frac{dx}{dt} + \frac{4}{3}x \qquad (3)$$

$$\frac{dy}{dt} = \frac{1}{3}\frac{d^2x}{dt^2} + \frac{4}{3}\frac{dx}{dt}$$

Substituting into equation (**2**) gives:

$$\frac{1}{3}\frac{d^2x}{dt^2} + \frac{4}{3}\frac{dx}{dt} = -3x + \frac{2}{3}\frac{dx}{dt} + \frac{8}{3}x$$

$$\frac{d^2x}{dt^2} + 2\frac{dx}{dt} + x = 0$$

Solve the auxiliary equation

$$m^2 + 2m + 1 = 0$$

$$(m+1)(m+1) = 0$$

$$m = -1$$

So $x = Ate^{-t} + Be^{-t}$

Differentiating with respect to t and substituting in equation (**3**) gives:

$$\frac{dx}{dt} = Ae^{-t} - Ate^{-t} - Be^{-t} = -Ate^{-t} + (A-B)e^{-t}$$

$$y = -\frac{1}{3}Ate^{-t} + \frac{1}{3}(A-B)e^{-t} + \frac{4}{3}Ate^{-t} + \frac{4}{3}Be^{-t}$$

$$= Ate^{-t} + \left(B + \frac{1}{3}A\right)e^{-t}$$

Using the initial conditions, at $t = 0$, $x = 10$ and $y = 20$, so:

$$Be^0 = 10 \Rightarrow B = 10$$

$$\left(B + \frac{1}{3}A\right)e^0 = 20 \Rightarrow \frac{1}{3}A = 20 - B = 10 \Rightarrow A = 30$$

Substituting for A and B gives the particular solutions

$$x = 30te^{-t} + 10e^{-t} \quad \text{and} \quad y = 30te^{-t} + 20e^{-t}$$

Which is of the required form with $A = 30, B = 10, C = 30$ and $D = 20$

b At time $t = 2.1$

$x = 30 \times 2.1e^{-2.1} + 10e^{-2.1} = 9$ to the nearest organism

$y = 30 \times 2.1e^{-2.1} + 20e^{-2.1} = 10$ to the nearest organism

c Since both expressions are dominated by the exponential, which decays to zero as t gets large, both organisms will eventually die out.

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18 a The system of equations is

$$\frac{dx}{dt} = 2 + \frac{1}{3}y - \frac{1}{2}x \qquad \textbf{(1)}$$

$$\frac{dy}{dt} = 1 + \frac{1}{2}x - \frac{2}{3}y \qquad \textbf{(2)}$$

Rearranging equation (**2**) and differentiating with respect to t gives:

$$x = 2\frac{dy}{dt} + \frac{4}{3}y - 2 \qquad \textbf{(3)}$$

$$\frac{dx}{dt} = 2\frac{d^2y}{dt^2} + \frac{4}{3}\frac{dy}{dt}$$

Substituting into equation (**1**) gives:

$$2\frac{d^2y}{dt^2} + \frac{4}{3}\frac{dy}{dt} = 2 + \frac{1}{3}y - \frac{dy}{dt} - \frac{2}{3}y + 1$$

$$\Rightarrow 2\frac{d^2y}{dt^2} + \frac{7}{3}\frac{dy}{dt} + \frac{1}{3}y = 3$$

$$\Rightarrow 6\frac{d^2y}{dt^2} + 7\frac{dy}{dt} + y = 9 \qquad \text{as required}$$

b Solving the auxiliary equation

$$6m^2 + 7m + 1 = 0$$

$$(6m+1)(m+1) = 0$$

$$m = -1 \text{ or } -\frac{1}{6}$$

So $y = Ae^{-t} + Be^{-\frac{1}{6}t}$

Use a particular integral in the form $y = \lambda$, so $\dot{y} = \ddot{y} = 0$

Substituting into $6\frac{d^2y}{dt^2} + 7\frac{dy}{dt} + y = 9$ gives $\lambda = 9$

$$y = Ae^{-t} + Be^{-\frac{1}{6}t} + 9$$

Differentiating with respect to t and substituting into equation (**3**)

$$\frac{dy}{dt} = -Ae^{-t} - \frac{1}{6}Be^{-\frac{1}{6}t}$$

$$x = -2Ae^{-t} - \frac{1}{3}Be^{-\frac{1}{6}t} + \frac{4}{3}Ae^{-t} + \frac{4}{3}Be^{-\frac{1}{6}t} + 12 - 2 = -\frac{2}{3}Ae^{-t} + Be^{-\frac{1}{6}t} + 10$$

Now use the initial conditions, at time $t = 0$, $x = y = 8$, so:

$$-\frac{2}{3}Ae^{0} + Be^{0} + 10 = 8 \Rightarrow 3B - 2A = -6 \qquad \textbf{(4)}$$

$$Ae^{0} + Be^{0} + 9 = 8 \Rightarrow -A = B + 1 \qquad \textbf{(5)}$$

Substituting equation (**5**) into equation (**1**) gives:

$$5B + 2 = -6 \Rightarrow B = -\frac{8}{5} \text{ and so } A = \frac{3}{5}$$

Substituting for A and B gives the particular solutions

$$x = -\frac{2}{5}e^{-t} - \frac{8}{5}e^{-\frac{1}{6}t} + 10 \quad \text{and} \quad y = \frac{3}{5}e^{-t} - \frac{8}{5}e^{-\frac{1}{6}t} + 9$$

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18 c As t becomes large $e^{-t} \to 0$, so there will be approximately 10 litres in tank A and 9 litres in tank B.

Challenge

a i From the question $\frac{dX}{dt} = -X$ so rearranging gives

$$\frac{dX}{X} = -dt$$
$$\Rightarrow \ln X = -t + c$$
$$\Rightarrow X = e^{-t+c} = Ce^{-t}$$

Using the initial condition at $t = 0, X = 300$ so $C = 300$

So $X = 300e^{-t}$ as required.

ii The amount of water in Y can be modelled as $\frac{dY}{dt} = X - Y$

Where X is the amount of water coming in from the tank above, and Y is the rate at which the water escapes the tank.

So $\frac{dY}{dt} = 300e^{-t} - Y$, which can be rearranged as $\frac{dY}{dt} + Y = 300e^{-t}$

Multiplying by the integrating factor, $e^{\int 1 dt} = e^{t}$, gives:

$$e^{t}\frac{dY}{dt} + e^{t}Y = 300$$
$$\Rightarrow \frac{d}{dt}\left(e^{t}Y\right) = 300$$
$$\Rightarrow e^{t}Y = 300t + c$$
$$\Rightarrow Y = 300te^{-t} + ce^{-t}$$

Using the initial condition at $t = 0, Y = 200 \Rightarrow c = 200$

When $X = Y$

$$e^{-t}(300t + 200) = 300e^{-t}$$
$$\Rightarrow 300t + 200 = 300$$
$$\Rightarrow t = \frac{100}{300} = \frac{1}{3} = 20 \text{ minutes}$$

iii As no water escapes tank Z, and the rate at which the water is added is equal to the rate at which it leaves container Y. So the amount of water in tank Z can be modelled as $\frac{dZ}{dt} = Y$

$$\frac{dZ}{dt} = Y = e^{-t}(300t + 200)$$
$$Z = 300\int te^{-t}dt + 200\int e^{-t}dt$$

The second integral can be solved directly and the first can be solved by integration by parts

$$Z = -300te^{-t} - 300e^{-t} - 200e^{-t} + c = -300te^{-t} - 500e^{-t} + c$$

Using the initial condition at $t = 0, Z = 100$ so $-500e^{-0} + c = 100 \Rightarrow c = 600$

So $Z = -300te^{-t} - 500e^{-t} + 600$, which can be written as $Z = 600 - 100e^{-t}(3t + 5)$

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Challenge

b i In this scenario $\frac{dX}{dt} = -2X$ as the second tap in tank X doubles the rate of flow

$$\frac{dX}{X} = -2dt$$
$$\Rightarrow \ln X = -2t + c$$
$$\Rightarrow X = e^{-2t+c} = Ce^{-2t}$$

Suppose initially tank X contains V gallons of water. Hence $V = Ce^0 \Rightarrow C = V$ and $X = Ve^{-2t}$

The amount of water in tank Y can be now modelled as $\frac{dY}{dt} = 2X - Y$

$$\frac{dY}{dt} = 2Ve^{-2t} - Y$$
$$\frac{dY}{dt} + Y = 2Ve^{-2t}$$

Multiplying by the integrating factor, $e^{\int 1dt} = e^t$, gives:

$$e^t\frac{dY}{dt} + e^tY = 2Ve^{-t}$$
$$\Rightarrow \frac{d}{dt}\left(e^tY\right) = 2Ve^{-t}$$
$$\Rightarrow e^tY = -2Ve^{-t} + c$$
$$\Rightarrow Y = -2Ve^{-2t} + ce^{-t}$$

Initially the tank is empty, so $-2V + c = 0 \Rightarrow c = 2V$

And thus $Y = 2V(-e^{-2t} + e^{-t})$

To find the maximum amount of water in tank Y, differentiate with respect to t and set to zero

$$\frac{dY}{dt} = 2V(2e^{-2t} - e^{-t})$$

When $\frac{dY}{dt} = 0$, $2V(2e^{-2t} - e^{-t}) = 0 \Rightarrow 2e^{-2t} = e^{-t} \Rightarrow e^{-t} = \frac{1}{2}$

So water in tank Y is at a maximum when $t = -\ln\frac{1}{2} = 0.6931\ldots = 42$ mins (to the nearest minute)

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Challenge

b ii The amount of water in tank Z can be modelled as before using $\frac{\mathrm{d}Z}{\mathrm{d}t} = Y$

$\frac{\mathrm{d}Z}{\mathrm{d}t} = 2V(-\mathrm{e}^{-2t} + \mathrm{e}^{-t}) \Rightarrow Z = 2V(0.5\mathrm{e}^{-2t} - \mathrm{e}^{-t}) + c$

Initially the tank is empty, so $2V(0.5\mathrm{e}^{0} - \mathrm{e}^{0}) + c = 0 \Rightarrow c = V$

So $Z = 2V(0.5\mathrm{e}^{-2t} - \mathrm{e}^{-t} + 0.5)$

When $Y = Z$

$2V(-\mathrm{e}^{-2t} + \mathrm{e}^{-t}) = 2V(0.5\mathrm{e}^{-2t} - \mathrm{e}^{-t} + 0.5)$

$\Rightarrow 1.5\mathrm{e}^{-2t} - 2\mathrm{e}^{-t} + 0.5 = 0$

$\Rightarrow 3\mathrm{e}^{-2t} - 4\mathrm{e}^{-t} + 1 = 0$

$\Rightarrow (3\mathrm{e}^{-t} - 1)(\mathrm{e}^{-t} - 1) = 0$

$\Rightarrow \mathrm{e}^{-t} = \frac{1}{3}$ or 1

$\mathrm{e}^{-t} = 1 \Rightarrow t = 0,$ which is the trivial case because initially both tanks are empty

$\mathrm{e}^{-t} = \frac{1}{3} \Rightarrow t = \ln 3$

So the tanks have the same amount of water after $\ln 3$ hours.

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Exam-style practice: Paper 1

1 a First we find y^2 and $\frac{dx}{dt}$.

$y = t - t^2 + 1 \Rightarrow y^2 = \left(t - t^2 + 1\right)^2$

$y^2 = t^4 - 2t^3 - t^2 + 2t + 1.$

$x = t + t^2 \Rightarrow \frac{dx}{dt} = 1 + 2t.$

We now find the limits in terms of t.

$x = 0 \Rightarrow t = 0$

$x = 3.75 = t + t^2$

$t^2 + t - 3.75 = 0$

$t = \frac{-1 \pm \sqrt{1+15}}{2}$

We neglect the negative solution, so the upper limit is $t = \frac{3}{2}$.

$V = \pi \int_0^{1.5} \left(t^4 - 2t^3 - t^2 + 2t + 1\right)(1 + 2t)\,dt$

$= \pi \int_0^{1.5} \left(2t^5 - 3t^4 - 4t^3 + 3t^2 + 4t + 1\right)dt$

$= \pi \left[\frac{t^6}{3} - \frac{3t^5}{5} - t^4 + t^3 + 2t^2 + t\right]_0^{1.5}$

$= 11.16 \text{ unit}^3.$

Since we have that a unit is 10 cm^3, this means:

$V = 11.16 \times (10\text{ cm})^3$

$\approx 11\,200 \text{ cm}^3$

b A criticism of the model is that it does not take into account the thickness of the clay.

2 a Substituting $(1, 4, 7)$ and the coefficients of $2x + 3y - z - 10 = 0$ into

$dist = \frac{|ax_0 + by_0 + cz_0 + d|}{\sqrt{a^2 + b^2 + c^2}}$

gives

$dist = \frac{|(2\times1) + (3\times4) + (-1\times7) - 10|}{\sqrt{2^2 + 3^2 + (-1)^2}}$

$= \frac{3}{\sqrt{14}}$

$= \frac{3}{14}\sqrt{14}$

Thus $a = \frac{3}{14}$.

b Since $\mathbf{v} = 5\mathbf{i} + \mathbf{j} - \mathbf{k}$ is perpendicular to $\mathbf{r} = \lambda(\mathbf{i} - 3\mathbf{j} + 2\mathbf{k}) + \mu(a\mathbf{i} + 2\mathbf{j} - 3\mathbf{k})$, we know that $\mathbf{r}.\mathbf{v} = \mathbf{0}$.

$\mathbf{r}.\mathbf{v}$

$= (\lambda(\mathbf{i} - 3\mathbf{j} + 2\mathbf{k}) + \mu(a\mathbf{i} + 2\mathbf{j} - 3\mathbf{k})).(5\mathbf{i} + \mathbf{j} - \mathbf{k})$

$= ((\lambda + \mu a)\mathbf{i} + (-3\lambda + 2\mu)\mathbf{j} + (2\lambda - 3\mu)\mathbf{k}).(5\mathbf{i} + \mathbf{j} - \mathbf{k})$

$= 5\lambda + 5\mu a - 3\lambda + 2\mu - 2\lambda + 3\mu$

$= 5\mu a + 5\mu = 0$

$\Rightarrow a = -1$

c The normal vector of the plane $\prod_1$ is $\mathbf{v}_1 = 2\mathbf{i} + 3\mathbf{j} - \mathbf{k}$, the plane $\prod_2$ has a perpendicular vector $\mathbf{v}_2 = 5\mathbf{i} + \mathbf{j} - \mathbf{k}$.

$\theta = \arccos\left(\frac{\mathbf{v}_1.\mathbf{v}_2}{|\mathbf{v}_1||\mathbf{v}_2|}\right)$

$= \arccos\left(\frac{(2\times5) + (3\times1) + (-1\times-1)}{\sqrt{2^2 + 3^2 + (-1)^2}\sqrt{5^2 + 1^2 + (-1)^2}}\right)$

$= \arccos\left(\frac{14}{\sqrt{14}\sqrt{27}}\right)$

$= \arccos\left(\frac{\sqrt{14}}{\sqrt{27}}\right)$

$= 43.9°$

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3 a The base case $n=1$,

$$\begin{pmatrix}1&2&2\\0&1&2\\0&0&1\end{pmatrix}=\begin{pmatrix}1&2\times1&2\times1^2\\0&1&2\times1\\0&0&1\end{pmatrix}$$ is true.

Assume the result is true for $n=k$.

For $n=k$, $\begin{pmatrix}1&2&2\\0&1&2\\0&0&1\end{pmatrix}^k=\begin{pmatrix}1&2k&2k^2\\0&1&2k\\0&0&1\end{pmatrix}$

For $n=k+1$,

$$\begin{pmatrix}1&2&2\\0&1&2\\0&0&1\end{pmatrix}^{k+1}=\begin{pmatrix}1&2&2\\0&1&2\\0&0&1\end{pmatrix}\begin{pmatrix}1&2k&2k^2\\0&1&2k\\0&0&1\end{pmatrix}$$

$$=\begin{pmatrix}1&2k+2&2k^2+4k+2\\0&1&2k+2\\0&0&1\end{pmatrix}$$

$$=\begin{pmatrix}1&2(k+1)&2(k+1)^2\\0&1&2(k+1)\\0&0&1\end{pmatrix}$$

Thus the result is true for $n=k+1$.
The result is true for $n=1$ and if it is true for $n=k$, then it is true for $n=k+1$.
By mathematical induction the result is true for all positive integers n.

b i M^{-1} only exists if the discriminant is non-zero. So for the discriminant to be zero, we find

$$\begin{aligned}discrim&=1\times\big((-2\times1)-(0\times-1)\big)\\&\quad+k\times\big((0\times3)-(2\times1)\big)\\&\quad+4\times\big((2\times-1)-(-2\times3)\big)\\&=-2-2k+16\\&=-2k+14=0\\&\Rightarrow k=7.\end{aligned}$$

3 b ii Using

$$A^{-1}=\begin{pmatrix}a&b&c\\d&e&f\\g&h&i\end{pmatrix}^{-1}$$

$$=\frac{1}{|A|}\begin{pmatrix}\begin{vmatrix}e&f\\h&i\end{vmatrix}&\begin{vmatrix}c&b\\i&h\end{vmatrix}&\begin{vmatrix}b&c\\e&f\end{vmatrix}\\\begin{vmatrix}f&d\\i&g\end{vmatrix}&\begin{vmatrix}a&c\\g&i\end{vmatrix}&\begin{vmatrix}c&a\\f&d\end{vmatrix}\\\begin{vmatrix}d&e\\g&h\end{vmatrix}&\begin{vmatrix}b&a\\h&g\end{vmatrix}&\begin{vmatrix}a&b\\d&e\end{vmatrix}\end{pmatrix}$$

we may compute

$$\begin{pmatrix}1&k&4\\2&-2&0\\3&-1&1\end{pmatrix}^{-1}=\frac{1}{14-2k}\begin{pmatrix}\begin{vmatrix}-2&0\\-1&1\end{vmatrix}&\begin{vmatrix}4&k\\1&-1\end{vmatrix}&\begin{vmatrix}k&4\\-2&0\end{vmatrix}\\\begin{vmatrix}0&2\\1&3\end{vmatrix}&\begin{vmatrix}1&4\\3&1\end{vmatrix}&\begin{vmatrix}4&1\\0&2\end{vmatrix}\\\begin{vmatrix}2&-2\\3&-1\end{vmatrix}&\begin{vmatrix}k&1\\-1&3\end{vmatrix}&\begin{vmatrix}1&k\\2&-2\end{vmatrix}\end{pmatrix}$$

$$=\frac{1}{14-2k}\begin{pmatrix}-2&-(k+4)&8\\-2&-11&8\\4&3k+1&-2(k+1)\end{pmatrix}$$

4 a Using $z=\cos\theta+i\sin\theta$ (since the modulus is 1 we don't need a coefficient) we may write

$$\begin{aligned}z^n&=(\cos\theta+i\sin\theta)^n\\&=\cos(n\theta)+i\sin(n\theta)\end{aligned}$$

and

$$\begin{aligned}z^{-n}&=(\cos\theta+i\sin\theta)^{-n}\\&=\cos(n\theta)-i\sin(n\theta)\end{aligned}$$

by De Moivre's theorem.
We find the difference of these terms and get

$$\begin{aligned}z^n-z^{-n}&=\cos(n\theta)+i\sin(n\theta)-(\cos(n\theta)-i\sin(n\theta))\\&=2i\sin(n\theta).\end{aligned}$$

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4 b We use

$$\left(z-z^{-1}\right)^4 = (2i\sin\theta)^4$$
$$= 16\sin^4\theta$$

(from part a with $n=1$) in order to obtain

$$8\sin^4\theta = \frac{1}{2}\left(z-z^{-1}\right)^4$$
$$= \frac{1}{2}\left(z^4-4z^2+6-4z^{-2}+z^{-4}\right)$$
$$= \frac{1}{2}\left(z^4+z^{-4}\right)-2\left(z^2+z^{-2}\right)+3$$
$$= \cos(4\theta)-4\cos(2\theta)+3.$$

Note that we have used the fact that $z^n+z^{-n}=2\cos(n\theta)$.

5 a The litres of liquid in the vat after t minutes is given by

$V = 500-15t+30t = 500+15t.$

Let x grams be the amount of sugar in the vat after t minutes.

Concentration of sugar after t minutes is

$Conc = \frac{x}{500+15t}$ grams per litre.

The rate of the sugar into the vat is

$Rate_{SugIn} = 30\times 25 = 750$ grams per minute.

The rate of the sugar out of the vat is

$Rate_{SugOut} = 15\times\frac{x}{500+15t} = \frac{3x}{100+3t}$

grams per minute.

Hence we have the rate of change of sugar in grams with respect to time in minutes as $\frac{dx}{dt} = 750-\frac{3x}{100+3t}$.

5 b We write down the equation above, in the form $\frac{dx}{dt}+Px=Q$.

$$\frac{dx}{dt}+\frac{3x}{100+3t}=750$$

Now use an integrating factor to solve the differential equation.

$$IF = e^{\int P\,dt}$$
$$= e^{\int\frac{3}{100+3t}dt}$$
$$= e^{\ln(100+3t)}$$
$$= 100+3t.$$

Multiplying our original equation through by the integrating factor gives

$$(100+3t)\frac{dx}{dt}+3x = 750(100+3t)$$
$$\frac{d}{dt}(100x+3tx) = 75\,000+2250t.$$

Integrating both sides with respect to t gives $100x+3tx = 75\,000t+1125t^2+c$.

Using the initial conditions $t=0,\ x=0$ we find that

$0+0=0+0+c$

$c=0.$

Thus the equation is

$100x+3tx = 75\,000t+1125t^2$.

So after 10 minutes

$100x+30x = 750\,000+112\,500$

$$x = \frac{862\,500}{130}\approx 6635g$$

c The model could be refined to reflect pressure caused by the volume of oil which would effect rate of leaking. Could take into account that sugar does not disperse uniformly throughout the vat on entry.

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6 a We convert $|z+12+5i|=13$ into Cartesian form and then to polar. Since we know that this locus of points is a circle with radius 13, centred at $(-12,-5)$, we know that the appropriate Cartesian equation is $(x+12)^2+(y+5)^2=169$.

We convert this equation to polar form via $x=r\cos\theta$

and

$y=r\sin\theta$,

giving

$(r\cos\theta+12)^2+(r\sin\theta+5)^2=169$

$r^2+24r\cos\theta+10r\sin\theta=0$

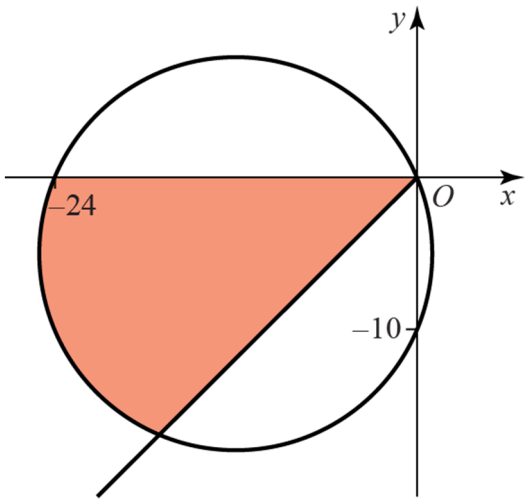
$r=-2(12\cos\theta+5\sin\theta)$ when $r\neq 0$.

b The set of points

$$A=\{z:|z+12+5i|\leqslant 13\}\cap\left\{z:-\pi\leqslant \arg z\leqslant -\frac{3}{4}\pi\right\}$$

defines the segment of a disc (filled circle) with radius 13, centred at $(-12,-5)$ between the half lines

$\theta=-\pi$ and $\theta=-\frac{3\pi}{4}$.



6 c To find the area of the region defined by A, we use the polar form $r=-2(12\cos\theta+5\sin\theta)$ to calculate

$$A=\frac{1}{2}\int_{-\pi}^{-\frac{3\pi}{4}}(-2)^2(12\cos\theta+5\sin\theta)^2\,\mathrm{d}\theta$$

$$=2\int_{-\pi}^{-\frac{3\pi}{4}}(144\cos^2\theta+120\cos\theta\sin\theta+25\sin^2\theta)\mathrm{d}\theta$$

$$=2\int_{-\pi}^{-\frac{3\pi}{4}}(72(1+\cos 2\theta)+60\sin 2\theta+\frac{25}{2}(1-\cos 2\theta))\mathrm{d}\theta$$

$$=2\left[(72(\theta+\tfrac{1}{2}\sin 2\theta)-30\cos 2\theta+\frac{25}{2}(\theta-\tfrac{1}{2}\sin 2\theta))\right]_{-\pi}^{-\frac{3\pi}{4}}$$

$$=\left(-\frac{507\pi}{4}+\frac{119}{2}-(-169\pi-60)\right)$$

$$=\frac{169}{4}\pi+\frac{239}{2}$$

≈ 252 units^2

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7 a Differentiating the equation $\frac{dx}{dt} = 0.3x + 0.2y + 1$ with respect to t gives the expression

$$\frac{d^2x}{dt^2} = 0.3\frac{dx}{dt} + 0.2\frac{dy}{dt}.$$

We remove y from the system of equations by

$$0.3 \times \frac{dx}{dt} - 0.2 \times \frac{dy}{dt}$$
$$= 0.3 \times (0.3x + 0.2y + 1) - 0.2 \times (-0.2x + 0.3y)$$
$$= (0.3^2 + 0.2^2)x + 0.3$$

and so we can write $\frac{dy}{dt}$ in terms of x as

$$0.2\frac{dy}{dt} = 0.3\frac{dx}{dt} - ((0.3^2 + 0.2^2)x + 0.3).$$

Combining the first expression we found for $\frac{d^2x}{dt^2}$ with the expression for $\frac{dy}{dt}$, we get

$$\frac{d^2x}{dt^2} = 0.3\frac{dx}{dt} + 0.2\frac{dy}{dt}$$
$$= 0.3\frac{dx}{dt} + 0.3\frac{dx}{dt} - ((0.3^2 + 0.2^2)x + 0.3)$$
$$= 0.6\frac{dx}{dt} - (0.13x + 0.3).$$

Multiplying through by 100 and rearranging to have all terms on the left-hand side gives us

$$100\frac{d^2x}{dt^2} - 60\frac{dx}{dt} + 13x + 30 = 0$$

7 b The auxiliary equation is $100m^2 - 60m + 13 = 0$, with solutions

$$m = \frac{60 \pm \sqrt{3600 - 5200}}{200}$$
$$= \frac{3}{10} \pm \frac{i}{5}.$$

Thus we have the complementary function $x_c = e^{0.3t}(A\cos 0.2t + B\sin 0.2t)$.

For the particular integral, we try $x = C$ with differentials $\frac{dx}{dt} = 0$ and $\frac{d^2x}{dt^2} = 0$.

Substituting these expressions into our differential equation gives

$$0 - 0 + 13C + 30 = 0$$
$$C = -\frac{30}{13}.$$

So the general solution for x is

$$x_G = e^{0.3t}(A\cos 0.2t + B\sin 0.2t) - \frac{30}{13}$$

c By differentiating

$$x_G = e^{0.3t}(A\cos 0.2t + B\sin 0.2t) - \frac{30}{13}$$

with respect to t, we obtain

$$\frac{dx_G}{dt} = 0.2e^{0.3t}(B\cos 0.2t - A\sin 0.2t)$$
$$+0.3e^{0.3t}(A\cos 0.2t + B\sin 0.2t)$$
$$= 0.2e^{0.3t}(B\cos 0.2t - A\sin 0.2t) + 0.3\left(x_G + \frac{30}{13}\right)$$

and since we have $\frac{dx}{dt} = 0.3x + 0.2y + 1$,

we can rewrite

$$\frac{dx_G}{dt} = 0.2e^{0.3t}(B\cos 0.2t - A\sin 0.2t) + 0.3\left(x_G + \frac{30}{13}\right)$$
$$= 0.3x_G + 0.2y_G + 1$$

After cancelling $0.3x_G$ from both sides, we solve for

$$0.2e^{0.3t}(B\cos 0.2t - A\sin 0.2t) + 0.3\left(\frac{30}{13}\right) = 0.2y_G + 1$$

$$0.2y_G = 0.2e^{0.3t}(B\cos 0.2t - A\sin 0.2t) + 0.3\left(\frac{30}{13}\right) - 1$$

$$y_G = e^{0.3t}(B\cos 0.2t - A\sin 0.2t) + \left(\frac{3}{2} \times \frac{30}{13}\right) - 5$$

$$y_G = e^{0.3t}(B\cos 0.2t - A\sin 0.2t) - \frac{20}{13}$$

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7 d Using the initial conditions $t = 0,\ x = 10,\ y = 5$ in the general solutions for x and y gives

$$10 = e^{0.}\left(A\cos 0 + B\sin 0\right) - \frac{30}{13}$$

$$A = \frac{160}{13}$$

and

$$5 = e^{0}\left(B\cos 0 - A\sin 0\right) - \frac{20}{13}$$

$$B = \frac{85}{13}.$$

Thus, we have particular solutions

$$x = e^{0.3t}\left(\frac{160}{13}\cos 0.2t + \frac{85}{13}\sin 0.2t\right) - \frac{30}{13}$$

$$y = e^{0.3t}\left(\frac{85}{13}\cos 0.2t - \frac{160}{13}\sin 0.2t\right) - \frac{20}{13}$$

e $$y(3) = e^{0.9}\left(\frac{85}{13}\cos 0.6 - \frac{160}{13}\sin 0.6\right) - \frac{20}{13}$$

$$\approx -5.36$$

The concentration on the right side of the model is predicted to be negative after 3 hours. This is clearly nonsense, so the model is not suitable.

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Exam-style practice Paper 2

1 a Using partial fractions, we find that
$$\frac{1}{(r+2)(r+4)} = \frac{1}{2(r+2)} - \frac{1}{2(r+4)}.$$
So we can now use the method of differences to find
$$\sum_{r=1}^{n} \frac{1}{(r+2)(r+4)} = \sum_{r=1}^{n} \left(\frac{1}{2(r+2)} - \frac{1}{2(r+4)} \right)$$
$$= \frac{1}{6} - \frac{1}{10} + \frac{1}{8} - \frac{1}{12} + \frac{1}{10} - \frac{1}{14} + \cdots$$
$$+ \frac{1}{2n} - \frac{1}{2(n+2)} + \frac{1}{2(n+1)} - \frac{1}{2(n+3)}$$
$$+ \frac{1}{2(n+2)} - \frac{1}{2(n+4)}$$
$$= \frac{1}{6} + \frac{1}{8} - \frac{1}{2(n+3)} - \frac{1}{2(n+4)}$$
$$= \frac{n(7n+25)}{24(n+3)(n+4)}.$$

Note that most middling terms have cancelled with each other.
$p = 7,$
$q = 25.$

1 b For the base case, $n = 1,$
$\mathrm{f}(1) = 2^3 + 3^3 = 35$ is divisible by 7.
We assume that the statement holds true for $n = k$. That is $\mathrm{f}(k) = 2^{k+2} + 3^{2k+1}$ is divisible by 7.
Now for $n = k+1,$
$$\mathrm{f}(k+1) = 2^{k+3} + 3^{2k+3}$$
$$= 2\left(2^{k+2}\right) + 3^2\left(3^{2k+1}\right)$$
$$= 2\left(2^{k+2}\right) + 9\left(3^{2k+1}\right)$$
$$= 2\left(2^{k+2}\right) + 2\left(3^{2k+1}\right) + 7\left(3^{2k+1}\right)$$
$$= 2\mathrm{f}(k) + 7\left(3^{2k+1}\right).$$
Since $\mathrm{f}(k)$ is divisible by 7, $\mathrm{f}(k+1)$ is also divisible by 7 and the statement holds for $n = k+1$.
The result is true for the base case $n = 1,$ and if it is true for $n = k$ then it is true for $n = k+1$. By mathematical induction, the result is true for all positive integers n.

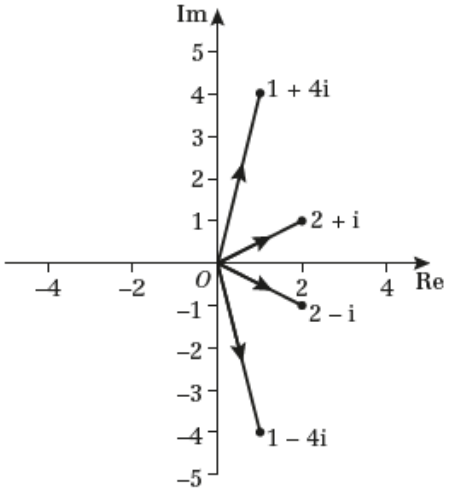
2 a Since all coefficients of $\mathrm{f}(x)$ are real, that means if there is a complex number as a root of the equation, there must also be the complex conjugate of that number as a root of the equation. In this case this means that since $1+4i$ is a root, $1-4i$ also is a root.

b $\mathrm{f}(z) = z^4 + az^3 + 30z^2 + bz + 85$
Since $\mathrm{f}(1+4i) = 0$ and $\mathrm{f}(1-4i) = 0,$
both $\left[z-(1+4i)\right]$ and $\left[z-(1-4i)\right]$
are factors of $\mathrm{f}(z)$
Therefore
$$\left[z-(1+4i)\right]\left[z-(1-4i)\right] = \left[z^2 - 2z + 17\right]$$

is also a factor of $\mathrm{f}(z)$
We write
$$\mathrm{f}(z) = \left[z^2 - 2z + 17\right]\left[z^2 + kz + 5\right]$$
Equating coefficients of z^2 gives
$30 = 17 + 5 - 2k$
So $k = -4$
Therefore
$$f(z) = \left[z^2 - 2z + 17\right]\left[z^2 - 4z + 5\right]$$

Solving $z^2 - 4z + 5 = 0$ leads to
$$z = \frac{4 \pm \sqrt{16-20}}{2} = 2 \pm i.$$
Thus we conclude that all the roots of the equation $\mathrm{f}(x)$ are
$1+4i,\ 1-4i,\ 2+i$ and $2-i.$

c



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3 First we need to find the point for which the tangent to the curve is perpendicular to the initial line. We form an expression for x and differentiate with respect to θ.

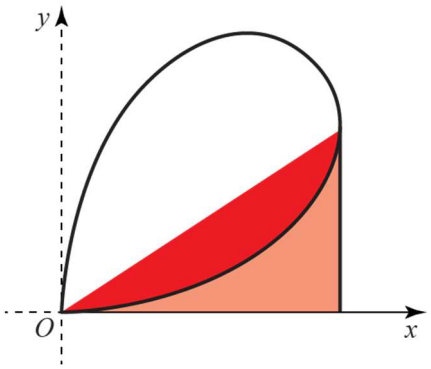
$$x = r\cos\theta$$
$$= 6\sin 2\theta\cos\theta$$
$$\frac{\mathrm{d}x}{\mathrm{d}\theta} = 12\cos 2\theta\cos\theta - 6\sin 2\theta\sin\theta$$
$$= 12(2\cos^2\theta - 1)\cos\theta - 12\cos\theta\sin^2\theta$$
$$= 36\cos^3\theta - 24\cos\theta$$
$$= 12\cos\theta(3\cos^2\theta - 2).$$

We now solve equal to 0 in order to find our required θ values. We choose to neglect the solutions arising from the $\cos\theta = 0$ factor, since a tangent at the origin is not what we are looking for even though it is perpendicular to the initial line.

So, $3\cos^2\theta - 2 = 0$ gives $\cos\theta = \pm\sqrt{\frac{2}{3}}$ and we choose to neglect the negative solution since $0 \leqslant \theta \leqslant \frac{\pi}{2}$.

Thus our tangent perpendicular to the initial line occurs at $\theta = \theta_A = \arccos\left(\sqrt{\frac{2}{3}}\right)$.

To find the area of the region, we will need to find the area of the sector that lies between $0 \leqslant \theta \leqslant \theta_A$ as shown in the diagram (red region).



$$A_{\text{sector}} = \frac{1}{2}\int_0^{\theta_A} (6\sin 2\theta)^2\,\mathrm{d}\theta$$
$$= 18\int_0^{\theta_A} (\sin^2 2\theta)\,\mathrm{d}\theta$$
$$= 9\int_0^{\theta_A} (1 - \cos 4\theta)\,\mathrm{d}\theta$$
$$= 9\left[\theta - \frac{\sin 4\theta}{4}\right]_0^{\theta_A}$$
$$= 9\theta_A - \frac{9}{4}\sin 4\theta_A$$
$$= 9\arccos\left(\sqrt{\frac{2}{3}}\right) - \frac{9}{4}\sin\left(4\arccos\left(\sqrt{\frac{2}{3}}\right)\right)$$
$$\approx 4.13.$$

Now we find the area of the right-angle triangle bounded by the horizontal axis, the tangent and the line OA.

Using the formula

$$A_{\text{tri}} = \frac{1}{2} \times \text{Base} \times \text{Height}$$
$$= \frac{1}{2}|x||y|$$
$$= \frac{1}{2}r^2|\cos\theta||\sin\theta|$$
$$= 18(\sin^2 2\theta)|\cos\theta||\sin\theta|$$

and substituting in $\theta = \theta_A$, we find that

$$A_{\text{tri}} = \frac{16\sqrt{2}}{3}.$$

So, our shaded region is

$$A = A_{\text{tri}} - A_{\text{sector}}$$
$$= \frac{16\sqrt{2}}{3} - 4.125\ldots$$
$$\approx 3.42 \text{ units}^2 \text{ (3sf)}.$$

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4 a We evaluate the differentials at 0 until we have three non-zero terms.

$f(x)=\cos x\sinh 2x \Rightarrow f(0)=0$

$f'(x)=2\cos x\cosh 2x-\sin x\sinh 2x \Rightarrow f'(0)=2$

$f''(x)=3\cos x\sinh 2x-4\sin x\cosh 2x \Rightarrow f''(0)=0$

$f'''(x)=2\cos x\cosh x-11\sin x\sinh x \Rightarrow f'''(0)=2$

$f^{(4)}(x)=-7\cos x\sinh 2x-24\sin x\cosh 2x \Rightarrow f^{(4)}(0)=0$

$f^{(5)}(x)=-38\cos x\cosh x-41\sin x\sinh x \Rightarrow f^{(5)}(0)=-38.$

Now we use the standard Maclaurin series expansion and obtain

$$\cos x\sinh 2x \approx 2x+\frac{2x^3}{3!}+\frac{-38x^5}{5!}$$
$$=2x+\frac{x^3}{3}-\frac{19x^5}{60}.$$

b Using the approximation,

$$f(0.1)=\cos 0.1\sinh(2\times 0.1)$$
$$\approx 2\times 0.1+\frac{0.1^3}{3}-\frac{19\times 0.1^5}{60}$$
$$=\frac{1201981}{6000000}.$$

$$\text{error}=\left|\frac{\frac{1201981}{6000000}-\cos 0.1\sinh(2\times 0.1)}{\cos 0.1\sinh(2\times 0.1)}\right|\times 100$$

$$=2.754\times 10^{-6}\ \%(4\text{sf}).$$

5 Since we know that $\int\frac{1}{a^2+x^2}\,dx=\frac{1}{a}\arctan\left(\frac{x}{a}\right)+C$, we can compute

$$\int_0^L\frac{1}{x^2+4}\,dx=\left[\frac{1}{2}\arctan\left(\frac{x}{2}\right)\right]_0^L$$
$$=\frac{1}{2}\arctan\left(\frac{L}{2}\right).$$

Now, since $\arctan\left(\frac{L}{2}\right)\to\frac{\pi}{2}$ as $L\to\infty$, we can conclude that the integral

$$\int_0^L\frac{1}{x^2+4}\,dx\to\frac{\pi}{4}\text{ as }L\to\infty.$$

6 The vertices $(0,2)$, $(k,0)$ and $(0,8)$ form a triangle with area $3k$.

The matrix $\begin{pmatrix}2&2\\-3&5\end{pmatrix}$ increases the area by a factor of $(2\times 5)-(2\times -3)=16$.

So, $\text{Area}(T)=\frac{456}{16}=\frac{57}{2}$.

When we set this equal to the area of T, we obtain

$$\frac{1}{2}\times(8-2)\times k=\frac{57}{2}$$
$$3k=\frac{57}{2}$$
$$k=\frac{57}{6}=\frac{19}{2}$$
$$k=9.5.$$

7 a $\int_{-1}^{1}\frac{1}{\sqrt{x^2+2x+2}}\,dx=\int_{-1}^{1}\frac{1}{\sqrt{(x+1)^2+1}}\,dx$

Set $u=x+1$, $du=dx$ then the integral becomes $\int_{-1}^{1}\frac{1}{\sqrt{(x+1)^2+1}}\,dx=\int_0^2\frac{1}{\sqrt{u^2+1}}\,du.$

Now set

$v=\operatorname{arsinh}u$

$\Rightarrow u=\sinh v,$

$du=\cosh v\,dv=\sqrt{1+\sinh^2 v}\,dv=\sqrt{1+u^2}\,dv.$

Thus the integral now becomes

$$\int_0^2\frac{1}{\sqrt{u^2+1}}\,du=\int_0^{\operatorname{arsinh}2}\frac{1}{\sqrt{u^2+1}}\sqrt{u^2+1}\,dv$$
$$=\int_0^{\operatorname{arsinh}2}1\,dv$$
$$=[v]_0^{\operatorname{arsinh}2}$$
$$=\operatorname{arsinh}2.$$

Now in order to find the mean value of $f(x)$ over $[-1,1]$ we calculate

$$\frac{1}{1-(-1)}\operatorname{arsinh}2\approx 0.722\ (3\text{ d.p.})$$

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7 b The mean value of $f(x)+2$ over $[-1,1]$ is

$$\frac{1}{1-(-1)}\int_{-1}^{1}\left(\frac{1}{\sqrt{x^2+2x+2}}+2\right)\mathrm{d}x$$

$$=\frac{1}{2}\int_{-1}^{1}\frac{1}{\sqrt{x^2+2x+2}}\,\mathrm{d}x+\frac{1}{2}\int_{-1}^{1}2\,\mathrm{d}x$$

$$\approx 0.722+\frac{1}{2}[2x]_{-1}^{1}$$

$$=0.722+\frac{4}{2}$$

$$=2.722 \text{ (3 d.p.)}$$

8 The point A has $x=4$ and so we find the y and z coordinates by solving

$$\frac{4-3}{-1}=\frac{y-2}{-2}=\frac{z-1}{3}.$$

$$\frac{y-2}{-2}=-1\Rightarrow y=4,$$

$$\frac{z-1}{3}=-1\Rightarrow z=-2.$$

So we have the coordinates $(4,4,-2)$ for the point A.

Next we find the perpendicular distance between A and $\prod$.

Substituting $(4,4,-2)$ and the coefficients of $2x-y+3z-4=0$ into

$$\text{dist}=\frac{|ax_0+by_0+cz_0+d|}{\sqrt{a^2+b^2+c^2}} \text{ gives } \frac{3\sqrt{14}}{7}$$

Thus there is a distance of $\frac{3\sqrt{14}}{7}$ between A and $\prod$.

There will be the same distance between A' and $\prod$.

This means the distance between A and A' is $\frac{6\sqrt{14}}{7}$.

9 a We look for a particular solution of the form

$$x=A+B\sin t+C\cos t,$$

$$\frac{\mathrm{d}x}{\mathrm{d}t}=B\cos t-C\sin t,$$

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2}=-B\sin t-C\cos t.$$

Now substituting these expressions into the differential equation:

$$-B\sin t-C\cos t$$

$$+2(B\cos t-C\sin t)$$

$$+3(A+B\sin t+C\cos t)$$

$$=21+15\cos t$$

which simplifies to

$$3A+2(B-C)\sin t+2(C+B)\cos t$$

$$=21+15\cos t$$

Comparing coefficients gives

$$A=7$$

$$B-C=0\Rightarrow B=C$$

$$2\times 2C=15\Rightarrow B=C=\frac{15}{4}.$$

Thus we have the particular solution

$$x=7+\frac{15}{4}(\sin t+\cos t).$$

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9 b The auxiliary equation is $m^2 + 2m + 3 = 0$,

with solutions $m = \dfrac{-2 \pm \sqrt{4-12}}{2}$

$= -1 \pm i\sqrt{2}.$

Thus we have the complementary function

$x_c = e^{-t}\left(D\cos\left(\sqrt{2}t\right) + E\sin\left(\sqrt{2}t\right)\right).$

We now add the complementary function and the particular integral we found in the previous part in order to obtain the general solution

$x_G = e^{-t}\left(D\cos\left(\sqrt{2}t\right) + E\sin\left(\sqrt{2}t\right)\right) + 7 + \frac{15}{4}(\sin t + \cos t).$

The first derivative of the general solution is

$\dfrac{\mathrm{d}x_G}{\mathrm{d}t} = e^{-t}\left(-\sqrt{2}D\sin\left(\sqrt{2}t\right) + \sqrt{2}E\cos\left(\sqrt{2}t\right)\right)$

$-e^{-t}\left(D\cos\left(\sqrt{2}t\right) + E\sin\left(\sqrt{2}t\right)\right)$

$+\dfrac{15}{4}(\cos t - \sin t).$

Now we use the initial conditions

$x(0) = 2,\ \dfrac{\mathrm{d}x}{\mathrm{d}t}(0) = 3$ in order to find D and E.

$x_G(0) = D + 7 + \dfrac{15}{4} = D + \dfrac{43}{4} = 2$

$D = -\dfrac{35}{4}$

$\dfrac{\mathrm{d}x_G}{\mathrm{d}t}(0) = \sqrt{2}E - D + \dfrac{15}{4} = 3$

$\sqrt{2}E = 3 - \dfrac{50}{4}$

$E = -\dfrac{19\sqrt{2}}{4}$

Thus we have the solution

$x = e^{-t}\left(-\dfrac{35}{4}\cos\left(\sqrt{2}t\right) - \dfrac{19\sqrt{2}}{4}\sin\left(\sqrt{2}t\right)\right)$

$+\dfrac{15}{4}(\sin t + \cos t) + 7$

c As $t \to \infty$, $x \to \dfrac{15}{4}(\sin t + \cos t) + 7$ which is an oscillation about $x = 7$.

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Review exercise 1

1 Using Euler's solution $e^{i\theta} = \cos\theta + i\sin\theta$,

$\cos 2x + i\sin 2x = e^{i2x}$

$\cos 9x - i\sin 9x = \cos(-9x) + i\sin(-9x) = e^{i(-9x)}$

For any angle, θ, $\cos\theta = \cos(-\theta)$ and $-\sin\theta = \sin(-\theta)$
You will find these relations useful when finding the arguments of complex numbers.

Hence

$$\frac{\cos 2x + i\sin 2x}{\cos 9x - i\sin 9x} = \frac{e^{i2x}}{e^{i(-9x)}} = e^{i(2x+9x)} = e^{i11x}$$

$$= \cos 11x + i\sin 11x$$

Manipulating the arguments in $e^{i\theta}$ you use the ordinary laws of indices.

This is the required form with $n = 11$.

2 a By de Moivre's theorem

$\cos 5\theta + i\sin 5\theta = (\cos\theta + i\sin\theta)^5 = (c + is)^5$, say

$= c^5 + 5c^4 is + 10c^3 i^2 s^2 + 10c^2 i^3 s^3 + 5ci^4 s^4 + i^5 s^5$

$= c^5 + i5c^4 s - 10c^3 s^3 - i10c^2 s^3 + 5cs^4 - is^5$

It is sensible to abbreviate $\cos\theta$ and $\sin\theta$ as c and s respectively when you have as many powers of $\cos\theta$ and $\sin\theta$ to write out as you have in this question.

Use the binomial expansion.

Equating real parts

$\cos 5\theta = c^5 - 10c^3 s^2 + 5cs^4$

Using $\cos^2\theta + \sin^2\theta = 1$

$\cos 5\theta = c^5 - 10c^3(1 - c^2) + 5c(1 - c^2)^2$

$= c^5 - 10c^3 + 10c^5 + 5c - 10c^3 + 5c^5$

$= 16c^5 - 20c^3 + 5c$

$= 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta$, as required

Use $i^2 = -1$, $i^3 = -i$, $i^4 = 1$ and $i^5 = i \times i^4 = i \times 1 = i$.

b Substitute $x = \cos\theta$ into $16x^5 - 20x^3 + 5x + 1 = 0$

$16\cos^5\theta - 20\cos^3\theta + 5\cos\theta + 1 = 0$

$16\cos^5\theta - 20\cos^3\theta + 5\cos\theta = -1$

Using the result of part **a**

$\cos 5\theta = -1$

$5\theta = \pi, 3\pi, 5\pi$

$\theta = \frac{\pi}{5}, \frac{3\pi}{5}, \frac{5\pi}{5}$

Additional solutions are found by increasing the angles in steps of 2π. You are asked for 3 answers, so you need 3 angles at this stage.

$x = \cos\theta = \cos\frac{\pi}{5}, \cos\frac{3\pi}{5}, \cos\pi$

$= 0.809, -0.309, -1$

The two approximate answers are given to 3 decimal places, as the question specified; the remaining answer –1 is exact.

3 a By de Moivre's theorem

$\cos 5\theta + i\sin 5\theta = (\cos\theta + i\sin\theta)^5 = (c + is)^5$, say

$= c^5 + 5c^4 is + 10c^3 i^2 s^2 + 10c^2 i^2 s^3 + 5ci^3 s^4 + i^5 s^5$

$= c^5 + i5c^4 s - 10c^3 s^2 - i10c^2 s^3 + 5cs^4 - is^5$

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3 a Equating imaginary parts

$$\begin{aligned}\sin 5\theta &= 5c^4s - 10c^2s^3 + s^5\\ &= s(5c^4 - 10c^2s^2 + s^4)\\ &= s(5c^4 - 10c^2(1-c^2) + (1-c^2)^2)\\ &= s(5c^4 - 10c^2 + 10c^4 + 1 - 2c^2 + c^4)\\ &= s(16c^4 - 12c^2 + 1)\\ &= \sin\theta(16\cos^4\theta - 12\cos^2\theta + 1), \text{ as required}\end{aligned}$$

Repeatedly using the identity $\cos^2\theta + \sin^2\theta = 1$, which in this context is $s^2 = 1 - c^2$.

b

$$\sin 5\theta + \cos\theta \sin 2\theta = 0$$

$$\sin\theta(16\cos^4\theta - 12\cos^2\theta + 1) + 2\sin\theta\cos^2\theta = 0$$

$$\sin\theta(16\cos^4\theta - 10\cos^2\theta + 1) = 0$$

$$\sin\theta(2\cos^2\theta - 1)(8\cos^2\theta - 1) = 0$$

Using the identity proved in part **a** and the identity $\sin 2\theta = 2\sin\theta\cos\theta$.

Hence $\sin\theta = 0,\ \cos^2\theta = \frac{1}{2},\ \cos^2\theta = \frac{1}{8}$

$$\sin\theta = 0 \Rightarrow \theta = 0$$

$$\cos^2\theta = \frac{1}{2} \Rightarrow \cos\theta = \pm\frac{1}{\sqrt{2}}$$

$$\cos\theta = \frac{1}{\sqrt{2}} \Rightarrow \theta = \frac{\pi}{4}$$

$$\cos\theta = -\frac{1}{\sqrt{2}} \Rightarrow \theta = \frac{3\pi}{4}$$

You must consider the negative as well as the positive square roots.

$$\cos^2\theta = \frac{1}{8} \Rightarrow \cos\theta = \pm\frac{1}{2\sqrt{2}}$$

$$\cos\theta = \frac{1}{2\sqrt{2}} \Rightarrow \theta = 1.209 \text{ (3 d.p.)}$$

$$\cos\theta = -\frac{1}{2\sqrt{2}} \Rightarrow \theta = 1.932 \text{ (3 d.p.)}$$

The question has specified no accuracy and any sensible accuracy would be accepted for the approximate answers.

The solutions of the equation are

$0,\ \frac{\pi}{4},\ \frac{3\pi}{4},\ 1.209$ (3 d.p.) and 1.932 (3 d.p.).

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4 a $\sin\theta = \dfrac{e^{i\theta} - e^{-i\theta}}{2i}$

Let $z = e^{i\theta}$

Putting $z = e^{i\theta}$ shortens the working.

then $\sin\theta = \dfrac{z - z^{-1}}{2i}$

$$\sin^5\theta = \left(\frac{z - z^{-1}}{2i}\right)^5$$

Use Pascal's triangle to remember the coefficients in $(a+b)^5 = a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5$.

$$= \frac{1}{(2i)^5}(z^5 - 5z^4 \times z^{-1} + 10z^3 \times z^{-2} - 10z^2 \times z^{-3} + 5z \times z^{-4} - z^{-5})$$

$$= \frac{1}{32i}(z^5 - 5z^3 + 10z - 10z^{-1} + 5z^{-3} - z^{-5})$$

$$= \frac{1}{16}\left(\frac{z^5 - z^{-5}}{2i} - \frac{5(z^3 - z^{-3})}{2i} + \frac{10(z - z^{-1})}{2i}\right)$$

The general relation is $\sin n\theta = \dfrac{e^{in\theta} - e^{-in\theta}}{2i} = \dfrac{z^n - z^{-n}}{2i}$

$$= \frac{1}{16}(\sin 5\theta - 5\sin 3\theta + 10\sin\theta), \text{ as required}$$

b $\displaystyle\int_0^{\frac{\pi}{2}} \sin^5\theta \, d\theta = \frac{1}{16}\int_0^{\frac{\pi}{2}} (\sin 5\theta - 5\sin 3\theta + 10\sin\theta)\, d\theta$

Each term on the right hand side of the identity shown in part **a** can be integrated using the formula $\displaystyle\int \sin n\theta \, d\theta = -\frac{\cos n\theta}{n}$.

$$= \frac{1}{16}\left[-\frac{1}{5}\cos 5\theta + \frac{5}{3}\cos 3\theta - 10\cos\theta\right]_0^{\frac{\pi}{2}}$$

$$= \frac{1}{16}\left(0 - \left(-\frac{1}{5} + \frac{5}{3} - 10\right)\right)$$

$$= \frac{1}{16} \times \frac{128}{15} = \frac{8}{15}, \text{ as required}$$

5 a $z = \cos\theta + i\sin\theta$

Using de Moivre's theorem

$z^n = (\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$ **(1)**

From **(1)**

$$z^{-n} = \frac{1}{z^n} = \frac{1}{\cos n\theta + i\sin n\theta}$$

$$= \frac{1}{\cos n\theta + i\sin n\theta} \times \frac{\cos n\theta - i\sin n\theta}{\cos n\theta - i\sin n\theta}$$

Multiply the numerator and denominator by $\cos n\theta - i\sin n\theta$, the conjugate complex number of $\cos n\theta + i\sin n\theta$.

$$= \frac{\cos n\theta - i\sin n\theta}{\cos^2 n\theta + \sin^2 n\theta} = \cos n\theta - i\sin n\theta \quad \textbf{(2)}$$

$$z^n + z^{-n} = \cos n\theta + i\sin n\theta + \cos n\theta - i\sin n\theta$$

Use $\cos^2 n\theta + \sin^2 n\theta = 1$.

$$= 2\cos n\theta, \text{ as required.}$$

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5 b $\cos\theta = \dfrac{e^{i\theta}+e^{i\theta}}{2} = \dfrac{z+z^{-1}}{2}$

$$\cos^6\theta = \left(\frac{z+z^{-1}}{2}\right)^6$$

$$= \frac{1}{64}(z^6 + 6z^5z^{-1} + 15z^4z^{-2} + 20z^3z^{-3} + 15z^2z^{-4} + 6z^1z^{-5} + z^{-6})$$

Pair the terms as shown.

$$= \frac{1}{64}(z^6 + 6z^4 + 15z^2 + 20 + 15z^{-2} + 6z^{-4} + z^{-6})$$

$$= \frac{1}{32}\left(\frac{z^6+z^{-6}}{2} + \frac{6(z^4+z^{-4})}{2} + \frac{15(z^2+z^{-2})}{2} + \frac{20}{2}\right)$$

You use $\dfrac{z^n+z^{-n}}{2} = \cos n\theta$ with $n = 6$, 4 and 2.

$$= \frac{1}{32}(\cos 6\theta + 6\cos 4\theta + 15\cos 2\theta + 10)$$

c $\displaystyle\int_0^{\frac{\pi}{2}} \cos^6\theta\,\mathrm{d}\theta = \frac{1}{32}\int_0^{\frac{\pi}{2}} (\cos 6\theta + 6\cos 4\theta + 15\cos 2\theta + 10)\,\mathrm{d}\theta$

$$= \frac{1}{32} - \left[\frac{1}{6}\sin 6\theta + \frac{6}{4}\sin 4\theta + \frac{15}{2}\sin 2\theta + 10\theta\right]_0^{\frac{\pi}{2}}$$

With the exception of 10θ all of these terms have value 0 at both the upper and the lower limit.

$$= \frac{1}{32}\times 10\times\frac{\pi}{2} = \frac{5\pi}{32}, \text{ as required}$$

6 $C + iS = \displaystyle\sum_{r=0}^{n-1} e^{ir\theta} = \frac{e^{in\theta}-1}{e^{i\theta}-1} = \frac{\cos n\theta + i\sin n\theta - 1}{\cos\theta + i\sin\theta - 1}$

$$= \frac{(\cos n\theta - 1 + i\sin n\theta)(\cos\theta - 1 - i\sin\theta)}{(\cos\theta - 1 + i\sin\theta)(\cos\theta - 1 - i\sin\theta)}$$

$$= \frac{\cos n\theta\cos\theta - \cos\theta - \cos n\theta + 1 + \sin n\theta\sin\theta}{\cos^2\theta - 2\cos\theta + 1 + \sin^2\theta}$$

$$+ i\frac{\sin\theta - \cos n\theta\sin\theta + \sin n\theta\cos\theta - \sin n\theta}{\cos^2\theta - 2\cos\theta + 1 + \sin^2\theta}$$

$$C = \mathrm{Re}(C+iS) = \frac{\cos(n-1)\theta - \cos\theta - \cos n\theta + 1}{2 - 2\cos\theta}$$

$$S = \mathrm{Im}(C+iS) = \frac{\sin(n-1)\theta + \sin\theta - \sin n\theta}{2 - 2\cos\theta}$$

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7 a Let $4+4\mathrm{i}=r(\cos\theta+\mathrm{i}\sin\theta)=r\cos\theta+\mathrm{i}r\sin\theta$

Equating real parts

$4=r\cos\theta$ **(1)**

Equating imaginary parts

$4=r\sin\theta$ **(2)**

Dividing **(2)** by **(1)**

$\tan\theta=1\Rightarrow\theta=\dfrac{\pi}{4}$

Substituting $\theta=\dfrac{\pi}{4}$ into **(1)**

$4=r\cos\dfrac{\pi}{4}\Rightarrow 4=r\times\dfrac{1}{\sqrt{2}}\Rightarrow r=4\sqrt{2}$

Hence

$4+4\mathrm{i}=4\sqrt{2}\left(\cos\dfrac{\pi}{4}+\mathrm{i}\sin\dfrac{\pi}{4}\right)$

$=2^{\frac{5}{2}}\mathrm{e}^{\mathrm{i}\frac{\pi}{4}}$

$z^5=2^{\frac{5}{2}}\mathrm{e}^{\mathrm{i}\frac{\pi}{4}},\ 2^{\frac{5}{2}}\mathrm{e}^{\mathrm{i}\frac{9\pi}{4}},2^{\frac{5}{2}}\mathrm{e}^{\mathrm{i}\frac{17\pi}{4}},\ 2^{\frac{5}{2}}\mathrm{e}^{\mathrm{i}\frac{25\pi}{4}},\ 2^{\frac{5}{2}}\mathrm{e}^{\mathrm{i}\frac{33\pi}{4}}$

$z=2^{\frac{1}{2}}\mathrm{e}^{\mathrm{i}\frac{\pi}{20}},\ 2^{\frac{1}{2}}\mathrm{e}^{\mathrm{i}\frac{9\pi}{20}},\ 2^{\frac{1}{2}}\mathrm{e}^{\mathrm{i}\frac{17\pi}{20}},\ 2^{\frac{1}{2}}\mathrm{e}^{\mathrm{i}\frac{25\pi}{20}},\ 2^{\frac{1}{2}}\mathrm{e}^{\mathrm{i}\frac{33\pi}{20}}$

This is the required form with $r=\sqrt{2}$ and

$k=\dfrac{1}{20},\dfrac{9}{20},\dfrac{17}{20},\dfrac{25}{20}\left(=\dfrac{5}{4}\right),\dfrac{33}{20}.$

Finding the roots of a complex number is usually easier if you obtain the number in the form $re^{i\theta}$. As you will use Euler's relation, the first step towards this is to get the complex number into the form $r(\cos\theta+\mathrm{i}\sin\theta)$.

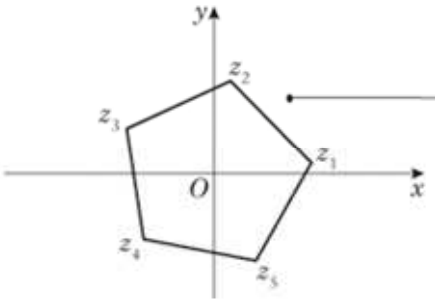
To take the fifth root, write $4\sqrt{2}=2^{\frac{5}{2}}$.

For example, if $z^5=2^{\frac{5}{2}}\mathrm{e}^{\mathrm{i}\frac{9\pi}{4}}$ then

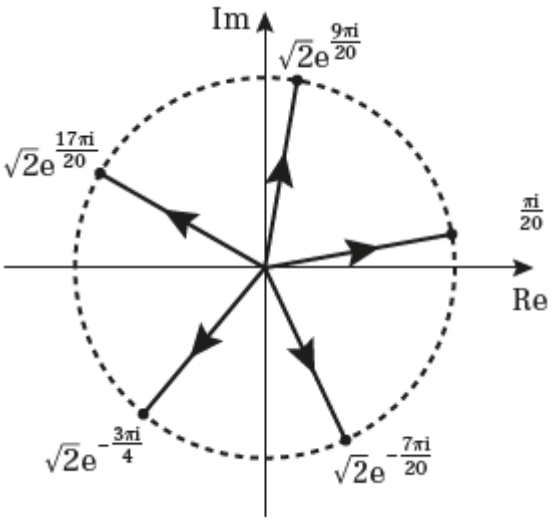
$z=\left(2^{\frac{5}{2}}\mathrm{e}^{\mathrm{i}\frac{9\pi}{4}}\right)^{\frac{1}{5}}=2^{\frac{5}{2}\times\frac{1}{5}}\mathrm{e}^{\mathrm{i}\frac{9\pi}{4}\times\frac{1}{5}}=2^{\frac{1}{2}}\mathrm{e}^{\mathrm{i}\frac{9\pi}{20}}.$

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7 b



The points representing the 5 roots are the vertices of a regular pentagon.



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8 a Let $32+32\sqrt{3}\mathrm{i} = r(\cos\theta + \mathrm{i}\sin\theta) = r\cos\theta + \mathrm{i}r\sin\theta$

Equating real parts

$32 = r\cos\theta$ **(1)**

Equating imaginary parts

$32\sqrt{3} = r\sin\theta$ **(2)**

Dividing **(2)** by **(1)**

$\tan\theta = \sqrt{3} \Rightarrow \theta = \dfrac{\pi}{3}$

Substituting $\theta = \dfrac{\pi}{3}$ into **(1)**

$32 = r\cos\dfrac{\pi}{3} \Rightarrow 32 = r \times \dfrac{1}{2} \Rightarrow r = 64$

Hence

$32+32\sqrt{3}\mathrm{i} = 64\left(\cos\dfrac{\pi}{3} + \mathrm{i}\sin\dfrac{\pi}{3}\right)$

$= 64\mathrm{e}^{\mathrm{i}\frac{\pi}{3}}$

$z^3 = 64\mathrm{e}^{\mathrm{i}\frac{\pi}{3}}, 64\mathrm{e}^{\mathrm{i}\frac{7\pi}{3}}, 64\mathrm{e}^{\mathrm{i}\frac{-5\pi}{3}}$

$z = 4\mathrm{e}^{\mathrm{i}\frac{\pi}{9}}, 4\mathrm{e}^{\mathrm{i}\frac{7\pi}{9}}, 4\mathrm{e}^{-\mathrm{i}\frac{5\pi}{9}}$

The solutions are $r\mathrm{e}^{\mathrm{i}\theta}$ where $r = 4$ and

$\theta = -\dfrac{5\pi}{9}, \dfrac{\pi}{9}, \dfrac{7\pi}{9}$

Finding the roots of a complex number is usually easier if you obtain the number in the form $r\mathrm{e}^{\mathrm{i}\theta}$. As you will use Euler's relation, the first step towards this is to get the complex number into the form $r(\cos\theta + \mathrm{i}\sin\theta)$.

Additional solutions are found by increasing or decreasing the arguments in steps of 2π. You area asked for 3 answers, so you need 3 arguments. Had you increased the argument $\frac{7\pi}{3}$ by 2π to $\frac{13\pi}{3}$, this would have given a correct solution to the equation but it would lead to $\theta = \frac{13\pi}{9}$, which does not satisfy the condition $\theta \le \pi$ in the question. So the third argument has to be found by subtracting 2π from $\frac{\pi}{3}$.

b $z = 4\mathrm{e}^{\mathrm{i}\frac{\pi}{9}}, 4\mathrm{e}^{\mathrm{i}\frac{7\pi}{9}}, 4\mathrm{e}^{-\mathrm{i}\frac{-5\pi}{9}}$

$z^9 = \left(4\mathrm{e}^{\mathrm{i}\frac{\pi}{9}}\right)^9, \left(4\mathrm{e}^{\mathrm{i}\frac{7\pi}{9}}\right)^9, \left(4\mathrm{e}^{-\mathrm{i}\frac{-5\pi}{9}}\right)^9$

$= 4^9\mathrm{e}^{\mathrm{i}\pi}, 4^9\mathrm{e}^{\mathrm{i}7\pi}, 4^9\mathrm{e}^{-\mathrm{i}5\pi}$

$\mathrm{e}^{\mathrm{i}\pi} = \cos\pi + \mathrm{i}\sin\pi = -1 + 0 = -1$. Similarly for the arguments 7π and -5π.

The value of all three of these expressions is $-4^9 = -2^{18}$

Hence the solutions satisfy $z^9 + 2^k = 0$, where $k = 18$.

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9 $\mathrm{i} = \cos\frac{\pi}{2} + \mathrm{i}\sin\frac{\pi}{2} = \mathrm{e}^{\mathrm{i}\frac{\pi}{2}}$

$z^5 = \mathrm{e}^{\mathrm{i}\frac{\pi}{2}}, \mathrm{e}^{\mathrm{i}\frac{5\pi}{2}}, \mathrm{e}^{\mathrm{i}\frac{9\pi}{2}}, \mathrm{e}^{\mathrm{i}\frac{13\pi}{2}}, \mathrm{e}^{\mathrm{i}\frac{17\pi}{2}},$

$z = \mathrm{e}^{\mathrm{i}\frac{\pi}{10}}, \mathrm{e}^{\mathrm{i}\frac{5\pi}{10}}, \mathrm{e}^{\mathrm{i}\frac{9\pi}{10}}, \mathrm{e}^{\mathrm{i}\frac{13\pi}{10}}, \mathrm{e}^{\mathrm{i}\frac{17\pi}{10}}$

Hence

$z = \cos\theta + \mathrm{i}\sin\theta$, where

$\theta = \frac{\pi}{10}, \frac{5\pi}{10}\left(=\frac{\pi}{2}\right), \frac{9\pi}{10}, \frac{13\pi}{10}, \frac{17\pi}{10}$

The modulus of the complex number i is 1 and its argument is $\frac{\pi}{2}$. So $\mathrm{i} = 1\mathrm{e}^{\mathrm{i}\frac{\pi}{2}}$.

Additional solutions are found by increasing the arguments in steps of 2π. As the equation is of degree 5, there are exactly 5 distinct answers.

For example, if $z^5 = \mathrm{e}^{\mathrm{i}\frac{9\pi}{2}}$ then $z = \left(\mathrm{e}^{\mathrm{i}\frac{9\pi}{2}}\right)^{\frac{1}{5}} = \left(\mathrm{e}^{\mathrm{i}\frac{9\pi}{10}}\right)$.

10 a $z^5 = 16 + 16i\sqrt{3} = 32\left(\cos\left(\frac{\pi}{3} + 2k\pi\right) + i\sin\left(\frac{\pi}{3} + 2k\pi\right)\right)$

as $\sqrt{16^2 + \left(16\sqrt{3}\right)^2} = 32, \arctan\frac{16\sqrt{3}}{16} = \frac{\pi}{3}$

$z = r(\cos\theta + i\sin\theta)$

$z^5 = r^5(\cos 5\theta + i\sin 5\theta)$

$r^5 = 32, r = 2$

$\theta = \frac{\pi}{15} + \frac{2k\pi}{5} = \frac{\pi}{15}, \frac{7\pi}{15}, \frac{13\pi}{15}, -\frac{\pi}{3}, -\frac{11\pi}{15}$

$z = 2e^{\frac{i\pi}{15}}, 2e^{\frac{7i\pi}{15}}, 2e^{\frac{13i\pi}{15}}, 2e^{-\frac{i\pi}{3}}, 2e^{-\frac{11i\pi}{15}}$

b The polygon is a regular pentagon.

11 a

$z^5 = 1$

$z = 1, e^{\frac{2i\pi}{5}}, e^{\frac{4i\pi}{5}}, e^{-\frac{2i\pi}{5}}, e^{-\frac{4i\pi}{5}}$

$\sum_{r=0}^{n-1} wz^r = \frac{w(z^n - 1)}{z - 1}$

$w = e^{-\frac{4i\pi}{5}}, z = e^{\frac{2i\pi}{5}}$

$1 + e^{\frac{2i\pi}{5}} + e^{\frac{4i\pi}{5}} + e^{-\frac{2i\pi}{5}} + e^{-\frac{4i\pi}{5}} = e^{-\frac{4i\pi}{5}} \frac{\left(e^{\frac{2i\pi}{5}}\right)^5 - 1}{e^{\frac{2i\pi}{5}} - 1}$

$= e^{-\frac{4i\pi}{5}} \frac{e^{2i\pi} - 1}{e^{\frac{2i\pi}{5}} - 1} = 0$

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11 b $(2-3)+i(1-0)=-1+i=\sqrt{2}e^{-\frac{i\pi}{4}}$

$z=2+i+\sqrt{2}e^{-\frac{i\pi}{4}}=3$

$z=2+i+\sqrt{2}e^{-\frac{i\pi}{4}}e^{\frac{2i\pi}{5}}=2+i+\sqrt{2}e^{\frac{3i\pi}{20}}$

$=3.26+1.64i$ (2 d.p.)

$z=2+i+\sqrt{2}e^{\frac{11i\pi}{20}}=1.78+2.40i$ (2 d.p.)

$z=2+i+\sqrt{2}e^{\frac{19i\pi}{20}}=0.60+1.22i$ (2 d.p.)

$z=2+i+\sqrt{2}e^{-\frac{13i\pi}{20}}=1.36-0.26i$ (2 d.p.)

Vertices of polygon are at

$(3,0),(3.26,1.64),(1.78,2.40),$

$(0.60,1.22),(1.36,-0.26)$

12 $\dfrac{2}{(r+1)(r+2)}=\dfrac{A}{r+1}+\dfrac{B}{r+2}$

$2=A(r+2)+B(r+1)$

$2=A=-B$

Let $\mathrm{f}(r)=\dfrac{1}{r+1}$

$$\sum_{r=1}^{n}\frac{2}{(r+1)(r+2)}=2\sum_{r=1}^{n}\left(\frac{1}{r+1}-\frac{1}{r+2}\right)$$

$$=2(f(1)-f(n+1))$$

$$=2\left(\frac{1}{2}-\frac{1}{n+2}\right)$$

$$=1-\frac{2}{n+2}=\frac{n}{n+2}$$

13 $\dfrac{2}{(r+1)(r+3)}=\dfrac{A}{r+1}+\dfrac{B}{r+3}$

$2=A(r+3)+B(r+1)$

$2=2A=-2B$

Let $\mathrm{f}(r)=\dfrac{1}{r+1}$

$$\sum_{r=1}^{n}\frac{2}{(r+1)(r+3)}=\sum_{r=1}^{n}\left(\frac{1}{r+1}-\frac{1}{r+3}\right)$$

$$=\mathrm{f}(1)+\mathrm{f}(2)-\mathrm{f}(n+1)-\mathrm{f}(n+2)$$

$$=\frac{1}{2}+\frac{1}{3}-\frac{1}{n+2}-\frac{1}{n+3}$$

$$=\frac{5(n+2)(n+3)-6(n+3)-6(n+2)}{6(n+2)(n+3)}$$

$$=\frac{n(5n+13)}{6(n+2)(n+3)}$$

Hence $a=5, b=13$ and $c=6$

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14 a $\text{LHS} = \dfrac{r+1}{r+2} - \dfrac{r}{r+1}$

$= \dfrac{(r+1)^2 - r(r+2)}{(r+1)(r+2)}$

$= \dfrac{r^2 + 2r + 1 - r^2 - 2r}{(r+1)(r+2)}$

$= \dfrac{1}{(r+1)(r+2)}$

$= \text{RHS, as required}$

To show that an algebraic identity is true, you should start from one side of the identity, here the left hand side (LHS), and use algebra to show that it is equal to the other side of the identity, here the right hand side (RHS).

b $\displaystyle\sum_{r=1}^{n} \frac{1}{(r+1)(r+2)} = \sum_{r=1}^{n} \left(\frac{r+1}{r+2} - \frac{r}{r+1} \right)$

You use the identity that you proved in part **a** to break up each term in the summation into two parts.

$= \cancel{\frac{2}{3}} - \frac{1}{2}$

This is the LHS of the identity with $r = 1$.

$+ \cancel{\frac{3}{4}} - \cancel{\frac{2}{3}}$

This is the LHS of the identity with $r = 2$.

$+ \cancel{\frac{4}{5}} - \cancel{\frac{3}{4}}$

This is the LHS of the identity with $r = 3$.

$\vdots$

$+ \cancel{\frac{n}{n+1}} - \cancel{\frac{n-1}{n}}$

This is the LHS of the identity with $r = n - 1$.

$\dfrac{r+1}{r+2} - \dfrac{r}{r+1} = \dfrac{n-1+1}{n-1+2} - \dfrac{n-1}{n-1+1}$

$= \dfrac{n}{n+1} - \dfrac{n-1}{n}$

$+ \cancel{\frac{n+1}{n+2}} - \cancel{\frac{n}{n+1}}$

This is the LHS of the identity with $r = n$.

$= \dfrac{n+1}{n+2} - \dfrac{1}{2}$

$= \dfrac{2(n+1) - (n+2)}{2(n+2)} = \dfrac{2n+2-n-2}{2(n+2)}$

$= \dfrac{n}{2(n+2)}$

The only terms which have not cancelled one another out are the $-\dfrac{1}{2}$ in the first line of the summation and the $\dfrac{n+1}{n+2}$ in the last line.

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15 a Let $\dfrac{2}{(x+1)(x+2)(x+3)} = \dfrac{A}{x+1} + \dfrac{B}{x+2} + \dfrac{C}{x+3}$

Multiplying throughout by $(x+1)(x+2)(x+3)$

$2 = A(x+2)(x+3) + B(x+1)(x+3) + C(x+1)(x+2)$

Substitute $x = -1$

$2 = A \times 1 \times 2 \Rightarrow A = 1$

Substitute $x = -2$

$2 = B \times -1 \times 1 \Rightarrow B = -2$

Substitute $x = -3$

$2 = C \times -2 \times -1 \Rightarrow C = 1$

When -1 is substituted for x then both $B(x+1)(x+3)$ and $C(x+1)(x+2)$ become zero.

Hence

$$f(x) = \frac{1}{x+1} - \frac{2}{x+2} + \frac{1}{x+3}$$

You use the partial fractions in part **a** to break up each term in the summation into three parts.

b Using the result in part **a** with $x = r$

$$\sum_{r=1}^{n} f(r) = \frac{1}{r+1} - \frac{2}{r+2} + \frac{1}{r+3}$$

$$= \frac{1}{2} - \frac{2}{3} + \cancel{\frac{1}{4}}$$
$$+ \frac{1}{3} - \cancel{\frac{2}{4}} + \cancel{\frac{1}{5}}$$
$$+ \cancel{\frac{1}{4}} - \cancel{\frac{2}{5}} + \cancel{\frac{1}{6}}$$
$$\vdots$$
$$+ \cancel{\frac{1}{n-1}} - \cancel{\frac{2}{n}} + \cancel{\frac{1}{n+1}}$$
$$+ \cancel{\frac{1}{n}} - \cancel{\frac{2}{n+1}} + \frac{1}{n+2}$$
$$+ \cancel{\frac{1}{n+1}} - \frac{2}{n+2} + \frac{1}{n+3}$$
$$= \frac{1}{2} - \frac{2}{3} + \frac{1}{3} + \frac{1}{n+2} - \frac{2}{n+2} + \frac{1}{n+3}$$
$$= \frac{1}{6} - \frac{1}{n+2} + \frac{1}{n+3}$$

Three terms at the beginning of the summation and three terms at the end have not been cancelled out.

This question asks for no particular form of the answer. You should collect together like terms but, otherwise, the expression can be left as it is. You do not have to express your answer as a single fraction unless the question asks you to do this.

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16 a $\dfrac{1}{(r-1)^2}-\dfrac{1}{r^2}=\dfrac{r^2-(r-1)^2}{r^2(r-1)^2}$

$=\dfrac{r^2-(r^2-2r+1)}{r^2(r-1)^2}$

$=\dfrac{2r-1}{r^2(r-1)^2}$

Methods for simplifying algebraic fractions can be found in Chapter 1 of book C3.

b $\displaystyle\sum_{r=2}^{n}\frac{2r-1}{r^2(r-1)^2}=\sum_{r=2}^{n}\left(\frac{1}{(r-1)^2}-\frac{1}{r^2}\right)$

$=\dfrac{1}{1^2}-\cancel{\dfrac{1}{2^2}}$

$+\cancel{\dfrac{1}{2^2}}-\cancel{\dfrac{1}{3^2}}$

$+\cancel{\dfrac{1}{3^2}}-\cancel{\dfrac{1}{4^2}}$

$\vdots$

$+\cancel{\dfrac{1}{(n-2)^2}}-\cancel{\dfrac{1}{(n-1)^2}}$

$+\cancel{\dfrac{1}{(n-1)^2}}-\dfrac{1}{n^2}$

$=\dfrac{1}{1^2}-\dfrac{1}{n^2}=1-\dfrac{1}{n^2}$, as required

This summation starts from $r=2$ and not from the more common $r=1$. It could not start from $r=1$ as $\dfrac{1}{(r-1)^2}$ is not defined for that value.

In this summation all of the terms cancel out with one another except for one term at the beginning and one term at the end.

17 a

$\dfrac{4}{r(r+2)}=\dfrac{A}{r}+\dfrac{B}{r+2}$

$4=A(r+2)+Br$

$4=2A=-2B$

Let $\mathrm{f}(r)=\dfrac{1}{r}$

$\displaystyle\sum_{r=1}^{n}\frac{4}{r(r+2)}=2\sum_{r=1}^{n}\left(\frac{1}{r}-\frac{1}{r+2}\right)$

$=2(\mathrm{f}(1)+\mathrm{f}(2)-\mathrm{f}(n+1)-\mathrm{f}(n+2))$

$=2\left(1+\dfrac{1}{2}-\dfrac{1}{n+1}-\dfrac{1}{n+2}\right)$

$=\dfrac{3(n+1)(n+2)-2(n+2)-2(n+1)}{(n+1)(n+2)}$

$=\dfrac{n(3n+5)}{(n+1)(n+2)}$

Hence $a=3$ and $b=5$

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17 b $\displaystyle\sum_{r=50}^{100}\frac{4}{r(r+2)}=\sum_{r=1}^{100}\frac{4}{r(r+2)}-\sum_{r=1}^{49}\frac{4}{r(r+2)}$

$$=\frac{100\times305}{101\times102}-\frac{49\times152}{50\times51}$$

$$=2.960\,590\ldots-2.920\,784$$

$$=0.0398\ (4\text{ d.p.})$$

$\displaystyle\sum_{r=50}^{100}f(r)=\sum_{r=1}^{100}f(r)-\sum_{r=1}^{49}f(r)$

You find the sum from the 50th to the 100th term by subtracting the sum from the first to the 49th term from the sum from the first to the 100th term.

It is a common error to subtract one term too many, in this case the 50th term. The sum you are finding starts with the 50th term. You must not subtract it from the series – you have to leave it in the series.

18 a $4r^2-1=(2r-1)(2r+1)$

Let

$$\frac{2}{4r^2-1}=\frac{2}{(2r-1)(2r+1)}=\frac{A}{2r-1}+\frac{B}{2r+1}$$

Multiply throughout by $(2r-1)(2r+1)$

$$2=A(2r+1)+B(2r-1)$$

Substitute $r=\frac{1}{2}$

$$2=2A\Rightarrow A=1$$

Substitute $r=-\frac{1}{2}$

$$2=-2B\Rightarrow B=-1$$

Hence

$$\frac{2}{4r^2-1}=\frac{1}{2r-1}-\frac{1}{2r+1}$$

$$\sum_{r=1}^{n}\frac{2}{4r^2-1}=\sum_{r=1}^{n}\left(\frac{1}{2r-1}-\frac{1}{2r+1}\right)$$

$$=\frac{1}{1}-\cancel{\frac{1}{3}}$$

$$+\cancel{\frac{1}{3}}-\cancel{\frac{1}{5}}$$

$$+\cancel{\frac{1}{5}}-\cancel{\frac{1}{7}}$$

$$\vdots$$

$$+\cancel{\frac{1}{2n-3}}-\cancel{\frac{1}{2n-1}}$$

$$+\cancel{\frac{1}{2n-1}}-\frac{1}{2n+1}$$

$$=1-\frac{1}{2n+1},\text{ as required}$$

This question gives you the option to choose your own method (the question has 'or otherwise') and, as you are given the answer, you could, if you preferred, use the method of mathematical induction.

If the method of differences is used, you begin by factorising $4r^2-1$, using the difference of two squares, and then express $\frac{2}{(2r-1)(2r+1)}$ in partial fractions.

With $r=1$,

$$\frac{1}{2r-1}-\frac{1}{2r+1}=\frac{1}{2\times1-1}-\frac{1}{2\times1+1}=\frac{1}{1}-\frac{1}{3}$$

With $r=n-1$,

$$\frac{1}{2r-1}-\frac{1}{2r+1}=\frac{1}{2\times(n-1)-1}-\frac{1}{2\times(n-1)+1}$$

$$=\frac{1}{2n-2-1}-\frac{1}{2n-2+1}=\frac{1}{2n-3}-\frac{1}{2n-1}$$

The only terms which are not cancelled out in the summation are the $\frac{1}{1}$ at the beginning and the $-\frac{1}{2n+1}$ at the end.

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18 b $\displaystyle\sum_{r=11}^{n}\frac{2}{4r^2-1}=\sum_{r=1}^{20}\frac{2}{4r^2-1}-\sum_{r=1}^{10}\frac{2}{4r^2-1}$

$\displaystyle=\left(1-\frac{1}{41}-1+\frac{1}{21}\right)$

$\displaystyle=-\frac{1}{41}+\frac{1}{21}=\frac{-21+41}{41\times 21}$

$\displaystyle=\frac{20}{861}$

You find the sum from the 11th to the 20th term by subtracting the sum from the first to the 10th term from the sum from the first to the 20th term.

The conditions of the question require an exact answer, so you must not use decimals.

19 a Using the binomial expansion

$(2r+1)^3=8r^3+12r^2+6r+1$ **(1)**

$(2r-1)^3=8r^3-12r^2+6r-1$ **(2)**

Subtracting **(2)** from **(1)**

$(2r+1)^3-(2r-1)^3=24r^2+2$ **(3)**

$A=24,\ B=2$

Subtracting the two expansions gives an expression in r^2. This enables you to sum r^2 using the method of differences.

b Using identity **(3)** in part **a**

$\displaystyle\sum_{r=1}^{n}(24r^2+2)=\sum_{r=1}^{n}\left((2r+1)^3-(2r-1)^3\right)$

$\displaystyle 24\sum_{r=1}^{n}r^2+\sum_{r=1}^{n}2=\sum_{r=1}^{n}\left((2r+1)^3-(2r-1)^3\right)$

$\displaystyle 24\sum_{r=1}^{n}r^2+2n=3^3-1^3$

$+5^3+3^3$

$+7^3-5^3$

$\vdots$

$+(2n-1)^3-(2n-3)^3$

$+(2n+1)^3-(2n-1)^3$

$\displaystyle\sum_{r=1}^{n}2=\underbrace{2+2+2+\ldots+2}_{n\text{ times}}=2n$

It is a common error to write $\displaystyle\sum_{r=1}^{n}2=2.$

The expression is $(2r+1)^3-(2r-1)^3$ with $n-1$ substituted for r. $(2(n-1)+1)^3-(2(n-1)-1)^3$

$=(2n-1)^3-(2n-3)^3$

$\displaystyle 24\sum_{r=1}^{n}r^2+2n=(2n+1)^3-1$

$\displaystyle 24\sum_{r=1}^{n}r^2=8n^2+12n^2+6n+1-1-2n$

$=8n^3+12n^2+4n=4n(2n^2+3n+1)$

$=4n(n+1)(2n+1)$

$\displaystyle\sum_{r=1}^{n}r^2=\frac{4n(n+1)(2n+1)}{24}=\frac{1}{6}n(n+1)(2n+1)$, as required.

Summing gives you an equation in $\sum r^2$, which you solve. You then factorise the result to give the answer in the form required by the question.

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19 c $(3r-1)^2 = 9r^2 - 6r + 1$

Hence

$$\sum_{r=1}^{40}(3r-1)^2 = 9\sum_{r=1}^{40} r^2 - 6\sum_{r=1}^{40} r + \sum_{r=1}^{40} 1$$

In the formula proved in part **b**, you replace the n by 40.

Using the result in part **b**.

$$9\sum_{r=1}^{40} r^2 = 9\times\frac{1}{6}\times 40\times 41\times 81 = 199\ 260$$

Using the standard result $\sum_{r=1}^{n} r = \frac{n(n+1)}{2}$,

$$6\sum_{r=1}^{40} r = 6\times\frac{40\times 41}{2} = 4920$$

$$\sum_{r=1}^{40} 1 = 40$$

$\sum_{r=1}^{40} 1 = 40$ is 40 ones added together which is, of course, 40.

Combining these results

$$\sum_{r=1}^{40}(3r-1)^2 = 199\ 260 - 4920 + 40 = 194\ 380$$

20 $\dfrac{1}{r(r+1)(r+2)} = \dfrac{A}{r} + \dfrac{B}{r+1} + \dfrac{C}{r+2}$

$1 = A(r+1)(r+2) + Br(r+2) + Cr(r+1)$

$r = 0 : 1 = 2A$

$r = 1 : 1 = -B$

$r = 2 : 1 = 2C$

Let $\mathrm{f}(r) = \dfrac{1}{r}$

$$\sum_{r=1}^{2n}\frac{1}{r(r+1)(r+2)} = \frac{1}{2}\sum_{r=1}^{2n}\left(\frac{1}{r} - \frac{1}{r+1} + \frac{1}{r+2} - \frac{1}{r+1}\right)$$

$$= \frac{1}{2}(f(1) - f(2n+1) + f(2n+2) - f(2))$$

$$= \frac{1}{2}\left(1 - \frac{1}{2} - \frac{1}{2n+1} + \frac{1}{2n+2}\right)$$

$$= \frac{1}{4}\left(1 - \frac{2}{2n+1} + \frac{1}{n+1}\right)$$

$$= \frac{1}{4}\left(\frac{(n+1)(2n+1) - 2(n+1) + (2n+1)}{(n+1)(2n+1)}\right)$$

$$= \frac{1}{4}\frac{2n^2+3n}{(n+1)(2n+1)} = \frac{n(2n+3)}{4(n+1)(2n+1)}$$

Hence $a = 2, b = 3$ and $c = 4$

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21 a $\text{RHS} = r - 1 + \dfrac{1}{r} - \dfrac{1}{r+1}$

$= \dfrac{(r-1)r(r+1) + (r+1) - r}{r(r+1)}$

$= \dfrac{r(r^2-1)+1}{r(r+1)}$

$= \dfrac{r^3 - r + 1}{r(r+1)} = \text{LHS, as required}$

To show that an algebraic identity is true, you should start from one side of the identity, here the right hand side (RHS), and use algebra to show that it is equal to the other side of the identity, here the left hand side (LHS).

b Using the result in part **a**

This summation is broken up into 3 separate summations. Only the third of these uses the method of differences.

$$\sum_{r=1}^{n} \frac{r^3 - r + 1}{r(r+1)} = \sum_{r=1}^{n} r - \sum_{r=1}^{n} 1 + \sum_{r=1}^{n} \left(\frac{1}{r} - \frac{1}{r+1} \right)$$

$$\sum_{r=1}^{n} r = \frac{n(n+1)}{2}$$

$$\sum_{r=1}^{n} 1 = n$$

$$\sum_{r=1}^{n} \left(\frac{1}{r} - \frac{1}{r+1} \right) = \frac{1}{1} - \cancel{\frac{1}{2}}$$
$$+ \cancel{\frac{1}{2}} - \cancel{\frac{1}{3}}$$
$$+ \cancel{\frac{1}{3}} - \cancel{\frac{1}{4}}$$
$$\vdots$$
$$+ \cancel{\frac{1}{n-1}} + \cancel{\frac{1}{n}}$$
$$+ \cancel{\frac{1}{n}} - \frac{1}{n+1}$$
$$= 1 + \frac{1}{n+1}$$

In the summation, using the method of differences, all of the terms cancel out with one another except for one term at the beginning and one term at the end.

Combining the three summations

$$\sum_{r=1}^{n} \frac{r^3 - r + 1}{r(r+1)} = \frac{n(n+1)}{2} - n + 1 - \frac{1}{n+1}$$

$$= \frac{n(n+1)^2 - 2n(n+1) + 2(n+1) - 2}{2(n+1)}$$

$$= \frac{n^3 + 2n^2 + n - 2n^2 - 2n + 2n + 2 - 2}{2(n+1)}$$

$$= \frac{n^3 + n}{2(n+1)} = \frac{n(n^2+1)}{2(n+1)}$$

To complete the question, you put the results of the three summations over a common denominator and simplify the resulting expression as far as possible.

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22 $\dfrac{2r+3}{r(r+1)} = \dfrac{A}{r} + \dfrac{B}{r+1}$

$2r+3 = A(r+1) + Br$

$3 = A, 1 = -B$

$$\frac{2r+3}{r(r+1)}\frac{1}{3^r} = \frac{1}{3^r}\left(\frac{3}{r} - \frac{1}{r+1}\right)$$

$$= \frac{1}{3^{r-1}}\frac{1}{r} - \frac{1}{3^r}\frac{1}{r+1}$$

Let $\mathrm{f}(r) = \dfrac{1}{3^{r-1}}\dfrac{1}{r}$

$$\sum_{r=1}^{n}\frac{2r+3}{r(r+1)}\frac{1}{3^r} = \sum_{r=1}^{n}\left(\mathrm{f}(r) - \mathrm{f}(r+1)\right)$$

$$= (\mathrm{f}(1) - \mathrm{f}(n+1))$$

$$= 1 - \frac{1}{3^n(n+1)}$$

23 $\cos x = 1 - \dfrac{x^2}{2!} + \dfrac{x^4}{4!} - \ldots$

$= 1 - \dfrac{x^2}{2}$, neglecting terms in x^3 and higher powers

$\sin x = x - \dfrac{x^3}{3!} + \dfrac{x^5}{5!} - \ldots$

$= x$, neglecting terms in x^3 and higher powers

The series of cos x and sin x are both given in the formulae book and may be quoted without proof, unless the question specifically asks for a proof.

$$11\sin x - 6\cos x + 5 = 11x - 6\left(1 - \frac{x^2}{2}\right) + 5$$

$$= 11x - 6 + 3x^2 + 5$$

$$= -1 + 11x + 3x^2$$

You substitute the abbreviated series into the expression and collect together terms.

$A = -1,\ B = 11,\ C = 3$

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24 $\text{LHS} = \ln(x^2 - x + 1) + \ln(x + 1) - 3\ln x$

$= \ln[(x^2 - x + 1)(x + 1)] - \ln x^3$

$= \ln\left(\dfrac{x^3 + 1}{x^3}\right) = \ln\left(1 + \dfrac{1}{x^3}\right)$

You collect together the three terms of the left hand side (LHS) of the expression into a single logarithm using all three log rules; $\log x + \log y = \log xy$

$\log x - \log y = \log\left(\dfrac{x}{y}\right)$,

and $n \log x = \log x^n$.

$(x^2 - x + 1)(x + 1) = x^3 + x^2 - x^2 - x + x + 1$

$= x^3 + 1$

Substituting $\dfrac{1}{x^3}$ for x and n for r in the series

This series is given in the formulae booklet. It is valid for $-1 < x \leqslant 1$ and, if $x > 1$, then $0 < \dfrac{1}{x^3} < 1$ so the series is valid for this question.

$\ln(1 + x) = x - \dfrac{x^2}{2} + \dfrac{x^3}{3} + \ldots + \dfrac{(-1)^{r+1}x^t}{r} + \ldots$

$\text{LHS} = \dfrac{1}{x^3} - \dfrac{1}{2x^6} + \ldots + \dfrac{(-1)^{n-1}}{nx^{3n}} + \ldots$, as required

$(-1)^{n+1} = (-1)^{n-1}$. If n is odd, both sides are 1. If n is even, both sides are –1.

25 $e^{-2x} = 1 + (-2x) + \dfrac{(-2x)^2}{2!} + \dfrac{(-2x)^3}{3!} + \ldots$

$= 1 - 2x + 2x^2 - \dfrac{4}{3}x^3 + \ldots$

Substituting $-2x$ for x in the formula $e^x = 1 + x + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + \ldots$ and ignoring terms in x^4 and higher powers.

$\cos 5x = 1 - \dfrac{(5x)^2}{2!} + \ldots$

$= 1 - \dfrac{25}{2}x^2 + \ldots$

Substituting $5x$ for x in the formula $\cos x = 1 - \dfrac{x^2}{2!} + \ldots$ and ignoring terms in x^4 and higher powers.

$e^{-2x}\cos 5x = \left(1 - 2x + 2x^2 - \dfrac{4}{3}x^3 + \ldots\right)\left(1 - \dfrac{25}{2}x^2 + \ldots\right)$

$= 1 - \dfrac{25}{2}x^2 - 2x + 25x^3 + 2x^2 - \dfrac{4}{3}x^3 + \ldots$

$= 1 - 2x + \left(-\dfrac{25}{2} + 2\right)x^2 + \left(25 - \dfrac{4}{3}\right)x^3 + \ldots$

$= 1 - 2x - \dfrac{21}{2}x^2 + \dfrac{71}{3}x^3 + \ldots$

When multiplying out the brackets, you discard terms in x^4 and higher powers. For example, multiplying $2x^2$ by $-\dfrac{25}{2}x^2$ gives $-25x^4$ and you just ignore his term.

$A = 1,\ B = -2,\ C = -\dfrac{21}{2}, D = \dfrac{71}{3}$

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26 a $(2x+3)^{-1} = 3^{-1}\left(1+\frac{2x}{3}\right)^{-1}$

Part **a** is a binomial series with a rational index.

$$= \frac{1}{3}\left(1-\frac{2x}{3}+\frac{(-1)(-2)}{2.1}\left(\frac{2x}{3}\right)^2+\frac{(-1)(-2)(-3)}{3.2.1}\left(\frac{2x}{3}\right)^3+\ldots\right)$$

$$= \frac{1}{3}\left(1-\frac{2}{3}x+\frac{4}{9}x^2-\frac{8}{27}x^3+\ldots\right)$$

$$= \frac{1}{3}-\frac{2}{9}x+\frac{4}{27}x^2-\frac{8}{81}x^3+\ldots$$

b $\frac{\sin 2x}{3+2x} = \sin 2x(3+2x)^{-1}$

$$= \left(2x-\frac{(2x)}{3!}+\ldots\right)\left(\frac{1}{3}-\frac{2}{9}x+\frac{4}{27}x^2-\frac{8}{81}x^3+\ldots\right)$$

$$= \left(2x-\frac{4}{3}x^3+\ldots\right)\left(\frac{1}{3}-\frac{2}{9}x+\frac{4}{27}x^2-\frac{8}{81}x^3+\ldots\right)$$

$$= \frac{2}{3}x-\frac{4}{9}x^2+\frac{8}{27}x^3-\frac{16}{81}x^4-\frac{4}{9}x^3+\frac{8}{27}x^4+\ldots$$

When multiplying out the brackets, you discard terms in x^4 and higher powers. For example, multiplying $-\frac{4}{3}x^3$ by $\frac{4}{27}x^2$ gives $-\frac{16}{81}x^5$ and you ignore this term.

$$= \frac{2}{3}x-\frac{4}{9}x^2+\left(\frac{8}{27}-\frac{4}{9}\right)x^3+\left(\frac{8}{27}-\frac{16}{81}\right)x^4+\ldots$$

$$= \frac{2}{3}x-\frac{4}{9}x^2-\frac{4}{27}x^3+\frac{8}{81}x^4+\ldots$$

27 a $\cos x = 1-\frac{x^2}{2!}+\frac{x^4}{4!}-\ldots$

$$= 1+\left(-\frac{x^2}{2}+\frac{x^4}{24}\right)$$ **(1),**

neglecting terms above x^4

The expression $-\frac{x^2}{2!}+\frac{x^4}{4!}$ is used to replace the x in the standard series for $\ln(1+x)$.

$$\ln(1+x) = x-\frac{x^2}{2}+\ldots$$

Using the expansion **(1)**

$$\ln(\cos x) = \ln\left(1+\left(-\frac{x^2}{2}+\frac{x^4}{24}\right)\right)$$

$$= \left(-\frac{x^2}{2}+\frac{x^4}{24}\right)-\frac{1}{2}\left(-\frac{x^2}{2}+\frac{x^4}{24}\right)^2+\ldots$$

$$= -\frac{x^2}{2}+\frac{x^4}{24}-\frac{x^4}{8}+\ldots$$

$$= -\frac{x^2}{2}-\frac{x^4}{12}-\ldots$$

$-\frac{1}{2}\left(-\frac{x^2}{2}+\frac{x^4}{24}\right)^2 = -\frac{x^4}{8}+\frac{x^6}{48}-\frac{x^8}{1152}$ but, as the expansion is only required up to the term in x^4, you only need the first of the three terms.

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27 b $\ln(\sec x) = \ln\left(\dfrac{1}{\cos x}\right) = \ln 1 - \ln\cos x$

$= -\ln\cos x$

Using the log rule $\log\left(\dfrac{a}{b}\right) = \log a - \log b$ and the fact that $\ln 1 = 0$.

Using the result to part **a**

$$\ln(\sec x) = -\left(-\frac{x^2}{2} - \frac{x^4}{12} - \ldots\right) = \frac{x^2}{2} + \frac{x^4}{12} + \ldots$$

28 a Let $u = 1 + \cos 2x$, then $f(x) = \ln u$

$$\frac{du}{dx} = -2\sin 2x$$

$$f'(x) = f'(u)\frac{du}{dx} = \frac{1}{u}\frac{du}{dx} = \frac{1}{1+\cos 2x} \times -2\sin 2x$$

$$= \frac{-4\sin x\cos x}{2\cos^2 x}$$

Using the identities $\sin 2x = 2\sin x\cos x$ and $\cos 2x = 2\cos^2 x - 1$.

$$= \frac{-2\sin x}{\cos x} = -2\tan x, \text{ as required}$$

b $f''(x) = -2\sec^2 x$

$f'''(x) = -4\sec^2 x\tan x$

$f''''(x) = -8\sec x\cdot\sec x\tan x\cdot\tan x - 4\sec^2 x\cdot\sec^2 x$

$= -8\sec^2 x\tan^2 x - 4\sec^4 x$

$= -[-4\sec^2 x\tan x\times -2\tan x + (-2\sec^2 x)^2]$

$= -[f'''(x)f'(x) + (f''(x))^2]$, as required

$f''''(x)$ is a symbol used for the fourth derivative of $f(x)$ with respect to x. The symbol $f^{(iv)}(x)$ is also used for the fourth derivative.

You use the product rule for differentiation $\dfrac{d}{dx}(uv) = v\dfrac{du}{dx} + u\dfrac{dv}{dx}$ with $u = -4\sec^2 x$ and $v = \tan x$. You also use the chain rule

$\dfrac{d}{dx}(\sec^2 x) = 2\sec x\dfrac{d}{dx}(\sec x)$

$= 2\sec x\times\sec x\tan x.$

c $f(0) = \ln(1+\cos 0) = \ln 2$

$f'(0) = -2\tan 0 = 0$

$f''(0) = -2\sec^2 0 = -2$

$f'''(0) = -4\sec^2 0\tan 0 = 0$

$f''''(0) = -[f'''(0)f'(0) + (f''(0))^2]$

$= -[0\times 0 + (-2)^2] = -4$

Using the result for part **b**.

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \frac{x^4}{4!}f^{(iv)}(0) + \ldots$$

$$= \ln + 2 + x\times 0 + \frac{x^2}{2}\times -2 + \frac{x^3}{6}\times 0 + \frac{x^4}{24}\times -4 + \ldots$$

$$= \ln 2 - x^2 - \frac{1}{6}x^4 + \ldots$$

The formula for Maclaurin's series is given in the formulae booklet. For this question, you need the terms up to and including the term in x^4.

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29 $\int_0^\infty e^{-x}\sin x\,dx = \left[-e^{-x}\cos x\right]_0^\infty - \int_0^\infty e^{-x}\cos x\,dx$

$\int_0^\infty e^{-x}\cos x\,dx = \left[e^{-x}\sin x\right]_0^\infty + \int_0^\infty e^{-x}\sin x\,dx$

$\int_0^\infty e^{-x}\sin x\,dx = \left[-e^{-x}\cos x\right]_0^\infty$

$-\left[e^{-x}\sin x\right]_0^\infty - \int_0^\infty e^{-x}\sin x\,dx$

$2\int_0^\infty e^{-x}\sin x\,dx = \left[-e^{-x}\cos x - e^{-x}\sin x\right]_0^\infty = 1$

$\int_0^\infty e^{-x}\sin x\,dx = \frac{1}{2}$

30 $\int_0^1 \frac{x^2}{\sqrt{1-x^2}}dx = \left[-x\sqrt{1-x^2}\right]_0^1 + \int_0^1 \sqrt{1-x^2}\,dx$

$\left(u = x, u' = 1, v' = \frac{x}{\sqrt{1-x^2}}, v = -\sqrt{1-x^2}\right)$

$\int_0^1 \frac{x^2}{\sqrt{1-x^2}}dx = \int_0^1 -\frac{1-x^2}{\sqrt{1-x^2}} + \frac{1}{\sqrt{1-x^2}}dx$

$= \left[\arcsin x\right]_0^1 + \int_0^1 \sqrt{1-x^2}\,dx$

$2\int_0^1 \frac{x^2}{\sqrt{1-x^2}}dx = \left[\arcsin x - x\sqrt{1-x^2}\right]_0^1$

$= \arcsin 1 = \frac{\pi}{2}$

$\int_0^1 \frac{x^2}{\sqrt{1-x^2}}dx = \frac{\pi}{4}$

31 a $\frac{1}{x(x+3)} = \frac{A}{x} + \frac{B}{x+3}$

$1 = A(x+3) + Bx, 1 = 3A = -3B$

$\int \frac{1}{x(x+3)}dx = \frac{1}{3}\int \frac{1}{x} - \frac{1}{x+3}dx$

$= \frac{1}{3}(\ln x - \ln(x+3)) = \frac{1}{3}\ln\frac{x}{x+3} + c$

b $\int_3^\infty \frac{1}{x(x+3)}dx = \lim_{t\to\infty}\int_3^t \frac{1}{x(x+3)}dx$

$= \lim_{t\to\infty}\left(\frac{1}{3}\ln\frac{t}{t+3} - \frac{1}{3}\ln\frac{3}{3+1}\right)$

$= \frac{1}{3}\ln 1 - \frac{1}{3}\ln\frac{1}{2} = \frac{1}{3}\ln 2$

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32 $\int_1^\infty x^3 e^{-x^4}\,\mathrm{d}x = \lim_{t\to\infty}\int_1^t x^3 e^{-x^4}\,\mathrm{d}x$

$\frac{\mathrm{d}}{\mathrm{d}x}\left(e^{-x^4}\right) = -4x^3 e^{-x^4}$

$\int_1^t x^3 e^{-x^4}\,\mathrm{d}x = -\frac{1}{4}\left[e^{-x^4}\right]_1^t = \frac{1}{4}\left(e^{-1} - e^{-t^4}\right)$

$\int_1^\infty x^3 e^{-x^4}\,\mathrm{d}x = \lim_{t\to\infty}\frac{1}{4}\left(e^{-1} - e^{-t^4}\right) = \frac{1}{4e}$

33 a $\int \frac{1}{(5-2x)^2}\,\mathrm{d}x = \frac{1}{2}\int 2(5-2x)^{-2}\,\mathrm{d}x = \frac{1}{2(5-2x)} + c$

b $\int_{-\infty}^3 \frac{1}{(5-2x)^2}\,\mathrm{d}x = \lim_{t\to\frac{5}{2}}\int_t^3 \frac{1}{(5-2x)^2}\,\mathrm{d}x + \lim_{s\to\frac{5}{2}}\int_{-\infty}^s \frac{1}{(5-2x)^2}\,\mathrm{d}x$

$\int_{-\infty}^s \frac{1}{(5-2x)^2}\,\mathrm{d}x = \frac{1}{2}\frac{1}{(5-2s)} \to \infty \text{ as } s \to \frac{5}{2}$

34 $\mathrm{f}(x) = x\cos 2x$

$\int \mathrm{f}(x)\mathrm{d}x = \frac{1}{2}x\sin 2x - \int \frac{1}{2}\sin 2x\mathrm{d}x$

$= \frac{1}{2}x\sin 2x + \frac{1}{4}\cos 2x + c$

$\bar{\mathrm{f}} = \frac{1}{\frac{\pi}{2}-0}\int_0^{\frac{\pi}{2}} \mathrm{f}(x)\mathrm{d}x = \frac{2}{\pi}\left[\frac{1}{2}x\sin 2x + \frac{1}{4}\cos 2x\right]_0^{\frac{\pi}{2}}$

$= \frac{2}{\pi}\left(\frac{1}{4}\cos\pi - \frac{1}{4}\cos 0\right) = \frac{2}{\pi}\left(-\frac{1}{2}\right) = -\frac{1}{\pi}$

35 a $\mathrm{f}(x) = \frac{3x}{(x-1)(2x-3)} = \frac{A}{x-1} + \frac{B}{2x-3}$

$3x = A(2x-3) + B(x-1), 3 = -A, 0 = -3A - B$

$\mathrm{f}(x) = -3\left(\frac{1}{x-1} - \frac{3}{2x-3}\right)$

$\bar{\mathrm{f}} = \frac{1}{5-2}\int_2^5 -3\left(\frac{1}{x-1} - \frac{3}{2x-3}\right)\mathrm{d}x$

$= \int_2^5 -\frac{1}{x-1} + \frac{3}{2x-3}\,\mathrm{d}x$

$= \left[-\ln(x-1) + \frac{3}{2}\ln(2x-3)\right]_2^5$

$= \left(-\ln 4 + \ln 1 + \frac{3}{2}\ln 7 - \frac{3}{2}\ln 1\right)$

$= \frac{1}{2}(\ln 7^3 - \ln 4^2) = \frac{1}{2}\ln\frac{343}{16}$

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35 b $\mathrm{f}(x)+\ln k$ has mean value $\bar{\mathrm{f}}+\ln k$

$$=\frac{1}{2}\ln\frac{343}{16}+\ln k=\frac{1}{2}\ln\frac{343k^2}{16}$$

36 a $\mathrm{f}(x)=x^2(x^3-1)^3$

$$\frac{\mathrm{d}}{\mathrm{d}x}(x^3-1)^4=4(x^3-1)^3(3x^2)=12x^2(x^3-1)^3$$

$$\bar{\mathrm{f}}=\frac{1}{3-1}\int_1^3 x^2(x^3-1)^3\,\mathrm{d}x=\frac{1}{24}\int_1^3 12x^2(x^3-1)^3\,\mathrm{d}x$$

$$=\frac{1}{24}\left[(x^3-1)^4\right]_1^3=\frac{57122}{3}$$

b $k\mathrm{f}(x)$ has mean value $k\bar{\mathrm{f}}$ over $[1,3]$

so $-2\mathrm{f}(x)$ has mean value $-\dfrac{114244}{3}$

37 $\bar{\mathrm{f}}=\dfrac{1}{3-1}\displaystyle\int_1^3\frac{1}{\sqrt{3-x}}\,\mathrm{d}x=-\int_1^3-\frac{1}{2\sqrt{3-x}}\,\mathrm{d}x$

$$=-\left[\sqrt{3-x}\right]_1^3=\sqrt{2}$$

38 $\mathrm{f}(x)=\ln kx$

$$\bar{\mathrm{f}}=\frac{1}{5-1}\int_1^5\ln kx\,\mathrm{d}x$$

$$\int\ln kx\,\mathrm{d}x=x\ln kx-\int 1\,\mathrm{d}x=x(\ln kx-1)+c$$

$$\bar{\mathrm{f}}=\frac{1}{4}\left[x(\ln kx-1)\right]_1^5=\frac{1}{4}(5(\ln 5k-1)-\ln k+1)$$

$$=\frac{1}{4}\ln\frac{(5k)^5}{k}-1=\frac{1}{4}\ln 5^5k^4-1=\frac{1}{4}\ln 5^9-1$$

$k=5$

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39 a $y = (\arcsin x)^2$

Let $u = \arcsin x$

$y = u^2$

$\dfrac{dy}{dx} = \dfrac{dy}{du} \times \dfrac{du}{dx}$

$\dfrac{dy}{du} = 2u$

$\dfrac{du}{dx} = \dfrac{1}{\sqrt{(1-x^2)}}$

This result is in the Edexcel formula booklet, which is provided for use with the paper. It is a good idea to quote any formulae you use in your solution.

Hence

$$\frac{dy}{dx} = 2u \times \frac{1}{\sqrt{(1-x^2)}} = \frac{2\arcsin x}{\sqrt{(1-x^2)}}$$

$\sqrt{(1-x^2)}\dfrac{dy}{dx} = 2\arcsin x$

Square both sides of this solution and use the given $y = (\arcsin x)^2$ to complete the solution.

$(1-x^2)\left(\dfrac{dy}{dx}\right)^2 = 4(\arcsin x)^2$

$= 4y$, as required

b Differentiating the result of part **a** implicitly with respect to x

$$-2x\left(\frac{dy}{dx}\right)^2 + (1-x^2)2\frac{dy}{dx}\frac{d^2y}{dx^2} = 4\frac{dy}{dx}$$

Using the chain rule

$$\frac{d}{dx}\left(\left(\frac{dy}{dx}\right)^2\right) = 2\frac{dy}{dx} \times \frac{d}{dx}\left(\frac{dy}{dx}\right) = 2\frac{dy}{dx} \times \frac{d^2y}{dx^2}.$$

$$-x\frac{dy}{dx} + (1-x^2)\frac{d^2y}{dx^2} = 2$$

Divide the equation throughout by $2\dfrac{dy}{dx}$.

$(1-x^2)\dfrac{d^2y}{dx^2} - x\dfrac{dy}{dx} = 2$, as required

40 a $y = \arctan 3x$

$\tan y = 3x$

Differentiating implicitly with respect to x

$\sec^2 y \dfrac{dy}{dx} = 3$

$$\frac{dy}{dx} = \frac{3}{\sec^2 y} = \frac{3}{1+\tan^2 y}$$

$= \dfrac{3}{1+9x^2}$, as required

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40 b Using integration by parts and the result in part **a**

$$\int 6x \arctan 3x \, dx = 3x^2 \arctan 3x - \int 3x^2 \times \frac{3}{1+9x^2} dx$$

$$= 3x^2 \arctan 3x - \int \frac{9x^2+1-1}{1+9x^2} dx$$

$$= 3x^2 \arctan 3x - \int 1 \, dx + \int \frac{1}{1+9x^2} dx$$

$$= 3x^2 \arctan 3x - x + \frac{1}{3} \arctan 3x$$

$$\left[3x^2 \arctan 3x - x + \frac{1}{3} \arctan 3x \right]_0^{\frac{\sqrt{3}}{3}}$$

$$= 3 \times \left(\frac{\sqrt{3}}{3} \right)^2 \arctan \sqrt{3} - \frac{\sqrt{3}}{3} + \frac{1}{3} \arctan \sqrt{3}$$

$$= \frac{4}{3} \arctan \sqrt{3} - \frac{\sqrt{3}}{3}$$

$$= \frac{4}{3} \times \frac{\pi}{3} - \frac{\sqrt{3}}{3} = \frac{1}{9}(4\pi - 3\sqrt{3}), \text{ as required}$$

You use $\int u \frac{dv}{dx} dx = uv - \int v \frac{du}{dx} dx$ with $u = \text{artanh } 3x$ and $\frac{dv}{dx} = 6x$. You know $\frac{du}{dx}$ from part **a**.

You have to integrate $\frac{9x^2}{1+9x^2}$. As the degree of the numerator is equal to the degree of the denominator, you must divide the denominator into the numerator before integrating.

The adaptation of the formula given in the Edexcel formulae booklet, $\int \frac{1}{a^2+x^2} dx = \frac{1}{a} \tan\left(\frac{x}{a}\right)$ to this integral is not straightforward.

$$\int \frac{1}{1+9x^2} dx = \frac{1}{9} \int \frac{1}{\frac{1}{9}+x^2} dx$$

$$= \frac{1}{9} \times \frac{1}{\frac{1}{3}} \arctan\left(\frac{x}{\frac{1}{3}} \right) = \frac{1}{3} \arctan 3x.$$ You may prefer to find such an integral using the substitution $3x = \tan\theta$.

41 a Let $y = f(x) = \arcsin x$

$\sin y = x$

Differentiating implicitly with respect to x

$$\cos y \frac{dy}{dx} = 1$$

$$\frac{dy}{dx} = \frac{1}{\cos y} = \frac{1}{\sqrt{(1-\sinh^2)y}} = \frac{1}{\sqrt{(1-x^2)}}$$

$$f'(x) = \frac{1}{\sqrt{(1-x^2)}}, \text{ as required}$$

Unless otherwise stated, arcsin x is taken to have the range $-\frac{\pi}{2} < \arcsin x < \frac{\pi}{2}$. These are the principal values of arcsin x. In this range, arcsin x is an increasing function of x, $\frac{dy}{dx}$ is positive and you can take the positive value of the square root.

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41 b $y = \arcsin 2x$

Let $u = 2x$, $\dfrac{\mathrm{d}u}{\mathrm{d}x} = 2$

$y = \arcsin u$

Using the chain rule

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}u} \times \frac{\mathrm{d}u}{\mathrm{d}x}$$
$$= \frac{1}{\sqrt{(1-u^2)}} \times 2 = \frac{2}{\sqrt{(1-4x^2)}}$$

c $x = \dfrac{1}{2}\sin\theta \Rightarrow \dfrac{\mathrm{d}x}{\mathrm{d}\theta} = \dfrac{1}{2}\cos\theta$

In this question it is convenient to carry out the substitution without returning to the original variable x. So at some stage you must change the x limits to θ limits.

At $x = \dfrac{1}{4}$, $\dfrac{1}{4} = \dfrac{1}{2}\sin\theta \Rightarrow \sin\theta = \dfrac{1}{2} \Rightarrow \theta = \dfrac{\pi}{6}$

At $x = 0$, $0 = \dfrac{1}{2}\sin\theta \Rightarrow \sin\theta = 0 \Rightarrow \theta = 0$

$$\int \frac{x \arcsin 2x}{\sqrt{(1-4x^2)}}\,\mathrm{d}x = \int \frac{\frac{1}{2}\sin\theta \arcsin(\sin\theta)}{\sqrt{(1-\sin^2\theta)}}\left(\frac{\mathrm{d}x}{\mathrm{d}\theta}\right)\mathrm{d}\theta$$
$$= \int \frac{\frac{1}{2}\sin\theta \times \theta}{\cos\theta}\left(\frac{1}{2}\cos\theta\right)\mathrm{d}\theta$$

By definition, $\arcsin(\sin\theta) = \theta$.

$$= \frac{1}{4}\int \theta \sin\theta\,\mathrm{d}\theta$$
$$= -\frac{1}{4}\theta\cos\theta + \frac{1}{4}\int \cos\theta\,\mathrm{d}\theta$$
$$= -\frac{1}{4}\theta\cos\theta + \frac{1}{4}\sin\theta$$

You use integration by parts, $\int u\dfrac{\mathrm{d}v}{\mathrm{d}\theta} = uv - \int v\dfrac{\mathrm{d}u}{\mathrm{d}\theta}\mathrm{d}\theta$, with $u = \theta$ and $\dfrac{\mathrm{d}v}{\mathrm{d}\theta} = \sin\theta$.

Hence

$$\int_0^{\frac{1}{4}} \frac{x \arcsin 2x}{\sqrt{(1-4x^2)}}\,\mathrm{d}x = \left[-\frac{1}{4}\theta\cos\theta + \frac{1}{4}\sin\theta\right]_0^{\frac{\pi}{6}}$$
$$= \left[-\frac{\pi}{24}\cos\frac{\pi}{6} + \frac{1}{4}\sin\frac{\pi}{6}\right] - [0]$$
$$= -\frac{\pi}{24} \times \frac{\sqrt{3}}{2} + \frac{1}{4} \times \frac{1}{2}$$
$$= \frac{1}{48}\left(6 - \pi\sqrt{3}\right), \text{ as required}$$

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42 $\dfrac{2x+1}{x(x^2+1)} = \dfrac{Ax+B}{x^2+1} + \dfrac{C}{x}$

$2x+1 = (Ax+B)x + C(x^2+1)$

Coefficients of $x^2 : 0 = A + C$,

$x : 2 = B$,

$1 : 1 = C$ so $A = -1, B = 2, C = 1$

$$\int \frac{2x+1}{x(x^2+1)}\,\mathrm{d}x = \int \frac{2}{x^2+1} - \frac{x}{x^2+1} + \frac{1}{x}\,\mathrm{d}x$$

$$= 2\arctan x - \frac{1}{2}\ln(x^2+1) + \ln x + c$$

Hence $A = 2$ and $B = -\dfrac{1}{2}$

43 a $\mathrm{f}(x) = \dfrac{3x^2+5x}{x^3-3x^2+5x-15} = \dfrac{3x^2+5x}{(x-3)(x^2+5)}$

$$= \frac{A}{x-3} + \frac{Bx+C}{x^2+5}$$

$3x^2+5x = A(x^2+5) + (Bx+C)(x-3)$

$x = 3 : 42 = 14A$

Coefficients of $x^2 : 3 = A + B$,

$x : 5 = C - 3B$,

$1 : 0 = 5A - 3C$ so $A = 3, B = 0, C = 5$

$$\mathrm{f}(x) = \frac{3}{x-3} + \frac{5}{x^2+5}$$

b $$\int \mathrm{f}(x)\,\mathrm{d}x = \int \frac{3}{x-3} + \frac{5}{x^2+5}\,\mathrm{d}x$$

$$= 3\ln(x-3) + 5\frac{1}{\sqrt{5}}\arctan\frac{x}{\sqrt{5}} + c$$

$$= 3\ln(x-3) + \sqrt{5}\arctan\frac{x}{\sqrt{5}} + c$$

Hence $P = 3, Q = \sqrt{5}$ and $R = \dfrac{1}{\sqrt{5}}$

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44 Use

$$V = \pi\int y^2\,\mathrm{d}x$$

$$= \pi\int_1^3 x^2\mathrm{e}^{2x}\,\mathrm{d}x$$

Let $u = x^2$ and $\frac{\mathrm{d}v}{\mathrm{d}x} = \mathrm{e}^{2x}$

$$u = x^2 \Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x} = 2x$$

$$v = \frac{1}{2}\mathrm{e}^{2x} \Leftarrow \frac{\mathrm{d}v}{\mathrm{d}x} = \mathrm{e}^{2x}$$

Complete the table for u, v, $\frac{\mathrm{d}u}{\mathrm{d}x}$ and $\frac{\mathrm{d}v}{\mathrm{d}x}$. Take care to differentiate u but integrate $\frac{\mathrm{d}v}{\mathrm{d}x}$.

$$\therefore V = \pi\left[x^2\cdot\frac{1}{2}\mathrm{e}^{2x}\right]_1^3 - \pi\int_1^3 \frac{1}{2}\mathrm{e}^{2x}\cdot 2x\,\mathrm{d}x.$$

i.e. $$V = \pi\left[\frac{9}{2}\mathrm{e}^6 - \frac{1}{2}\mathrm{e}^2\right] - \pi\int_1^3 x\mathrm{e}^{2x}\,\mathrm{d}x.$$

This integral is simpler than V but still not one you can write down. Use integration by parts again with $u = x$ and $\frac{\mathrm{d}v}{\mathrm{d}x} = \mathrm{e}^{2x}$.

$$u = x \Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x} = 1$$

$$v = \frac{1}{2}\mathrm{e}^{2x} \Rightarrow \frac{\mathrm{d}v}{\mathrm{d}x} = \mathrm{e}^{2x}$$

Complete a new table for the new u, v, $\frac{\mathrm{d}u}{\mathrm{d}x}$ and $\frac{\mathrm{d}v}{\mathrm{d}x}$.

$$\therefore V = \pi\left[\frac{9}{2}\mathrm{e}^6 - \frac{1}{2}\mathrm{e}^2\right] - \pi\left[x\cdot\frac{1}{2}\mathrm{e}^{2x}\right]_1^3 + \pi\int_1^3 \frac{1}{2}\mathrm{e}^{2x}\cdot 1\,\mathrm{d}x$$

Apply the integration by parts formula a second time.

$$= \pi\left[\frac{9}{2}\mathrm{e}^6 - \frac{1}{2}\mathrm{e}^2\right] - \pi\left[\frac{3}{2}\mathrm{e}^6 - \frac{1}{2}\mathrm{e}^2\right] + \pi\left[\frac{1}{4}\mathrm{e}^{2x}\right]_1^3$$

$$= \frac{13}{4}\pi\mathrm{e}^6 - \frac{\pi}{4}\mathrm{e}^2$$

45 a $$\text{Area} = \int_0^{2\pi} 3\sin\frac{x}{2}\,\mathrm{d}x$$

Recall (5) in the introduction to integrating. Integration a sin function gives a change of sign and a cos function.

$$= \left[3\times -2\cos\frac{x}{2}\right]_0^{2\pi}$$

$$= \left[-6\cos\frac{x}{2}\right]_0^{2\pi}$$

The 2 here is obtained from dividing by $\frac{1}{2}$ which anses from the chain rule.

$$= [6-(-6)]$$

$$\text{Area} = 12$$

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45 b Volume $= \pi \int_0^{2\pi} 9\sin^2 \frac{x}{2}\,dx$

You cannot integrate $\sin^2 \frac{x}{2}$, but you can write this in terms of $\cos x$.

Recall $\cos 2A = 1 - 2\sin^2 A$

So $\sin^2 A = \frac{1}{2}(1 - 2\cos 2A)$

So $\sin^2 \frac{x}{2} = \frac{1}{2}(1 - \cos x)$

$\therefore$ Volume $= \frac{9\pi}{2} \int_0^{2\pi} (1 - \cos x)\,dx$

You can now integrate each term directly.

$= \frac{9\pi}{2}[x + \sin x]_0^{2\pi}$

$= \frac{9\pi}{2} \times (2\pi - 0)$

volume $= 9\pi^2$

46 Volume $= \pi \int_0^{\pi} (x\sqrt{\sin x})^2\,dx$

Use $v = \pi \int y^2\,dx$.

$= \pi \int_0^{\pi} x^2 \sin x\,dx$

Use integration by parts.

Let $u = x^2$ and $\frac{dv}{dx} = \sin x$.

$u = x^2 \Rightarrow \frac{du}{dx} = 2x$

$v = -\cos x \Leftarrow \frac{dv}{dx} = \sin x$

Complete the table for $u, v, \frac{du}{dx}$ and $\frac{dv}{dx}$.

$\therefore$ Volume $= \pi\left[[-x^2 \cos x]_0^{\pi} - \int_0^{\pi} -2x\cos x\,dx\right]$

$= \pi\left(\pi^2 + \int_0^{\pi} 2x\cos x\,dx\right)$

This integral is simpler than the original one but you will need to use integration by parts again, with $u = 2x$ and $\frac{dv}{dx} = \cos x$.

$u = 2x \Rightarrow \frac{du}{dx} = 2$

$v = \sin x \Leftarrow \frac{dv}{dx} = \cos x$

$\therefore$ Volume $= \pi\left[\pi^2 + [2x\sin x]_0^{\pi} - \int_0^{\pi} 2\sin x\,dx\right]$

This term becomes zero as $2\pi \sin \pi - 0 = 0$.

$= \pi\left(\pi^2 + [2\cos x]_0^{\pi}\right)$

$= \pi\left(\pi^2 + [-2 - 2]\right)$

$= \pi(\pi^2 - 4)$

$= \pi^3 - 4\pi$

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47 $V = \pi\int_a^5 x^2 dy = \pi\int_a^5 \frac{1}{(3y-1)^2}\,\mathrm{d}y$

$= \pi\left[-\frac{1}{3(3y-1)}\right]_a^5 = \pi\left(-\frac{1}{42} + \frac{1}{3(3a-1)}\right) = \frac{3\pi}{70}$

$-5 + \frac{70}{3a-1} = 9,\ 14 = \frac{70}{3a-1}$

$42a = 84,\ a = 2$

48 a $x = y\cos y$

$x = 0$ when $y\cos y = 0$

$y = 0$ or $\cos y = 0, y = \frac{\pi}{2}, \frac{3\pi}{2}, \ldots$

$k = \frac{\pi}{2}$

b $V = \pi\int_0^{\frac{\pi}{2}} x^2\mathrm{d}y = \pi\int_0^{\frac{\pi}{2}} y^2\cos^2 y\mathrm{d}y$

$= \pi\int_0^{\frac{\pi}{2}} y^2\left(\frac{1}{2}(1+\cos 2y)\right)\mathrm{d}y$

$u = y^2, u' = 2y, v' = \frac{1}{2}(1+\cos 2y), v = \frac{1}{2}y + \frac{1}{4}\sin 2y$

$V = \pi\left[\frac{1}{2}y^3 + \frac{1}{4}y^2\sin 2y\right]_0^{\frac{\pi}{2}} - \pi\int_0^{\frac{\pi}{2}} y^2 + \frac{1}{2}y\sin 2y\mathrm{d}y$

$\int_0^{\frac{\pi}{2}} y^2 + \frac{1}{2}y\sin 2y\mathrm{d}y = \left[\frac{y^3}{3} - \frac{1}{4}y\cos 2y\right]_0^{\frac{\pi}{2}} + \int_0^{\frac{\pi}{2}} \frac{1}{4}\cos 2y\mathrm{d}y$

$V = \pi\left[\frac{y^3}{2} + \frac{y^2}{4}\sin 2y - \frac{y^3}{3} + \frac{1}{4}y\cos 2y + \frac{1}{8}\sin 2y\right]_0^{\frac{\pi}{2}}$

$= \pi\left[\frac{y^3}{6} + \left(\frac{y^2}{4} + \frac{1}{8}\right)\sin 2y + \frac{1}{4}y\cos 2y\right]_0^{\frac{\pi}{2}}$

$= \frac{\pi^4}{48} + \frac{\pi^2}{8}\cos\pi = \frac{\pi^4}{48} - \frac{\pi^2}{8}$

Hence $a = \frac{1}{48}$ and $b = -\frac{1}{8}$

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49 $V = \pi\int_{f^{-1}(\ln 2)}^{f^{-1}(b)} y^2 \frac{\mathrm{d}x}{\mathrm{d}t}\mathrm{d}t$, let $b = \ln a$

$V = \pi\int_2^a (t^2-2)\frac{1}{t}\mathrm{d}t = \pi\int_2^a t^3 - 4t + \frac{4}{t}\mathrm{d}t$

$= \pi\left[\frac{t^4}{4} - 2t^2 + 4\ln t\right]_2^a = \pi\left(\frac{a^4}{4} - 2a^2 + 4\ln a + 4 - 4\ln 2\right)$

$= \pi\left(2a^2\left(\frac{a^2}{8} - 1\right) + 4 + 4\ln\frac{a}{2}\right)$

$a = 4, V = \pi(36 + 4\ln 2)$

$b = \ln a = \ln 4$

50 a $V = \pi\int_0^4 y^2\mathrm{d}x = \pi\int_0^4 \frac{25}{1+4x}\mathrm{d}x$

$= \pi\left[\frac{25}{4}\ln(1+4x)\right]_0^4 = \frac{25\pi}{4}\ln 17$

Volume scale factor is $(2\text{cm})^3$

$V = \frac{25}{4}\ln 17 \times 8\text{cm}^3 = 50\pi\ln 17\text{cm}^3$

b Non-zero thickness of the vase

51 a $x = \sqrt{y}\left(3 - \frac{1}{2}e^y\right)$

$x = 0 \Rightarrow y = 0$ or $3 - \frac{1}{2}e^y = 0, e^y = 6$

Intersection at $(0, \ln 6)$

b $V = \pi\int_0^{\ln 6} x^2\mathrm{d}y = \pi\int_0^{\ln 6} y\left(3 - \frac{1}{2}e^y\right)^2\mathrm{d}y$

$= \pi\int_0^{\ln 6} y\left(9 - 3e^y + \frac{1}{4}e^{2y}\right)\mathrm{d}y$

$\int ye^{ay}\mathrm{d}y = \frac{1}{a}ye^{ay} - \int\frac{1}{a}e^{ay}\mathrm{d}y = \frac{1}{a^2}e^{ay}(ay-1)$

$V = \pi\int_0^{\ln 6} 9y - 3ye^y + \frac{1}{4}ye^{2y}\mathrm{d}y$

$= \pi\left[\frac{9y^2}{2} - 3ye^y + 3e^y + \frac{1}{4}\left(\frac{1}{2}ye^{2y} - \frac{1}{4}e^{2y}\right)\right]_0^{\ln 6}$

$= \pi\left(\frac{9(\ln 6)^2}{2} - 18\ln 6 + 18 + \frac{18}{4}\ln 6 - \frac{36}{6} - \left(3 - \frac{1}{16}\right)\right)$

$= \pi\left(\frac{9(\ln 6)^2}{2} - \frac{27}{2}\ln 6 + \frac{205}{16}\right)$

Volume scale factor is $(1\text{cm})^3$

$V = 9.65\text{cm}^3$ (2 d.p.)

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51 c Filament may be wasted

52 a $y = 0 = 3\sin 2t, 2t = \pi$

$$V = \pi\int_0^{\frac{\pi}{2}} y^2 \frac{\mathrm{d}x}{\mathrm{d}t}\mathrm{d}t = \pi\int_0^{\frac{\pi}{2}} (3\sin 2t)^2 (2\cos t)\mathrm{d}t$$

Volume scale factor is $(0.5\text{cm})^3$

$$V = \frac{9\pi}{4}\int_0^{\frac{\pi}{2}} \sin^2 2t\cos t\mathrm{d}t \text{ cm}^3$$

$$= \frac{9\pi}{4}\int_0^{\frac{\pi}{2}} 4\sin^2 t\cos^2 t\cos t\mathrm{d}t \text{ cm}^3$$

$$= 9\pi\int_0^{\frac{\pi}{2}} \sin^2 t\cos^3 t\mathrm{d}t \text{ cm}^3$$

b $V = 9\pi\int_0^{\frac{\pi}{2}} \sin^2 t\cos^3 t\mathrm{d}t \text{ cm}^3$

$$= 9\pi\int_0^{\frac{\pi}{2}} \sin^2 t(1-\sin^2 t)\cos t\mathrm{d}t \text{ cm}^3$$

$$= 9\pi\left[\frac{\sin^3 t}{3} - \frac{\sin^5 t}{5}\right]_0^{\frac{\pi}{2}} \text{cm}^3$$

$$= 9\pi\left(\frac{1}{3} - \frac{1}{5}\right)\text{cm}^3 = \frac{6\pi}{5}\text{cm}^3$$

Challenge

1 a $\omega = e^{\frac{2\pi i}{3}}, \omega^{3k} = e^{2\pi ik} = 1$

$$n = 0: \frac{1^n + \omega^n + \left(\omega^2\right)^n}{3} = \frac{1+1+1}{3} = 1$$

$$n = 3k: \frac{1^{3k} + \omega^{3k} + \left(\omega^2\right)^{3k}}{3} = \frac{1^{3k} + \omega^{3k} + \left(\omega^{3k}\right)^2}{3}$$

$$= \frac{1+1+1}{3} = 1$$

$$n = 3k+1: \frac{1^{3k+1} + \omega^{3k+1} + \left(\omega^2\right)^{3k+1}}{3} = \frac{1+\omega^{3k+1}+\omega^{6k+2}}{3}$$

$$= \frac{1+\omega^{3k}\omega + \left(\omega^{3k}\right)^2\omega^2}{3} = \frac{1+\omega+\omega^2}{3} = 0$$

$$n = 3k+2: \frac{1^{3k+1} + \omega^{3k+2} + \left(\omega^2\right)^{3k+2}}{3} = \frac{1+\omega^{3k+2}+\omega^{6k+4}}{3}$$

$$= \frac{1+\omega^{3k}\omega^2 + \left(\omega^{3k}\right)^2\omega^3\omega}{3} = \frac{1+\omega^2+\omega}{3} = 0$$

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Challenge

1 b $\mathrm{f}(x)=\sum a_n x^n$

$\mathrm{f}(1)=\sum a_n, \mathrm{f}(\omega)=\sum a_n\omega^n, \mathrm{f}(\omega^2)=\sum a_n\left(\omega^2\right)^n$

$$\frac{f(1)+\mathrm{f}(\omega)+\mathrm{f}(\omega^2)}{3}=\sum a_n\frac{1^n+\omega^n+\left(\omega^2\right)^n}{3}$$

$$=\sum a_n 1_{\{n=3k\}},$$

where $1_{\{n=3k\}}=1$ if $n=3k$ and 0 otherwise

c $\mathrm{f}(x)=(1+x)^{45}=\sum_{r=0}^{45}\binom{45}{r}x^r$

$$S=\sum_{r=0}^{45}\binom{45}{r}1_{\{r=3k\}}=\sum_{r=0}^{15}\binom{45}{3r}$$

$$=\frac{\mathrm{f}(1)+\mathrm{f}(\omega)+\mathrm{f}(\omega^2)}{3}=\frac{2^{45}+(1+\omega)^{45}+(1+\omega^2)^{45}}{3}$$

$$=\frac{2^{45}+(-\omega^2)^{45}+(-\omega)^{45}}{3}=\frac{2^{45}-(\omega^3)^{30}-(\omega^3)^{15}}{3}$$

$$=\frac{2^{45}-2}{3}\text{ as }1+\omega+\omega^2=0\text{ and }\omega^3=0$$

2 $\mathrm{f}(x)=\sqrt{\frac{x+3}{x^3}}=y$

$$V=\pi\int_2^\infty y^2dx=\pi\lim_{t\to\infty}\int_2^t\frac{x+3}{x^3}\mathrm{d}x$$

$$=\pi\lim_{t\to\infty}\int_2^t\frac{1}{x^2}+\frac{3}{x^3}\mathrm{d}x=\pi\lim_{t\to\infty}\left[-\frac{1}{x}-\frac{3}{2x^2}\right]_2^t$$

$$=\pi\lim_{t\to\infty}\left(-\frac{1}{t}-\frac{3}{2t^2}+\frac{1}{2}+\frac{3}{8}\right)=\frac{7\pi}{8}$$

3 a $\mathrm{f}(x)=\frac{A}{1+x^2}$

$$\int_{-\infty}^{\infty}\mathrm{f}(x)\mathrm{d}x=A\int_{-\infty}^{\infty}\frac{1}{1+x^2}\mathrm{d}x=A\left[\arctan x\right]_{-\infty}^{\infty}$$

$$=A\left(\frac{\pi}{2}-\left(-\frac{\pi}{2}\right)\right)=A\pi=1, A=\frac{1}{\pi}$$

b $$\int_{-\infty}^{\infty}x^2\mathrm{f}(x)\mathrm{d}x=\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{x^2}{1+x^2}\mathrm{d}x$$

$$=\frac{1}{\pi}\lim_{s,t\to\infty}\int_{-s}^{t}1-\frac{1}{1+x^2}\mathrm{d}x$$

$$=\frac{1}{\pi}\lim_{s,t\to\infty}\left[x-\arctan x\right]_{-s}^{t}\to\infty$$

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Challenge

3 c $\lim_{a\to\infty}\int_{-a}^{a}\frac{x}{1+x^2}\,\mathrm{d}x=\lim_{a\to\infty}\left[\frac{1}{2}\ln\left(1+x^2\right)\right]_{-a}^{a}$

$$=\lim_{a\to\infty}\left(\frac{1}{2}\ln\left(1+a^2\right)-\frac{1}{2}\ln\left(1+(-a)^2\right)\right)=0$$

$\lim_{a\to\infty}\int_{-a}^{2a}\frac{x}{1+x^2}\,\mathrm{d}x=\lim_{a\to\infty}\left[\frac{1}{2}\ln\left(1+x^2\right)\right]_{-a}^{2a}$

$$=\lim_{a\to\infty}\left(\frac{1}{2}\ln\left(1+(2a)^2\right)-\frac{1}{2}\ln\left(1+(-a)^2\right)\right)$$

$$=\lim_{a\to\infty}\frac{1}{2}\ln\frac{1+4a^2}{1+a^2}=\frac{1}{2}\ln 4=\ln 2$$

$\lim_{a\to\infty}\int_{-a}^{a}\frac{x}{1+x^2}\,\mathrm{d}x=\int_{-\infty}^{\infty}\frac{x}{1+x^2}\,\mathrm{d}x$

$\lim_{a\to\infty}\int_{-a}^{2a}\frac{x}{1+x^2}\,\mathrm{d}x=\int_{-\infty}^{\infty}\frac{x}{1+x^2}\,\mathrm{d}x$

but $\lim_{a\to\infty}\int_{-a}^{a}\frac{x}{1+x^2}\,\mathrm{d}x\neq\lim_{a\to\infty}\int_{-a}^{2a}\frac{x}{1+x^2}\,\mathrm{d}x$

so mean does not exist

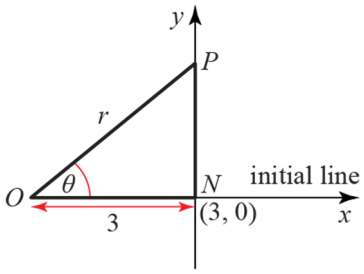
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Review exercise 2

1 a $r = 2$

You can just write the answer to part **a** down. The equation $r = k$ is the equation of a circle centre O and radius k, for any positive k.

b



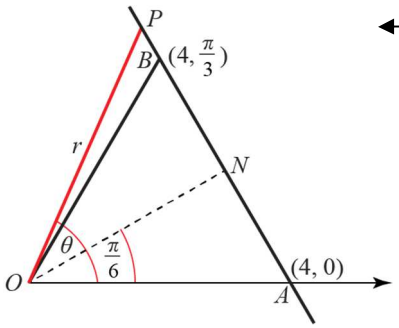
For any point P on the line

$$\frac{3}{r} = \cos\theta$$

$$r = \frac{3}{\cos\theta} = 3\sec\theta$$

If the point (3, 0) is labelled N, trigonometry on the right-angled triangle ONP gives the polar equation of the line.

c



In this diagram, the point (4, 0) is labelled A, the point $\left(4, \frac{\pi}{3}\right)$ is labelled B and the foot of the perpendicular from O to AB is labelled N. The triangle OAB is equilateral and $\angle AON = \frac{1}{2}60^\circ = 30^\circ = \frac{\pi}{6}$ radians.

In the triangle ONA

$$\frac{ON}{OA} = \frac{ON}{4} = \cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}$$

$$ON = 2\sqrt{3}$$

In the triangle ONP,

$$\frac{ON}{OP} = \cos\left(\theta - \frac{\pi}{6}\right)$$

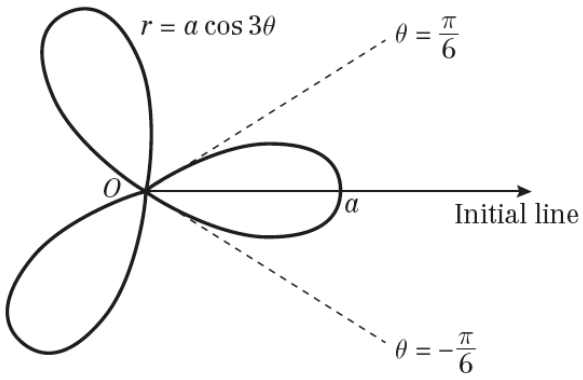
$$\frac{2\sqrt{3}}{t} = \cos\left(\theta - \frac{\pi}{6}\right)$$

$$r = 2\sqrt{3}\sec\left(\theta - \frac{\pi}{6}\right)$$

This relation is true for any point P on the line and, as $OP = r$ this gives you the polar equation of the line.

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2 a



At $\theta = -\frac{\pi}{6}$, $r = 0$. As θ increases, r increases until $\theta = 0$. For $\theta = 0$, $a\cos 6\theta$ has its greatest value of a. Then, as θ increases, r decreases to 0 at $\theta = \frac{\pi}{6}$. Between $\theta = \frac{\pi}{6}$ and $\theta = \frac{\pi}{2}$, $\cos 6\theta$ is negative and, as $r \geqslant 0$, the curve does not exist. The pattern repeats itself in the other intervals where the curve exists.

b $A = \frac{1}{2}\int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} r^2 \mathrm{d}\theta$

Using $\cos 2A = 2\cos^2 A - 1$ with $A = 3\theta$.

$$\frac{1}{2}\int a^2\cos^2 3\theta \mathrm{d}\theta = \frac{a^2}{2}\int\left(\frac{1}{2}\cos 6\theta + \frac{1}{2}\right)\mathrm{d}\theta$$

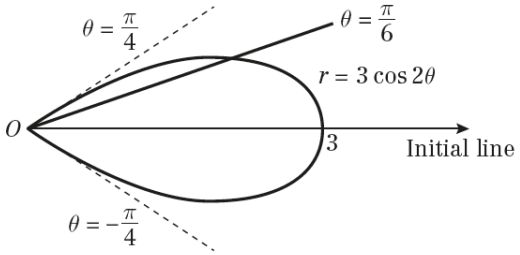
$$= \frac{a^2}{4}\left(\frac{\sin 6\theta}{6} + \theta\right)$$

$\sin\left(6\times\frac{\pi}{6}\right) = \sin\pi = 0$

$$A = \frac{a^2}{4}\left[\frac{\sin 6\theta}{6} + \theta\right]_{-\frac{\pi}{6}}^{\frac{\pi}{6}} = \frac{a^2}{4}\left[\frac{1}{6}(0-0) + \frac{\pi}{6} - \left(-\frac{\pi}{6}\right)\right]$$

$$= \frac{a^2}{4}\times\frac{\pi}{3} = \frac{\pi}{12}a^2$$

3 a



At $\theta = -\frac{\pi}{4}$, $r = 0$, As θ increases, r increases until $\theta = 0$. For $\theta = 0$, $3\cos 2\theta$ has its greatest value of 3. After that, as θ increases, r decreases to 0 at $\theta = \frac{\pi}{4}$.

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3 b $A = \frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} r^2 \mathrm{d}\theta$

$$\frac{1}{2}\int r^2 \mathrm{d}\theta = \frac{1}{2}\int 9\cos^2 2\theta \mathrm{d}\theta$$

$$= \frac{9}{2}\int\left(\frac{\cos 4\theta}{2} + \frac{1}{2}\right)\mathrm{d}\theta = \frac{9}{4}\int(\cos 4\theta + 1)\mathrm{d}\theta$$

$$= \frac{9}{4}\left[\frac{\sin 4\theta}{4} + \theta\right]$$

$$A = \frac{9}{4}\left[\frac{\sin 4\theta}{4} + \theta\right]_{\frac{\pi}{6}}^{\frac{\pi}{4}}$$

$$= \frac{9}{4}\left[\frac{1}{4}\left(0 - \frac{\sqrt{3}}{2}\right) + \left(\frac{\pi}{4} - \frac{\pi}{6}\right)\right]$$

$$= -\frac{9\sqrt{3}}{32} + \frac{3\pi}{16} = \frac{3}{32}(2\pi - 3\sqrt{3})$$

Using $\cos 2A = 2\cos^2 A - 1$ with $A = 2\theta$.

$\sin\left(4 \times \frac{\pi}{6}\right) = \sin\frac{2\pi}{3} = \frac{\sqrt{3}}{2}$

c Let $y = r\sin\theta = 3\cos 2\theta \sin\theta$

$$\frac{\mathrm{d}y}{\mathrm{d}\theta} = -6\sin 2\theta \sin\theta + 3\cos 2\theta \cos\theta = 0$$

$$2\sin 2\theta \sin 2\theta = \cos 2\theta \cos\theta$$

$$\frac{\sin 2\theta \sin\theta}{\cos 2\theta \cos\theta} = \tan 2\theta \tan\theta = \frac{1}{2}$$

$$\frac{2\tan^2\theta}{1 - \tan^2\theta} = \frac{1}{2}$$

$$4\tan^2\theta = 1 - \tan^2\theta$$

$$5\tan^2\theta = 1$$

$$\tan\theta = \frac{1}{\sqrt{5}}$$

Where the tangent at a point is parallel to the initial line, the distance y from the point to the initial line has a stationary value. You find the polar coordinate θ of such a point by finding the value of θ for which $y = r\sin\theta$ has a stationary value.

Using $\tan 2\theta = \frac{2\tan\theta}{1 - \tan^2\theta}$.

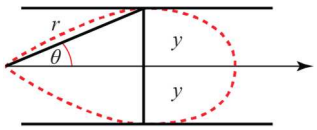
One value of $\tan\theta$ is sufficient to complete the question. r is not needed.

The distance between the two tangents is given by

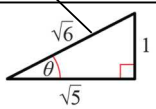
$$2y = 2r\sin\theta = 6\cos 2\theta \sin\theta = 6(2\cos^2\theta - 1)\sin\theta$$

$$= 6 \times \left(2 \times \frac{5}{6} - 1\right) \times \frac{1}{\sqrt{6}} = 6 \times \frac{2}{3} \times \frac{1}{\sqrt{6}}$$

$$= \frac{2\sqrt{6}}{3}$$



This sketch shows you that the distance between the two tangents parallel to the initial line is given by $2y = 2r\sin\theta$.

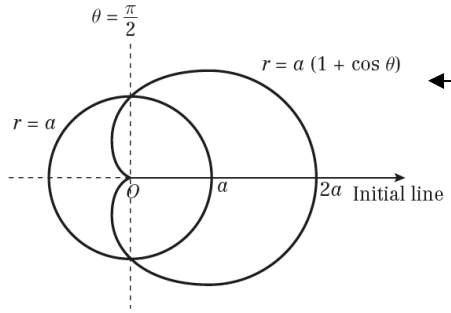


As $(\sqrt{5})^2 + 1^2 = (\sqrt{6})^2$, if $\tan\theta = \frac{1}{\sqrt{5}}$,

then $\sin\theta = \frac{1}{\sqrt{6}}$ and $\cos\theta = \frac{\sqrt{5}}{\sqrt{6}}$.

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4 a



$r = a(1+\cos\theta)$ is a cardioid and $r = a$ is a circle centre O, radius a.

b Let $y = r\sin\theta = a(1+\cos\theta)\sin\theta$

$$= a\sin\theta + a\cos\theta\sin\theta = a\sin\theta + \frac{a}{2}\sin 2\theta$$

$$\frac{dy}{d\theta} = a\cos\theta + a\cos 2\theta = 0$$

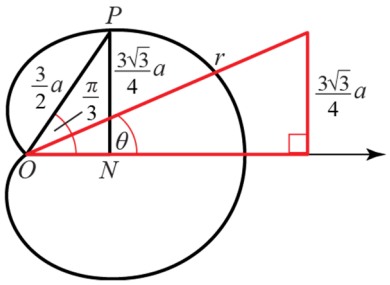
Where the tangent at a point is parallel to the initial line, the distance y from the point to the initial line has a stationary value. You find the polar coordinates θ of such points by finding the values of θ for which $y = r\sin\theta$ has stationary values.

$$\cos 2\theta + \cos\theta = 2\cos^2\theta - 1 + \cos\theta = 0$$

$$2\cos^2\theta + \cos\theta - 1 = (2\cos\theta - 1)(\cos\theta + 1) = 0$$

$$\cos\theta = \frac{1}{2},\ \cos\theta = -1$$

$$\theta = \pm\frac{\pi}{3},\ \theta = \pi$$



$$\text{At } \theta = \frac{\pi}{3},$$

$$r = a\left(1+\cos\frac{\pi}{3}\right) = a\left(1+\frac{1}{2}\right) = \frac{3}{2}a$$

$$\text{And } y = r\sin\frac{\pi}{3} = \frac{3}{2}a \times \frac{\sqrt{3}}{2} = \frac{3\sqrt{3}}{4}a$$

You find the distance (labelled PN in the diagram above) from the point where the tangent meets the curve to the initial line.

The polar equation of the tangent is given by

$$r\sin\theta = \frac{3\sqrt{3}}{4}a$$

$$r = \frac{3a\sqrt{3}}{4}\operatorname{cosec}\theta$$

The polar equation is found by trigonometry in the triangle marked in red on the diagram above.

Similarly at $\theta = -\frac{\pi}{3}$, the equation of the tangent is $r = -\frac{3a\sqrt{3}}{4}\operatorname{cosec}\theta$.

At $\theta = \pi$, the equation of the tangent is

$$\theta = \pi.$$

It is easy to overlook this case. The half-line $\theta = \pi$ does touch the cardioid at the pole.

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4 c The circle and the cardioids meet when

$a = a(1+\cos\theta) \Rightarrow \cos\theta = \theta$

$\theta = \pm\frac{\pi}{2}$

To find the area of the cardioid between

$\theta = -\frac{\pi}{2}$ and $\theta = \frac{\pi}{2}$

$A = 2\times\frac{1}{2}\int_0^{\frac{\pi}{2}} r^2 \mathrm{d}\theta$

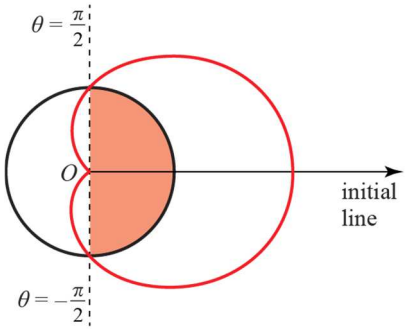
The total area is twice the area above the initial line.

$\int r^2 \mathrm{d}\theta = \int a^2(1+\cos\theta)^2 \mathrm{d}\theta = \int a^2(1+2\cos\theta+\cos^2\theta)\mathrm{d}\theta$

$= a^2\int\left(1+2\cos\theta+\frac{1}{2}\cos 2\theta+\frac{1}{2}\right)\mathrm{d}\theta$

$= a^2\int\left(\frac{3}{2}+2\cos\theta+\frac{1}{2}\cos 2\theta\right)\mathrm{d}\theta$

$= a^2\left[\frac{3}{2}\theta+2\sin\theta+\frac{1}{4}\sin 2\theta\right]$



$A = a^2\left[\frac{3}{2}\theta+2\sin\theta+\frac{1}{4}\sin 2\theta\right]_0^{\frac{\pi}{2}}$

$= a^2\left(\frac{3\pi}{4}+2\right)$

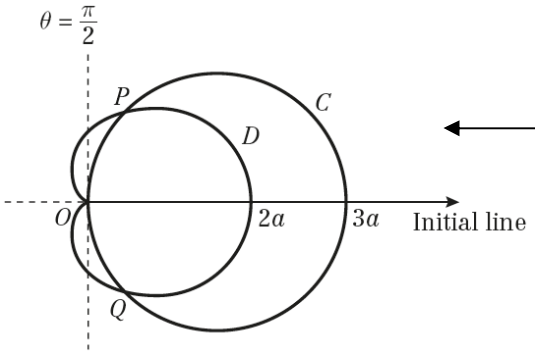
The required area is A less half of the circle

$\left(\frac{3\pi}{4}+2\right)a^2 - \frac{1}{2}\pi a^2 = \frac{1}{4}\pi a^2 + 2a^2$

$= \left(\frac{\pi+8}{4}\right)a^2$, as required.

The area you are asked to find is inside the cardioid and outside the circle. You find it by subtracting the shaded semi-circle from the area of the cardioid bounded by the half-lines $\theta = \frac{\pi}{2}$ and $\theta = -\frac{\pi}{2}$.

5 a



The curve C is a circle of diameter $3a$ and the curve D is a cardioid.
The points of intersection of C and D have been marked on the diagram. The question does not specify which is P and which is Q. They could be interchanged. This would make no substantial difference to the solution of the question.

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5 b The points of intersection of C and D are given by

$3\cancel{a}\cos\theta = \cancel{a}(1+\cos\theta)$

$2\cos\theta = 1 \Rightarrow \cos\theta = \frac{1}{2}$

$\theta = \pm\frac{\pi}{3}$

In this question $-\frac{\pi}{2} \leqslant \theta < \frac{\pi}{2}$.

Where $\cos\theta = \frac{1}{2}$

$r = 3a\cos\frac{\pi}{3} = 3a \times \frac{1}{2} = \frac{3}{2}a$

$P: \left(\frac{3}{2}a, \frac{\pi}{3}\right)$, $Q: \left(\frac{3}{2}a, -\frac{\pi}{3}\right)$

c The area between D, the initial line and OP is given by

$A_1 = \frac{1}{2}\int_0^{\frac{\pi}{3}} r^2 \mathrm{d}\theta$

$\int r^2 \,\mathrm{d}\theta = \int a^2(1+\cos\theta)^2 \mathrm{d}\theta = a^2\int(1+2\cos\theta+\cos^2\theta)\mathrm{d}\theta$

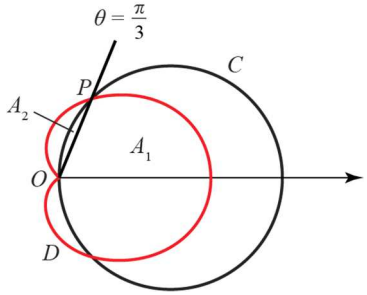
$= a^2\int\left(1+2\cos\theta+\frac{1}{2}\cos 2\theta+\frac{1}{2}\right)\mathrm{d}\theta$

$= a^2\int\left(\frac{3}{2}+2\cos\theta+\frac{1}{2}\cos 2\theta\right)\mathrm{d}\theta$

$= a^2\left[\frac{3}{2}\theta+2\sin\theta+\frac{1}{4}\sin 2\theta\right]$

$A_1 = \frac{1}{2}\times a^2\left[\frac{3}{2}\theta+2\sin\theta+\frac{1}{4}\sin 2\theta\right]_0^{\frac{\pi}{3}}$

$= \frac{a^2}{2}\left[\frac{\pi}{2}+\sqrt{3}+\frac{\sqrt{3}}{8}\right] = \frac{a^2}{16}(4\pi+9\sqrt{3})$



By the symmetry of the figure, to find the area inside C but outside D, you subtract two areas A_1 and two areas A_2 from the area inside C. C is a circle of radius $\frac{3a}{2}$.

d Let the smaller area enclosed by C and the half-line $\theta = \frac{\pi}{3}$ be A_2.

$R = \pi\left(\frac{3a}{2}\right)^2 - 2A_1 - 2A_2$

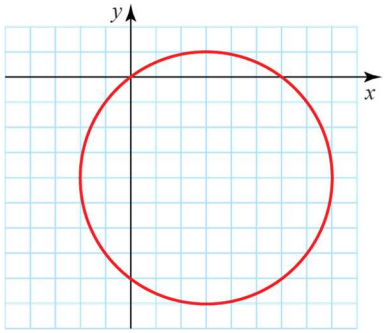
$= \frac{9a^2\pi}{4} - \frac{2a^2}{16}(4\pi+9\sqrt{3}) - \frac{6a^2}{16}(2\pi-3\sqrt{3})$

This is twice the area you are given in the question.

$= \frac{9a^2\pi}{4} - \frac{\pi a^2}{2} - \cancel{\frac{9\sqrt{3}a^2}{8}} - \frac{3\pi a^2}{4} + \cancel{\frac{9\sqrt{3}a^2}{8}} = \pi a^2$, as required.

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6 a The locus is a circle of centre $3-4i$ and radius 5, so the Argand diagram is the following:



b Suppose z is such that $|z-3+4i|=5$. Then assuming $z=r(\cos\theta+i\sin\theta)$ we get that $|r\cos\theta-3+i(r\sin\theta+4)|=5$; but the magnitude of this complex number is given by $\sqrt{(r\cos\theta-3)^2+(r\sin\theta+4)^2}$, so we get the following:

$r^2\cos^2\theta-6r\cos\theta+9+r^2\sin^2\theta+8r\sin\theta+16=25$

$r^2+25-6r\cos\theta+8r\sin\theta=25$

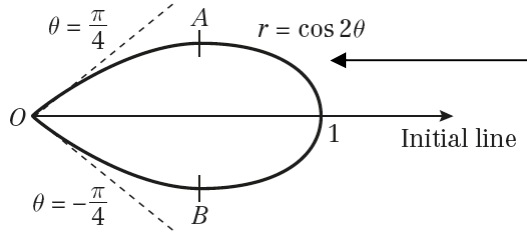
$r=6\cos\theta-8\sin\theta$

c The area of A is the area of the circle minus the areas that are enclosed in the fourth Cartesian quadrant. Now consider the 63circular sector enclosed between the radii that intersect the circle on the real line. The intersections between the circle and the real line are the origin and 6. Then since the circle has centre $3-4i$, if we interpret this in the Cartesian plane we can find the angle between the radii: it is $\arccos\left(\frac{16-9}{25}\right)$, as the cosine of the angle is given by the ratio between the inner product and the product of the magnitudes of the two radii, seen as vector. Then the area of the circular sector is $\frac{\arccos\left(\frac{7}{25}\right)\cdot 25}{2}$. From this we subtract the area of the triangle formed by the two real points and the circle, which is 12. With the same procedure applied to the complex line we find that the arc between the origin and $-8i$ encloses an area of $\frac{\arccos\left(-\frac{7}{25}\right)\cdot 25}{2}-12$. Then the area of A is:

$$25\pi-\left(\frac{\arccos\left(-\frac{7}{25}\right)\cdot 25}{2}-12\right)-\left(\frac{\arccos\left(\frac{7}{25}\right)\cdot 25}{2}-12\right)$$

$$=78.5-11.2-4.1=63.3$$

7 a



At $\theta=-\frac{\pi}{4}$, $r=0$. As θ increases, r increases until $\theta=0$. For $\theta=0$, $\cos 2\theta$ has its greatest value of 1. After that, as θ increases, r decreases to 0 at $\theta=\frac{\pi}{4}$.

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7 b $y = r\sin\theta = \cos 2\theta \sin\theta$

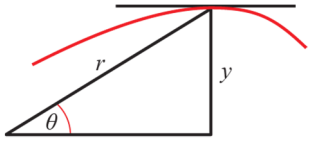
$$\frac{dy}{d\theta} = -2\sin 2\theta \sin\theta + \cos 2\theta \cos\theta = 0$$

$$-4\sin\theta\cos\theta\sin\theta + (1 - 2\sin^2\theta)\cos\theta = 0$$

$$\cos\theta(-4\sin^2\theta + 1 - 2\sin^2\theta) = 0$$

At A and B, $\cos\theta \neq 0$

$$6\sin^2\theta = 1$$



Where the tangent at a point is parallel to the initial line, the distance y from the point to the initial line has a stationary value. The diagram above shows that $y = r\sin\theta$. You find the polar coordinates θ of the points by finding the values of θ for which $r\sin\theta$ has a maximum or minimum value.

$$\sin\theta = \pm\frac{1}{\sqrt{6}}$$

$$\theta = \pm 0.420\,534\ldots$$

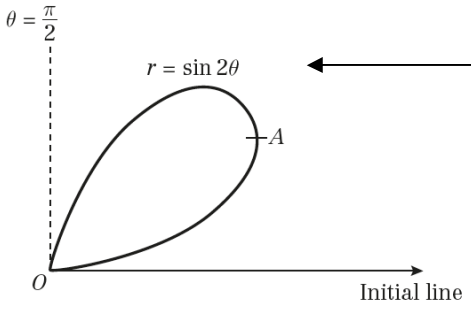
$$r = \cos 2\theta = 1 - 2\sin^2\theta = 1 - \frac{2}{6} = \frac{2}{3}$$

To 3 significant figures, the polar coordinates of A and B are

(0.667, 0.421) and (0.667, −0.421).

r has an exact value but the question specifically asks for 3 significant figures. Unless the question specifies otherwise, in polar coordinates, you should always give the value of the angle in radians.

8 a



At $\theta = 0$, $r = 0$. As θ increases, r increases until $\theta = \frac{\pi}{4}$. For $\theta = \frac{\pi}{4}$, $\sin 2\theta$ has its greatest value of 1. After that, as θ increases, r decreases to $\sin\left(2 \times \frac{\pi}{2}\right) = \sin\pi = 0$ at $\theta = \frac{\pi}{2}$.

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8 b $x = r\cos\theta = \sin 2\theta\cos\theta$

$$\frac{dx}{d\theta} = 2\cos 2\theta\cos\theta - \sin 2\theta\sin\theta$$
$$= 2(2\cos^2\theta - 1)\cos\theta - 2\sin\theta\cos\theta\sin\theta$$
$$= 2(2\cos^2\theta - 1)\cos\theta - 2\sin^2\theta\cos\theta$$
$$= 4\cos^3\theta - 2\cos\theta - 2(1-\cos^2\theta)\cos\theta$$
$$= 6\cos^3\theta - 4\cos\theta = 0$$
$$2\cos\theta(3\cos^2\theta - 2) = 0$$

At A, $\cos\theta \neq 0$

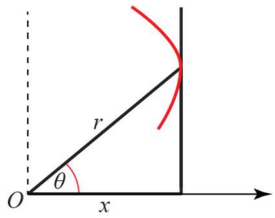
$$\cos^2\theta = \frac{2}{3}$$
$$\cos\theta = \left(\frac{2}{3}\right)^{\frac{1}{2}}, \text{ for } 0 \leqslant \theta \leqslant \frac{\pi}{2}$$
$$\theta = 0.615\,479\ldots$$

By calculator

$r = \sin 2\theta = 0.942\,809\ldots$

To 3 significant figures, the coordinates of A are

(0.943, 0.615)

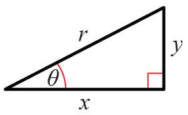


Where the tangent at a point is parallel to $\theta = \frac{\pi}{2}$ (which is the same as being perpendicular to the initial line), the distance x from the half line $\theta = \frac{\pi}{2}$ has a stationary value. The diagram above shows that $x = r\cos\theta$. You find the polar coordinates θ of such points by finding the values of θ for which $r\cos\theta$ has a maximum or minimum value.

9 a $r = 6\cos\theta$

Multiplying the equation by r

$$r^2 = 6r\cos\theta$$
$$x^2 + y^2 = 6x$$
$$x^2 - 6x + 9 + y^2 = 0$$
$$(x-3)^2 + y^2 = 9$$
$$r = 3\sec\left(\frac{\pi}{3} - \theta\right)$$
$$3 = r\cos\left(\frac{\pi}{3} - \theta\right) = r\cos\frac{\pi}{3}\cos\theta + r\sin\frac{\pi}{3}\sin\theta$$
$$= \frac{1}{2}r\cos\theta + \frac{\sqrt{3}}{2}r\sin\theta$$
$$= \frac{1}{2}x + \frac{\sqrt{3}}{2}y$$
$$x + \sqrt{3}y = 6$$

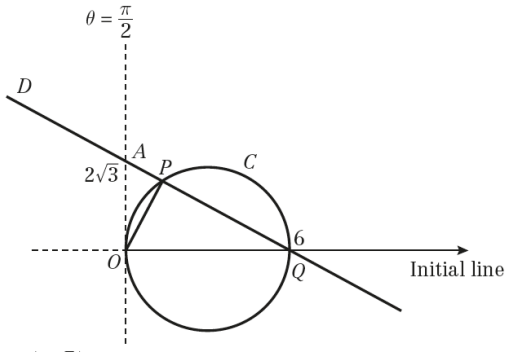


This diagram illustrates the relations between polar and Cartesian coordinates. The relations you need to solve the question are $r^2 = x^2 + y^2$, $x = r\cos\theta$ and $y = r\sin\theta$.

This is an acceptable answer but putting the equation into a form which shows that the curve is a circle, centre (3, 0) and radius 3, helps you to draw the sketch in part **b**.

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9 b



The initial line is the positive x-axis and the half-line $u = \frac{\pi}{2}$ is the positive y-axis.

At $x = 0$, $x + \sqrt{3}y = 6$ gives $y = \frac{6}{\sqrt{3}} = 2\sqrt{3}$.

c By inspection, the polar coordinates of Q are (6, 0)

The question does not say which point is P and which is Q. You can choose which is which.

$\angle OPQ = 90^\circ$

The angle in a semi-circle is a right angle.

In the triangle OAQ

$$\tan AQO = \frac{OA}{OQ} = \frac{2\sqrt{3}}{6} = \frac{\sqrt{3}}{3} \Rightarrow \angle AQO = 30^\circ$$

In the triangle OPQ

$$OP = OQ \sin PQO = 6 \sin 30^\circ = 3$$

$$\angle POQ = 180^\circ - 90^\circ - 30^\circ = 60^\circ = \frac{\pi}{3}$$

Hence the polar coordinates of P are

$$(OP, \angle POQ) = \left(3, \frac{\pi}{3}\right)$$

10 $A = \frac{1}{2}\int_0^{\frac{\pi}{2}} r^2 \mathrm{d}\theta$

You need to know the formula for the area of polar curves $A = \frac{1}{2}\int r^2 \mathrm{d}\theta$. In this question, the diagram shows that the limits are 0 and $\frac{\pi}{2}$.

$$\frac{1}{2}\int r^2 \mathrm{d}\theta = \frac{1}{2}\int a^2 \sin 2\theta \, \mathrm{d}\theta$$

$$= \frac{a^2}{2}\left[-\frac{\cos 2\theta}{2}\right]$$

$$A = \frac{a^2}{4}\left[-\cos 2\theta\right]_0^{\frac{\pi}{2}} = \frac{a^2}{4}[1 - (-1)]$$

$\cos\left(2 \times \frac{\pi}{2}\right) = \cos \pi = -1$ and $\cos 0 = 1$.

$$= \frac{1}{2}a^2$$

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$$A = \frac{1}{2}\int_{\frac{\pi}{8}}^{\frac{\pi}{4}} r^2 \mathrm{d}\theta$$

The lower limit, $\frac{\pi}{8}$, is given by the polar equation of m. The upper limit, $\frac{\pi}{4}$, can be identified from the domain of definition, $0 \leqslant \theta \leqslant \frac{\pi}{4}$ given in the question and the diagram.

11 $\frac{1}{2}\int_{\frac{\pi}{8}}^{\frac{\pi}{4}} r^2 \mathrm{d}\theta = \frac{1}{2}\int_{\frac{\pi}{8}}^{\frac{\pi}{4}} 16a^2\cos^2 2\theta \mathrm{d}\theta$

$$= 8a^2\int_{\frac{\pi}{8}}^{\frac{\pi}{4}} \cos^2 2\theta \mathrm{d}\theta$$

$$= 8a^2\int_{\frac{\pi}{8}}^{\frac{\pi}{4}} \left(\frac{1}{2} + \frac{1}{2}\cos 4\theta\right)\mathrm{d}\theta$$

Using $\cos 2A = 2\cos^2 A - 1$ with $A = 2\theta$.

$$= 4a^2\left[\theta + \frac{\sin 4\theta}{4}\right]_{\frac{\pi}{8}}^{\frac{\pi}{4}}$$

$$A = 4a^2\left[\theta + \frac{\sin 4\theta}{4}\right]_{\frac{\pi}{8}}^{\frac{\pi}{4}}$$

$$= 4a^2\left[\left(\frac{\pi}{4} - \frac{\pi}{8}\right) + \left(0 - \frac{1}{4}\right)\right]$$

$$= 4a^2\left[\frac{\pi}{8} - \frac{1}{4}\right]$$

$$= \frac{1}{2}a^2(\pi - 2)$$

$\sin\left(4 \times \frac{\pi}{4}\right) = \sin\pi = 0$ and $\sin\left(4 \times \frac{\pi}{8}\right) = \sin\frac{\pi}{2} = 1$.

12 $A = 2 \times \frac{1}{2}\int_0^{\pi} r^2 \mathrm{d}\theta$

The method used here is to find twice the area above the initial line.

$$= \int_0^{\pi} a^2\left(1 + \frac{1}{2}\cos\theta\right)^2 \mathrm{d}\theta$$

$$= a^2\int_0^{\pi}\left(1 + \cos\theta + \frac{1}{4}\cos^2\theta\right)\mathrm{d}\theta$$

Use $\cos 2\theta = 2\cos^2\theta - 1$.

$$= a^2\int_0^{\pi}\left(1 + \cos\theta + \frac{1}{8}\cos 2\theta + \frac{1}{8}\right)\mathrm{d}\theta$$

$$= a^2\int_0^{\pi}\left(\frac{9}{8} + \cos\theta + \frac{1}{8}\cos 2\theta\right)\mathrm{d}\theta$$

$$= a^2\left[\frac{9}{8}\theta + \sin\theta + \frac{\sin 2\theta}{16}\right]_0^{\pi}$$

As $\sin\pi = \sin 2\pi = 0$ and $\sin 0 = 0$, all of the terms are zero at both the lower and the upper limit except for $\frac{9}{8}\theta$, which has a non-zero value at π.

$$= a^2 \times \frac{9}{8}\pi = \frac{9}{8}\pi a^2$$

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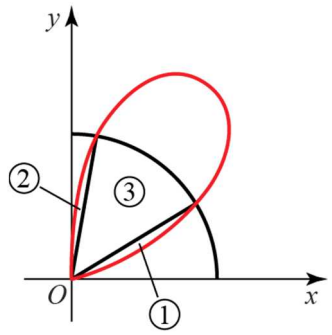
13 a The curves intersect at

$$\frac{1}{2} = \sin 2\theta$$

$$2\theta = \frac{\pi}{6}, \frac{5\pi}{6}$$

$$\theta = \frac{\pi}{12}, \frac{5\pi}{12}$$

b



The shaded area can be broken up into three parts. You can find the small areas labelled (**1**) and (**2**), which are equal in area, by integration. The larger area is a sector of a circle and you find this using $A = \frac{1}{2}r^2\theta$, where θ is in radians.

The area of the sector (**3**) is given by

$$A_3 = \frac{1}{2} \times \left(\frac{1}{2}\right)^2 \times \frac{\pi}{3} = \frac{\pi}{24}$$

The radius of the sector is $\frac{1}{2}$ and the angle is $\frac{5\pi}{12} - \frac{\pi}{12} = \frac{\pi}{3}$.

The area of (**1**) is given by

$$A_1 = \frac{1}{2}\int_0^{\frac{\pi}{12}} r^2 \mathrm{d}\theta$$

Using $\cos 2A = 1 - 2\sin^2 A$ with $A = 2\theta$.

$$\frac{1}{2}\int \sin^2 2\theta \,\mathrm{d}\theta = \frac{1}{2}\int\left(\frac{1}{2} - \frac{1}{2}\cos 4\theta\right)\mathrm{d}\theta$$

$$= \frac{1}{4}\left[\theta - \frac{\sin 4\theta}{4}\right]$$

$$\sin\left(4 \times \frac{\pi}{12}\right) = \sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$$

$$A_1 = \frac{1}{4}\left[\theta - \frac{\sin 4\theta}{4}\right]_0^{\frac{\pi}{12}}$$

$$= \frac{1}{4}\left[\frac{\pi}{12} - 0 - \frac{1}{4}\left(\frac{\sqrt{3}}{2} - 0\right)\right]$$

$$= \frac{1}{4}\left[\frac{\pi}{12} - \frac{\sqrt{3}}{8}\right]$$

The area of the shaded region is given by

$$2 \times A_1 + A_3 = \frac{1}{2}\left[\frac{\pi}{12} - \frac{\sqrt{3}}{8}\right] + \frac{\pi}{24} = \frac{\pi}{12} - \frac{\sqrt{3}}{16}$$

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14 a Let $y = r\sin\theta$

Where the tangent at a point is parallel to the initial line, the distance y from the point to the initial line has a stationary value. You find the polar coordinate θ of the point by finding the value θ for which $y = r\sin\theta$ has a stationary value.

$$y = a(3+\sqrt{5}\cos\theta)\sin\theta$$

$$= 3a\sin\theta + \sqrt{5}a\cos\theta\sin\theta = 3a\sin\theta + \frac{\sqrt{5}a}{2}\sin 2\theta$$

$$\frac{\mathrm{d}y}{\mathrm{d}\theta} = 3a\cos\theta + \sqrt{5}a\cos 2\theta = 0$$

$$3\cos\theta + \sqrt{5}(2\cos^2\theta - 1) = 0$$

$$2\sqrt{5}\cos^2\theta + 3\cos\theta - \sqrt{5} = 0$$

$$\cos\theta = -3 \pm \frac{\sqrt{(9+40)}}{4\sqrt{5}}$$

As $|\cos\theta| \leqslant 1$, you reject the value $-\frac{10}{4\sqrt{5}} \approx -1.118$.

$$= \frac{-3+7}{4\sqrt{5}} = \frac{1}{\sqrt{5}}$$

By calculator

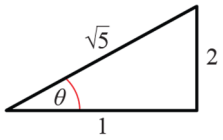
$\theta = \pm 1.107$ (3 d.p.)

At $\cos\theta = \frac{1}{\sqrt{5}}$

$$r = a(3+\sqrt{5}\cos\theta) = a\left(3+\sqrt{5}\times\frac{1}{\sqrt{5}}\right) = 4a$$

The polar coordinates are

P: $(4a, 1.107)$, Q: $(4a, -1.107)$



As $1^2 + 2^2 = (\sqrt{5})^2$, the diagram illustrates that if $\cos\theta = \frac{1}{\sqrt{5}}$ then $\sin\theta = \frac{2}{\sqrt{5}}$.

b $PQ = 2y = 2r\sin\theta$

$$= 2\times 4a\times\frac{2}{\sqrt{5}} = \frac{16}{\sqrt{5}}a = 20\text{m, given}$$

$$a = \frac{20\sqrt{5}}{16}\text{m} = \frac{5\sqrt{5}}{4}\text{m}$$

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14 c Total area $= 2 \times \frac{1}{2}\int_{\alpha}^{\beta} r^2 \mathrm{d}\theta$

$r = a(3 + \sqrt{5}\cos\theta)$

so area $= a^2 \int_0^{\pi} (3 + \sqrt{5}\cos\theta)^2 \mathrm{d}\theta$

$= a^2 \int_0^{\pi} (9 + 6\sqrt{5}\cos\theta + 5\cos^2\theta)\mathrm{d}\theta$

$= a^2 \left[9\theta + 6\sqrt{5}\sin\theta + \frac{5}{4}\sin 2\theta + \frac{5}{2}\theta \right]_0^{\pi}$

$= a^2 \left[\frac{23}{2}\theta + 6\sqrt{5}\sin\theta + \frac{5}{4}\sin 2\theta \right]_0^{\pi}$

$= a^2 \left[\frac{23}{2}\pi \right]$

$= \frac{23}{2}\pi a^2$

$a = \frac{5\sqrt{5}}{4}$

so area $= \frac{23}{2}\left(\frac{5\sqrt{5}}{4} \right)^2 \pi$

$= \frac{2875}{32}\pi \text{ m}^2$

15 a Area $= \frac{1}{2}\int_{-\pi}^{\pi} r^2 \mathrm{d}\theta$

$= \frac{1}{2}\int_{-\pi}^{\pi} a^2(1 + \cos\theta)^2 \mathrm{d}\theta$

$= \frac{a^2}{2}\int_{-\pi}^{\pi} (1 + 2\cos\theta + \cos^2\theta)\,\mathrm{d}\theta$

$= \frac{a^2}{2}\int_{-\pi}^{\pi} \mathrm{d}\theta + a^2\int_{-\pi}^{\pi} \cos\theta\,\mathrm{d}\theta + \frac{a^2}{2}\int_{-\pi}^{\pi} \cos^2\theta\,\mathrm{d}\theta$

$= a^2\pi + a^2\left[\sin\theta\right]_{-\pi}^{\pi} + \frac{a^2}{4}\int_{-\pi}^{\pi} \mathrm{d}\theta + \frac{a^2}{4}\int_{-\pi}^{\pi} \cos 2\theta\,\mathrm{d}\theta$ using the identity $\cos 2\theta \equiv 2\cos^2\theta - 1$

$= a^2\pi + \frac{a^2\pi}{2} + \frac{a^2}{4}\left[\frac{1}{2}\sin 2\theta \right]_{-\pi}^{\pi}$

$= \frac{3\pi a^2}{2}$

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15 b At A and B, $\frac{\mathrm{d}}{\mathrm{d}\theta}(r\cos\theta) = 0$.

$$\frac{\mathrm{d}}{\mathrm{d}\theta}(r\cos\theta) = \frac{\mathrm{d}}{\mathrm{d}\theta}(a\cos\theta(1+\cos\theta))$$
$$= a\frac{\mathrm{d}}{\mathrm{d}\theta}(\cos\theta + \cos^2\theta)$$
$$= -a\sin\theta + a\frac{\mathrm{d}}{\mathrm{d}\theta}(\cos^2\theta)$$
$$= -a\sin\theta - 2a\cos\theta\sin\theta, \text{ using the product rule}$$
$$= -a\sin\theta(1+2\cos\theta)$$

Setting this equal to 0 gives:

$\sin\theta(1+2\cos\theta) = 0$

$\sin\theta = 0$ or $\cos\theta = -\frac{1}{2}$

$\therefore\ \theta = 0$ or $\pm\frac{2\pi}{3}$

But $\theta = 0$ is where C intersects the initial line so $\theta = \pm\frac{2\pi}{3}$ at A and B.

$\theta = \pm\frac{2\pi}{3} \Rightarrow r = a\left(1+\cos\frac{2\pi}{3}\right) = \frac{a}{2}$

So the polar coordinates of A and B are A: $\left(\frac{a}{2}, \frac{2\pi}{3}\right)$ and B: $\left(\frac{a}{2}, -\frac{2\pi}{3}\right)$.

c When C intersects the initial line, $r = 2a$.

Therefore, length WX $= 2a +$ length of x–component of vector $\overrightarrow{\mathrm{OA}}$

$= 2a + \frac{2}{a}\cos\frac{\pi}{3} = 2a + \frac{a}{4} = \frac{9a}{4}$.

d Area WXYZ $= \frac{9a}{4} \times \frac{3a\sqrt{3}}{2} = \frac{27\sqrt{3}a^2}{8}$.

e Area wasted = Area WXYZ − Area inside C

$$= \frac{27\sqrt{3}a^2}{8} - \frac{3\pi a^2}{2}$$

So when $a = 10$ cm, the area of card wasted is

$\frac{2700\sqrt{3}}{8} - \frac{300\pi}{2} = 113\ \mathrm{cm}^2$ (to 3s.f.)

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16 a C_1 and C_2 intersect where

$3\cancel{a}(1-\cos\theta) = \cancel{a}(1+\cos\theta)$

$3-3\cos\theta = 1+\cos\theta$

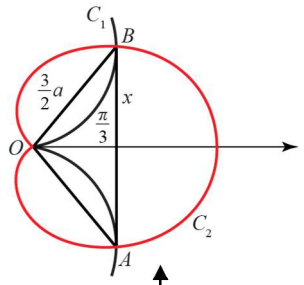
$4\cos\theta = 2 \Rightarrow \cos\theta = \frac{1}{2}$

$\theta = \pm\frac{\pi}{3}$

Where $\cos\theta = \frac{1}{2}$

$r = a(1+\cos\theta) = a\left(1+\frac{1}{2}\right) = \frac{3}{2}a$

$A: \left(\frac{3}{2}a, -\frac{\pi}{3}\right)$, $B: \left(\frac{3}{2}a, \frac{\pi}{3}\right)$



Referring to the diagram,

$\frac{x}{\frac{3}{2}a} = \sin\frac{\pi}{3} \Rightarrow x = \frac{3}{2}a\sin\frac{\pi}{3}$ and

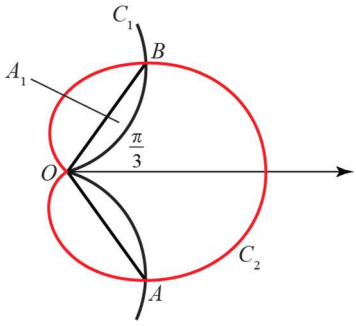
$AB = 2x$.

b $AB = 2\times\frac{3}{2}a\sin\frac{\pi}{3} = 3a\times\frac{\sqrt{3}}{2}$

$= \frac{3\sqrt{3}}{2}a$, as required.

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16 c The area A_1 enclosed by OB and C_1 is given by



$$A_1 = \frac{1}{2}\int_0^{\frac{\pi}{3}} r^2 \mathrm{d}\theta$$

$$\int r^2 \mathrm{d}\theta = \int 9a^2(1-\cos\theta)^2 \mathrm{d}\theta = \int 9a^2(1-2\cos\theta+\cos^2\theta)\mathrm{d}\theta$$

$$= 9a^2\int\left(1-2\cos\theta+\frac{1}{2}\cos 2\theta+\frac{1}{2}\right)\mathrm{d}\theta$$

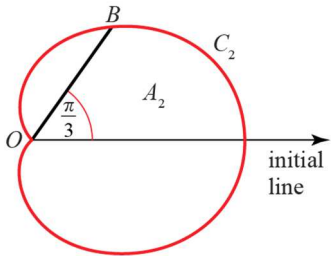
$$= 9a^2\int\left(\frac{3}{2}-2\cos\theta+\frac{1}{2}\cos 2\theta\right)\mathrm{d}\theta$$

$$= 9a^2\left[\frac{3}{2}\theta-2\sin\theta+\frac{1}{4}\sin 2\theta\right]$$

$$A_1 = \frac{1}{2}\times 9a^2\left[\frac{3}{2}\theta-2\sin\theta+\frac{1}{4}\sin 2\theta\right]_0^{\frac{\pi}{3}}$$

$$= \frac{9}{2}a^2\left[\frac{\pi}{2}-\sqrt{3}+\frac{\sqrt{3}}{8}\right] = \frac{9a^2}{16}(4\pi-7\sqrt{3})$$

The area A_2 enclosed by the initial line, C_2 and OB is given by



$$A_2 = \frac{1}{2}\int_0^{\frac{\pi}{3}} r^2 \mathrm{d}\theta$$

$$\int r^2 \mathrm{d}\theta = \int a^2(1+\cos\theta)^2 \mathrm{d}\theta = a^2\int(1+2\cos\theta+\cos^2\theta)\mathrm{d}\theta$$

$$= a^2\int\left(1+2\cos\theta+\frac{1}{2}\cos 2\theta+\frac{1}{2}\right)\mathrm{d}\theta$$

$$= a^2\int\left(\frac{3}{2}+2\cos\theta+\frac{1}{2}\cos 2\theta\right)\mathrm{d}\theta$$

$$= a^2\left[\frac{3}{2}\theta+2\sin\theta+\frac{1}{4}\sin 2\theta\right]$$

$$A_2 = \frac{1}{2}\times a^2\left[\frac{3}{2}\theta+2\sin\theta+\frac{1}{4}\sin 2\theta\right]_0^{\frac{\pi}{3}}$$

$$= \frac{a^2}{2}\left[\frac{\pi}{2}+\sqrt{3}+\frac{\sqrt{3}}{8}\right] = \frac{a^2}{16}(4\pi+9\sqrt{3})$$

The required area R is given by

$$R = 2(A_2 - A_1)$$

$$= 2\left[\frac{a^2}{16}(4\pi+9\sqrt{3})-\frac{9a^2}{16}(4\pi-7\sqrt{3})\right]$$

$$= \frac{2a^2}{16}\left[4\pi+9\sqrt{3}-(36\pi-63\sqrt{3})\right]$$

$$= \frac{a^2}{8}\left[72\sqrt{3}-32\pi\right] = (9\sqrt{3}-4\pi)a^2$$

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16 d $\frac{3\sqrt{3}}{2}a = 4.5\,\text{cm}$

$a = \frac{9}{3\sqrt{3}}\,\text{cm} = \sqrt{3}\,\text{cm}$

You use the result from part **b** to find a and substitute the value of a into the result of part **c**.

The area of the badge is

$(9\sqrt{3} - 4\pi)a^2 = (9\sqrt{3} - 4\pi) \times 3\,\text{cm}^2$

$= 9.07\,\text{cm}^2$ (3 s.f.)

17 a $A: (5a, 0),\ B: (3a, 0)$

For A, at $\theta = 0$, $r = a(3 + 2\cos 0) = a(3 + 2) = 5a$.
For B, at $\theta = 0$, $r = a(5 - 2\cos 0) = a(5 - 2) = 3a$.

b The curves intersect where

$a(3 + 2\cos\theta) = a(5 - 2\cos\theta)$

$4\cos\theta = 2 \Rightarrow \cos\theta = \frac{1}{2}$

$\theta = \frac{\pi}{3}, \frac{5\pi}{3}$

In this question $0 \leq \theta < 2\pi$.

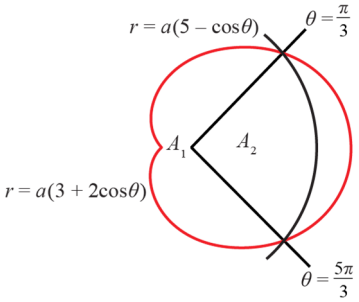
Where $\cos\theta = \frac{1}{2}$

$r = a(3 + 2\cos\theta) = a\left(3 + 2 \times \frac{1}{2}\right) = 4a$

$C: \left(4a, \frac{5\pi}{3}\right),\ D: \left(4a, \frac{\pi}{3}\right)$

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17 c The area A1 enclosed by $r = a(3+2\cos\theta)$ and the half-lines $\theta = \frac{\pi}{3}$ and $\theta = \frac{5\pi}{3}$ is given by



The shaded area in the question is the sum of the two areas A_1 and A_2 shown in the diagram above. It is important that you carefully distinguish which curve is which.

$$A_1 = 2\times\frac{1}{2}\int_{\frac{\pi}{3}}^{\pi} r^2\,\mathrm{d}\theta$$

$$\int r^2\,\mathrm{d}\theta = \int a^2(3+2\cos\theta)^2\,\mathrm{d}\theta$$
$$= a^2\int(9+12\cos\theta+4\cos^2\theta)\,\mathrm{d}\theta$$
$$= a^2\int(9+12\cos\theta+2\cos 2\theta+2)\,\mathrm{d}\theta$$
$$= a^2\int(11+12\cos\theta+2\cos 2\theta)\,\mathrm{d}\theta$$
$$= a^2\left[11\theta+12\sin\theta+\sin 2\theta\right]$$

$$A_1 = a^2\left[11\theta+12\sin\theta+\sin 2\theta\right]_{\frac{\pi}{3}}^{\pi}$$
$$= a^2\left[11\left(\pi-\frac{\pi}{3}\right)+12\left(0-\frac{\sqrt{3}}{2}\right)+\left(0-\frac{\sqrt{3}}{2}\right)\right]$$
$$= a^2\left[\frac{22\pi}{3}-\frac{13\sqrt{3}}{2}\right]$$

The area A_2 enclosed by $r = a(5-2\cos\theta)$ and the half-lines $\theta = \frac{5\pi}{3}$ and $\theta = \frac{\pi}{3}$ is given by

$$A_2 = 2\times\frac{1}{2}\int_0^{\frac{\pi}{3}} r^2\,\mathrm{d}\theta$$

$$\int r^2\,\mathrm{d}\theta = \int a^2(5-2\cos\theta)^2\,\mathrm{d}\theta = a^2\int(25-20\cos\theta+4\cos^2\theta)\,\mathrm{d}\theta$$
$$= a^2\int(25-20\cos\theta+2\cos 2\theta+2)\,\mathrm{d}\theta$$
$$= a^2\int(27-20\cos\theta+2\cos 2\theta)\,\mathrm{d}\theta$$
$$= a^2\left[27\theta-20\sin\theta+\sin 2\theta\right]$$

The double angle formulae, here $\cos 2\theta = 2\cos^2\theta - 1$, are used in all questions involving the areas of cardioids.

$$A_2 = a^2\left[27\theta-20\sin\theta+\sin 2\theta\right]_0^{\frac{\pi}{3}}$$
$$= a^2\left[27\times\frac{\pi}{3}-20\times\frac{\sqrt{3}}{2}+\frac{\sqrt{3}}{2}\right]$$
$$= a^2\left[\frac{27\pi}{3}-\frac{19\sqrt{3}}{2}\right]$$

The area of the overlapping region is given by

$$A_1 + A_2 = a^2\left(\frac{22\pi}{3}-\frac{13\sqrt{3}}{2}+\frac{27\pi}{3}-\frac{19\sqrt{3}}{2}\right)$$
$$= a^2\left(\frac{49\pi}{3}-16\sqrt{3}\right)$$
$$= \frac{a^2}{3}(49\pi-48\sqrt{3}),\ \text{as required.}$$

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18 $2\tanh x - 1 = 0$

$$2\left(\frac{e^x - e^{-x}}{e^x + e^{-x}}\right) = 1$$

$\tanh x = \dfrac{\sinh x}{\cosh x} = \dfrac{\frac{e^x - e^{-x}}{2}}{\frac{e^x + e^{-x}}{2}} = \dfrac{e^x - e^{-x}}{e^x + e^{-x}}$

$$2e^x - 2e^{-x} = e^x + e^{-x}$$

You multiply both sides of this equation by e^x.

$$e^x = 3e^{-x}$$

$$e^{2x} = 3$$

You take the logarithms of both sides of this equation and use the property that $\ln e^{2x} = 2x$.

$$2x = \ln 3$$

$$x = \frac{1}{2}\ln 3$$

19

$$5 = 3\cosh x$$

The wording of the question requires you to use the definition $\cosh x = \dfrac{e^x + e^{-x}}{2}$.

$$5 = 3\left(\frac{e^x + e^{-x}}{2}\right)$$

$$10 = 3e^x + 3e^{-x}$$

You multiply this equation throughout by e^x.

$$3e^x - 10 + 3e^{-x} = 0$$

$$3e^{2x} - 10e^x + 3 = 0$$

You may find it helpful to substitute $y = e^x$ and then, factorise, $3y^2 - 10y + 3 = (3y - 1)(y - 3) = 0$. This gives $y = \frac{1}{3}$ and $y = 3$ and, hence, $e^x = \frac{1}{3}$ and $e^x = 3$. With practice, the substitution can be omitted.

$$(3e^x - 1)(e^x - 3) = 0$$

$$e^x = \frac{1}{3},\ e^x = 3$$

$$x = \ln\left(\frac{1}{3}\right),\ \ln 3$$

The answer $x = -\ln 3$ would also be acceptable.

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20 The curves intersect when

$$5\sinh x = 4\cosh x$$

$$5\left(\frac{e^x - e^{-x}}{2}\right) = 4\left(\frac{e^x + e^{-x}}{2}\right)$$

$$5e^x - 5e^{-x} = 4e^x + 4e^{-x}$$

You use the definitions $\sinh x = \frac{e^x - e^{-x}}{2}$ and $\cosh x = \frac{e^x + e^{-x}}{2}$.

$$e^x = 9e^{-x}$$

$$e^{2x} = 9$$

$$2x = \ln 9$$

$$x = \frac{1}{2}\ln 9 = \ln\sqrt{9} = \ln 3$$

Using the law of logarithms $n\ln a = \ln a^n$ with $n = \frac{1}{2}$ and $a = 9$.

$$y = 5\sinh(\ln 3) = 5\left(\frac{e^{\ln 3} - e^{-\ln 3}}{2}\right) = \frac{5}{2}\times\left(3 - \frac{1}{3}\right)$$

$$= \frac{5}{2}\times\frac{8}{3} = \frac{20}{3}$$

$$p = 3,\ q = \frac{20}{3}$$

$e^{\ln 3} = 3$ and $e^{-\ln 3} = e^{\ln 1 - \ln 3} = e^{\ln\frac{1}{3}} = \frac{1}{3}$, using $\ln 1 = 0$ and the law of logarithms $\ln a - \ln b = \ln\frac{a}{b}$.

21 $5\cosh x - 2\sinh x = 11$

$$5\left(\frac{e^x + e^{-x}}{2}\right) - 2\left(\frac{e^x - e^{-x}}{2}\right) = 11$$

$$5e^x + 5e^{-x} - 2e^x + 2e^{-x} = 22$$

$$3e^x - 22 + 7e^{-x} = 0$$

$$3e^{2x} - 22e^x + 7 = 0$$

$$(3e^x - 1)(e^x - 7) = 0$$

$$e^x = \frac{1}{3},\ 7$$

$$x = \ln\frac{1}{3},\ \ln 7$$

You use the definitions $\sinh x = \frac{e^x - e^{-x}}{2}$ and $\cosh x = \frac{e^x + e^{-x}}{2}$.

You multiply this equation throughout by e^x.

You may find it helpful to substitute $y = e^x$ and then, factorising $3y^2 - 22y + 7 = (3y - 1)(y - 7) = 0$. This gives $y = \frac{1}{3}$ and $y = 7$ and, hence, $e^x = \frac{1}{3}$ and $e^x = 7$. With practice, the substitution can be omitted.

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22 $6\sinh 2x + 9\cosh 2x = 7$

$$6\left(\frac{e^{2x} - e^{-2x}}{2}\right) + 9\left(\frac{e^{2x} + e^{-2x}}{2}\right) = 7$$

$$6e^{2x} - 6e^{-2x} + 9e^{2x} + 9e^{-2x} = 14$$

$$15e^{2x} - 14 + 3e^{-2x} = 0$$

$$15e^{4x} - 14e^{2x} + 3 = 0$$

$$(3e^{2x} - 1)(5e^{2x} - 3) = 0$$

You use the definitions $\sinh x = \frac{e^x - e^{-x}}{2}$ and $\cosh x = \frac{e^x + e^{-x}}{2}$ replacing x by $2x$.

You multiply this equation throughout by e^{2x}.

$$e^{2x} = \frac{1}{3}, \frac{3}{5}$$

$$2x = \ln\frac{1}{3}, \ln\frac{3}{5}$$

$$x = \frac{1}{2}\ln\frac{1}{3}, \frac{1}{2}\ln\frac{3}{5}$$

$$p = \frac{1}{3}, \frac{3}{5}$$

You take the logarithms of both sides of this equation and use the property that $\ln e^{2x} = 2x$.

23 a $\sinh x + 2\cosh x = k$

$$\frac{e^x - e^{-x}}{2} + 2\left(\frac{e^x + e^{-x}}{2}\right) = k$$

$$e^x - e^{-x} + 2e^x + 2e^{-x} = 2k$$

$$3e^x - 2k + e^{-x} = 0$$

$$3e^{2x} - 2ke^x + 1 = 0$$

You use the definitions $\sinh x = \frac{e^x - e^{-x}}{2}$ and $\cosh x = \frac{e^x + e^{-x}}{2}$.

Let $y = e^x$

$$3y^2 - 2ky + 1 = 0$$

$$y = \frac{2k \pm \sqrt{(4k^2 - 12)}}{6}$$

$$= \frac{k \pm \sqrt{(k^2 - 3)}}{3} \quad (\mathbf{1})$$

Using the quadratic formula $y = \frac{-b \pm \sqrt{(b^2 - 4ac)}}{2a}$.

For real y

$k^2 - 3 \geqslant 0 \Rightarrow k \geqslant \sqrt{3},\ k \leqslant -\sqrt{3}$

As $y = e^x > 0$ for all real x, $k \leqslant -\sqrt{3}$ is rejected.

$k \geqslant \sqrt{3}$.

If $x \leqslant -\sqrt{3}$, then both $\frac{k + \sqrt{k^2 - 3}}{3}$ and $\frac{k - \sqrt{(k^2 - 3)}}{3}$ are negative.

b Using (**1**) above with $k = 2$

$$y = e^x = \frac{2 \pm \sqrt{(4 - 3)}}{3} = \frac{2 \pm 1}{3}$$

$$e^x = 1, \frac{1}{3} \Rightarrow x = \ln 1, \ln\frac{1}{3} = 0, -\ln 3$$

You could solve the equation in part **b** without using part **a** but it is efficient to use the work you have already done.

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24 a $\cosh^2 x - \sinh^2 x = \left(\frac{e^x + e^{-x}}{2}\right) - \left(\frac{e^x - e^{-x}}{2}\right)^2$

$= \frac{e^{2x} + 2 + e^{-2x} - (e^{2x} - 2 + e^{-2x})}{4}$

$= \frac{4}{4} = 1$, as required.

$(e^x + e^{-x})^2 = (e^x)^2 + 2e^x \cdot e^{-x} + (e^{-x})^2$
$= e^{2x} + 2 + e^{-2x}$

b Rewriting the equation in terms of the exponential definitions of the hyperbolic functions, it becomes:

$\frac{1}{\frac{e^x - e^{-x}}{2}} - \frac{2}{\frac{e^x - e^{-x}}{e^x + e^{-x}}} = 2$

$\frac{2}{e^x - e^{-x}} - \frac{2(e^x + e^{-x})}{e^x - e^{-x}} = 2$

$2 - 2e^x - 2e^{-x} = 2e^x - 2e^{-x}$

$4e^x = 2$

$x = \ln\frac{1}{2} = -\ln 2$

25 a $2\cosh^2 x - 1 = 2\left(\frac{e^x + e^{-x}}{2}\right)^2 - 1$

$= 2 \times \frac{e^{2x} + 2 + e^{-2x}}{4} - 1$

$= \frac{2e^{2x}}{4} + \frac{4}{4} + \frac{2e^{-2x}}{4} - 1$

$= \frac{e^{2x} + e^{-2x}}{2} = \cosh 2x$, as required

$(e^x + e^{-x})^2 = (e^x)^2 + 2e^x \cdot e^{-x} + (e^{-x})^2$
$= e^{2x} + 2 + e^{-2x}$

b Using the result in part **a**

$\cosh 2x - 5\cosh x = 2$

$2\cosh^2 x - 1 - 5\cosh x = 2$

$2\cosh^2 x - 5\cosh x - 3 = 0$

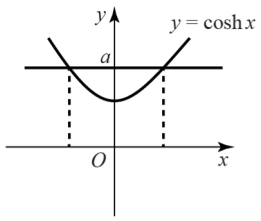
$(2\cosh x + 1)(\cosh x - 3) = 0$

$\cosh x = -\frac{1}{2}$, $\cosh x = 3$

$\cosh x = -\frac{1}{2}$ is impossible

$x = \pm \operatorname{arcosh} 3 = \pm \ln(3 + \sqrt{8})$

If $\operatorname{arcosh} x > 0$ then $\operatorname{arcosh} x = \ln(x + \sqrt{(x^2 - 1)})$.
However if $\cosh x = a$, where $a > 0$, then there are two answers $x = \pm \ln(a + \sqrt{(a^2 - 1)})$



These answers can also be written as $x = \ln(3 \pm \sqrt{8})$.

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26 a $4\cosh^3 x - 3\cosh x = 4\left(\frac{e^x + e^{-x}}{2}\right)^3 - 3\left(\frac{e^x + e^{-x}}{2}\right)$

$= \frac{e^x + 3e^x + 3e^{-x} + e^{-3x}}{2} - \frac{3e^x + 3e^{-x}}{2}$

$= \frac{e^{3x} + e^{-3x}}{2} = \cosh 3x$, as required.

Using the binomial expansion
$(e^x + e^{-x})^3$
$= (e^x)^3 + 3(e^x)^2 \cdot e^{-x}$
$+ 3e^x(e^{-x})^2 + (e^{-x})^3$
$= e^{3x} + 3e^x + 3e^{-x} + e^{-3x}$.

b $\cosh 3x = 5\cosh x$

Using the result in part **a**

$4\cosh^3 x - 3\cosh x = 5\cosh x$

$4\cosh^3 x - 8\cosh x = 0$

$4\cosh x(\cosh^2 x - 2) = 0$

As for all x, $\cosh x \geqslant 1$,

$\cosh x = \sqrt{2}$

$x = \pm \ln(\sqrt{2} + 1)$

$-\ln(\sqrt{2} + 1) = \ln\left(\frac{1}{\sqrt{2} + 1}\right) = \ln\left(\frac{1}{\sqrt{2+1}} \times \frac{\sqrt{2} - 1}{\sqrt{2} - 1}\right)$

$= \ln\left(\frac{\sqrt{2} - 1}{1}\right) = \ln(\sqrt{2} - 1)$

There are 3 possible answers to this cubic, $\cosh x = 0$, $\cosh x = -\sqrt{2}$ and $\cosh x = \sqrt{2}$. As for all real x, $\cosh x \geqslant 1$ only the last of the three gives real values of x.

Using $\operatorname{arcosh} x = \ln\left(x + \sqrt{(x^2 - 1)}\right)$

The solutions of $\cosh 3x = 5\cosh x$, as natural logarithms, are $x = \ln(\sqrt{2} \pm 1)$.

27 a $\cosh A \cosh B - \sinh A \sinh B$

$= \left(\frac{e^A + e^{-A}}{2}\right)\left(\frac{e^B + e^{-B}}{2}\right) - \left(\frac{e^A - e^{-A}}{2}\right)\left(\frac{e^B - e^{-B}}{2}\right)$

$= \frac{1}{4}\left(e^{A+B} + e^{-A+B} + e^{A-B} + e^{-A-B} - e^{A+B} + e^{-A+B} + e^{A-B} - e^{-A-B}\right)$

$= \frac{1}{4}\left(2e^{-A+B} + 2e^{A-B}\right) = \frac{e^{A-B} + e^{-(A-B)}}{2}$

$= \cosh(A - B)$, as required.

When multiplying out the brackets you must be careful to obtain all eight terms with the correct signs.

You use the definition $\cosh x = \frac{e^x + e^{-x}}{2}$ with $x = A + B$.

b $\cosh x \cosh 1 - \sinh x \sinh 1 = \sinh x$

$\cosh x \cosh 1 = \sinh x(1 + \sinh 1)$

$\tanh x = \frac{\cosh 1}{1 + \sinh 1}$

$\tanh x = \frac{\frac{e+e^{-1}}{2}}{1 + \frac{e-e^{-1}}{2}} = \frac{e + e^{-1}}{2 + e - e^{-1}} = \frac{e^2 + 1}{e^2 + 2e - 1}$, as required.

You expand $\cosh(x - 1)$ using the result of part **a**.

Divide both sides of this equation by $\cosh x(1 + \sinh 1)$ and use $\tanh x = \frac{\sinh x}{\cosh x}$.

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28 a Let $y = \operatorname{arsinh} x$

Then $x = \sinh y = \dfrac{e^y - e^{-y}}{2}$

$2x = e^y - e^{-y}$

$e^{2y} - 2xe^y - 1 = 0$

$e^y = \dfrac{2x + \sqrt{(4x^2 + 4)}}{2}$

$= \dfrac{2x + 2\sqrt{(x^2 + 1)}}{2} = x + \sqrt{(x^2 + 1)}$

You multiply this equation throughout by e^y and treat the result as a quadratic in e^y.

The quadratic formula has $\pm$ in it. However $x - \sqrt{(x^2 + 1)}$ is negative for all real x and does not have a real logarithm, so you can ignore the negative sign.

Taking the natural logarithms of both sides,

$y = \ln\left[x + \sqrt{(x^2 + 1)}\right]$, as required.

b $\operatorname{arsinh}(\cot\theta) = \ln\left[\cot\theta + \sqrt{(1 + \cot^2\theta)}\right]$

$= \ln(\cot\theta + \operatorname{cosec}\theta)$

$= \ln\left(\dfrac{\cos\theta}{\sin\theta} + \dfrac{1}{\sin\theta}\right) = \ln\left(\dfrac{\cos\theta + 1}{\sin\theta}\right)$

$= \ln\left(\dfrac{2\cos^2\frac{\theta}{2}}{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}\right)$

$= \ln\left(\dfrac{\cos\frac{\theta}{2}}{\sin\frac{\theta}{2}}\right) = \ln\left(\cot\dfrac{\theta}{2}\right)$, as required.

Using the result of part **a** with $x = \cot\theta$.

Using $\operatorname{cosec}^2\theta = 1 + \cot^2\theta$.

You use both double angle formulae $\cos 2x = 2\cos^2 x - 1$ and $\sin 2x = 2\sin x\cos x$ with $2x = \theta$.

29 a Let $y = \operatorname{artanh} x$

$x = \tanh y = \dfrac{e^y - e^{-y}}{e^y + e^{-y}}$

$= \dfrac{e^y - e^{-y}}{e^y + e^{-y}} \times \dfrac{e^y}{e^y}$

$= \dfrac{e^{2y} - 1}{e^{2y} + 1}$

$xe^{2y} + x = e^{2y} - 1$

$e^{2y}(1 - x) = 1 + x$

$e^{2y} = \dfrac{1 + x}{1 - x}$

$2y = \ln\left(\dfrac{1 + x}{1 - x}\right)$

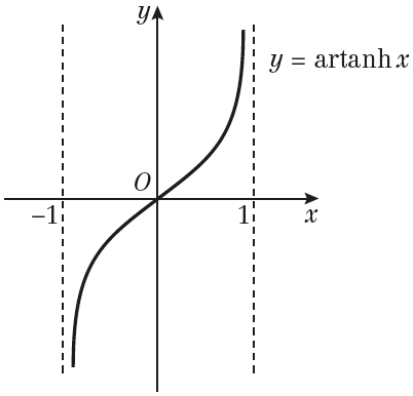
$y = \operatorname{artanh} x = \dfrac{1}{2}\ln\left(\dfrac{1 + x}{1 - x}\right)$ for $|x| < 1$ as required.

You have been asked to prove a standard result in this question. You should learn the proof of this and other similar results as part of your preparation for the examination.

Make e^{2y} the subject of the formulas and then take logarithms.

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29 b



You need to be able to sketch the graphs of the hyperbolic and inverse hyperbolic functions. When you sketch a graph you should show any important features of the curve. In this case, you should show the asymptotes $x=-1$ and $x=1$ of the curve.

c $x=\tanh\left[\ln\sqrt{(6x)}\right]$

$$\ln\sqrt{(6x)}=\operatorname{artanh}x=\frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$$

You use the result in part **a**.

$$\ln\sqrt{(6x)}=\ln\sqrt{\left(\frac{1+x}{1-x}\right)}$$

$$\sqrt{(6x)}=\sqrt{\left(\frac{1+x}{1-x}\right)}$$

As you have squared this equation, you might have introduced an incorrect solution. It would be sensible to check on your calculator that $x=\frac{1}{2},\frac{1}{3}$ are solutions of $x=\tanh\left[\ln\sqrt{(6x)}\right]$. In this case, both are correct.

Squaring

$$6x=\frac{1+x}{1-x}$$

$$6x-6x^2=1+x$$

$$6x^2-5x+1=(3x-1)(2x-1)=0$$

$$x=\frac{1}{2},\frac{1}{3}$$

30 a

$$\ln\left(\frac{1-\sqrt{(1-x^2)}}{x}\right)+\ln\left(\frac{1+\sqrt{(1-x^2)}}{x}\right)$$

$$=\ln\left(\frac{1-\sqrt{(1-x^2)}}{x}\right)\left(\frac{1+\sqrt{(1-x^2)}}{x}\right)$$

There are a number of different ways of starting this question. The method used here begins by using the log rule $\log a+\log b=\log ab$.

$$=\ln\frac{1-(1-x^2)}{x^2}=\ln\frac{x^2}{x^2}=\ln 1=0$$

This is the difference of two squares $a^2-b^2=(a-b)(a+b)$ with $a=1$ and $b=\sqrt{(1-x^2)}$.

Hence

$$\ln\left(\frac{1-\sqrt{(1-x^2)}}{x}\right)=-\ln\left(\frac{1+\sqrt{(1-x^2)}}{x}\right),\text{ as required.}$$

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30 b Let $y = \operatorname{arcosh}\frac{1}{x} = \operatorname{arsech} x$

$$\operatorname{sech} y = x$$

$$\frac{2}{e^{y}+e^{-y}} = x$$

$$2 = xe^{y} + xe^{-y}$$

Multiply throughout by e^{y} and treat the result as a quadratic in e^{y}.

$$xe^{2y} - 2e^{y} + x = 0$$

$$e^{y} = \frac{2 \pm \sqrt{(4-4x^2)}}{2x} = \frac{1 \pm \sqrt{(1-x^2)}}{x}$$

$$y = \ln\left(\frac{1+\sqrt{(1-x^2)}}{x}\right), \ln\left(\frac{1-\sqrt{(1-x^2)}}{x}\right)$$

$$= \pm\ln\left(\frac{1+\sqrt{(1-x^2)}}{x}\right), \text{ using the result of a}$$

Either of the two answers is possible but it is conventional to take $0 < \operatorname{arsech} x \leqslant 1$.

$$y = \operatorname{arsech} x = \ln\left(\frac{1+\sqrt{(1-x^2)}}{x}\right), \text{ as required.}$$

c Using $\operatorname{sech}^2 x = 1 - \tanh^2 x$

$$3\tanh^2 x - 4\operatorname{sech} x + 1 = 0,$$

$$3 - 3\operatorname{sech}^2 x - 4\operatorname{sech} x + 1 = 0$$

$$3\operatorname{sech}^2 x + 4\operatorname{sech} x - 4 = 0$$

$$(3\operatorname{sech} x - 2)(\operatorname{sech} x + 2) = 0$$

$\operatorname{sech} x = -2$ is impossible with real values of x.

$$\operatorname{sech} x = \frac{2}{3}$$

$$x = \pm\ln\left(\frac{1+\sqrt{\left(1-\frac{4}{9}\right)}}{\frac{2}{3}}\right) = \pm\ln\left(\frac{3+\sqrt{5}}{2}\right)$$

$x = \ln\left(\frac{3 \pm \sqrt{5}}{2}\right)$ is another correct form of the answer.

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31 a $\cosh 3\theta = \cosh(2\theta + \theta)$

$= \cosh 2\theta \cosh\theta + \sinh 2\theta \sinh\theta$

$\cosh\theta = c$ and $\sinh\theta = s$

$\cosh 3\theta = (2c^2 - 1)c + 2sc \times s$

$= 2c^3 - c + 2s^2 c$

$= 2c^3 - c + 2(c^2 - 1)c$

$= 2c^3 - c + 2c^2 - 2c$

$= 4\cosh^3\theta - 3\cosh\theta$

In a complicated calculation like this, it is sensible to use the abbreviated notation suggested here but, if you intend to use a notation like this, you should state the notation in the solution so that the marker knows what you are doing.

You use the 'double angle' for hyperbolics $\cosh 2\theta = 2\cosh^2\theta - 1$ and $\sinh 2\theta = 2\sinh\theta\cosh\theta$ and the identity $\cosh^2\theta - \sinh^2\theta = 1$. The signs in these formulae can be worked out using Osborn's rule.

$\cosh 5\theta = \cosh(3\theta + 2\theta) = \cosh 3\theta \cosh 2\theta + \sinh 3\theta \sinh 2\theta$

$\cosh 3\theta \cosh 2\theta = (4c^3 - 3c)(2c^2 - 1)$

$= 8c^5 - 10c^3 + 3c$

$\sinh 3\theta \sinh 2\theta = \sin(2\theta + \theta)\sinh 2\theta$

$= (\sinh 2\theta \cosh\theta + \cosh 2\theta \sinh\theta)\sinh 2\theta$

$= (2sc \times c + (2c^2 - 1)s)2sc$

$= 2(4c^2 - 1)s^2 c$

$= 2(4c^2 - 1)(c^2 - 1)c$

$= 8c^5 - 10c^3 + 2c$

Combining the results

$\cosh 5\theta = 8c^2 - 10c^3 + 3c + 8c^5 - 10c^3 + 2c$

$= 16\cosh^5\theta - 20\cosh^3\theta + 5\cosh\theta$

b $2\cosh 5x + 10\cosh 3x + 20\cosh x = 243,$

Letting $\cosh x = c$ and using the results in **a**

$32c^5 - 40c^3 + 10c + 40c^3 - 30c + 20c = 243$

$c^5 = \frac{243}{32} \Rightarrow c = \frac{3}{2}$

$x = \pm \operatorname{arcosh}\frac{3}{2} \approx \pm 0.96$

You can use an inverse hyperbolic button on your calculator to find $\operatorname{arcosh}\frac{3}{2}$.

32 a $\cosh 2x = \frac{e^{2x} + e^{-2x}}{2} = \frac{e^{2\ln k} + e^{-2\ln k}}{2}$

$= \frac{e^{\ln k^2} + e^{\ln\frac{1}{k^2}}}{2} = \frac{1}{2}\left(k^2 + \frac{1}{k^2}\right)$

$= \frac{1}{2}\left(\frac{k^4 + 1}{k^2}\right) = \frac{k^4 + 1}{2k^2}$, as requred.

Using the law of logarithms $n\ln x = \ln x^n$, with $n = -2$, $-2\ln k = \ln k^{-2} = \ln\frac{1}{k^2}$.

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32 b $f(x) = px - \tanh 2x$

For a stationary value

$f'(x) = p - 2\text{sech}^2 2x = 0$

$$p = 2\text{sech}^2 2x = \frac{2}{\cosh^2 2x}$$

Using the result of part **a** with $k = 2$

If $x = \ln 2$

$$\cosh 2x = \frac{2^4 + 1}{2 \times 2^2} = \frac{17}{8}$$

Hence

$$p = \frac{2}{\left(\frac{17}{8}\right)^2} = \frac{128}{289}$$

There is no 'hence' in this question but using the result in part **a** shortens the working. The question requires an exact fraction for the answer and you should not use a calculator other than, possibly, for multiplying and dividing fractions.

33 a $y = -x + \tanh 4x$

$$\frac{dy}{dx} = -1 + 4\text{sech}^2 4x = 0$$

$$\text{sech}^2 4x = \frac{1}{4} \Rightarrow \cosh^2 4x = 4$$

$$\cosh 4x = 2$$

$$4x = \text{arcosh}\, 2 = \ln(2 + \sqrt{3})$$

$$x = \frac{1}{4}\ln(2 + \sqrt{3})$$

As $\cosh x \geqslant 1$ for all real x, $\cosh 4x = -2$ is impossible.

For $x \geqslant 0$, there is only one value of x which gives a stationary value. The question tells you that the curve has a maximum point so, in this question, you need not show that this point is a maximum by, for example, examining the second derivative.

b $\tanh^2 4x = 1 - \text{sech}^2 4x = 1 - \frac{1}{4} = \frac{3}{4}$

As $x \geqslant 0$, $\tanh 4x = \frac{\sqrt{3}}{2}$

At $x = \frac{1}{4}\ln(2 + \sqrt{3})$

$$y = -x + \tanh 4x = -\frac{1}{4}\ln(2 + \sqrt{3}) + \frac{\sqrt{3}}{2}$$

$$= \frac{1}{4}\left\{2\sqrt{3} - \ln(2 + \sqrt{3})\right\}, \text{ as required.}$$

You need a value for $\tanh 4x$ and this is easiest found using the hyperbolic identity $\text{sech}^2 x = 1 - \tanh^2 x$.

34 a Deriving the function we get $f'(x) = 2\cosh^2 2x + 2\sinh^2 2x$; thus $f'(0) = 2$ is the first non-zero term. Deriving $f'(x)$ we find $f''(x) = 2\left[4\cosh 2x \sinh 2x + 4\sinh 2x \cosh 2x\right] = 16f(x)$
So the corresponding term in the Maclaurin series will be zero, but on the other hand we already know that $f^{(3)}(x) = \left(16f(x)\right)'$ which is $16f'(x)$ by linearity of the derivative operator, so $f^{(3)}(0) = 32$. Similarly, we find that $f^{(4)}(0) = 0$ and $f^{(5)}(0) = 512$, so the first terms of the series expansion are as follows:

$$f(x) = 2x + \tfrac{16}{3}x^3 + \tfrac{64}{15}x^5 + \ldots$$

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34 b From **a** it is easy to see that $f^{(n)}(x) = 16^{n-2} f'(x)$ for n odd and $f^{(n)}(x) = 16^{n-2} f(x)$ for n even; however, all the even terms of the Maclaurin series are zero. So a formula for the nth non-zero term is $\frac{2^{4(n-1)}}{n!} f'(0) x^{2n-1}$; since $f'(0) = 2$ we can rewrite this as $\frac{2^{4n-3}}{(2n-1)!} x^{2n-1}$.

35 a First of all we note that $f(0) = 1$, so this is the first non-zero term of the series expansion. The first derivative of the function is given by $\sinh x \cos 2x - 2 \sin 2x \cosh x$; of course, since one summand features the term $\sinh x$, and the other one features $\sin x$, $f'(0) = 0$. The second derivative is given by $-4 \sinh x \sin 2x - 3 \cosh x \cos 2x$, so $f''(0) = -3$: then the beginning of the series expansion is $1 - \frac{3}{2} x^2$.

b This expression gives $f(0.1) \sim 0.985$. The correct value of the function is about 0.9849710. Then the error is given by $\frac{0.9849710 - 0.985}{0.9849710} \sim 0.000029$, which corresponds to an error of the 0.0029%.

36 $x = \frac{a}{\sinh \theta} = a(\sinh \theta)^{-1}$

$\frac{dx}{d\theta} = -a(\sinh \theta)^{-2} \cosh \theta = -\frac{a \cosh \theta}{\sinh^2 \theta}$

When substituting remember to substitute for the dx as well as the rest of the integral.

$$\int \frac{1}{x\sqrt{(x^2 + a^2)}} dx = \int \frac{1}{\frac{a}{\sinh\theta}\sqrt{\left(\frac{a^2}{\sinh^2\theta} + a^2\right)}} \times \frac{dx}{d\theta} d\theta$$

$$= \int \frac{\frac{-a\cosh\theta}{\sinh^2\theta}}{\frac{a^2\sqrt{1+\sinh^2\theta}}{\sinh^2\theta}} d\theta = \frac{-1}{a} \int \frac{\cosh\theta}{\cosh\theta} d\theta$$

Use $1 + \sinh^2\theta = \cosh^2\theta$ to simplify this expression.

$$= -\frac{1}{a} \int \frac{\cosh\theta}{\cosh\theta} d\theta = -\frac{1}{a} \int 1 \, d\theta$$

$$= -\frac{1}{a}\theta + \text{constant}$$

$$= -\frac{1}{a} \text{arsinh}\left(\frac{a}{x}\right) + \text{constant, as required.}$$

As $x = \frac{a}{\sinh\theta}$, then $\sinh\theta = \frac{a}{x}$ and $\theta = \text{arsinh}\left(\frac{a}{x}\right)$.

37 a Let $y = \text{artanh}\, x$

$\tanh y = x$

Differentiate implicitly with respect to x

To differentiate a function f(y) with respect to x you use a version of the chain rule $\frac{d}{dx}(f(y)) = f'(y) \times \frac{dy}{dx}$.

$$\text{sech}^2 y \frac{dy}{dx} = 1 \Rightarrow \frac{dy}{dx} = \frac{1}{\text{sech}^2 y} = \frac{1}{1 - \tanh^2 y}$$

$$= \frac{1}{1 - x^2}, \text{ as required}$$

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37 b Using integration by parts and the result in part **a**

$$\int \text{artanh}\, x \,\mathrm{d}x = \int 1 \times \text{artanh}\, x \,\mathrm{d}x$$

$$= x\,\text{artanh}\, x - \int \frac{x}{1-x^2}\,\mathrm{d}x$$

$$= x\,\text{artanh}\, x + \frac{1}{2}\ln(1-x^2) + A$$

You use $\int u\frac{\mathrm{d}v}{\mathrm{d}x}\mathrm{d}x = uv - \int v\frac{\mathrm{d}u}{\mathrm{d}x}\mathrm{d}x$ with $u = \text{artanh}\, x$ and $\frac{\mathrm{d}v}{\mathrm{d}x} = 1$. You know $\frac{\mathrm{d}u}{\mathrm{d}x}$ from part **a**.

This solution uses the result $\int \frac{\mathrm{f}'(x)}{\mathrm{f}(x)}\mathrm{d}x = \ln \mathrm{f}(x)$. So $\int \frac{-2x}{1-x^2}\mathrm{d}x = \ln(1-x^2)$ and you multiply this by $-\frac{1}{2}$ to complete the solution. This is a question where there are a number of possible alternative forms of the answer.

38 a Let $y = \text{arsinh}\, x$ then $x = \sinh y = \frac{\mathrm{e}^y - \mathrm{e}^{-y}}{2}$

$$2x = \mathrm{e}^y - \mathrm{e}^{-y}$$

$$\mathrm{e}^{2y} - 2x\mathrm{e}^y - 1 = 0$$

$$\mathrm{e}^y = \frac{2x + \sqrt{(4x^2-4)}}{2}$$

$$= \frac{2x + 2\sqrt{(x^2+1)}}{2} = x + \sqrt{(x^2+1)}$$

You multiply this equation throughout by e^y and treat the result as a quadratic in e^y.

The quadratic formula has $\pm$ in it. However $x - \sqrt{(x^2+1)}$ is negative for all real x and does not have a real logarithm, so you can ignore the negative sign.

Taking the natural logarithms of both sides, $y = \ln\left[x + \sqrt{(x^2+1)}\right]$, as required.

b $y = \text{arsinh}\ x$

$\sinh y = x$

Differentiating implicitly with respect to x

$$\cosh y \frac{\mathrm{d}y}{\mathrm{d}x} = 1 \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{\cosh y}$$

$$\cosh^2 y = 1 + \sinh^2 y = 1 + x^2 \Rightarrow \cosh y = \sqrt{(1+x^2)}$$

Hence $\frac{\mathrm{d}}{\mathrm{d}x}(\text{arsinh}\, x) = \frac{1}{\sqrt{(1+x^2)}} = (1+x^2)^{-\frac{1}{2}}$, as required.

arsinh x is an increasing function of x for all x. So its gradient is always positive and you need not consider the negative square root.

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38 c $y = (\text{arsinh}\, x)^2$

$$\frac{dy}{dx} = 2\text{arsinh}\, x(1+x^2)^{-\frac{1}{2}}$$

$$\frac{d^2y}{dx^2} = 2(1+x^2)^{-\frac{1}{2}}(1+x^2)^{-\frac{1}{2}} + 2\text{arsinh}\, x \times \left(-\frac{1}{2}\right)(2x)(1+x^2)^{-\frac{3}{2}}$$

$$= 2(1+x^2)^{-1} - 2x\,\text{arsinh}\, x(1+x^2)^{-\frac{3}{2}}$$

You use the product rule for differentiation $\frac{d}{dx}(uv) = v\frac{du}{dx} + u\frac{dv}{dx}$ with $u = 1\ 2\text{arsinh}\, x$ and $v = (1+x^2)^{-\frac{1}{2}}$.

Substituting for $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ into

$$(1+x^2)\frac{d^2y}{dx^2} + x\frac{dy}{dx} - 2$$

$$= (1+x^2)\left(2(1+x^2)^{-1} - 2\text{arsinh}\, x(1+x^2)^{-\frac{3}{2}}\right) + x \times 2\text{arsinh}\, x(1+x^2)^{-\frac{1}{2}} - 2$$

$$= 2 - 2x\,\text{arsinh}\, x(1+x^2)^{-\frac{1}{2}} + 2x\,\text{arsinh}\, x(1+x^2)^{-\frac{1}{2}} - 2$$

$= 0$, as required.

d $$\int_0^1 \text{arsinh}\, x\, dx = \int_0^1 1 \times \text{arsinh}\, x\, dx$$

$$= \left[x\,\text{arsinh}\, x\right]_0^1 - \int_0^1 \frac{x}{\sqrt{(1+x^2)}}\, dx$$

$$= \text{arsinh}\, 1 - \left[\sqrt{(1+x^2)}\right]_0^1$$

$$= \ln(1+\sqrt{2}) - \sqrt{2} + 1$$

$\frac{d}{dx}\left((1+x^2)^{\frac{1}{2}}\right) = \frac{1}{2} \times 2x \times (1+x^2)^{-\frac{1}{2}} = \frac{x}{\sqrt{(1+x^2)}}$

so $\int \frac{x}{\sqrt{(1+x^2)}}\, dx = \sqrt{(1+x^2)}$.

39 a $4x^2 + 4x + 5 = (px+q)^2 + r$

$$= p^2x^2 + 2pqx + q^2 + r$$

Equating coefficients of x^2

$4 = p^2 \Rightarrow p = 2$

The conditions of the question allow $p = -2$ as an answer, but the negative sign would make the integrals following awkward, so choose the positive root.

Equating coefficients of x

$4 = 2pq = 4q \Rightarrow q = 1$

Equating constant coefficients

$5 = q^2 + r = 1 + r \Rightarrow r = 4$

$p = 2, q = 1, r = 4$

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39 b $\displaystyle\int \frac{1}{4x^2+4x+5}\,\mathrm{d}x = \int \frac{1}{(2x+1)^2+4}\,\mathrm{d}x$

If you know a formula of the type $\displaystyle\int \frac{1}{a^2x^2+b^2}\,\mathrm{d}x = \frac{1}{ab}\arctan\left(\frac{ax}{b}\right)$, or you are confident at writing down integrals by inspection, you may be able to find this integral without working. It is, however, very easy to make errors with the constant and get, for example, the common error $\frac{1}{2}\arctan\left(\frac{2x+1}{2}\right)+C$.

Let $2x+1 = 2\tan\theta$

$$2\frac{\mathrm{d}}{\mathrm{d}\theta} = 2\sec^2\theta \Rightarrow \frac{\mathrm{d}x}{\mathrm{d}\theta} = \sec^2\theta$$

$$\int \frac{1}{(2x+1)^2+4}\,\mathrm{d}x = \int \frac{1}{4\tan^2\theta+4}\left(\frac{\mathrm{d}x}{\mathrm{d}\theta}\right)\mathrm{d}\theta$$

$$= \int \frac{1}{4\cancel{\sec^2\theta}}(\cancel{\sec^2\theta})\,\mathrm{d}\theta$$

$$= \frac{1}{4}\theta + C$$

$$= \frac{1}{4}\arctan\left(\frac{2x+1}{2}\right)+C$$

c $\displaystyle\int \frac{2}{\sqrt{(4x^2+4x+5)}}\,\mathrm{d}x = \int \frac{2}{\sqrt{\left((2x+1)^2+4\right)}}\,\mathrm{d}x$

As in part **b**, you may be able to write down this integral without working.

Let $2x+1 = 2\sinh\theta$

$$2\frac{\mathrm{d}x}{\mathrm{d}\theta} = 2\cosh\theta \Rightarrow \frac{\mathrm{d}x}{\mathrm{d}\theta} = \cosh\theta$$

$$\int \frac{2}{\sqrt{\left((2x+1)^2+4\right)}}\,\mathrm{d}x = \int \frac{2}{\sqrt{(4\sinh^2\theta+4)}}\left(\frac{\mathrm{d}x}{\mathrm{d}\theta}\right)\mathrm{d}\theta$$

$$= \int \frac{2}{2\cosh\theta}(\cosh\theta)\,\mathrm{d}\theta = \int 1\,\mathrm{d}\theta$$

$$= \theta + C = \operatorname{arsinh}\left(\frac{2x+1}{2}\right)+C$$

Using $\operatorname{arsinh} x = \ln\left(x+\sqrt{(x^2+1)}\right)$

$$\int \frac{2}{\sqrt{(4x^2+4x+5)}}\,\mathrm{d}x = \ln\left[\left(\frac{2x+1}{2}\right)+\sqrt{\left(\frac{(2x+1)^2}{4}+1\right)}\right]+C$$

$$= \ln\left[\left(\frac{2x+1}{2}\right)+\sqrt{\left(\frac{4x^2+4x+1+4}{4}\right)}\right]+C$$

$$= \ln\left[\left(\frac{2x+1}{2}\right)+\frac{1}{2}\sqrt{(4x^2+4x+5)}\right]+C$$

$$= \ln\left[(2x+1)+\sqrt{(4x^2+4x+5)}\right]-\ln 2+C$$

$-\ln 2+$ an arbitrary constant is another arbitrary constant.

$$= \ln\left[(2x+1)+\sqrt{(4x^2+4x+5)}\right]+k, \text{ as required.}$$

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40 Solve the integral as follows:

$$\int \frac{x+2}{\sqrt{4x^2+9}}\,\mathrm{d}x$$
$$= \int \frac{x}{\sqrt{4x^2+9}}\,\mathrm{d}x + \int \frac{2}{\sqrt{4x^2+9}}\,\mathrm{d}x$$

Then the first integral is a standard integral of the form $kf'(x)(f(x))^n$, and we can easily see that its value is $\frac{\sqrt{4x^2+9}}{4}$, as the derivative of $\sqrt{4x^2+9}$ is $\frac{4x}{\sqrt{4x^2+9}}$. For the second integral we make a substitution: if $t = \frac{2}{3}x$, then:

$$\int \frac{2}{\sqrt{4x^2+9}}\,\mathrm{d}x$$
$$= \int \frac{2}{\sqrt{4\frac{9}{4}t^2+9}}\,\tfrac{3}{2}\mathrm{d}t$$
$$= \int \frac{1}{\sqrt{t^2+1}}\,\mathrm{d}t = \operatorname{arsinh} t = \operatorname{arsinh}\tfrac{2}{3}x$$

Then we can conclude that the integral of the function is $\frac{4x^2+9}{4} + \operatorname{arsinh}\frac{2}{3}x + C$.

41 Since $(x-2)^2 = x^2 - 4x + 4$, we can rewrite the integral as $= \int_2^5 \frac{1}{2\sqrt{\frac{(x-2)^2}{4}+1}}\,\mathrm{d}x$. So if we put $t = \frac{x-2}{2}$ we get that the integral is:

$$\int_0^{\frac{3}{2}} \frac{1}{2\sqrt{t^2+1}}\,2\,\mathrm{d}t$$
$$= \int_0^{\frac{3}{2}} \frac{1}{\sqrt{t^2+1}}\,\mathrm{d}t$$
$$= \operatorname{arsinh}\tfrac{3}{2}$$

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42 $\int x\,\text{arcosh}\,x\,\mathrm{d}x = \frac{x^2}{2}\text{arcosh}\,x - \int \frac{x^2}{2\sqrt{(x^2-1)}}\mathrm{d}x$

This solution uses integration by parts, $\int u\frac{\mathrm{d}v}{\mathrm{d}x}\mathrm{d}x = uv - \int v\frac{\mathrm{d}u}{\mathrm{d}x}\mathrm{d}x$, with $u = \text{arcosh}x$ and $\frac{\mathrm{d}v}{\mathrm{d}x} = x$. There are other possible approaches to this question, for example, substituting $u = \text{arcosh}x$.

To find the remaining integral, let $x = \cosh\theta$.

$\frac{\mathrm{d}x}{\mathrm{d}\theta} = \sinh\theta$

$$\int \frac{x^2}{2\sqrt{(x^2-1)}}\mathrm{d}x = \int \frac{\cosh^2\theta}{2\sqrt{\cosh^2\theta - 1}}\left(\frac{\mathrm{d}x}{\mathrm{d}\theta}\right)\mathrm{d}\theta$$

$$= \int \frac{\cosh^2\theta}{2\sinh\theta}\sinh\theta\,\mathrm{d}\theta = \frac{1}{2}\int \cosh^2\theta\,\mathrm{d}\theta$$

$$= \frac{1}{4}\int (\cosh 2\theta + 1)\mathrm{d}\theta$$

Using the identity $\cosh 2\theta = 2\cosh^2\theta - 1$.

$$= \frac{\sinh 2\theta}{8} + \frac{\theta}{4} = \frac{\sinh\theta\cosh\theta}{4} + \frac{\theta}{4}$$

$$= \frac{\left[\sqrt{(x^2-1)}\right]x}{4} + \frac{1}{4}\text{arcosh}x$$

$\sinh\theta = \sqrt{(\cosh^2\theta - 1)} = \sqrt{(x^2-1)}$

Hence the area, A, of R is given by

$$A = \left[\frac{x^2}{2}\text{arcosh}x - \frac{1}{4}x\sqrt{(x^2-1)} - \frac{1}{4}\text{arcosh}x\right]_1^2$$

$$= \left[\left(\frac{x^2}{2} - \frac{1}{4}\right)\text{arcosh}x - \frac{1}{4}x\sqrt{(x^2-1)}\right]_1^2$$

$$= \left[\frac{7}{4}\text{arcosh}2 - \frac{\sqrt{3}}{2}\right] - [0]$$

As $\text{arcosh}\,1 = 0$ and $\sqrt{(1^2-1)} = 0$, both terms are zero at the lower limit.

$$= \frac{7}{4}\ln\left(2+\sqrt{3}\right) - \frac{\sqrt{3}}{2}, \text{as required.}$$

43 Calculate the area of the section of the greenhouse with the integral:

$$\int_{-2}^{2} \frac{10}{\sqrt{x^2+9}}\,\mathrm{d}x$$

$$= 10\int_{-2}^{2} \frac{1}{3\sqrt{\left(\frac{x}{3}\right)^2 + 1}}\,\mathrm{d}x$$

Substitute $t = \frac{x}{3}$ to get:

$$10\int_{-\frac{2}{3}}^{\frac{2}{3}} \frac{1}{\sqrt{t^2+1}}\,\mathrm{d}x$$

$$= 10\left[\text{arsinh}\tfrac{2}{3} - \left(\text{arsinh}\left(-\tfrac{2}{3}\right)\right)\right]$$

$$= 10\left[2\text{arsinh}\tfrac{2}{3}\right] = 20\text{arsinh}\tfrac{2}{3}$$

Then the volume of the greenhouse is given by $20\text{arsinh}\frac{2}{3}\cdot 55$ which, to the third significant figure, is 688 m^3.

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44 The integrating factor is

$$e^{\int \frac{4}{x}dx} = e^{4\ln x} = e^{\ln x^4} = x^4$$

Multiply the equation throughout by x^4

$$x^4\frac{dy}{dx} + 4x^3y = 6x^5 - 5x^4$$

$$\frac{d}{dx}(x^4y) = 6x^5 - 5x^4$$

$$x^4y = \int (6x^5 - 5x^4)dx = x^6 - x^5 + C$$

$$y = x^2 - x + \frac{C}{x^4}$$

If the differential equation has the form $\frac{dy}{dx} + Py = Q$, the integrating factor is $e^{\int P dx}$.

For any function $f(x)$, $e^{\ln f(x)} = f(x)$.

It is important that you remember to add the constant of integration. When you divide by x^4, the constant becomes a function of x and its omission would be a significant error.

45 The integrating factor is

$$e^{\int -\frac{1}{x}dx} = e^{-\ln x} = e^{\ln \frac{1}{x}} = \frac{1}{x}$$

Multiply the equation throughout by $\frac{1}{x}$

$$\frac{1}{x}\frac{dy}{dx} - \frac{y}{x^2} = x$$

$$\frac{d}{dx}\left(\frac{y}{x}\right) = x$$

$$\frac{y}{x} = \frac{x^2}{2} + C$$

$$y = \frac{x^3}{2} + Cx$$

For all n, $n\ln x = \ln x^n$, so for $n = -1$, $-\ln x = \ln x^{-1} = \ln\frac{1}{x}$.

The product rule for differentiating, in this case $\frac{d}{dx}\left(y \times \frac{1}{x}\right) = \frac{dy}{dx} \times \frac{1}{x} + y \times -\frac{1}{x^2}$, enables you to write the differential equation as an exact equation, where one side is the exact derivative of a product and the other side can be integrated with respect to x.

46 $(x+1)\frac{dy}{dx} + 2y = \frac{1}{x}$

$$\frac{dy}{dx} + \frac{2}{x+1}y = \frac{1}{x(x+1)}$$

The integrating factor is

$$e^{\int \frac{2}{x+1}dx} = e^{2\ln(x+1)} = e^{\ln(x+1)^2} = (x+1)^2$$

Multiply throughout by $(x+1)^2$

$$(x+1)^2\frac{dy}{dx} + 2(x+1)y = \frac{x+1}{x}$$

$$\frac{d}{dx}\left((x+1)^2y\right) = 1 + \frac{1}{x}$$

$$(x+1)^2y = \int\left(1 + \frac{1}{x}\right)dx = x + \ln x + C$$

$$y = \frac{x + \ln x + C}{(x+1)^2}$$

If the equation is in the form $R\frac{dy}{dx} + Sy = T$, you must begin by dividing throughout by R, in this case $(x + 1)$, before finding the integrating factor.

To integrate $\frac{x+1}{x}$, write $\frac{x+1}{x} = \frac{x}{x} + \frac{1}{x} = 1 + \frac{1}{x}$.

You divide throughout by $(x + 1)^2$ to obtain the equation in the form $y = f(x)$. This is required by the wording of the question.

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47 The integrating factor is $e^{\int \tan x \, dx}$

$$\int \tan x \, dx = \int \frac{\sin x}{\cos x} dx = -\ln \cos x = \ln \frac{1}{\cos x} = \ln \sec x$$

Here we have that $\int \frac{f'(x)}{f(x)} dx = \ln f(x)$. As $-\sin x$ is the derivative of $\cos x$, $\int \frac{-\sin x}{\cos x} dx = \ln \cos x$.

Hence

$$e^{\int \tan x \, dx} = e^{\ln \sec x} = \sec x$$

Multiply the differential equation throughout by $\sec x$

$$\sec x \frac{dy}{dx} + y \sec x \tan x = e^{2x} \sec x \cos x = e^{2x}$$

$\sec x \cos x = \frac{1}{\cos x} \times \cos x = 1$

$$\frac{d}{dx}(y \sec x) = e^{2x}$$

$$y \sec x = \int e^{2x} dx = \frac{e^{2x}}{2} + C$$

Multiply throughout by $\cos x$

$$y = \left(\frac{e^{2x}}{2} + C \right) \cos x$$

$y = 2$ at $x = 0$

$$2 = \frac{1}{2} + C \Rightarrow C = \frac{3}{2}$$

The condition $y = 2$ at $x = 0$ enables you to evaluate the constant of integration and find the particular solution of the differential equation for these values.

$$y = \frac{1}{2}\left(e^{2x} + 3\right) \cos x$$

48 The integrating factor is $e^{\int 2 \cot 2x \, dx}$

$$\int 2 \cot 2x \, dx = \int \frac{2 \cos 2x}{\sin 2x} dx = \ln \sin 2x$$

Hence

$$e^{\int 2 \cot 2x \, dx} = e^{\ln \sin 2x} = \sin 2x$$

Multiply the differential equation throughout by $\sin 2x$

$$\sin 2x \frac{dy}{dx} + 2y \cos 2x = \sin x \sin 2x$$

Using the identity $\sin 2x = 2 \sin x \cos x$.

$$\frac{d}{dx}(y \sin 2x) = 2 \sin^2 x \cos x$$

As $\frac{d}{dx}(\sin^3 x) = 3 \sin^2 x \cos x$, then $\int \sin^2 x \cos dx = \frac{\sin^3 x}{3}$. It saves time to find integrals of this type by inspection. However, you can use the substitution $\sin^y = s$ if you find inspection difficult.

$$y \sin 2x = \frac{2 \sin^3 x}{2} + C$$

$$y = \frac{2 \sin^3 x}{3 \sin 2x} + \frac{C}{\sin 2x}$$

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49 $(1+x)\dfrac{dy}{dx} - xy = xe^{-x}$

$$\frac{dy}{dx} - \frac{xy}{1+x} = \frac{xe^{-x}}{1+x} \qquad (1)$$

If the equation is in the form $R\dfrac{dy}{dx} + Sy = T$, you must begin by dividing throughout by R, in this case $(1 + x)$ before finding the integrating factor.

The integrating factor is $e^{\int -\frac{x}{1+x}dx}$

$$\frac{x}{1+x} = \frac{1+x-1}{1+x} = 1 - \frac{1}{1+x}$$

To integrate an expression in which the degree of the numerator is greater or equal to the degree of the denominator, you must transform the expression into one with a proper fraction. This can be done using partial fractions, long division or, as here, using decomposition.

Hence

$$\int \frac{x}{1+x}dx = x - \ln(1+x)$$

and the integrating factor is

$$e^{-x+\ln(1+x)} = e^{-x}e^{\ln(1+x)} = e^{-x} + (1+x)$$

Multiplying (**1**) throughout by $(1+x)e^{-x}$

$$(1+x)e^{-x}\frac{dy}{dx} - xe^{-x}y = xe^{-2x}$$

$$\frac{d}{dx}\left(y(1+x)e^{-x}\right) = xe^{-2x}$$

$$y(1+x)e^{-x} = \int xe^{-2x}dx$$

You integrate $x\,e^{-2x}$ using integration by parts.

$$= -\frac{xe^{-2x}}{2} + \int \frac{e^{-2x}}{2}dx = -\frac{xe^{-2x}}{2} - \frac{e^{-2x}}{4} + C$$

$$\frac{e^{-2x}}{e^{-x}} = e^{-2x-(-x)} = e^{-2x+x} = e^{-x}$$

$$y = -\frac{xe^{-2x}}{2(1+x)e^{-x}} - \frac{e^{-2x}}{4(1+x)e^{-x}} + \frac{C}{(1+x)e^{-x}}$$

$$= -\frac{xe^{-x}}{2(1+x)} - \frac{e^{-x}}{4(1+x)} + \frac{Ce^{x}}{(1+x)}$$

$y = 1$ at $x = 0$

$$1 = 0 - \frac{1}{4} + C \Rightarrow C = \frac{5}{4}$$

$$y = \frac{5e^{x}}{4(1+x)} - \frac{xe^{-x}}{2(1+x)} - \frac{e^{-x}}{4(1+x)}$$

This expression could be put over a common denominator but, other than requiring that y is expressed in terms of x, the question asks for no particular form and this is an acceptable answer.

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50 a Dividing throughout by $\cos x$

$$\frac{dy}{dx} + \frac{\sin x}{\cos x} y = \cos^2 x \qquad (1)$$

$$\int \frac{\sin x}{\cos x} dx = -\int \frac{-\sin x}{\cos x} dx = -\ln \cos x = \ln \frac{1}{\cos x} = \ln \sec x$$

Hence the integrating factor is $e^{\ln \sec x} = \sec x$

Multiply (**1**) by $\sec x$

$$\sec x \frac{dy}{dx} + \sec x \frac{\sin x}{\cos x} y = \cos^2 x \sec x$$

$$\sec x \frac{dy}{dx} + (\sec x \tan x) y = \cos x$$

$$\frac{d}{dx}(y \sec x) = \cos x$$

$$y \sec x = \int \cos x \, dx = \sin x + C$$

Multiplying throughout by $\cos x$

$$y = \sin x \cos x + C \cos x$$

Here we use $\int \frac{f'(x)}{f(x)} dx = \ln f(x)$. As $-\sin x$ is the derivative if $\cos x$, $-\int \frac{-\sin x}{\cos x} dx = -\ln \cos x$.

As $\ln 1 = 0$,

$$-\ln \cos x = \ln 1 - \ln \cos x = \ln \frac{1}{\cos x},$$

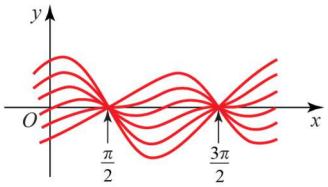
using the log law $\ln a - \ln b = \ln \frac{a}{b}$.

b Where $\cos x = 0$ and $0 \leqslant x \leqslant 2\pi$

$$x = \frac{\pi}{2}, \frac{3\pi}{2}$$

The points $\left(\frac{\pi}{2}, 0\right)$ and $\left(\frac{3\pi}{2}, 0\right)$ lie on all of the solution curves of the differential equation.

In general, for a given value of x, different values of c give different values of y. However, if $\cos x = 0$, the c will have no effect and y will be zero for any value of c.



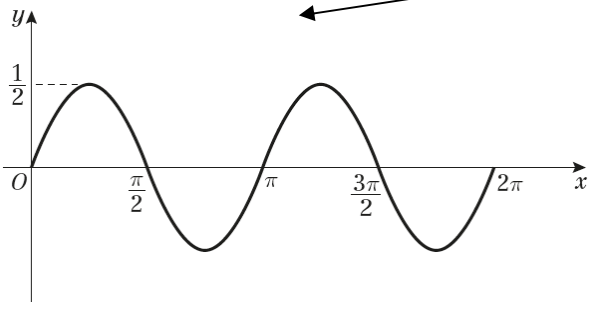
c $y = \sin x \cos x + \cos x$

At $x = 0, y = 0$

$0 = 0 + C \Rightarrow C = 0$

$$y = \sin x \cos x = \frac{1}{2} \sin 2x$$

Using the identity $\sin 2x = 2 \sin x \cos x$. $\sin 2x$ is a function with period π. So the curve makes two complete oscillations in the interval $0 \leqslant x \leqslant 2\pi$



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51 a The integrating factor is

$$e^{\int 2dx} = e^{2x}$$

Multiplying the differential equation throughout by e^{2x}

$$e^{2x}\frac{dy}{dx} + 2e^{2x}y = xe^{2x}$$

$$\frac{d}{dx}(ye^{2x}) = xe^{2x}$$

$$ye^{2x} = \int xe^{2x}dx$$

Integrate by parts.

$$= \frac{xe^{2x}}{2} - \int \frac{e^{2x}}{2}dx = \frac{xe^{2x}}{2} - \frac{e^{2x}}{4} + C$$

$$y = \frac{x}{2} - \frac{1}{4} + Ce^{-2x}$$

b $y = 1$ at $x = 0$

$$1 = 0 - \frac{1}{4} + C \Rightarrow C = \frac{5}{4}$$

$$y = \frac{x}{2} - \frac{1}{4} + \frac{5e^{-2x}}{4}$$

This is the particular solution of the differential equation for $y = 1$ at $x = 0$. You are asked to sketch this in part **c**.

For a minimum $\frac{dy}{dx} = 0$

$$\frac{dy}{dx} = \frac{1}{2} - \frac{5e^{-2x}}{2} = 0 \Rightarrow 5e^{-2x} = 1 \Rightarrow e^{2x} = 5$$

$$\ln e^{2x} = \ln 5 \Rightarrow 2x = \ln 5$$

$$x = \frac{1}{2}\ln 5$$

At the minimum, the differential equation reduces to

$$2y = x$$

The differential equation is $\frac{dy}{dx} + 2y = x$. At the minimum, $\frac{dy}{dx} = 0$ and so $2y = x$. If you did not see this you could, of course, substitute $x = \frac{1}{2}\ln 5$ into the particular solution and find y. This would take longer but would gain full marks.

Hence

$$y = \frac{1}{2}x = \frac{1}{4}\ln 5$$

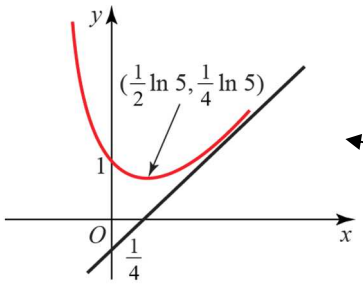
$$\frac{d^2y}{d^2x} = 5e^{-2x} > 0 \text{ for any real } x$$

This confirms the point is a minimum.

The coordinates of the minimum are $\left(\frac{1}{2}\ln 5, \frac{1}{4}\ln 5\right)$.

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51 c



As x increases, $e^{-2x} \to 0$ and so $\frac{x}{2} - \frac{1}{4} + \frac{5e^{-2x}}{4} \to \frac{x}{2} - \frac{1}{4}$. This means that $y = \frac{x}{2} - \frac{1}{4}$ is an asymptote of the curve. This has been drawn on the graph. It is not essential to do this, but if you recognise that this line is an asymptote, it helps you to draw the correct shape of the curve.

52 a Rewrite the equation as $\frac{dy}{y} = \sinh x \, dx$. Integrate this to get $\ln y = \cosh x + C$, and therefore $y = Ce^{\cosh x}$ is the general solution of the equation.

b When $x = 0$, the function takes the value Ce. Then the particular solution that satisfies $f(0) = e^3$ is $y = e^{2+\cosh x}$.

53 The auxiliary equation is

$$m^2 + 4m + 5 = 0$$
$$m^2 + 4m + 4 = -1$$
$$(m+2)^2 = -1$$
$$m = -2 \pm i$$

The general solution is

$\theta = e^{-2t}(A\cos t + B\sin t)$

If the solutions to the auxiliary equation are $\alpha \pm i\beta$, you may quote the result that the general solution of the differential equation is $e^{\alpha t}.(A\cos \beta t + B\sin \beta t)$.

$t = 0, \theta = 3$

$3 = A$

Using $\sin 0 = 0$ and $\cos t\, 0 = 1$.

$\frac{d\theta}{dt} = -2e^{-2t}(A\cos t + B\sin t) + e^{-2t}(-A\sin t + B\cos t)$

$t = 0, \frac{d\theta}{dt} = -6$

$-6 = -2A + B$

$B = 2A - 6 = 0$

As $A = 3$

The particular solution is

$\theta = 3e^{-2t}\cos t$

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54 a $y = 3x\sin 2x \Rightarrow \dfrac{\mathrm{d}y}{\mathrm{d}x} = 3\sin 2x + 6x\cos 2x$

$\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2} = 6\cos 2x + 6\cos 2x - 12x\sin 2x$

$= 12\cos 2x - 12x\sin 2x$

Use the product rule for differentiating.

Substituting into the differential equation

$12\cos 2x - \cancel{12x\sin 2x} + \cancel{12x\sin 2x} = k\cos 2x$

Hence

$k = 12$

b The auxiliary equation is

$m^2 + 4 = 0$

$m = \pm 2\mathrm{i}$

The complementary function is given by

$y = A\cos 2x + B\sin 2x$

If the solutions to the auxiliary equation are $m = \pm \alpha\mathrm{i}$, you may quote the result that the complementary function is $A\cos \alpha x + B\sin \alpha x$.

From **a**, the general solution is

$y = A\cos 2x + B\sin 2x + 3x\sin 2x$

Part **a** of the question gives you that $3x\sin 2x$ is a particular integral of the differential equation and general solution = complementary function + particular integral.

$x = 0, y = 2$

$2 = A$

$x = \dfrac{\pi}{4}, y = \dfrac{\pi}{2}$

$\dfrac{\pi}{2} = A\cos\dfrac{\pi}{2} + B\sin\dfrac{\pi}{2} + 3\times\dfrac{\pi}{4}\sin\dfrac{\pi}{2}$

$\dfrac{\pi}{2} = B + \dfrac{3\pi}{4} \Rightarrow B = -\dfrac{\pi}{4}$

Use $\cos\dfrac{\pi}{2} = 0$ and $\sin\dfrac{\pi}{2} = 1$.

The particular solution is

$y = 2\cos 2x - \dfrac{\pi}{4}\sin 2x + 3x\sin 2x$

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55 a $y = a + bx \Rightarrow \dfrac{\mathrm{d}y}{\mathrm{d}x} = b$ and $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2} = 0$

Substituting into the differential equation

$0 - 4b + 4a + 4bx = 16 + 4x$

Equating the coefficients of x

$4b - 4 \Rightarrow b = 1$

Equating the constant coefficients

$-4b + 4a = 16$

$-4 + 4a = 16 \Rightarrow a = 5$ ← Use $b = 1$.

$a = 5, b = 1$

b The auxiliary equation is

$m^2 - 4m + 4 = 0$

$(m-2)^2 = 0$

$m = 2$, repeated

The complementary function is given by

$y = \mathrm{e}^{2x}(A + Bx)$ ← If the auxiliary equation has a repeated root α, then the complementary function is $\mathrm{e}^{\alpha x}(A + Bx)$. You can quote this result.

The general solution is

$y = \mathrm{e}^{2x}(A + Bx) + 5 + x$ ← general solution = complementary function + particular integral.

$y = 8,\ x = 0$

$8 = A + 5 \Rightarrow A = 3$

$\dfrac{\mathrm{d}y}{\mathrm{d}x} = 2\mathrm{e}^{2x}(A + Bx) + B\mathrm{e}^{2x} + 1$

$\dfrac{\mathrm{d}y}{\mathrm{d}x} = 9, x = 0$

$9 = 2A + B + 1 \Rightarrow B = 8 - 2A = 2$ ← Use $A = 3$

The Particular solution is

$y = \mathrm{e}^{2x}(3 + 2x) + 5 + x$

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56 a The auxiliary equation is

$$m^2 + 4m + 5 = 0$$
$$m^2 + 4m + 4 = -1$$
$$(m+2)^2 = -1$$
$$m = -2 \pm \mathrm{i}$$

The complementary function is given by
$y = e^{-2x}(A\cos x + B\sin x)$

For a particular integral, let $y = p\cos 2x + q\sin 2x$

If the right hand side of the second order differential equation is a sine or cosine function, then you should try a particular integral of the form $p\cos\omega x + q\sin\omega x$, with an appropriate ω. Here $\omega = 2$.

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -2p\sin 2x + 2q\cos 2x$$
$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = -4p\cos 2x - 4q\sin 2x$$

Substituting into the differential equation

$$-4p\cos 2x - 4q\sin 2x - 8p\sin 2x + 8q\cos 2x + 5p\cos 2x + 5q\sin 2x = 65\sin 2x$$
$$(-4p + 8q + 5p)\cos 2x + (-4q - 8p + 5q)\sin 2x = 65\sin 2x$$

Equating the coefficients of $\cos 2x$ and $\sin 2x$

The coefficients of $\cos 2x$ and $\sin 2x$ can be equated separately. The coefficient of $\cos 2x$ on the right hand side of this equation is zero.

$\cos 2x$: $\quad -4p + 8q + 5p = 0 \Rightarrow p + 8q = 0 \quad$ **(1)**

$\sin 2x$: $\quad -4q - 8p + 5q = 65 \Rightarrow -8p + q = 65 \quad$ **(2)**

$$8p + 64q = 0 \quad \textbf{(3)}$$
$$65q = 65 \Rightarrow q = 1$$

Multiply **(1)** by 8 and add the result to **(2)**.

Substitute $q = 1$ into **(1)**

$$p + 8 = 0 \Rightarrow p = -8$$

A particular integral is $-8\cos 2x + \sin 2x$

The general solution is

$$y = e^{-2x}(A\cos x + B\sin x) + \sin 2x - 8\cos 2x$$

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56 b As $x \to \infty, e^{-2x} \to 0$ and, hence,

$y \to \sin 2x - 8\cos 2x$

Let

$\sin 2x - 8\cos 2x = R\sin(2x - \alpha)$

$= R\sin 2x\cos\alpha - R\cos 2x\sin\alpha$

Equating the coefficients of cos 2x and sin 2x

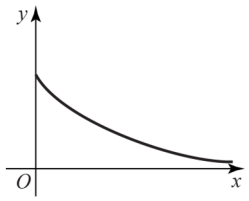
$1 = R\cos\alpha \ldots$ **(4)**

$8 = R\sin\alpha \ldots$ **(5)**

$R^2\cos^2\alpha + R^2\sin^2\alpha = 1^2 + 8^2 = 65$

$R^2 = 65 \Rightarrow R\sqrt{65}$

$\frac{R\sin\alpha}{R\cos\alpha} = \frac{8}{1} \Rightarrow \tan\alpha = 8$



The graph of e^{-2x} against x has this shape. As x becomes larger e^{-2x} is close to zero, so e^{-2x} ($A\cos x + B\sin x$) is also small.

Add (**4**) squared to (**5**) squared and use the identity $\cos^2\alpha + \sin^2\alpha = 1$.

Divide (**5**) by (**4**).

Hence, for large x, y can be approximated by the sine function $\sqrt{65}\sin(2x - a)$, where $\tan\alpha = 8$ $(a \approx 82.9^\circ)$

57 a The auxiliary equation is

$m^2 + 2m + 2 = 0$

$m^2 + 2m + 1 = -1$

$(m+1)^2 = -1$

$m = -1 \pm i$

The complementary function is

$y = e^{-t}(A\cos t + B\sin t)$

Try a particular integral $y = ke^{-t}$

$\frac{dy}{dt} = -ke^{-t}, \frac{d^2y}{dt^2} = ke^{-t}$

If the right hand side of the differential equation is λe^{at+b}, where λ is any constant, then a possible form of the particular integral is ke^{at+b}.

Substituting into the differential equation

$ke^{-t} - 2ke^{-t} + 2ke^{-t} = 2e^{-t}$

$k - 2k + 2k = 2 \Rightarrow k = 2$

Divide throughout by e^{-t}.

A particular integral is $2e^{-t}$

The general solution is

$y = e^{-t}(A\cos t + B\sin t) + 2e^{-t}$

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57 b $y = 1, \quad t = 0$

$1 = A + 2 \Rightarrow A = -1$

Substitute the boundary condition $y = 1, t = 0$ into the general solution gives you an equation for one arbitrary constant.

$\frac{\mathrm{d}y}{\mathrm{d}t} = -\mathrm{e}^{-t}(A\cos t + B\sin t) + \mathrm{e}^{-t}(-A\sin t + B\cos t) - 2\mathrm{e}^{-t}$

Use the product rule for differentiating.

$\frac{\mathrm{d}y}{\mathrm{d}t} = 1, \ t = 0$

$1 = -A + B - 2 \Rightarrow B = 3 + A = 2$

As $A = -1$.

The particular solution is

$y = \mathrm{e}^{-t}(2\sin t - \cos t) + 2\mathrm{e}^{-t}$

58 a The auxiliary equation is

$m^2 + 2m + 5 = 0$

You may use any appropriate method to solve the quadratic. Completing the square works efficiently when the coefficient of m is given.

$m^2 + 2m + 1 = -4$

$(m+1)^2 = -4$

$m = -1 \pm 2\mathrm{i}$

The general solution is

$x = \mathrm{e}^{-t}(A\cos 2t + B\sin 2t)$

b $x = 1, \ t = 0$

$1 = A$

Use the product rule for differentiation.

$\frac{\mathrm{d}x}{\mathrm{d}t} = -\mathrm{e}^{-t}(A\cos 2t + B\sin 2t) + 2\mathrm{e}^{-t}(-A\sin 2t + B\cos 2t)$

$\frac{\mathrm{d}x}{\mathrm{d}t} = 1, \ t = 0$

$1 = -A + 2B \Rightarrow 2B = A + 1 = 2 \Rightarrow B = 1$

The particular solution is

$x = \mathrm{e}^{-t}(\cos 2t + \sin 2t)$

Both A and B are 1.

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58 c The curve crosses the t-axis where

$e^{-t}(\cos 2t + \sin 2t) = 0$

$\cos 2t + \sin 2t = 0$

$\sin 2t = -\cos 2t$

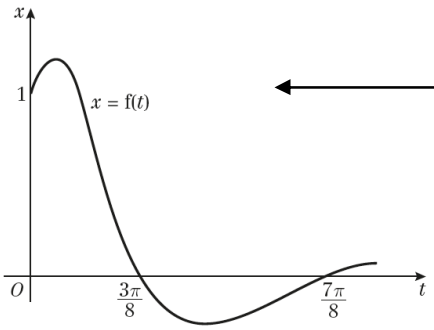
$\tan 2t = -1$

$2t = \frac{3\pi}{4}, \frac{7\pi}{4}$

$t = \frac{3\pi}{8}, \frac{7\pi}{8}$

e^{-t} can never be zero.

Divide both sides by $\cos 2t$ and use the identity $\tan\theta = \frac{\sin\theta}{\cos\theta}$.



The boundary conditions give you that at $t = 0$, $x = 1$ and the curve has a positive gradient. The curve must then turn down and cross the axis at the two points where $t = \frac{3\pi}{8}$ and $\frac{7\pi}{8}$.

59 a The auxiliary equation is

$2m^2 + 7m + 3 = 0$

$(2m+1)(m+3) = 0$

$m = -\frac{1}{2}, -3$

The complementary function is given by

$y = Ae^{-\frac{1}{2}t} + Be^{-3t}$

If the auxiliary equation has two real solutions α and β, the complementary function is $y = Ae^{\alpha t} + Be^{\beta t}$. You can quote this result.

For a particular integral, try $y = at^2 + bt + c$

$\frac{dy}{dt} = 2at + b, \frac{d^2y}{dt^2} = 2a$

If the right hand side of the differential equation is a polynomial of degree n, then you can try a particular integral of the same degree. Here the right-hand side is a quadratic, so you try the general quadratic $at^2 + bt + c$.

Substitute into the differential equation

$4a + 14at + 7b + 3at^2 + 3bt + 3c = 3t^2 + 11t$

$3at^2 + (14a + 3b)t + 4a + 7b + 3c = 3t^2 + 11t$

Equating the coefficients of t^2

$3a = 3 \Rightarrow a = 1$

Use $a = 1$.

Equating the coefficients of t

$14a + 3b = 11 \Rightarrow 3b = 11 - 14a = -3 \Rightarrow b = -1$

Use $a = 1$ and $b = -1$.

Equating the constant coefficients

$4a + 7b + 3c = 0 \Rightarrow 3c = -4a - 7b = 3 \Rightarrow c = 1$

A particular integral is $t^2 - t + 1$.

The general solution is $y = Ae^{-\frac{1}{2}t} + Be^{-3t} + t^2 - t + 1$.

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59 b $y = 1,\ t = 0$

$1 = A + B + 1 \Rightarrow A + B = 0$ **(1)**

$\dfrac{dy}{dt} = -\dfrac{1}{2}Ae^{-\frac{1}{2}t} - 3Be^{-3t} + 2t - 1$

Differentiate the general solution in part **a** with respect to t.

$\dfrac{dy}{dt} = 1,\ t = 0$

$1 = -\dfrac{1}{2}A - 3B - 1 \Rightarrow \dfrac{1}{2}A + 3B = -2$ **(2)**

$A + 6B = -4$ **(3)**

$5B = -4 \Rightarrow B = -\dfrac{4}{5}$

Multiply (**2**) by 2 and then subtract (**1**) from (**3**).

Substituting $B = \dfrac{4}{5}$ into **(1)**

$A - \dfrac{4}{5} = 0 \Rightarrow A = \dfrac{4}{5}$

The particular solution is $y = \dfrac{4}{5}\left(e^{-\frac{1}{2}t} - e^{-3t}\right) + t^2 - t + 1.$

c When $t = 1,\ y = \dfrac{4}{5}\left(e^{-\frac{1}{2}} - e^{-3}\right) + 1 = 1.45$ (3 s.f.)

60 a Let $y = \lambda x \cos 3x$

$\dfrac{dy}{dx} = \lambda \cos 3x - 3\lambda x \sin 3x$

Use the product rule for differentiation

$\dfrac{d}{dx}(x \sin 3x) = \dfrac{d}{dx}(x)\sin 3x + x\dfrac{d}{dx}(\sin 3x)$

$= \sin 3x + 3x \cos 3x$

$\dfrac{d^2y}{dx^2} = -3\lambda \sin 3x - 3\lambda \sin 3x - 9\lambda x \cos 3x$

$= -6\lambda \sin 3x - 9\lambda x \cos 3x$

Substituting into the differential equation

$-6\lambda \sin 3x - \cancel{9\lambda x \cos 3x} + \cancel{9\lambda x \cos 3x} = -12 \sin 3x$

Hence

$\lambda = 2$

b The auxiliary equation is

$m^2 + 9 = 0 \Rightarrow m^2 = -9$

$m = \pm 3i$

The complementary function is given by

$y = A\cos 3x + B\sin 3x$

The general solution is

$y = A\cos 3x + B\sin 3x + 2x\cos 3x$

Part **a** shows that $2x\cos 3x$ is a particular integral of the differential equation and general solution = complementary function + particular integral

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60 c $y = 1,\ x = 0$

$1 = A$

$\dfrac{\mathrm{d}y}{\mathrm{d}x} = -3A\sin 3x + 3B\cos 3x + 2\cos 3x - 6x\sin 3x$

Differentiate the general solution in part **b** with respect to x.

$\dfrac{\mathrm{d}y}{\mathrm{d}x} = 2,\ x = 0$

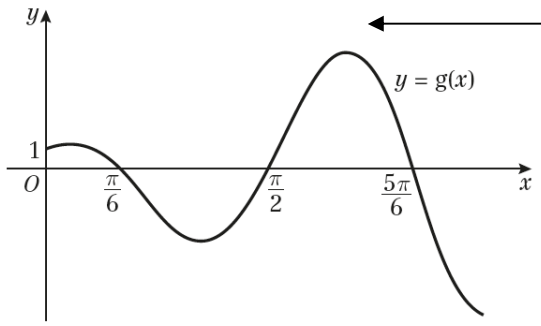
$2 = 3B + 2 \Rightarrow B = 0$

The particular solution is

$y = \cos 3x + 2x\cos 3x = (1 + 2x)\cos 3x$

d For $x > 0$, the curve crosses the x-axis at $\cos 3x = 0$

$3x = \dfrac{\pi}{2}, \dfrac{3\pi}{2}, \dfrac{5\pi}{2} \Rightarrow x = \dfrac{\pi}{6}, \dfrac{\pi}{2}, \dfrac{5\pi}{6}$



The boundary conditions give you that at $x = 0,\ y = 1$ and the curve has a positive gradient. The curve must then turn down and cross the axis at the three points where $x = \dfrac{\pi}{6}, \dfrac{\pi}{2}$ and $\dfrac{5\pi}{6}$.

The $(1 + 2x)$ factor in the general solution means that the size of the oscillations increases as x increases.

61 a If $y = Kt^2\mathrm{e}^{3t}$

$\dfrac{\mathrm{d}y}{\mathrm{d}t} = 2Kt\,\mathrm{e}^{3t} + 3Kt^2\,\mathrm{e}^{3t}$

$\dfrac{\mathrm{d}^2y}{\mathrm{d}t^2} = 2Kt\,\mathrm{e}^{3t} + 3Kt^2\,\mathrm{e}^{3t} + 6Kt\,\mathrm{e}^{3t} + 9Kt^2\,\mathrm{e}^{3t}$

$= 2K\,\mathrm{e}^{3t} + 12Kt\,\mathrm{e}^{3t} + 9Kt^2\,\mathrm{e}^{3t}$

e^{3t} cannot be zero, so you can divide throughout by e^{3t}.

Substituting into the differential equation

$2K\,\mathrm{e}^{3t} + \cancel{12Kt\,\mathrm{e}^{3t}} + \cancel{9Kt^2\,\mathrm{e}^{3t}} - \cancel{12Kt\,\mathrm{e}^{3t}} - \cancel{18Kt\,\mathrm{e}^{3t}} + \cancel{9Kt\,\mathrm{e}^{3t}} = 4\,\mathrm{e}^{3t}$

Hence

$2K = 4 \Rightarrow K = 2$

$2t^2\,\mathrm{e}^{3t}$ is a particular integral of the differential equation.

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61 b The auxiliary equation is

$$m^2 - 6m + 9 = 0$$
$$(m-3)^2 = 0$$
$$m = 3, \text{ repeated}$$

The complementary function is given by

$$y = e^{3t}(A + Bt)$$

If the auxiliary equation has a repeated root α, then the complementary function is $e^{\alpha t}(A + Bt)$. You can quote this result.

The general solution is

$$y = e^{3t}(A + Bt) + 2t^2 e^{3t} = (A + Bt + 2t^2)e^{3t}$$

c $y = 3,\ t = 0$

$$3 = A$$

$$\frac{dy}{dt} = (B + 4t)e^{3t} + 3(A + Bt + 2t^2)e^{3t}$$

$$\frac{dy}{dt} = 1,\ t = 0$$

As $A = 3$.

$$1 = B + 3A \Rightarrow B = 1 - 3A \Rightarrow B = -8$$

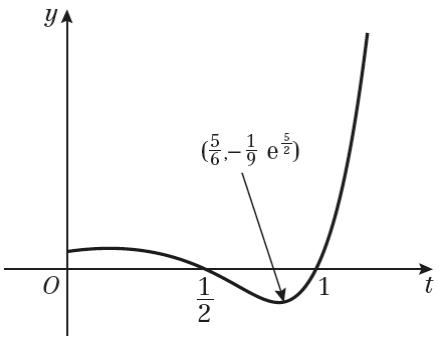
The particular solution is

$$y = (3 - 8t + 2t^2)e^{3t}$$

d This particular solution crosses the t-axis where

$$1 - 3t + 2t^2 = (1 - 2t)(1 - t) = 0$$

$$t = \frac{1}{2}, 1$$



For a minimum $\frac{dy}{dt} = 0$

$$(-3 + 4t)e^{3t} + (1 - 3t + 2t^2)3e^{3t} = 0$$

e^{3t} cannot be zero, so you can divide throughout by e^{3t}.

$$-3 + 4t + 3 - 9t + 6t^2 = 0$$

$$6t^2 - 5t = t(6t - 5) = 0 \Rightarrow t = 0, \frac{5}{6}$$

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61 d From the diagram $t = \frac{5}{6}$ gives the minimum

It is clear from the diagram that there is a minimum point between $t = \frac{1}{2}$ and $t = 1$. You do not have to consider the second derivative to show that it is a minimum.

At $t = \frac{5}{6}$

$$y = \left(1 - 3 \times \frac{5}{6} + 2 \times \left(\frac{5}{6}\right)^2\right) e^{3 \times \frac{5}{6}} = -\frac{1}{9} e^{\frac{5}{2}}$$

The coordinates of the minimum point are

$$\left(\frac{5}{6}, -\frac{1}{9} e^{\frac{5}{2}}\right).$$

62 a The auxiliary equation is

$$2m^2 + 5m + 2 = 0$$

$$(2m + 1)(m + 2) = 0$$

$$m = -\frac{1}{2}, -2$$

If the auxiliary equation has two real solutions α and β, the complementary function is $x = Ae^{\alpha t} + Be^{\beta t}$. You can quote this result.

The complementary function is given by

$$x = Ae^{-\frac{1}{2}t} + Be^{-2t}$$

For a particular integration, try $x = pt + q$

$$\frac{dx}{dt} = p, \frac{d^2x}{dt^2} = 0$$

If the right hand side of the differential equation is a polynomial of degree n, then you can try a particular integral of the same degree. Here the right-hand side is linear, so you try the general linear function $pt + q$.

Substituting into the differential equation

$$0 + 5p + 2pt + 2q = 2t + 9$$

Equating the coefficients of t

$$2p = 2 \Rightarrow p = 1$$

Equating the constant coefficients

$$5p + 2q = 9 \Rightarrow q = \frac{9 - 5p}{2} \Rightarrow q = 2$$

A particular integral is $t + 2$

The general solution is

$$x = Ae^{-\frac{1}{2}t} + Be^{-2t} + t + 2$$

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62 b $x = 3,\ t = 0$

$3 = A + B + 2 \Rightarrow A + B = 1$ **(1)**

$\dfrac{\mathrm{d}x}{\mathrm{d}t} = -\dfrac{1}{2}A\mathrm{e}^{-\frac{1}{2}t} - 2B\mathrm{e}^{-2t} + 1$

Differentiating the general solution in part **a**.

$\dfrac{\mathrm{d}x}{\mathrm{d}t} = -1,\ t = 0$

$-1 = -\dfrac{1}{2}A - 2B + 1 \Rightarrow \dfrac{1}{2}A + 2B = 2$ **(2)**

$A + 4B = 4$ **(3)**

$3B = 3 \Rightarrow B = 1$

Multiplying (**2**) by 2 and subtracting (**1**) from (**3**).

Substituting $B = 1$ into (**1**)

$A + 1 = 1 \Rightarrow A = 0$

The particular solution is

$x = \mathrm{e}^{-2t} + t + 2$

c For a minimum

$\dfrac{\mathrm{d}x}{\mathrm{d}t} = -2\mathrm{e}^{-2t} + 1 = 0$

$\mathrm{e}^{-2t} = \dfrac{1}{2}$

You take logarithms of both sides of this equation and use $\mathrm{e}^{\ln \mathrm{f}(x)} = \mathrm{f}(x)$.

$-2t = \ln\dfrac{1}{2} = -\ln 2$

$\ln\dfrac{1}{2} = \ln 1 - \ln 2 = -\ln 2,$ as $\ln 1 = 0.$

$t = \dfrac{1}{2}\ln 2$

$\dfrac{\mathrm{d}^2x}{\mathrm{d}t^2} = 4\mathrm{e}^{-2t} > 0,$ for any real t

Hence the stationary value is a minimum value

$\mathrm{e}^{-\ln 2} = \mathrm{e}^{\ln 1 - \ln 2} = \mathrm{e}^{\ln\frac{1}{2}} = \dfrac{1}{2}$

When $t = \dfrac{1}{2}\ln 2$

$x = \mathrm{e}^{-\ln 2} + \dfrac{1}{2}\ln 2 + 2 = \dfrac{1}{2} + \dfrac{1}{2}\ln 2 + 2 = \dfrac{5}{2} + \dfrac{1}{2}\ln 2$

The minimum distance is $\dfrac{1}{2}(5 + \ln 2)$ m, as required.

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63 a If $x = At^2 e^{-t}$

$$\frac{dx}{dt} = 2At e^{-t} - At^2 e^{-t}$$

$$\frac{d^2x}{dt^2} = 2A e^{-t} - 2At e^{-t} - 2At e^{-t} + At^2 e^{-t}$$

$$= 2A e^{-t} - 4At e^{-t} + At^2 e^{-t}$$

Substituting into the differential equation

$$2A e^{-t} - 4At e^{-t} + At^2 e^{-t} + 4At^2 e^{-t} - 2At^2 e^{-t} + At^2 e^{-t} = e^{-t}$$

e^{-t} cannot be zero, so you can divide throughout by e^{-t}.

Hence

$$2A = 1 \Rightarrow A = \frac{1}{2}$$

b The auxiliary equation is

$$m^2 + 2m + 1 = (m+1)^2 = 0$$

$$m = -1, \text{ repeated}$$

The complementary function is given by

$$x = e^{-t}(A + Bt)$$

If the auxiliary equation has a repeated root α, then the complementary function is $e^{\alpha t}(A + Bt)$. You can quote this result.

The general solution is

$$x = e^{-t}(A + Bt) + \frac{1}{2}t^2 e^{-t} = \left(A + Bt + \frac{1}{2}t^2\right)e^{-t}$$

$$x = 1, t = 0$$

$$1 = A$$

From part **a**, $\frac{1}{2}t^2 e^{-t}$ is a particular integral of the differential equation.

$$\frac{dx}{dt} = (B + t)e^{-t} - \left(A + Bt + \frac{1}{2}t^2\right)e^{-t}$$

$$\frac{dx}{dt} = 0, \ t = 0$$

$$0 = B - A \Rightarrow B = A = 1$$

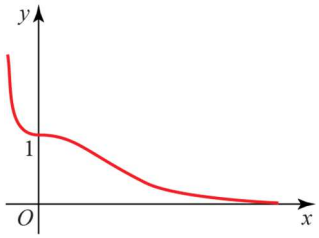
The particular solution is

$$x = \left(1 + t + \frac{1}{2}t^2\right)e^{-t}$$

c $$\frac{dx}{dt} = (1 + t)e^{-t} - \left(1 + t + \frac{1}{2}t^2\right)e^{-t}$$

$$= -\frac{1}{2}t^2 e^{-t} \leqslant 0, \text{ for all real } t.$$

When $t = 0$, $x = 1$ and x has a negative gradient for all positive t, x is a decreasing function of t. Hence, for $t \geqslant 0$, $x \leqslant 1$, as required.



The graph of x against t, shows the curve crossing the x-axis at $x = 1$ and then decreasing. For all positive t, x is less than 1.

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64 a $y = kx \Rightarrow \dfrac{dy}{dx} = k \Rightarrow \dfrac{d^2y}{dx^2} = 0$

Substituting into $\dfrac{d^2y}{dx^2} + y = 3x$

$0 + kx = 3x$

$k = 3$

b The auxiliary equation is

$m^2 + 1 = 0 \Rightarrow m = \pm i$

The complementary function is given by

$y = A\sin x + B\cos x$

and the general solution is

$y = A\sin x + B\cos x + 3x$

$y = 0,\ x = 0$

$0 = B + 0 \Rightarrow B = 0$

The most general solution is

$y = A\sin x + 3x$

In part **b**, only one condition is given, so only one of the arbitrary constants can be found. The solution is a family of functions, some of which are illustrated in the diagram below.

c At $x = \pi$

$y = A\sin\pi + 3\pi = 3\pi$

This is independent of the value of A.

Hence, all curves given by the solution in part **a** pass through $(\pi, 3\pi)$.

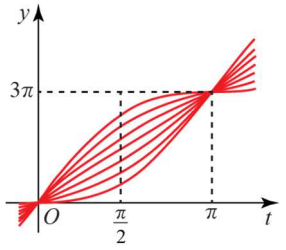
$\dfrac{dy}{dx} = A\cos x + 3$

At $x = \dfrac{\pi}{2}$

$\dfrac{dy}{dx} = A\cos\dfrac{\pi}{2} + 3 = 3$

This is independent of the value of A.

Hence, all curves given by the solution in part **a** have an equal gradient of 3 at $x = \dfrac{\pi}{2}$.



As is illustrated by this diagram, the family of curves $y = A\sin x + 3x$ all go through $(0, 0)$ and $(\pi, 3\pi)$. The tangent to the curves at $x = \dfrac{\pi}{2}$ are all parallel to each other.

d $y = 0,\ x = \dfrac{\pi}{2}$

Substituting into $y = A\sin x + 3x$

$0 = A\sin\dfrac{\pi}{2} + \dfrac{3\pi}{2} = A + \dfrac{3\pi}{2} \Rightarrow A = -\dfrac{3\pi}{2}$

The particular solution is

$y = 3x - \dfrac{3\pi}{2}\sin x$

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64 e For a minimum

$$\frac{dy}{dx} = 3 - \frac{3\pi}{2}\cos x = 0$$

$$\cos x = \frac{2}{\pi} \Rightarrow x = \arccos\left(\frac{2}{\pi}\right)$$

$$\frac{d^2y}{dx^2} = \frac{3\pi}{2}\sin x$$

In the interval $0 \leqslant x \leqslant \frac{\pi}{2}$,

$\cos x = \frac{2}{\pi}$ has an infinite number of solutions. This shows that the solution in the first quadrant gives a minimum as $\sin x$ is positive in that quadrant.

$$\frac{d^2y}{dx^2} > 0 \Rightarrow \text{minimum}$$

$$\sin^2 x = 1 - \cos^2 x = 1 - \frac{4}{\pi^2} = \frac{\pi^2 - 4}{\pi^2}$$

In the interval $0 \leqslant x \leqslant \frac{\pi}{2}$

$$\sin x = +\left(\frac{\pi^2 - 4}{\pi^2}\right)^{\frac{1}{2}} = \frac{\sqrt{\pi^2 - 4}}{\pi}$$

$$y = 3\arccos\left(\frac{2}{\pi}\right) - \frac{3\pi}{2} \times \frac{\sqrt{\pi^2 - 4}}{\pi}$$

$$= 3\arccos\left(\frac{2}{\pi}\right) - \frac{3}{2}\sqrt{\pi^2 - 4}, \text{ as required.}$$

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65 a $\displaystyle\int \frac{2}{120-t}\,\mathrm{d}t = -2\ln(120-t) = \ln(120-t)^{-2} = \ln\frac{1}{(120-t)^2}$

Using the log law $n\log a = \log a^n$, with $n = -2$.

Hence the integrating factor is

$$\mathrm{e}^{\int \frac{2}{120-t}\mathrm{d}t} = \mathrm{e}^{\ln\frac{1}{(120-t)^2}} = \frac{1}{(120-t)^2}$$

Multiply the equation throughout by $\dfrac{1}{(120-t)^2}$

$$\frac{1}{(120-t)^2}\frac{\mathrm{d}s}{\mathrm{d}t} + \frac{2}{(120-t)}S = \frac{1}{4(120-t)^2}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(S(120-t)^{-2}\right)$$
$$= \frac{\mathrm{d}S}{\mathrm{d}t}(120-t)^{-2} - S\times(-2)(120-t)^{-3}$$
$$= \frac{1}{(120-t)^2}\frac{\mathrm{d}S}{\mathrm{d}t} + \frac{2}{(120-t)^3}S$$

This product enables you to write the differential equation as a complete equation.

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{S}{(120-t)^2}\right) = \frac{1}{4}(120-t)^{-2}$$

Integrating both sides with respect to t

$$\frac{S}{(120-t)^2} = \frac{1}{4}\int (120-t)^{-2}\mathrm{d}t = -\frac{1}{4}\frac{(120-t)^{-1}}{-1} + C$$

$$\frac{S}{(120-t)^2} = \frac{1}{4(120-t)} + C$$

Multiply this equation by $(120-t)^2$.

$$S = \frac{120-t}{4} + C(120-t)^2$$

$S = 6$ when $t = 0$

Remember to multiply the C by $(120-t)^2$. It is a common error to obtain C instead of $C(120-t)^2$ at this stage.

$$6 = 30 + C\times 120^2 \Rightarrow C = -\frac{24}{120^2} = -\frac{1}{600}$$

$$S = \frac{120-t}{4} - \frac{(120-t)^2}{600}$$

b For a maximum value

$$\frac{\mathrm{d}S}{\mathrm{d}t} = -\frac{1}{4} + \frac{2(120-t)}{600} = 0$$

$$240 - 2t = 150 \Rightarrow t = \frac{240-150}{2} = 45$$

$$\frac{\mathrm{d}^2S}{\mathrm{d}t^2} = -\frac{1}{300} < 0 \Rightarrow \text{maximum}$$

Maximum values is given by

$$S = \frac{120-45}{4} - \frac{(120-45)^2}{600} = \frac{75}{4} - \frac{75}{8} = \frac{75}{8} = 9\frac{3}{8}$$

The maximum mass of salt predicted is $9\frac{3}{8}$ kg.

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66 a Three quarters of the nutrient is $\frac{3}{4} \times 100m_0 = 75m_0$

At time t, the nutrient consumed is $5(m - m_0)$

Hence

$5(m - m_0) = 75m_0$

$5m - 5m_0 = 75m_0 \Rightarrow 5m = 80m_0$

$m = \frac{80m_0}{5} = 16m_0$, as required.

b Rate of increase of mass $= \mu \times \text{mass} \times \text{nutrient remaining}$

$\frac{dm}{dt} = \mu \times m \times [100m_0 - 5(m - m_0)]$

$\frac{dm}{dt} = \mu m(100m_0 - 5m + 5m_0)$

$= \mu m(105m_0 - 5m)$

$= 5\mu m(21m_0 - m)$, as required.

The nutrient remaining is the nutrient consumed, $5(m - m_0)$, subtracted from the original nutrient $100m_0$.

c $\frac{dm}{dt} = 5\mu m(21m_0 - m)$

This is a separable equation.

$\int 5\mu \, dt = \int \frac{1}{m(21m_0 - m)} dm$

Separating the variables.

Let $\frac{1}{m(21m_0 - m)} = \frac{A}{m} + \frac{B}{21m_0 - m}$

To integrate the right hand side of this equation, you must break the expression up into partial fractions using one of the methods you learnt in C4.

Multiplying throughout by $m(21m_0 - m)$

$1 = A(21m_0 - m) + Bm$

Let $m = 0$

$1 = A \times 21m_0 \Rightarrow A = \frac{1}{21m_0}$

Let $m = 21m_0$

$1 = B \times 21m_0 \Rightarrow B = \frac{1}{21m_0}$

Hence

$5\mu t = \frac{1}{21m_0} \int \left(\frac{1}{m} + \frac{1}{21m_0 - m} \right) dm$

$105\mu m_0 t = \int \left(\frac{1}{m} + \frac{1}{21m_0 - m} \right) dm = \ln m - \ln(21m_0 - m) + C$

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66 c When $t = 0$, $m = m_0$

$0 = \ln m_0 - \ln 20m_0 + C$

$C = \ln 20m_0 - \ln m_0 = \ln \dfrac{20m_0}{m_0} = \ln 20$

$105\mu m_0 t = \ln m - \ln(21m_0 - m) + \ln 20 = \ln\left(\dfrac{20m}{21m_0 - m}\right)$

From part **a**, when $t = T$, $m = 16m_0$

$105\mu m_0 T = \ln\left(\dfrac{20 \times 16m_0}{21m_0 - 16m_0}\right) = \ln\left(\dfrac{320m_0}{5m_0}\right)$

$= \ln 64$, as required.

Initially, the mass of the embryo is m_0. This enables you to find the particular solution of the differential equation. The initial conditions are often known in scientific applications of mathematics.

Combining the logarithms at this stage simplifies the next stage of the calculation. The form of the simplification is

$\ln a - \ln b + \ln c = \ln \dfrac{ac}{b}$

67 a $t\dfrac{\mathrm{d}v}{\mathrm{d}t} - v = t$

Divide throughout by t

$\dfrac{\mathrm{d}v}{\mathrm{d}t} - \dfrac{v}{t} = -1$ **(1)**

The integrating factor is

$e^{\int -\frac{1}{t}\mathrm{d}t} = e^{-\ln t} = e^{\ln \frac{1}{t}} = \dfrac{1}{t}$

Multiply **(1)** throughout by $\dfrac{1}{t}$

$\dfrac{1}{t}\dfrac{\mathrm{d}v}{\mathrm{d}t} - \dfrac{v}{t^2} = \dfrac{1}{t}$

$\dfrac{\mathrm{d}}{\mathrm{d}t}\left(\dfrac{v}{t}\right) = \dfrac{1}{t}$

$\dfrac{v}{t} = \int \dfrac{1}{t}\mathrm{d}t = \ln t + c$

$v = t(\ln t + c)$, as required.

The product rule for differentiating, in this case $\dfrac{\mathrm{d}}{\mathrm{d}t}(v \times t^{-1}) = \dfrac{\mathrm{d}v}{\mathrm{d}t} \times t^{-1} + v \times (-1)t^{-2}$, enables you to write the differential equation as an exact equation, where one side is the exact derivative of a product and the other side can be integrated with respect to t.

b $v = 3$ when $t = 2$

$3 = 2(\ln 2 + c) = 2\ln 2 + 2c \Rightarrow c = 1.5 - \ln 2$

$v = t(\ln t + 1.5 - \ln 2)$

When $t = 4$

$v = 4(\ln 4 + 1.5 - \ln 2)$

≈ 8.77

Use your calculator to evaluate this expression.

The speed of the particle when $t = 4$ is $8.77\,\mathrm{m\,s^{-1}}$ (3 s.f.)

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68 $a = \dfrac{dv}{dt} = e^{2t}$

$v = \int e^{2t}\, dt = \dfrac{1}{2}e^{2t} + A$

To find the acceleration, you integrate the velocity with respect to time. Remember to include a constant of integration.

When $t = 0, v = 0$

$0 = \dfrac{1}{2} + A \Rightarrow A = -\dfrac{1}{2}$

Hence $v = \dfrac{1}{2}(e^{2t} - 1)$, as required.

69 a $a = \dfrac{dv}{dt} = \dfrac{1}{2}e^{-\frac{1}{6}t}$

$v = \int \dfrac{1}{2}e^{-\frac{1}{6}t}\, dt = -3e^{-\frac{1}{6}t} + A$

Using $\int e^{kt}\, dt = \dfrac{1}{k}e^{kt} + A$, then

$\int \dfrac{1}{2}e^{-\frac{1}{2}}\, dt = \dfrac{1}{2 \times (1 - \frac{1}{6})}e^{-\frac{1}{6}} + A$

$= -3e^{-\frac{1}{6}} + A.$

When $t = 0,\ v = 10$

$10 = -3 + A \Rightarrow A = 13$

Hence $v = 13 - 3e^{-\frac{1}{6}t}$

b When $t = 3$

$v = 13 - 3e^{-\frac{1}{2}} = 11.180\ldots$

The speed of P when $t = 3$ is $11.2\,\text{m s}^{-1}$ (3 s.f.).

c As $t \to \infty, e^{-\frac{1}{6}t} \to 0$ and $v \to 13$.

The limiting value of v is 13.

As t gets large, $e^{-\frac{1}{6}t}$ gets very small. For example, if $t = 120$, then $e^{-\frac{1}{6}t} \approx 2.06 \times 10^{-9}$. In this question, as t gets larger, v gets closer and closer to 13 and so 13 is the limiting value of v.

70 a $a = \dfrac{dv}{dt} = -4e^{2t}$

$v = \int 4e^{-2t}\, dt = 2^{-2t} + A$

At $t = 0,\ v = 1$

$1 = 2 + A \Rightarrow A = -1$

Hence $v = 2e^{-2t} - 1$

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70 b P is instantaneously at rest when $v = 0$.

$$0 = 2e^{-2t} - 1$$

To find the speed when P is instantaneously at rest, you will need to know the value of t when $v = 0$.

$$e^{-2t} = \frac{1}{2} \Rightarrow e^{2t} = 2$$

Take natural logarithms of both sides of this equation and use the property that, for any x, $\ln(e^x) = x$

$$2t = \ln 2 \Rightarrow t = \frac{1}{2}\ln 2$$

$$x = \int v \, dt = \int (2e^{-2t}) \, dt$$

$$= -e^{-2t} - t + B$$

When $t = 0$, $x = 0$

$$0 = -1 - 0 + B \Rightarrow B = 1$$

Using $e^0 = 1$. It is a common error to obtain $B = 0$ by carelessly writing $e^0 = 0$.

Hence $x = 1 - e^{-2t} - t$

When $t = \frac{1}{2}\ln 2$

$$x = 1 - e^{-2\left(\frac{1}{2}\ln 2\right)} - \frac{1}{2}\ln 2 = 1 - e^{-\ln 2} - \frac{1}{2}\ln 2$$

Using the law of logarithms, $\ln a^n = n \ln a$ with $n = -1$,

$-\ln 2 = (-1)\ln 2 = \ln 2^{-1} = \ln \frac{1}{2}$. Then as

for any x, $e^{\ln x} = x$, $e^{\ln \frac{1}{2}} = \frac{1}{2}$.

$$= 1 - e^{\ln \frac{1}{2}} - \frac{1}{2}\ln 2 = 1 - \frac{1}{2} - \frac{1}{2}\ln 2$$

$$= \frac{1}{2} - \frac{1}{2}\ln 2 = \frac{1}{2}(1 - \ln 2)$$

The distance of P from O when P comes to instantaneous rest is $\frac{1}{2}(1 - \ln 2)$m.

71 a Rewrite the equation as $\frac{dv}{dt} + \frac{3v}{t+3} = 9.8$. This equation has integrating factor $e^{\int \frac{3}{t+3} dt} = e^{3\ln(t+3)} = (t+3)^3$. Multiplying both factors by this yields:

$$(t+3)^3 \frac{dv}{dt} + (t+3)^2 3v = 9.8(t+3)^3$$

$$\frac{d}{dt}\left(v(t+3)^3\right) = 9.8(t+3)^3$$

$$v(t+3)^3 = 9.8\frac{(t+3)^4}{4} + C$$

$$v = 9.8\frac{t+3}{4} + \frac{C}{(t+3)^3}$$

$$v = \frac{9.8(t+3)^4 + 4C}{4(t+3)^3}$$

$$v = \frac{49(t+3)^4 + 20C}{20(t+3)^3}$$

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71 a Since the droplet starts falling from rest, the equation must satisfy the condition $v(0)=0$, so $49\cdot 3^4+20C=0$. Then $C=-\frac{3969}{20}$, and the velocity can be expressed as $v=\frac{49(t+3)^4-3969}{20(t+3)^3}$.

b After six seconds, the velocity of the droplets is given by $\frac{49(9)^4-3969}{20(9)^3}=21.78\text{ms}^{-1}$.

c According to this model the velocity of the droplets is always increasing (it is clear that the limit for $t\to\infty$ is infinity, so for every possible value of velocity there is a time t such that that value is reached). However, it is known from fluid dynamics that a particle falling through air must at some point reach the so-called terminal velocity and stop accelerating.

72 a The amount of liquid inside the bottle at the minute t is given by $400-30t+40t=400+10t$. Of this amount we know that x is acid, so the concentration of acid in the bottle is given simply by $\frac{x}{400+10t}$. Now the liquid that leaks out of this bottle has this concentration of water, so at each minute the rate of acid leaking out is $30\frac{x}{400+10t}=\frac{3x}{40+t}$; the quantity of acid is 10% of 40ml, so 4ml. Then the equation that describes the rate of change in the quantity of acid at minute t is exactly $\frac{dx}{dt}=4-\frac{3x}{40+t}$.

b Solve the equation, by finding the integrating factor $e^{\int\frac{3}{40+t}\,dt}=(40+t)^3$. Then the equation is equivalent to $\frac{d}{dt}x(40+t)^3=4(40+t)^3$, which is solved by $x=(40+t)^4+\frac{C}{(40+t)^3}$. By forcing the condition $x(0)=0$ we find that the value of the constant C is $-2\,560\,000$, and we easily compute $x(7)=22.3\text{ml}$.

c Of course, acid cannot disperse immediately on entry, so a correct model would also take into account the velocity at which this process takes place.

73 a From the differential equation relating position and acceleration it is clear that it is a simple harmonic motion.

b The auxiliary equation for this equation is $m^2+49=0$, so the general solution is $x=A\cos 7t+B\sin 7t$. By considering the initial condition for position and velocity we have that $A=0.3$ and $7B=0$, so the equation of the motion is $x=0.3\cos 7t$.

c The period is $\frac{2\pi}{7}$, as is known by a simple analysis of the cosine function. Since the maximum of the cosine function is 1, the maximum of x is 0.3. Differentiating we find that the maximum speed is 7(0.3) = 2.1 m per second.

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74 a The auxiliary equation is $m^2 + 1.6 = 0$, so the general solution of the differential equation is: $x = A\cos\left(\sqrt{1.6}t\right) + B\sin\left(\sqrt{1.6}t\right)$. Assuming the initial conditions $x(0) = 0$ and $\frac{\mathrm{d}x}{\mathrm{d}t}(0) = 1$ we see that the two coefficients have to satisfy the conditions $A = 0$ and $\sqrt{1.6}B = 1$, so the correct solution is $x = \frac{\sqrt{1.6}}{1.6}\sin\left(\sqrt{1.6}t\right)$. Calculating $\frac{\sqrt{1.6}}{1.6}$ yields the maximum displacement: it is 0.791 m to (3 s. f.).

b The boat is at its maximum displacement when $\sqrt{1.6}t = \frac{\pi}{2}$, and at its minimum when $\sqrt{1.6}t = \frac{3\pi}{2}$. Then the time elapsing between these two moments is given by:

$$\frac{3\pi}{2\sqrt{1.6}} - \frac{\pi}{2\sqrt{1.6}}$$
$$= \frac{\pi}{\sqrt{1.6}} = 2.48$$

So the time is 2.48 minutes.

c It is very unlikely that for large values of t the boat keeps moving with such regularity, surely other factors should be taken into account.

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75 a $\dfrac{d^2x}{dt^2}+2\dfrac{dx}{dx}+2x=12\cos 2t-6\sin 2t$

Auxiliary equation: $m^2+2m+2=0$

An expression for x is needed. Refer to book FP2 Chapter 5 for the method of solving these equations.

$$m=\frac{-2\pm\sqrt{(4-8)}}{2}=-1\pm i$$

Complementary function is

$x=e^{-t}(A\cos t+B\sin t)$

Let $x=p\cos 2t+q\sin 2t$

Try this for a particular integrate.

$\dot{x}=-2p\sin 2t+2q\cos 2t$

$\ddot{x}=-4p\cos 2t-4q\sin 2t$

$\therefore -4p\cos 2t-4q\sin 2t$

$+2(-2p\sin 2t+2q\cos 2t)$

Substitute the expressions for x, $\dot{x}$ and $\ddot{x}$ into the differential equation.

$+2(p\cos 2t+q\sin 2t)$

$=12\cos 2t-6\sin 2t$

$\cos 2t(-4p+4q+2p)$

$+\sin 2t(-4q-4p+2q)$

$=12\cos 2t-6\sin 2t$

$-4p+4q+2p=12$

Equate coefficients of cos 2*t*...

$-2p+4q=12$ **(1)**

$-4q-4p+2q=-6$

$-4p-2q=-6$

... and of sin 2*t*.

$-2p-q=-3$ **(2)**

$5q=15$

Solve **(1)** and **(2)**.

$q=3, p=0$

$\therefore x=e^{-t}(A\cos t+B\sin t)+3\sin 2t$

$t=0,\ x=0\therefore 0=A$

Use the initial conditions given in the question to obtain values for A and B.

$\dot{x}=-e^{-t}B\sin t+e^{-t}B\cos t+6\cos 2t$

$t=0,\ \dot{x}=0\therefore 0=B+6$

$B=-6$

$\therefore x=3\sin 2t-6e^{-t}\sin t$

b $\dot{x}=6e^{-t}\sin t-6e^{-t}\cos t+6\cos 2t$

$$t=\frac{\pi}{4},\dot{x}=6\left[e^{-\frac{\pi}{4}}\sin\frac{\pi}{4}-e^{-\frac{\pi}{4}}\cos\frac{\pi}{4}+\cos\frac{\pi}{2}\right]$$

$$=0$$

$\sin\dfrac{\pi}{4}=\cos\dfrac{\pi}{4}\cos\dfrac{\pi}{2}=0$

$\therefore P$ comes to instantaneous test when $t=\dfrac{\pi}{4}$

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75 c $x = 3\sin 2t - 6e^{-t}\sin t$

$t = \frac{\pi}{4}$ $x = 3\sin\frac{\pi}{2} - 6e^{-\frac{\pi}{4}}\sin\frac{\pi}{4}$

$= 3 - 6e^{-\frac{\pi}{4}} \times \frac{1}{\sqrt{2}}$

$= 1.07 (3\text{s.f.})$

d $t \to \infty$

Large values of t needed, so let $t \to \infty$

$x \approx 3\sin 2t$

$\therefore$ approximate period is π

$e^{-t} \to 0$ as $t \to \infty$

76 a The value of the constants can be found by forcing the conditions $x(0) = 0$ and $\frac{dx}{dt}(0) = U$: these give us that $A = 0$ and $-\omega A + B\omega = U$, so the correct equation for this motion is $x = e^{-\omega t}\left(\frac{U}{\omega}\sin\omega t\right)$.

b The velocity of the particle is given by the following expression:

$\frac{dx}{dt} = -\omega e^{-\omega t}\left(\frac{U}{\omega}\sin\omega t\right) + e^{-\omega t}\left(\frac{U}{\omega}\omega\cos\omega t\right) =$

$= -Ue^{-\omega t}\sin\omega t + Ue^{-\omega t}\cos\omega t =$

$= Ue^{-\omega t}(\cos\omega t - \sin\omega t)$

The least value for which this is zero is $\omega t = \frac{\pi}{4}$, so the time is $t = \frac{\pi}{4\omega}$.

77 To find a particular integral for this equation we look at functions of the form $x = \lambda + \mu t$; by deriving it we find that the coefficients must satisfy the conditions $10k^2\mu = 10k^2 V$ and $2k\mu + 10k^2\lambda = 0$, so the particular integral will be $x = -\frac{V}{5k} + Vt$. As for the general solution, the auxiliary equation associated with the homogeneous equation is $m^2 + 2km + 10k^2 = 0$, and solutions for this equation are $-k \pm 3ik$, so the general solution for the homogeneous equation is $x = e^{-kt}(A\cos 3kt + B\sin 3kt)$. Then the general solution of the equation is $x = -\frac{V}{5k} + Vt + e^{-kt}(A\cos 3kt + B\sin 3kt)$. To find A and B we force the conditions $x(0) = 0$ and $\frac{dx}{dt}(0) = 0$, which yield $-\frac{V}{5k} + A = 0$ and $V - kA + 3Bk = 0$. By solving these simple equations we find that the expression for x is:

$x = -\frac{V}{5k} + Vt + e^{-kt}\left(\frac{V}{5k}\cos 3kt - \frac{4V}{15k}\sin 3kt\right)$.

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78 a A particular integral for this equation is given by the constant function $x = 0.3$. To find the general solution, consider the auxiliary equation $m^2 + 6m + 8 = 0$; this gives us that the general solution of the homogeneous equation is $x = Ae^{-2t} + Be^{-4t}$, so the general solution is $x = 0.3 + Ae^{-2t} + Be^{-4t}$. To find the coefficients we force the conditions $x(0) = 0$ and $\frac{dx}{dt}(0) = 0$, which yield $A + B + 0.3 = 0$ and $-2A - 4B = 0$, so we can easily see that the solution of the equation that describes the motion of P is $x = 0.3 - 0.6e^{-2t} + 0.3e^{-4t}$.

b Deriving the expression for x gives us that at time t the speed of the particle is $1.2\left(e^{-2t} - e^{-4t}\right)$. This is a positive number for every $t > 0$, so the particle never stops moving downward.

79 a Compute the second derivative of x:

$$\frac{d^2x}{dt^2} = \frac{d}{dt}\left(\frac{dx}{dt}\right)$$
$$= \frac{d}{dt}(0.1x + 0.1y)$$
$$= 0.1\frac{dx}{dt} + 0.1\frac{dy}{dt}$$
$$= 0.01x + 0.01y - 0.0025x + 0.02y$$
$$= 0.0075x + 0.03y$$

Now we can substitute the derivatives of x in the equation and we get:

$$0.0075x + 0.03y - 0.03x - 0.03y + 0.0225x$$
$$= -0.0225x + 0.0225x = 0$$

b The auxiliary equation is $m^2 - 0.3m + 0.0225 = 0$, which can be rewritten as $(m - 0.15)^2 = 0$, so the general solution is $x = (A + Bt)e^{0.15t}$.

c We must solve $\frac{dy}{dt} = -0.025x + 0.2y$. This is $\frac{dy}{dt} - 0.2y = -0.025x$; the integrating factor for this equation is $e^{-0.2t}$, so we get $\frac{d}{dt}ye^{-0.2t} = -0.025xe^{-0.2t}$. In order to solve this, we must first solve $\int -0.025xe^{-0.2t}\,dt$, as follows:

$$\int -0.025xe^{-0.2t}\,dt =$$
$$= -0.025\int xe^{-0.2t}\,dt =$$
$$= -0.025\int (A + Bt)e^{-0.05t}\,dt =$$
$$= -0.025\left(A\int e^{-0.05t}\,dt + B\int te^{-0.05t}\,dt\right) =$$
$$= -0.025\left(-A\frac{e^{-0.05t}}{0.05} + B\left(-\frac{te^{-0.05t}}{0.05} - \frac{e^{-0.05t}}{0.0025}\right)\right) =$$
$$= 0.5Ae^{-0.05} + 0.5Bte^{-0.05} + 10Be^{-0.05}$$

$m^2 - m - 6 = 0$

Multiply this by the integrating factor to get an expression for y:

$$y = 0.5Ae^{0.15t} + 0.5Bte^{0.15t} + 10Be^{0.15t}$$

d Use the given initial conditions to find the value of the constants. In particular, if we fix $t = 0$ at the beginning of 2010, we get that $A = 20$ and $0.5A + 10B = 100$, so the coefficients are 20 and 9. Then after seven months the number of angler fish is given by $(20 + 9 \cdot 7)e^{0,15 \cdot 7} = 237$.

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79 e The model does not give any bounds on the growth of the number of fish, so it models a situation in which the number is always increasing – unlikely to be a good model for large values of t.

80 a Compute the second derivative of x:

$$\frac{d^2x}{dt^2} = \frac{d}{dt}\left(\frac{dx}{dt}\right) =$$
$$= \frac{d}{dt}(2x + y + 1) =$$
$$= 2\frac{dx}{dt} + \frac{dy}{dt} =$$
$$= 4x + 2y + 2 + 4x - y + 1 =$$
$$= 8x + y + 3$$

Now we can substitute the derivatives of x in the equation and we get:

$$8x + y + 3 - 2x - y - 1 - 6x =$$
$$= 6x - 6x + 3 - 1 = 2$$

b First we solve the differential equation that models x; we easily see that a particular integral is given by the constant function $x = -\frac{1}{3}$. Solving the auxiliary equation $m^2 - m - 6 = 0$ then tells us that the general solution for the equation is $x = -\frac{1}{3} + Ae^{3t} + Be^{-2t}$. Then we use this to find an expression for y: we need to solve the equation, $\frac{dy}{dt} + y = 4x + 1$ which has integrating factor e^t and is therefore equivalent to $\frac{d}{dt}ye^t = (4x + 1)e^t$. Substitute the value of x to solve this:

$$\int (4x + 1)e^t \, dt =$$
$$= \int \left(-\tfrac{1}{3} + 4Ae^{3t} + 4Be^{-2t}\right)e^t \, dt =$$
$$= \int -\frac{e^t}{3} + 4Ae^{4t} + 4Be^{-t} \, dt =$$
$$= -\frac{e^t}{3} + Ae^{4t} - 4Be^{-t}$$

Therefore:

$$ye^t = -\frac{e^t}{3} + Ae^{4t} - 4Be^{-t}$$
$$y = -\tfrac{1}{3} + Ae^{3t} - 4Be^{-2t}$$

Now we use the initial conditions to determine the coefficients: we have that $-\frac{1}{3} + A + B = 20$ and $-\frac{1}{3} + A - 4B = 60$, and so $B = -8$ and $A = \frac{85}{3}$.

Hence $x = -\frac{1}{3} + \frac{85}{3}e^{3t} - 8e^{-2t}$, $y = -\frac{1}{3} + \frac{85}{3}e^{3t} + 32e^{-2t}$

c Since both expressions feature the term e^{3t}, for large values of t the amount of gas should go to infinity in both tanks, which cannot happen.

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Challenge

1 Let $n = 1$

The result $\mathrm{M}^n = \mathrm{M}$ becomes $\mathrm{M}^1 = \mathrm{M}$, which is true.

Assume the result is true for $n = k$.

You can prove this result using mathematical induction, a method of proof you learnt in the FP1 module. The prerequisites in the FP3 specification state that a knowledge of FP1 is assumed and may be tested.

That is

$$\mathrm{M}^k = \mathrm{M} = \begin{pmatrix} \cosh^2 x & \cosh^2 x \\ -\sinh^2 x & -\sinh^2 x \end{pmatrix}$$

$$\mathrm{M}^{k+1} = \mathrm{M}^k\mathrm{M}$$

$$= \begin{pmatrix} \cosh^2 x & \cosh^2 x \\ -\sinh^2 x & -\sinh^2 x \end{pmatrix} \cdot \begin{pmatrix} \cosh^2 x & \cosh^2 x \\ -\sinh^2 x & -\sinh^2 x \end{pmatrix}$$

$$= \begin{pmatrix} \cosh^4 x - \cosh^2 x \sinh^2 & x\cosh^4 x - \cosh^2 x \sinh^2 x \\ -\sinh^2 x \cosh^2 x + \sinh^4 x - \sinh^2 x \cosh^2 + \sinh^4 x \end{pmatrix}$$

You use the identity $\cosh^2 x - \sinh^2 x = 1$ to simplify the terms in the matrix.

$$\cosh^4 x - \cosh^2 x \sinh^2 x = \cosh^2 x\left(\cosh^2 x - \sinh^2 x\right)$$

$$= \cosh^2 x$$

$$-\sinh^2 x \cosh^2 x + \sinh^4 x = \sinh^2 x\left(-\cosh^2 x + \sinh^2 x\right)$$

$$= -\sinh^2 x$$

Hence $\mathrm{M}^{k+1} = \begin{pmatrix} \cosh^2 x & \cosh^2 x \\ -\sinh^2 x & -\sinh^2 x \end{pmatrix}$

and this is the result for $n = k + 1$.

The result is true for $n = 1$, and, if it is true for $n = k$, then it is true for $n = k + 1$.
By mathematical induction the result is true for all positive integers n.

2 a Turn one of these into a second order differential equation by noting that $x = \dfrac{\mathrm{d}y}{\mathrm{d}t} = \dfrac{\mathrm{d}}{\mathrm{d}t}\left(\dfrac{\mathrm{d}x}{\mathrm{d}t}\right) = \dfrac{\mathrm{d}^2x}{\mathrm{d}t^2}$.

Of course this is solved by $x = Ae^t + Be^{-t}$. Now we use this to turn $\dfrac{\mathrm{d}y}{\mathrm{d}t} = x$ into $\dfrac{\mathrm{d}y}{\mathrm{d}t} = Ae^t + Be^{-t}$, an equation that can be easily solved finding $y = Ae^t - Be^{-t}$. With the given initial conditions we find that x and y must satisfy $A - B = 0$ and $A + B = 1$, so $A = B = \frac{1}{2}$. Substituting this into the expressions for x and y yields $x = \cosh t$ and $y = \sinh t$.

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Challenge

2 b Turn one of these into a second order differential equation by noting that q satisfies the equation $\frac{\mathrm{d}^2q}{\mathrm{d}t^2}-2\frac{\mathrm{d}q}{\mathrm{d}t}+2q=0$, which is solved by $q=e^t\left(A\cos t+B\sin t\right)$. Now consider the equation $\frac{\mathrm{d}p}{\mathrm{d}t}=p-q$, which is equivalent to $\frac{\mathrm{d}}{\mathrm{d}t}pe^{-t}=-qe^{-t}$; with the expression for q that we found this can be easily rewritten as $\frac{\mathrm{d}}{\mathrm{d}t}pe^{-t}=-\left(A\cos t+B\sin t\right)$, and therefore $pe^{-t}=-\left(A\sin t-B\cos t\right)$. Then an expression for p is $p=-e^t\left(A\sin t-B\cos t\right)$. Then it is easy to solve the equation for r:

$$\frac{\mathrm{d}r}{\mathrm{d}t}=p+2q+r$$
$$\frac{\mathrm{d}r}{\mathrm{d}t}-r=p+2q$$
$$\frac{\mathrm{d}}{\mathrm{d}t}re^{-t}=\left(p+2q\right)e^{-t}$$
$$re^{-t}=\int 2A\cos t+2B\sin t-A\sin t+B\cos t\ \mathrm{d}t$$
$$re^{-t}=A\left(2\sin t+\cos t\right)+B\left(\sin t-2\cos t\right)+C$$
$$r=Ae^t\left(2\sin t+\cos t\right)+Be^t\left(\sin t-2\cos t\right)+Ce^t$$

Now we can use the initial conditions to find the coefficients: from the expression for q we get $A=1$, from the expression for p we get $B=1$ and from the expression for r we get $C=2$. Then the particular solution for r satisfying the given conditions is $r=e^t\left(3\sin t-\cos t+2\right)$.

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Challenge

3 The gradient of the line l is clearly the tangent of the angle $\alpha+\theta$. This gradient can be expressed as follows: since the Cartesian coordinates of P are $(r\cos\theta, r\sin\theta)$, the gradient is $\dfrac{\frac{\mathrm{d}r}{\mathrm{d}\theta}\sin\theta + r\cos\theta}{\frac{\mathrm{d}r}{\mathrm{d}\theta}\cos\theta - r\sin\theta}$.

Now obviously $\alpha = \alpha+\theta-\theta$, so $\tan\alpha = \tan\big((\alpha+\theta)-\theta\big)$. By a formula, this can be solved as follows:

$$\tan\big((\alpha+\theta)-\theta\big) = \frac{\tan(\alpha+\theta)-\tan\theta}{1+\tan(\alpha+\theta)\tan\theta} =$$

$$= \frac{\dfrac{\frac{\mathrm{d}r}{\mathrm{d}\theta}\sin\theta + r\cos\theta}{\frac{\mathrm{d}r}{\mathrm{d}\theta}\cos\theta - r\sin\theta} - \tan\theta}{1+\dfrac{\frac{\mathrm{d}r}{\mathrm{d}\theta}\sin\theta + r\cos\theta}{\frac{\mathrm{d}r}{\mathrm{d}\theta}\cos\theta - r\sin\theta}\tan\theta} =$$

$$= \frac{\frac{\mathrm{d}r}{\mathrm{d}\theta}\sin\theta\cos\theta + r\cos^2\theta - \frac{\mathrm{d}r}{\mathrm{d}\theta}\cos\theta\sin\theta + r\sin^2\theta}{\frac{\mathrm{d}r}{\mathrm{d}\theta}\cos^2\theta - r\sin\theta\cos\theta + \frac{\mathrm{d}r}{\mathrm{d}\theta}\sin^2\theta + r\cos\theta\sin\theta} =$$

$$= \frac{r\cos^2\theta + r\sin^2\theta}{\frac{\mathrm{d}r}{\mathrm{d}\theta}\cos^2\theta + \frac{\mathrm{d}r}{\mathrm{d}\theta}\sin^2\theta} = \frac{r}{\frac{\mathrm{d}r}{\mathrm{d}\theta}}$$

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