Mixed Exercise

1 a $\overrightarrow{AB} = \begin{pmatrix}1\\2\\2\end{pmatrix} - \begin{pmatrix}1\\-1\\3\end{pmatrix} = \begin{pmatrix}0\\3\\-1\end{pmatrix}$

$\therefore \mathbf{r} = \begin{pmatrix}1\\-1\\3\end{pmatrix} + \lambda\begin{pmatrix}0\\3\\-1\end{pmatrix} = \mathbf{i} - \mathbf{j} + 3\mathbf{k} + \lambda(3\mathbf{j} - \mathbf{k})$

b $\mathbf{c} = \overrightarrow{OA} + \frac{2}{3}\overrightarrow{AB}$

$= \begin{pmatrix}1\\-1\\3\end{pmatrix} + \frac{2}{3}\begin{pmatrix}0\\3\\-1\end{pmatrix} = \begin{pmatrix}1\\1\\\frac{7}{3}\end{pmatrix}$

$= \mathbf{i} + \mathbf{j} + \frac{7}{3}\mathbf{k}$

2 $A(7,-1,2)$ and $B(-1,3,8)$

$\overrightarrow{AB} = \begin{pmatrix}-8\\4\\6\end{pmatrix} = -2\begin{pmatrix}4\\-2\\-3\end{pmatrix}$

$\therefore$ **A** vector equation is

$\mathbf{r} = \begin{pmatrix}7\\-1\\2\end{pmatrix} + \lambda\begin{pmatrix}4\\-2\\-3\end{pmatrix}$

$\therefore \frac{x-7}{4} = \frac{y+1}{-2} = \frac{z-2}{-3}$

3 $(2\mathbf{i} + 3\mathbf{j} - 4\mathbf{k}) + \lambda(2\mathbf{j} + 3\mathbf{k})$

4 a $\frac{x-3}{2} = \frac{y+2}{1} = \frac{z-1}{-1}$

b $\frac{0-3}{2} = \frac{a+2}{1} = \frac{b-1}{-1}$

$\Rightarrow a = -\frac{7}{2},\ b = \frac{5}{2}$

© Pearson Education Ltd 2017. Copying permitted for purchasing institution only. This material is not copyright free.

5 Use the equation $\mathbf{r} = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ 1 \\ -2 \end{pmatrix}$

When $\lambda = 2$, $\mathbf{r} = \begin{pmatrix} 7 \\ 4 \\ -5 \end{pmatrix}$, so the point $(7, 4, -5)$ lies on l

$\begin{pmatrix} 9 \\ 3 \\ -6 \end{pmatrix} = 3\begin{pmatrix} 3 \\ 1 \\ -2 \end{pmatrix}$ so these two vectors are parallel.

So an alternative form of the equation is $\mathbf{r} = \begin{pmatrix} 7 \\ 4 \\ -5 \end{pmatrix} + \lambda \begin{pmatrix} 9 \\ 3 \\ -6 \end{pmatrix}$

6 a $\mathbf{r} = \begin{pmatrix} -2 \\ 2 \\ -3 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 3 \\ 4 \end{pmatrix}$

b Set $\lambda = 2$

7 a $\overrightarrow{AB} = \begin{pmatrix} -1 \\ 4 \\ -3 \end{pmatrix}$, $\overrightarrow{AC} = \begin{pmatrix} 2 \\ 3 \\ 3 \end{pmatrix}$

$\therefore \mathbf{r} = \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 4 \\ -3 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ 3 \\ 3 \end{pmatrix}$

b $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 4 \\ -3 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ 3 \\ 3 \end{pmatrix}$

$x - 2 = -\lambda + 2\mu$ (1)

$y + 1 = 4\lambda + 3\mu$ (2)

$z - 2 = -3\lambda + 3\mu$ (3)

$(3) - (2)$: $z - y - 3 = -7\lambda \Rightarrow \lambda = \dfrac{y - z + 3}{7}$

$\Rightarrow \mu = \dfrac{3y + 4z - 5}{21}$

Into (1):

$x - 2 = -\dfrac{y + 3 - z}{7} + 2\left(\dfrac{3y + 4z - 5}{21}\right)$

$\Rightarrow 21(x - 2) = -3(y + 3 - z) + 2(3y + 4z - 5)$

$\Rightarrow 21x - 3y - 11z = 23$

© Pearson Education Ltd 2017. Copying permitted for purchasing institution only. This material is not copyright free.

8 Three points on the plane are $A(6,0,0)$, $B(8,-3,0)$ and $C(10,0,3)$

$$\overrightarrow{AB} = \begin{pmatrix} 2 \\ -3 \\ 0 \end{pmatrix}, \ \overrightarrow{AC} = \begin{pmatrix} 4 \\ 0 \\ 3 \end{pmatrix}$$

$$\therefore \mathbf{r} = \begin{pmatrix} 6 \\ 0 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -3 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} 4 \\ 0 \\ 3 \end{pmatrix}$$

9 a $\overrightarrow{ML} = \begin{pmatrix} 4 \\ 7 \\ 7 \end{pmatrix} - \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \\ 5 \end{pmatrix}$

$$\overrightarrow{MN} = \begin{pmatrix} 2 \\ 4 \\ 6 \end{pmatrix} - \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 4 \end{pmatrix}$$

b $\dfrac{\overrightarrow{ML}.\overrightarrow{MN}}{\left|\overrightarrow{ML}\right|\left|\overrightarrow{MN}\right|} = \dfrac{27}{5\sqrt{2}\ 3\sqrt{2}} = \dfrac{9}{10}$

10 a $\mathbf{a} = \begin{pmatrix} 9 \\ -2 \\ 1 \end{pmatrix}, \mathbf{b} = \begin{pmatrix} 6 \\ 2 \\ 6 \end{pmatrix}, \mathbf{c} = \begin{pmatrix} 3 \\ p \\ q \end{pmatrix}$

$$\mathbf{b} - \mathbf{a} = \begin{pmatrix} 6 \\ 2 \\ 6 \end{pmatrix} - \begin{pmatrix} 9 \\ -2 \\ 1 \end{pmatrix} = \begin{pmatrix} -3 \\ 4 \\ 5 \end{pmatrix}$$

Equation of l:

$$\mathbf{r} = \begin{pmatrix} 9 \\ -2 \\ 1 \end{pmatrix} + t \begin{pmatrix} -3 \\ 4 \\ 5 \end{pmatrix}$$

b Since C lies on l,

$$\begin{pmatrix} 3 \\ p \\ q \end{pmatrix} = \begin{pmatrix} 9 \\ -2 \\ 1 \end{pmatrix} + t \begin{pmatrix} -3 \\ 4 \\ 5 \end{pmatrix}$$

$3 = 9 - 3t$

$3t = 6$

$t = 2$

So $p = -2 + 4t = 6$ and $q = 1 + 5t = 11$

© Pearson Education Ltd 2017. Copying permitted for purchasing institution only. This material is not copyright free.

10 c $\cos\theta = \dfrac{\mathbf{OC}.\,\mathbf{AB}}{|\mathbf{OC}||\mathbf{AB}|}$

$\mathbf{OC}.\ \mathbf{AB} = \begin{pmatrix} 3 \\ 6 \\ 11 \end{pmatrix} \cdot \begin{pmatrix} -3 \\ 4 \\ 5 \end{pmatrix} = -9 + 24 + 55 = 70$

$|\mathbf{OC}| = \sqrt{3^2 + 6^2 + 11^2} = \sqrt{166}$

$|\mathbf{AB}| = \sqrt{(-3)^2 + 4^2 + 5^2} = \sqrt{50}$

$\cos\theta = \dfrac{70}{\sqrt{166}\sqrt{50}}$

$\theta = 39.8^\circ$ (1 d.p.)

d If **OD** and **AB** are perpendicular, **d** . (**b** – a) = 0

Since **d** lies on **AB**, use $\mathbf{d} = \begin{pmatrix} 9-3t \\ -2+4t \\ 1+5t \end{pmatrix}$

$\begin{pmatrix} 9-3t \\ -2+4t \\ 1+5t \end{pmatrix} \cdot \begin{pmatrix} -3 \\ 4 \\ 5 \end{pmatrix} = 0$

$-3(9-3t) + 4(-2+4t) + 5(1+5t) = 0$

$-27 + 9t - 8 + 16t + 5 + 25t = 0$

$50t = 30$

$t = \dfrac{3}{5}$

$\mathbf{d} = \begin{pmatrix} 9 - \frac{9}{5} \\ -2 + \frac{12}{5} \\ 1+3 \end{pmatrix} = \frac{36}{5}\mathbf{i} + \frac{2}{5}\mathbf{j} + 4\mathbf{k}$

11 a $\overrightarrow{AB} = \begin{pmatrix} 5 \\ 0 \\ -3 \end{pmatrix} - \begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix} = \begin{pmatrix} 4 \\ -2 \\ 0 \end{pmatrix}$

$\therefore\ \mathbf{r} = \begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix} + \lambda \begin{pmatrix} 4 \\ -2 \\ 0 \end{pmatrix}$

b Set $\mu = -3$

© Pearson Education Ltd 2017. Copying permitted for purchasing institution only. This material is not copyright free.

11 c Let $\mathbf{a} = \begin{pmatrix} 4 \\ -2 \\ 0 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix}$

$\mathbf{a} \cdot \mathbf{b} = \begin{pmatrix} 4 \\ -2 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix} = 4 + 4 + 0 = 8$

$|\mathbf{a}| = \sqrt{4^2 + (-2)^2} = \sqrt{20}$

$|\mathbf{b}| = \sqrt{1^2 + (-2)^2 + 2^2} = 3$

$\therefore \cos\theta = \dfrac{8}{3\sqrt{20}}$

$\Rightarrow \theta = 53.4^o$ (1.d.p.)

d Let G be a general point on l_1

$\overrightarrow{\mathbf{CG}} = \begin{pmatrix} 1+4\lambda \\ -2-2\lambda \\ 2 \end{pmatrix}$

When **C** is closest to l_1, $\overrightarrow{\mathbf{CG}}$ is perpendicular to l_1.

$\therefore \begin{pmatrix} 1+4\lambda \\ -2-2\lambda \\ 2 \end{pmatrix} \cdot \begin{pmatrix} 4 \\ -2 \\ 0 \end{pmatrix} = 0$

$\Rightarrow \lambda = -\dfrac{2}{5}$

$\therefore \overrightarrow{\mathbf{CG}} = \begin{pmatrix} -\frac{3}{5} \\ -\frac{6}{5} \\ 2 \end{pmatrix}$

$\Rightarrow |\overrightarrow{\mathbf{CG}}| = \sqrt{\left(-\frac{3}{5}\right)^2 + \left(-\frac{6}{5}\right)^2 + 2^2} = \dfrac{\sqrt{145}}{5}$

© Pearson Education Ltd 2017. Copying permitted for purchasing institution only. This material is not copyright free.

12 a Line l_1: $\mathbf{r} = \begin{pmatrix} 3 \\ 4 \\ -5 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix}$

Line l_2: $\mathbf{r} = \begin{pmatrix} 9 \\ 1 \\ -2 \end{pmatrix} + \mu \begin{pmatrix} 4 \\ 1 \\ -1 \end{pmatrix}$

Using the direction vectors:

$\begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} 4 \\ 1 \\ -1 \end{pmatrix} = 4 - 2 - 2 = 0$

Since the scalar product is zero, the directions are perpendicular.

b At an intersection point: $\begin{pmatrix} 3+\lambda \\ 4-2\lambda \\ -5+2\lambda \end{pmatrix} = \begin{pmatrix} 9+4\mu \\ 1+\mu \\ -2-\mu \end{pmatrix}$

$3+\lambda = 9+4\mu$ $(\times 2)$

$4-2\lambda = 1+\mu$

$6+2\lambda = 18+8\mu$

$4-2\lambda = 1+\mu$

Adding: $10 = 19+9\mu$

$\Rightarrow 9\mu = -9$

$\Rightarrow \mu = -1$

$3+\lambda = 9-4$

$\Rightarrow \lambda = 2$

Intersection point: $\begin{pmatrix} 3+\lambda \\ 4-2\lambda \\ -5+2\lambda \end{pmatrix} = \begin{pmatrix} 5 \\ 0 \\ -1 \end{pmatrix}$

Position vector of **A** is a = 5**i** – **k**.

c Position vector of **B**: $\mathbf{b} = 10\mathbf{j} - 11\mathbf{k} = \begin{pmatrix} 0 \\ 10 \\ -11 \end{pmatrix}$

For l_1, to give zero as the x component, $\lambda = -3$.

$\mathbf{r} = \begin{pmatrix} 3 \\ 4 \\ -5 \end{pmatrix} - 3\begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 10 \\ -11 \end{pmatrix}$

So **B** lies on l_1.

For l_2, to give – 11 as the z component, $\mu = 9$.

$\mathbf{r} = \begin{pmatrix} 9 \\ 1 \\ -2 \end{pmatrix} + 9\begin{pmatrix} 4 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 45 \\ 10 \\ -11 \end{pmatrix}$

So **B** does not lie on l_2.

So Only one of the submarines passes through **B**.

© Pearson Education Ltd 2017. Copying permitted for purchasing institution only. This material is not copyright free.

d $|\mathbf{AB}| = \sqrt{(0-5)^2 + (10-0)^2 + [-11-(-1)]^2}$

$= \sqrt{(-5)^2 + 10^2 + (-10)^2}$

$= \sqrt{225} = 15$

Since 1 unit represents 100 m, the distance **AB** is $15 \times 100 = 1500$ m $= 1.5$ km.

13 Use the formula $\left|\dfrac{(\mathbf{a}-\mathbf{c})\cdot(\mathbf{b}\times\mathbf{d})}{|\mathbf{b}\times\mathbf{d}|}\right|$

Write the first equation in the form $\mathbf{r} = 3\mathbf{i} - \mathbf{k} + s\mathbf{j}$

with $\mathbf{a} = 3\mathbf{i} - \mathbf{k}, \mathbf{b} = \mathbf{j}, \mathbf{c} = 9\mathbf{i} - 2\mathbf{j} - \mathbf{k}$ and $\mathbf{d} = \mathbf{i} - 2\mathbf{j} + \mathbf{k}$.

Then $\mathbf{a} - \mathbf{c} = -6\mathbf{i} + 2\mathbf{j}$

and $\mathbf{b} \times \mathbf{d} = \mathbf{i} - \mathbf{k}$

$\therefore$ The shortest distance is $\left|\dfrac{(-6\mathbf{i} + 2\mathbf{j})\cdot(\mathbf{i} - \mathbf{k})}{\sqrt{1^2 + (-1)^2}}\right|$

$= \left|\dfrac{-6}{\sqrt{2}}\right|$

$= 3\sqrt{2}$ or 4.24

14 Use the formula $\left|\dfrac{(\mathbf{a}-\mathbf{c})\cdot(\mathbf{b}\times\mathbf{d})}{|\mathbf{b}\times\mathbf{d}|}\right|$

with $\mathbf{a} = -3\mathbf{i} + \mathbf{k}, \mathbf{b} = 3\mathbf{i} - \mathbf{j} + \mathbf{k}, \mathbf{c} = 3\mathbf{i} - 2\mathbf{j} + \mathbf{k}$ and $\mathbf{d} = \mathbf{i} + 2\mathbf{j}$

Then $\mathbf{a} - \mathbf{c} = -6\mathbf{i} + 2\mathbf{j}$ and $\mathbf{b} \times \mathbf{d} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & -1 & 1 \\ 1 & 2 & 0 \end{vmatrix}$

$= -2\mathbf{i} + \mathbf{j} + 7\mathbf{k}$

So shortest distance $= \left|\dfrac{(-6\mathbf{i} + 2\mathbf{j})\cdot(-2\mathbf{i} + \mathbf{j} + 7\mathbf{k})}{\sqrt{(-2)^2 + 1^2 + 7^2}}\right|$

$= \dfrac{12 + 2}{\sqrt{54}}$

$= \dfrac{14}{\sqrt{54}}$

$= \dfrac{14}{3\sqrt{6}}$

$= \dfrac{14\sqrt{6}}{18}$

$= \dfrac{7\sqrt{6}}{9}$ or 1.91 (3 s.f.)

© Pearson Education Ltd 2017. Copying permitted for purchasing institution only. This material is not copyright free.

15 a The line has direction $2\mathbf{i} - \mathbf{k}$, and this is a direction in the plane.
Another vector in the plane is $4\mathbf{i} + 3\mathbf{j} + \mathbf{k} - (1 + \mathbf{j} - 2\mathbf{k})$
i.e. $3\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$
The normal to the plane is in direction
$(2\mathbf{i} - \mathbf{k}) \times (3\mathbf{i} + 2\mathbf{j} + 3\mathbf{k})$
i.e. $\mathbf{n} = 2\mathbf{i} - 9\mathbf{j} + 4\mathbf{k}$
$\therefore$ The Plane has equation
$\mathbf{r} \cdot (2\mathbf{i} - 9\mathbf{j} + 4\mathbf{k}) = (4\mathbf{i} + 3\mathbf{j} + \mathbf{k}) \cdot (2\mathbf{i} - 9\mathbf{j} + 4\mathbf{k})$
i.e. $\mathbf{r} \cdot (2\mathbf{i} - 9\mathbf{j} + 4\mathbf{k}) = 8 - 27 + 4$
i.e. $\mathbf{r} \cdot (2\mathbf{i} - 9\mathbf{j} + 4\mathbf{k}) = -15$

The equation of the line includes the position vector of another point on the plane and includes a direction vector in the plane.

b The line has direction $(2\mathbf{i} + \mathbf{j} - 3\mathbf{k})$
Another vector in the plane is $3\mathbf{i} + 5\mathbf{j} + \mathbf{k} - (\mathbf{i} + 2\mathbf{j} + 2\mathbf{k})$
i.e. $(2\mathbf{i} + 3\mathbf{j} - \mathbf{k})$

$\therefore$ the normal to the plane is $\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 1 & -3 \\ 2 & 3 & -1 \end{vmatrix} = 8\mathbf{i} - 4\mathbf{j} + 4\mathbf{k}$

$\therefore$ Equation of the plane is

$$\begin{aligned} \mathbf{r} \cdot (8\mathbf{i} - 4\mathbf{j} + 4\mathbf{k}) &= (3\mathbf{i} + 5\mathbf{j} + \mathbf{k}) \cdot (8\mathbf{i} - 4\mathbf{j} + 4\mathbf{k}) \\ &= 24 - 20 + 4 \\ &= 8 \end{aligned}$$

i.e. $\mathbf{r} \cdot (2\mathbf{i} - \mathbf{j} + \mathbf{k}) = 2$

c $7\mathbf{i} + 8\mathbf{j} + 6\mathbf{k}$ is the position vector of a point on the plane.
$2\mathbf{i} - \mathbf{j} + \mathbf{k}$ is the position vector of another point on the plane.
The vector joining these points is $5\mathbf{i} + 9\mathbf{j} + 5\mathbf{k}$
This lies in the plane.
A second vector which lies in the plane is $\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$.

The normal to the plane $\mathbf{n} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 5 & 9 & 5 \\ 1 & 2 & 2 \end{vmatrix}$

i.e. $\mathbf{n} = 8\mathbf{i} - 5\mathbf{j} + \mathbf{k}$
$\therefore$ The equation of the plane is

$$\begin{aligned} \mathbf{r} \cdot (8\mathbf{i} - 5\mathbf{j} + \mathbf{k}) &= (7\mathbf{i} + 8\mathbf{j} + 6\mathbf{k}) \cdot (8\mathbf{i} - 5\mathbf{j} + \mathbf{k}) \\ &= 56 - 40 + 6 \end{aligned}$$

$\therefore \mathbf{r} \cdot (8\mathbf{i} - 5\mathbf{j} + \mathbf{k}) = 22$

You need 2 directions in the plane. One is the direction of the line. The other is the vector joining the two points that are given in the plane i.e. (7, 8, 6) and (2, –1, 1).

© Pearson Education Ltd 2017. Copying permitted for purchasing institution only. This material is not copyright free.

16 The line is in the direction $3\mathbf{i}+\mathbf{j}+2\mathbf{k}$. This lies in the plane.

$(2, -4, 1)$ is a point on the line. This also lies in the plane, as does the point (1, 1, 1).

$\therefore \begin{pmatrix}2\\-4\\1\end{pmatrix}-\begin{pmatrix}1\\1\\1\end{pmatrix}-\begin{pmatrix}1\\-5\\0\end{pmatrix}$ is a direction in the plane.

First obtain the equation of the plane in the form, $\mathbf{r}\cdot\mathbf{n}=\mathbf{a}\cdot\mathbf{n}$, then convert to Cartesian form.

The normal to the plane $\mathbf{n}=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\1&-5&0\\3&1&2\end{vmatrix}$

$=-10\mathbf{i}-2\mathbf{j}+16\mathbf{k}$

$\therefore$ The equation of the plane is

$\mathbf{r}\cdot(-10\mathbf{i}-2\mathbf{j}+16\mathbf{k})=(\mathbf{i}+\mathbf{j}+\mathbf{k})\cdot(-10\mathbf{i}-2\mathbf{j}+16\mathbf{k})$

i.e: $-10x-2y+16z=4$

This is a Cartesian equation of the plane.

17 a $\overrightarrow{\mathbf{AB}}=(3\mathbf{i}+\mathbf{j}+4\mathbf{k})-(\mathbf{i}+3\mathbf{j}+3\mathbf{k})$

$=2\mathbf{i}-2\mathbf{j}+\mathbf{k}$

$\overrightarrow{\mathbf{AC}}=(2\mathbf{i}+4\mathbf{j}+\mathbf{k})-(\mathbf{i}+3\mathbf{j}+3\mathbf{k})$

$=\mathbf{i}+\mathbf{j}-2\mathbf{k}$

A vector normal to this plane **ABC** is in the direction $\overrightarrow{\mathbf{AB}}\times\overrightarrow{\mathbf{AC}}$

i.e. $\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\2&-2&1\\1&1&-2\end{vmatrix}=3\mathbf{i}+5\mathbf{j}+4\mathbf{k}$

A unit vector normal to the plane is $\dfrac{1}{\sqrt{3^2+5^2+4^2}}(3\mathbf{i}+5\mathbf{j}+4\mathbf{k})$

$=\dfrac{1}{\sqrt{50}}(3\mathbf{i}+5\mathbf{j}+4\mathbf{k})$

b The equation of the plane may be written as

$\mathbf{r}\cdot(3\mathbf{i}+5\mathbf{j}+4\mathbf{k})=(\mathbf{i}+3\mathbf{j}+3\mathbf{k})\cdot(3\mathbf{i}+5\mathbf{j}+4\mathbf{k})$

$=3+15+12$

$=30$

i.e. $3x+5y+4z=30$

c The perpendicular distance from the origin to the plane is

$\dfrac{30}{\sqrt{3^2+5^2+4^2}}=\dfrac{30}{\sqrt{50}}=\dfrac{30\sqrt{50}}{50}=3\sqrt{2}.$

© Pearson Education Ltd 2017. Copying permitted for purchasing institution only. This material is not copyright free.

18 a The plane with vector equation
$\mathbf{r} = \mathbf{i} + s\mathbf{j} + t(\mathbf{i} - \mathbf{k})$
is perpendicular to $\mathbf{i} + \mathbf{k}$, as $(\mathbf{i} + \mathbf{k}) \cdot \mathbf{j} = 0$ and $(\mathbf{i} + \mathbf{k}) \cdot (\mathbf{i} - \mathbf{k}) = 1 - 1 = 0$
The plane also has equation
$\mathbf{r} \cdot (\mathbf{i} + \mathbf{k}) - \mathbf{i} \cdot (\mathbf{i} + \mathbf{k})$, as $\mathbf{i}$ is the position vector of a point on the plane.
i.e. $\mathbf{r} \cdot (\mathbf{i} + \mathbf{k}) = 1$

The perpendicular distance from the origin to this plane is $\dfrac{1}{\sqrt{1^2 + 1^2}} = \dfrac{1}{\sqrt{2}} = \dfrac{\sqrt{2}}{2}$ or 0.707 (3 s.f.)

b The Cartesian form of the equation of the plane is $x + z = 1$.

19 a $\overrightarrow{\mathbf{AB}} = \overrightarrow{\mathbf{OB}} - \overrightarrow{\mathbf{OA}} = (5\mathbf{i} - 2\mathbf{j} + \mathbf{k}) - (\mathbf{i} + \mathbf{j} + \mathbf{k})$
$= 4\mathbf{i} - 3\mathbf{j}$
$\overrightarrow{\mathbf{AC}} = \overrightarrow{\mathbf{OC}} - \overrightarrow{\mathbf{OA}} = (3\mathbf{i} + 2\mathbf{j} + 6\mathbf{k}) - (\mathbf{i} + \mathbf{j} + \mathbf{k})$
$= 2\mathbf{i} + \mathbf{j} + 5\mathbf{k}$
Perpendicular vector to the plan is in direction
$$\overrightarrow{\mathbf{AB}} \times \overrightarrow{\mathbf{AC}} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 4 & -3 & 0 \\ 2 & 1 & 5 \end{vmatrix} = -15\mathbf{i} - 20\mathbf{j} + 10\mathbf{k}$$
Or a multiple of $3\mathbf{i} + 4\mathbf{j} - 2\mathbf{k}$

b The equation on of plane containing **A**, **B** and **C** is
$\mathbf{r} \cdot (-15\mathbf{i} - 20\mathbf{j} + 10\mathbf{k}) = (\mathbf{i} + \mathbf{j} + \mathbf{k}) \cdot (-15\mathbf{i} - 20\mathbf{j} + 10\mathbf{k})$
i.e. $-15x - 20y + 10z = -25$
or $3x + 4y - 2z - 5 = 0$

20 a $(2\mathbf{i} - 2\mathbf{j} + 3\mathbf{k}) \cdot (\mathbf{i} + 5\mathbf{j} + 3\mathbf{k}) = 2 - 10 + 9$
$= 1$
$\therefore$ (2, –2, 3) lies on the plane Π_2

b $(2\mathbf{i} - \mathbf{j} + \mathbf{k}) \cdot (\mathbf{i} + 5\mathbf{j} + 3\mathbf{k}) = 2 - 5 + 3$
$= 0$
$\therefore$ the normal to plane Π_1 is perpendicular to the normal to plane Π_2.
$\therefore$ Π_1 is perpendicular to Π_2.

c $\mathbf{r} = 2\mathbf{i} - 2\mathbf{j} + 3\mathbf{k} + \lambda(2\mathbf{i} - \mathbf{j} + \mathbf{k})$

d This line meets the plane Π_1 when
$\left[(2 + 2\lambda)\mathbf{i} + (-2 - \lambda)\mathbf{j} + (3 + \lambda)\mathbf{k}\right] \cdot (2\mathbf{i} - \mathbf{j} + \mathbf{k}) = 0$
i.e. $4 + 4\lambda + 2 + \lambda + 3 + \lambda = 0$
i.e. $6\lambda + 9 = 0$
$\therefore \lambda = -\dfrac{3}{2}$

Substitute $\therefore \lambda = -\dfrac{3}{2}$ into the equation of the line: then $\mathbf{r} = -\mathbf{i} - \dfrac{1}{2}\mathbf{j} + \dfrac{3}{2}\mathbf{k}$

i.e. The line meets Π_1 at the point $\left(-1, -\dfrac{1}{2}, \dfrac{3}{2}\right)$

© Pearson Education Ltd 2017. Copying permitted for purchasing institution only. This material is not copyright free.

20 e The distance required is

$$\sqrt{(2-(-1))^2+\left(-2-\left(-\frac{1}{2}\right)\right)^2+\left(3-\frac{3}{2}\right)^2}=\sqrt{9+2\frac{1}{4}+2\frac{1}{4}}=\sqrt{13\frac{1}{2}}$$

$$=3.67 \text{ (3 s.f.)}$$

21 a $l_1: \mathbf{r}=\begin{pmatrix}1\\1\\0\end{pmatrix}+\lambda\begin{pmatrix}2\\1\\-2\end{pmatrix}=\begin{pmatrix}1+2\lambda\\1+\lambda\\-2\lambda\end{pmatrix}$ and $l_2: \mathbf{r}=\begin{pmatrix}1\\4\\-4\end{pmatrix}+\mu\begin{pmatrix}-3\\0\\1\end{pmatrix}=\begin{pmatrix}1-3\mu\\4\\-4+\mu\end{pmatrix}$

If l_1 and l_2 intersect, then $\begin{pmatrix}1+2\lambda\\1+\lambda\\-2\lambda\end{pmatrix}=\begin{pmatrix}1-3\mu\\4\\-4+\mu\end{pmatrix}$

$1+2\lambda=1-3\mu$ **(1)**

$1+\lambda=4$ **(2)**

Equation **(2)** gives $\lambda=3$

Substituting into **(1)** gives $7=1-3\mu$, so $\mu=-2$

Check for consistency: $-2\lambda=-6$ and $-4+\mu=-6$

$2+4\lambda=-5+2\mu$, so these equations are consistent.

Therefore l_1 and l_2 intersect.

b Substitute $\lambda=3$ into line l_1, so $\mathbf{r}=\begin{pmatrix}1\\1\\0\end{pmatrix}+3\begin{pmatrix}2\\1\\-2\end{pmatrix}=\begin{pmatrix}7\\4\\-6\end{pmatrix}$

Therefore the position vector of their point of intersection is $\begin{pmatrix}7\\4\\-6\end{pmatrix}$

c The cosine of the acute angle θ between the lines is the cosine of the acute angle between their respective direction vectors.

$$\begin{pmatrix}2\\1\\-2\end{pmatrix}.\begin{pmatrix}-3\\0\\1\end{pmatrix}=(2\times-3)+(1\times0)+(-2\times1)=-8$$

$$\left|\begin{pmatrix}2\\1\\-2\end{pmatrix}\right|=\sqrt{2^2+1^2+(-2)^2}=\sqrt{9}=3 \text{ and } \left|\begin{pmatrix}-3\\0\\1\end{pmatrix}\right|=\sqrt{(-3)^2+0^2+1^2}=\sqrt{10}$$

So $\cos\theta=\left|\frac{-8}{3\sqrt{10}}\right|=\frac{8}{3\sqrt{10}}$

Therefore $\cos\theta=\frac{8}{3\sqrt{10}}\times\frac{\sqrt{10}}{\sqrt{10}}=\frac{8\sqrt{10}}{30}=\frac{4\sqrt{10}}{15}$, as required.

© Pearson Education Ltd 2017. Copying permitted for purchasing institution only. This material is not copyright free.

22 a $\begin{pmatrix}6\\8\\5\end{pmatrix}+\lambda\begin{pmatrix}1\\-1\\1\end{pmatrix}=\begin{pmatrix}3\\a\\2\end{pmatrix}$

$\lambda=-3 \Rightarrow a=11$

$\begin{pmatrix}6\\8\\5\end{pmatrix}+\lambda\begin{pmatrix}1\\-1\\1\end{pmatrix}=\begin{pmatrix}8\\6\\b\end{pmatrix}$

$\lambda=2 \Rightarrow b=7$

b $\left(\left(\begin{pmatrix}6\\8\\5\end{pmatrix}+\lambda\begin{pmatrix}1\\-1\\1\end{pmatrix}\right)\cdot\begin{pmatrix}1\\-1\\1\end{pmatrix}=0\right.$

$\Rightarrow\begin{pmatrix}6\\8\\5\end{pmatrix}\cdot\begin{pmatrix}1\\-1\\1\end{pmatrix}+\lambda\begin{pmatrix}1\\-1\\1\end{pmatrix}\cdot\begin{pmatrix}1\\-1\\1\end{pmatrix}=0$

$\Rightarrow 6-8+5+\lambda(1+1+1)=0$

$\Rightarrow \lambda=-1$

$\therefore$ Position vector of **P** is $\begin{pmatrix}6\\8\\5\end{pmatrix}-\begin{pmatrix}1\\-1\\1\end{pmatrix}=\begin{pmatrix}5\\9\\4\end{pmatrix}$

$\therefore \mathbf{P}(5,9,4)$

c $\sqrt{5^2+9^2+4^2}=\sqrt{122}$

23 a $\overrightarrow{\mathbf{AB}}=\begin{pmatrix}5\\2\\6\end{pmatrix}-\begin{pmatrix}6\\3\\4\end{pmatrix}=\begin{pmatrix}-1\\-1\\2\end{pmatrix}$

b $\mathbf{r}=\begin{pmatrix}6\\3\\4\end{pmatrix}+\lambda\begin{pmatrix}-1\\-1\\2\end{pmatrix}$

© Pearson Education Ltd 2017. Copying permitted for purchasing institution only. This material is not copyright free.

23 c $\overrightarrow{\mathbf{CP}} = \begin{pmatrix} 2-\lambda \\ -7-\lambda \\ 2+2\lambda \end{pmatrix}$

$\begin{pmatrix} 2-\lambda \\ -7-\lambda \\ 2+2\lambda \end{pmatrix} \cdot \begin{pmatrix} -1 \\ -1 \\ 2 \end{pmatrix} = 6\lambda + 9 = 0$

$\Rightarrow \lambda = -\frac{3}{2}$

$\therefore$ The position vector of **P** is

$\begin{pmatrix} 6 \\ 3 \\ 4 \end{pmatrix} - \frac{3}{2}\begin{pmatrix} -1 \\ -1 \\ 2 \end{pmatrix} = \begin{pmatrix} \frac{15}{2} \\ \frac{9}{2} \\ 1 \end{pmatrix}$

$\therefore \mathbf{P}\left(\frac{15}{2}, \frac{9}{2}, 1\right)$ or $\mathbf{P}(7.5, 4.5, 1)$

24 a $\begin{pmatrix} 3+2\lambda \\ -2+\lambda \\ 4-\lambda \end{pmatrix} = \begin{pmatrix} 1+\mu \\ 12-2\mu \\ 8-\mu \end{pmatrix}$

j and **k** components $\Rightarrow \lambda = 2,\ \mu = 6$

Check **i** component: $3+2(2) = 7 = 1+6$

So the equations are consistent

Therefore the lines meet, and they meet at

$\begin{pmatrix} 3+2(2) \\ -2+2 \\ 4-2 \end{pmatrix} = \begin{pmatrix} 7 \\ 0 \\ 2 \end{pmatrix} \quad \therefore A(7, 0, 2)$

b Let $\mathbf{a} = \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 1 \\ -2 \\ -1 \end{pmatrix}$

$\mathbf{a} \cdot \mathbf{b} = \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -2 \\ -1 \end{pmatrix} = 2 - 2 + 1 = 1$

$|\mathbf{a}| = \sqrt{2^2 + 1^2 + (-1)^2} = \sqrt{6}$

$|\mathbf{b}| = \sqrt{6}$

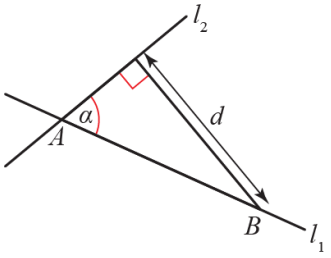
$\therefore \cos\theta = \frac{1}{6}$

$\Rightarrow \theta = 80.4^o$ (1 d.p.)

c Set $\lambda = 2$

© Pearson Education Ltd 2017. Copying permitted for purchasing institution only. This material is not copyright free.

24 d The shortest distance of **B** to the line l_2 is given by $\mathbf{d} = \left|\overrightarrow{\mathbf{BA}}\right| \sin\alpha$



$$\overrightarrow{\mathbf{OA}} = \begin{pmatrix} 7 \\ 0 \\ 2 \end{pmatrix} \text{ and } \overrightarrow{\mathbf{OB}} = \begin{pmatrix} 5 \\ -1 \\ 3 \end{pmatrix}$$

$$\overrightarrow{\mathbf{BA}} = \overrightarrow{\mathbf{OA}} - \overrightarrow{\mathbf{OB}} = \begin{pmatrix} 7 \\ 0 \\ 2 \end{pmatrix} - \begin{pmatrix} 5 \\ -1 \\ 3 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}$$

$$\left|\overrightarrow{\mathbf{BA}}\right| = \sqrt{2^2 + 1^2 + (-1)^2} = \sqrt{6}$$

So $\mathbf{d} = \sqrt{6}\sin 80.4^\circ = 2.42$

25 a $\overrightarrow{AP} = \mathbf{a} = \begin{pmatrix} 3 \\ 0 \\ -3 \end{pmatrix}$

$$\mathbf{n} = \begin{pmatrix} 2 \\ -2 \\ 3 \end{pmatrix}$$

$$\mathbf{a} \cdot \mathbf{n} = \begin{pmatrix} 3 \\ 0 \\ -3 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ -2 \\ 3 \end{pmatrix} = -3$$

$$|\mathbf{a}| = \sqrt{3^2 + (-3)^2} = \sqrt{18}$$

$$|\mathbf{n}| = \sqrt{2^2 + (-2)^2 + 3^2} = \sqrt{17}$$

$$\therefore \sin\theta = \frac{|-3|}{\sqrt{18}\sqrt{17}} = \frac{3}{\sqrt{18}\sqrt{17}}$$

$\Rightarrow \theta = 10^o$ to the nearest degree

b $\dfrac{|1(2) + 2(-2) + 2(3) - 1|}{\sqrt{2^2 + (-2)^2 + 3^2}} = \dfrac{3}{\sqrt{17}} = \dfrac{3\sqrt{17}}{17}$

© Pearson Education Ltd 2017. Copying permitted for purchasing institution only. This material is not copyright free.

26 a Let l_1 denote the path of the first aeroplane, and l_2 the second.

$$l_1: \mathbf{r} = \begin{pmatrix} 120 \\ -80 \\ 13 \end{pmatrix} + \lambda \begin{pmatrix} 80 \\ 100 \\ -8 \end{pmatrix}$$

$$l_2: \mathbf{r} = \begin{pmatrix} -20 \\ 35 \\ 5 \end{pmatrix} + \mu \begin{pmatrix} 10 \\ -2 \\ 0.1 \end{pmatrix}$$

$$\begin{pmatrix} 120+80\lambda \\ -80+100\lambda \\ 13-8\lambda \end{pmatrix} = \begin{pmatrix} -20+10\mu \\ 35-2\mu \\ 5+0.1\mu \end{pmatrix}$$

i and **k** components $\Rightarrow \lambda = \frac{3}{4},\ \mu = 20$

Check **j** component: $-80+100\left(\frac{3}{4}\right) = -5 = 35-2(20)$

So the equations are consistent

Therefore the paths intersect, and they intersect at

$$\begin{pmatrix} 120 \\ -80 \\ 13 \end{pmatrix} + \frac{3}{4}\begin{pmatrix} 80 \\ 100 \\ -8 \end{pmatrix} = \begin{pmatrix} 180 \\ -5 \\ 7 \end{pmatrix} \Rightarrow (180, -5, 7)$$

b The planes pass through the same point, but not necessarily at the same time.

Challenge

1 a $-2(-2x + y - 3z) = (-2) \times (-5)$ gives $4x - 2y + 6z = 10$

b matrix **A** is singular if det **A** = 0

$4(c + 3b) + 2(-2c + 3a) + 6(-2b - a) = 6a - 6a + 12b - 12b + 4c - 4c = 0$

c i $a = 2n$, $b = -n$, $c = 30$ where $n \in \mathbb{R}$, $n \neq 3$

ii $a = 6$, $b = -3$ and $c = 9$

© Pearson Education Ltd 2017. Copying permitted for purchasing institution only. This material is not copyright free.

Challenge

2 Coordinates $A(4,-4,5), B(0,4,1)$ and $C(0,0,5)$ lie on the circumference of the circle.

Let O be the centre of the circle.
Then *A*, *B*, *C* and *O* lie on the same plane.
Without loss of generality, suppose the plane has equation $ax+by+cz=1$

Then substituting point *A* gives: $4a-4b+5c=1$ **(1)**
substituting point *B* gives: $4b+c=1$ **(2)**
substituting point *C* gives: $5c=1$ **(3)**

From equation **(3)**, $c=\frac{1}{5}$

Substituting into **(2)** gives $4b+\frac{1}{5}=1$

$4b=\frac{4}{5}$

$b=\frac{1}{5}$

Substituting *b* and *c* in equation **(1)** gives $4a-\frac{4}{5}+\frac{5}{5}=1$

$4a=\frac{4}{5}$

$a=\frac{1}{5}$

So the equation of the plane is $\frac{x}{5}+\frac{y}{5}+\frac{z}{5}=1$

Or $x+y+z=5$ **(4)**

The centre of the circle lies on the intersection of any two perpendicular bisectors chosen from *AB*, *BC* or *AC*.
Each perpendicular bisector must also lie on the plane $x+y+z=5$
To find the equation of the perpendicular bisector of **AB**:

The mid-point of **AB** is $\mathbf{M}\left(\frac{4+0}{2},\frac{-4+4}{2},\frac{5+1}{2}\right)=\mathbf{M}(2,0,3)$

Therefore the equation of the line **MO** may be written as $\mathbf{r}=\begin{pmatrix}2\\0\\3\end{pmatrix}+\lambda\begin{pmatrix}\alpha\\\beta\\\gamma\end{pmatrix}$

$\overrightarrow{\mathbf{AB}}=\begin{pmatrix}-4\\8\\4\end{pmatrix}$ and $\overrightarrow{\mathbf{AB}}$ is perpendicular to the direction of **MO**, so $\overrightarrow{\mathbf{AB}}\cdot\begin{pmatrix}\alpha\\\beta\\\gamma\end{pmatrix}=0$

i.e. $\begin{pmatrix}-4\\8\\4\end{pmatrix}\cdot\begin{pmatrix}\alpha\\\beta\\\gamma\end{pmatrix}=0$, so $-4\alpha+8\beta-4\gamma=0$ **(5)**

The line $\mathbf{r}=\begin{pmatrix}2\\0\\3\end{pmatrix}+\lambda\begin{pmatrix}\alpha\\\beta\\\gamma\end{pmatrix}$ lies on the plane $x+y+z=5$ for all values of λ.

Choosing $\lambda=1$, you have $\mathbf{r}=\begin{pmatrix}2+\alpha\\\beta\\3+\gamma\end{pmatrix}$

© Pearson Education Ltd 2017. Copying permitted for purchasing institution only. This material is not copyright free.

Challenge
2 continued

Solving $\mathbf{r} = \begin{pmatrix} 2+\alpha \\ \beta \\ 3+\gamma \end{pmatrix}$ and $x+y+z=5$ simultaneously gives $(2+\alpha)+\beta+(3+\gamma)=5,$ or

$\alpha+\beta+\gamma=0$ **(6)**

Equation **(6)** is equivalent to $4\alpha+4\beta+4\gamma=0$

Adding this to equation **(5)** gives $12\beta=0,$ so $\beta=0$ and $\alpha=-\gamma$

Therefore $\mathbf{r} = \begin{pmatrix} 2+\alpha \\ \beta \\ 3+\gamma \end{pmatrix} = \begin{pmatrix} 2+\alpha \\ 0 \\ 3-\alpha \end{pmatrix}$

Now to find the equation of the perpendicular bisector of **BC**:

The mid-point of **BC** is $\mathbf{M}_1\left(\frac{0+0}{2}, \frac{4+0}{2}, \frac{1+5}{2}\right) = \mathbf{M}_1(0,2,3)$

Therefore the equation of the line $\mathbf{M}_1\mathbf{O}$ may be written as $\mathbf{r} = \begin{pmatrix} 0 \\ 2 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} \alpha_1 \\ \beta_1 \\ \gamma_1 \end{pmatrix}$

$\overrightarrow{\mathbf{BC}} = \begin{pmatrix} 0 \\ -4 \\ 4 \end{pmatrix}$ and $\overrightarrow{\mathbf{BC}}$ is perpendicular to the direction of $\mathbf{M}_1\mathbf{O}$, so $\overrightarrow{\mathbf{BC}} \cdot \begin{pmatrix} \alpha_1 \\ \beta_1 \\ \gamma_1 \end{pmatrix} = 0$

i.e. $\begin{pmatrix} 0 \\ -4 \\ 4 \end{pmatrix} \cdot \begin{pmatrix} \alpha_1 \\ \beta_1 \\ \gamma_1 \end{pmatrix} = 0,$ so $-4\beta_1+4\gamma_1=0,$ or $\beta_1=\gamma_1$ **(7)**

The line $\mathbf{r} = \begin{pmatrix} 0 \\ 2 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} \alpha_1 \\ \beta_1 \\ \gamma_1 \end{pmatrix}$ lies on the plane $x+y+z=5$ for all values of λ.

Choosing $\lambda=1,$ you have $\mathbf{r} = \begin{pmatrix} \alpha_1 \\ 2+\beta_1 \\ 3+\gamma_1 \end{pmatrix}$

Solving $\mathbf{r} = \begin{pmatrix} \alpha_1 \\ 2+\beta_1 \\ 3+\gamma_1 \end{pmatrix}$ and $x+y+z=5$ simultaneously gives $\alpha_1+(2+\beta_1)+(3+\gamma_1)=5,$ or

$\alpha_1+\beta_1+\gamma_1=0$ **(8)**

Solving equations **(7)** and **(8)** simultaneously gives $\alpha_1=-\beta_1-\gamma_1=-\gamma_1-\gamma_1=-2\gamma_1$

Therefore $\mathbf{r} = \begin{pmatrix} \alpha_1 \\ 2+\beta_1 \\ 3+\gamma_1 \end{pmatrix} = \begin{pmatrix} -2\gamma_1 \\ 2+\gamma_1 \\ 3+\gamma_1 \end{pmatrix}$

© Pearson Education Ltd 2017. Copying permitted for purchasing institution only. This material is not copyright free.

Challenge

2 continued

Now consider the equations of the two perpendicular bisectors:

$\mathbf{r} = \begin{pmatrix} 2+\alpha \\ 0 \\ 3-\alpha \end{pmatrix}$ and $\mathbf{r} = \begin{pmatrix} -2\gamma_1 \\ 2+\gamma_1 \\ 3+\gamma_1 \end{pmatrix}$

These meet at the centre of the circle, *O*.

Equating the **j** components gives $0 = 2+\gamma_1$, so $\gamma_1 = -2$

Therefore the two bisectors must meet at the position vector $\mathbf{r} = \begin{pmatrix} -2(-2) \\ 2+(-2) \\ 3+(-2) \end{pmatrix} = \begin{pmatrix} 4 \\ 0 \\ 1 \end{pmatrix}$

The centre of the circle is therefore $O(4,0,1)$

The radius is the distance between O and any of the points *A*, *B* or *C*

Consider the point $C(0,0,5)$

Then $OC = \sqrt{(0-4)^2+(0-0)^2+(5-1)^2} = \sqrt{32} = 4\sqrt{2}$

Therefore the circle has radius $4\sqrt{2}$

© Pearson Education Ltd 2017. Copying permitted for purchasing institution only. This material is not copyright free.