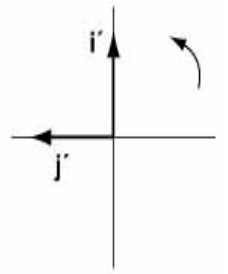
Mixed exercise 7

1 a



$$\mathbf{Y} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

b $\mathbf{AB} = \mathbf{Y} \quad \Rightarrow \quad \mathbf{A} = \mathbf{YB}^{-1}.$

$$\mathbf{B} = \begin{pmatrix} 3 & 2 \\ 2 & 1 \end{pmatrix} \Rightarrow \det \mathbf{B} = 3 - 4 = -1$$

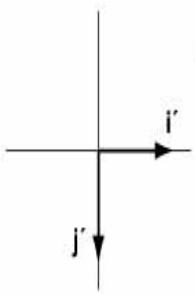
$$\therefore \mathbf{B}^{-1} = \frac{1}{-1}\begin{pmatrix} 1 & -2 \\ -2 & 3 \end{pmatrix} = \begin{pmatrix} -1 & 2 \\ 2 & -3 \end{pmatrix}$$

$$\therefore \mathbf{A} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} -1 & 2 \\ 2 & -3 \end{pmatrix}$$

$$= \begin{pmatrix} -2 & 3 \\ -1 & 2 \end{pmatrix}$$

c $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ (Identity matrix); four successive anticlockwise rotations of 90° about (0, 0).

2 Reflection in *x*-axis



$$\Rightarrow \mathbf{R} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Enlargement S. F. 2 centre (0, 0)

$$\Rightarrow \mathbf{E} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$$

a $\mathbf{C} = \mathbf{ER} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

$$= \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix}$$

Reflection in the *x*-axis and enlargement scale factor 2, centre (0, 0).

b $\mathbf{C}^{-1} = \frac{1}{-4}\begin{pmatrix} -2 & 0 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}$

Reflection in the *x*-axis and enlargement scale factor $\frac{1}{2}$. Centre (0, 0)

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3 a $\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}M = P \Rightarrow M = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}^{-1} P$

$\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}^{-1} = \frac{1}{-1}\begin{pmatrix}0 & -1\\ -1 & 0\end{pmatrix} = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$

$M = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}0 & k\\ k & 0\end{pmatrix} = \begin{pmatrix}k & 0\\ 0 & k\end{pmatrix}$

b $\begin{pmatrix}k & 0\\ 0 & k\end{pmatrix}\begin{pmatrix}-3\\ -2\end{pmatrix} = \begin{pmatrix}9\\ 6\end{pmatrix} \Rightarrow -3k = 9$

$\Rightarrow k = -3$

4 a $\det \mathbf{M} = \left(2\sqrt{3}\right)^2 + 4 = 16$

$\Rightarrow$ scale factor = 4

b $\mathbf{M} = \begin{pmatrix}4 & 0\\ 0 & 4\end{pmatrix}R$

$\Rightarrow R = \begin{pmatrix}\frac{1}{4} & 0\\ 0 & \frac{1}{4}\end{pmatrix}\begin{pmatrix}2\sqrt{3} & -2\\ 2 & 2\sqrt{3}\end{pmatrix} = \begin{pmatrix}\frac{\sqrt{3}}{2} & -\frac{1}{2}\\ \frac{1}{2} & \frac{\sqrt{3}}{2}\end{pmatrix}$

$= \begin{pmatrix}\cos 30^\circ & -\sin 30^\circ\\ \sin 30^\circ & \cos 30^\circ\end{pmatrix}$

So angle is 30° anti-clockwise.

c $\mathbf{M}^{-1} = \frac{1}{16}\begin{pmatrix}2\sqrt{3} & 2\\ -2 & 2\sqrt{3}\end{pmatrix}$

$\mathbf{M}\begin{pmatrix}x\\ y\end{pmatrix} = \begin{pmatrix}a\\ b\end{pmatrix} \Rightarrow \begin{pmatrix}x\\ y\end{pmatrix} = \mathbf{M}^{-1}\begin{pmatrix}a\\ b\end{pmatrix}$

$= \frac{1}{16}\begin{pmatrix}2\sqrt{3} & 2\\ -2 & 2\sqrt{3}\end{pmatrix}\begin{pmatrix}a\\ b\end{pmatrix} = \begin{pmatrix}\frac{\sqrt{3}a}{8} + \frac{b}{8}\\ -\frac{a}{8} + \frac{\sqrt{3}b}{8}\end{pmatrix}$

So coordinates of P are $\left(\frac{\sqrt{3}a}{8} + \frac{b}{8}, -\frac{a}{8} + \frac{\sqrt{3}b}{8}\right)$

5 a **A** represents a reflection in the line $y = x$;
B represents a rotation through 270° anticlockwise about (0, 0)

b $\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix} = \begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix}$

$\begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}p\\ q\end{pmatrix} = \begin{pmatrix}-p\\ q\end{pmatrix}$

So coordinates of image are $(-p, q)$

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6 Area of $\mathbf{T} = \frac{1}{2}|k-6| \times 5 = \frac{5}{2}|k-6|$

$\det \mathbf{M} = 8 - 3 = 5$

$\Rightarrow 5 \times \frac{5}{2}|k-6| = 110$

$\Rightarrow |k-6| = 8.8$

$\Rightarrow k = 14.8 \text{ or } -2.8$

7 a $P = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 4 & 0 \\ 0 & 3 \end{pmatrix} = \begin{pmatrix} -4 & 0 \\ 0 & 3 \end{pmatrix}$

b $\det \mathbf{P} = -12$

So area of $T = 60 \div 12 = 5$

8 a $\mathbf{A} = \begin{pmatrix} \cos 150^\circ & -\sin 150^\circ & 0 \\ \sin 150^\circ & \cos 150^\circ & 0 \\ 0 & 0 & 1 \end{pmatrix}$

So **A** represents a 150° anti-clockwise rotation about the z-axis.

b

$$\begin{pmatrix} -\frac{\sqrt{3}}{2} & -\frac{1}{2} & 0 \\ \frac{1}{2} & -\frac{\sqrt{3}}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} a \\ b \\ -a \end{pmatrix} = \begin{pmatrix} -\frac{a\sqrt{3}}{2} - \frac{b}{2} \\ \frac{a}{2} - \frac{b\sqrt{3}}{2} \\ -a \end{pmatrix}$$

So coordinates of image are $\left(-\frac{a\sqrt{3}}{2} - \frac{b}{2}, \frac{a}{2} - \frac{b\sqrt{3}}{2}, -a\right)$

9 a

$$\mathbf{P}^{-1} = \frac{1}{a^2}\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} = \begin{pmatrix} \frac{1}{a} & 0 \\ 0 & \frac{1}{a} \end{pmatrix}$$

b $\mathbf{P}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 4 \\ 7 \end{pmatrix}$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \mathbf{P}^{-1}\begin{pmatrix} 4 \\ 7 \end{pmatrix} = \begin{pmatrix} \frac{1}{a} & 0 \\ 0 & \frac{1}{a} \end{pmatrix}\begin{pmatrix} 4 \\ 7 \end{pmatrix} = \begin{pmatrix} \frac{4}{a} \\ \frac{7}{a} \end{pmatrix}$$

So coordinates of **A** are $\left(\frac{4}{a}, \frac{7}{a}\right)$

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10 The matrix representing a 90° rotation about the origin is $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\mathbf{P} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} -1 & 2 \\ -5 & 8 \end{pmatrix} = \begin{pmatrix} 5 & -8 \\ -1 & 2 \end{pmatrix}$$

$$\begin{pmatrix} 5 & -8 \\ -1 & 2 \end{pmatrix}^{-1} = \frac{1}{2}\begin{pmatrix} 2 & 8 \\ 1 & 5 \end{pmatrix} = \begin{pmatrix} 1 & 4 \\ \frac{1}{2} & \frac{5}{2} \end{pmatrix}$$

11 a $\mathbf{P} = \mathbf{AB} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 4 & -1 \\ 3 & -2 \end{pmatrix} = \begin{pmatrix} -4 & 1 \\ -3 & 2 \end{pmatrix}$

b $\det \mathbf{P} = -8 - -3 = -5$

So area of $T = 35 \div 5 = 7$

c

$$\mathbf{Q} = \mathbf{P}^{-1} = \frac{1}{-5}\begin{pmatrix} 2 & -1 \\ 3 & -4 \end{pmatrix} = -\begin{pmatrix} -\frac{2}{5} & \frac{1}{5} \\ -\frac{3}{5} & \frac{4}{5} \end{pmatrix}$$

12

$$\mathbf{M}^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ 0 & \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \end{pmatrix}$$

$$\mathbf{M}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \Rightarrow \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \mathbf{M}^{-1}\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & -\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ 0 & \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \end{pmatrix}\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ -\sqrt{2} \\ 0 \end{pmatrix}$$

$\Rightarrow a = 0, b = -\sqrt{2}, c = 0$

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13 a

$$\mathbf{A}^{-1} = \begin{pmatrix} -\frac{4}{3} & \frac{1}{6} & \frac{2}{3} \\ -\frac{7}{3} & \frac{1}{6} & \frac{2}{3} \\ \frac{2}{3} & \frac{1}{6} & -\frac{1}{3} \end{pmatrix}$$

$$\mathbf{A}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ 7 \\ 4 \end{pmatrix} \Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \mathbf{A}^{-1}\begin{pmatrix} 3 \\ 7 \\ 4 \end{pmatrix}$$

$$= \begin{pmatrix} -\frac{4}{3} & \frac{1}{6} & \frac{2}{3} \\ -\frac{7}{3} & \frac{1}{6} & \frac{2}{3} \\ \frac{2}{3} & \frac{1}{6} & -\frac{1}{3} \end{pmatrix}\begin{pmatrix} 3 \\ 7 \\ 4 \end{pmatrix} = \begin{pmatrix} -\frac{1}{6} \\ -\frac{19}{6} \\ \frac{11}{6} \end{pmatrix}$$

So coordinates of **P** are $\left(-\frac{1}{6}, -\frac{19}{6}, \frac{11}{6}\right)$

b $\mathbf{BA} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$

$$\Rightarrow \mathbf{B} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} -\frac{4}{3} & \frac{1}{6} & \frac{2}{3} \\ -\frac{7}{3} & \frac{1}{6} & \frac{2}{3} \\ \frac{2}{3} & \frac{1}{6} & -\frac{1}{3} \end{pmatrix}$$

$$= \begin{pmatrix} -\frac{4}{3} & \frac{1}{6} & \frac{2}{3} \\ -\frac{2}{3} & -\frac{1}{6} & \frac{1}{3} \\ -\frac{7}{3} & \frac{1}{6} & \frac{2}{3} \end{pmatrix} = \frac{1}{6}\begin{pmatrix} -8 & 1 & 4 \\ -4 & -1 & 2 \\ -14 & 1 & 4 \end{pmatrix}$$

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Challenge

1 $\mathbf{M} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$

2 a Let the point be $P = (a, b)$: $\begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0+b \\ 0+b \end{pmatrix} = \begin{pmatrix} b \\ b \end{pmatrix}$

So P' is (b, b); its x- and y-coordinates are equal, so it is on $y = x$

b $\begin{pmatrix} 0 & 1 \\ 0 & m \end{pmatrix}$

c If c does not equal 0, then the line $ax + by = c$ does not go through the origin. Hence the origin cannot be mapped to itself, and the transformation is not linear.

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