Exercise 9F

1 Let A be a general point on the first line and B be a general point on the second line,

then $\overrightarrow{AB} = \begin{pmatrix} -2 \\ +2 \\ 0 \end{pmatrix} + t\begin{pmatrix} -3 \\ -4 \\ 5 \end{pmatrix}$, where $t = \mu - \lambda$.

Find the minimum value of the quadratic by using calculus, or completion of the square.

Let the distance $AB = x$

then $x^2 = (-2-3t)^2 + (2-4t)^2 + (5t)^2 = 8 - 4t + 50t^2$

The minimum value of x^2 occurs when $t = \frac{1}{25}$

So $x^2 = 8 - \frac{4}{25} + \frac{50}{625}$

$= \frac{198}{25}$

$\therefore x = \frac{\sqrt{198}}{5}$ or 2.81 (3 s.f.)

2 Let the lines respectively be l_1 and l_2

Let A and B be general points on l_1 and l_2 respectively.

$\therefore \overrightarrow{AB} = \begin{pmatrix} 5+\lambda+4\mu \\ -3-\lambda-2\mu \\ 1+\lambda+3\mu \end{pmatrix}$

$\overrightarrow{AB}$ perpendicular to l_1

$\therefore \begin{pmatrix} 5+\lambda+4\mu \\ -3-\lambda-2\mu \\ 1+\lambda+3\mu \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix} = 0$

$\Rightarrow \lambda + 3\mu = -3$

$\overrightarrow{AB}$ perpendicular to l_2

$\begin{pmatrix} 5+\lambda+4\mu \\ -3-\lambda-2\mu \\ 1+\lambda+3\mu \end{pmatrix} \cdot \begin{pmatrix} 4 \\ -2 \\ 3 \end{pmatrix} = 0$

$\Rightarrow 9\lambda + 29\mu = -29$

Solving these simultaneous equations,

$\lambda = 0,\ \mu = -1$

$\therefore \overrightarrow{AB} = \begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix}$

$\Rightarrow |\overrightarrow{AB}| = \sqrt{1^2 + 1^2 + (-2)^2} = \sqrt{6}$

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3 a $\begin{pmatrix} 7+6\lambda \\ 3+2\lambda \\ 1-4\lambda \end{pmatrix} = \begin{pmatrix} -1 \\ 1-2\mu \\ 2 \end{pmatrix}$

i component $\Rightarrow \lambda = -\frac{4}{3}$

k component $\Rightarrow \lambda = -\frac{1}{4}$

So the equations aren't consistent

Therefore, l_1 and l_2 do not intersect

Let A and B be general points on l_1 and l_2 respectively.

$$\therefore \overrightarrow{AB} = \begin{pmatrix} -8-6\lambda \\ -2-2\lambda-2\mu \\ 1+4\lambda \end{pmatrix}$$

$\overrightarrow{AB}$ is perpendicular to l_1

$$\therefore \begin{pmatrix} -8-6\lambda \\ -2-2\lambda-2\mu \\ 1+4\lambda \end{pmatrix} \cdot \begin{pmatrix} 6 \\ 2 \\ -4 \end{pmatrix} = 0$$

$$\Rightarrow 14\lambda + \mu = -14$$

$\overrightarrow{AB}$ is perpendicular to l_2

$$\begin{pmatrix} -8-6\lambda \\ -2-2\lambda-2\mu \\ 1+4\lambda \end{pmatrix} \cdot \begin{pmatrix} 0 \\ -2 \\ 0 \end{pmatrix} = 0$$

$$\Rightarrow \lambda + \mu = -1$$

Solving these simultaneous equations,

$\lambda = -1,\ \mu = 0$

$$\therefore \overrightarrow{AB} = \begin{pmatrix} -2 \\ 0 \\ -3 \end{pmatrix}$$

$$\Rightarrow |\overrightarrow{AB}| = \sqrt{(-2)^2 + (-3)^2} = \sqrt{13}$$

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3 b Assume that l_1 and l_2 meet:

$$\begin{pmatrix} 2+2\lambda \\ 1-2\lambda \\ -2+2\lambda \end{pmatrix} = \begin{pmatrix} 1+\mu \\ -1-\mu \\ 3+\mu \end{pmatrix}$$

i.e. $2+2\lambda = 1+\mu$ **(1)**

$1-2\lambda = -1-\mu$ **(2)**

$-2+2\lambda = 3+\mu$ **(3)**

Adding equations **(1)** and **(2)** gives 3 = 0

This is a contradiction.

$\therefore$ Lines do not meet.

The lines are in fact parallel as $2\mathbf{i} - 2\mathbf{j} + 2\mathbf{k}$ is a multiple of $\mathbf{i} - \mathbf{j} + \mathbf{k}$.

The distance between them is found by considering A on line l_1 and B on line l_2.

Then $\overrightarrow{AB} = -\mathbf{i} - 2\mathbf{j} + 5\mathbf{k} + t(\mathbf{i} - \mathbf{j} + \mathbf{k})$

$$\left|\overrightarrow{AB}\right|^2 = x^2 = (-1+t)^2 + (-2-t)^2 + (5+t)^2$$
$$= 1 - 2t + t^2 + 4 + 4t + t^2 + 25 + 10t + t^2$$
$$= 30 + 12t + 3t^2$$

The minimum value of x^2 occurs when $\frac{\mathrm{d}(x^2)}{\mathrm{d}t} = 0$

$$\frac{\mathrm{d}(x)^2}{\mathrm{d}t} = 12 + 6t$$

When $\frac{\mathrm{d}(x)^2}{\mathrm{d}t} = 0, t = -2$

$$\therefore x^2 = 30 - 24 + 12$$
$$= 18$$
$$\therefore x = \sqrt{18} = 3\sqrt{2} \text{ or } 4.24 \text{ (3 s.f.)}$$

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3 c Let l_1 and l_2, then

$$\begin{pmatrix} 1+2\lambda \\ 1+\lambda \\ 5-2\lambda \end{pmatrix} - \begin{pmatrix} -1+\mu \\ -1+\mu \\ 2+\mu \end{pmatrix} \quad \textbf{(1)} \quad \textbf{(2)} \quad \textbf{(3)}$$

Subtract **(1)** – **(2)**

Then $\lambda = 0$

Substitute into equation **(1)**

Then $\mu = 2$

But $\lambda = 0$, $\mu = 2$ does not satisfy equation **(3)**

So the lines do not meet.

They are skew.

Using the formula distance $= \left| \dfrac{(\mathbf{a}-\mathbf{c})\cdot(\mathbf{b}\times\mathbf{d})}{|\mathbf{b}\times\mathbf{d}|} \right|$

$\mathbf{a}-\mathbf{c} = 2\mathbf{i}+2\mathbf{j}+3\mathbf{k}$

$\mathbf{b}\times\mathbf{d} = (2\mathbf{i}+\mathbf{j}-2\mathbf{k})(\mathbf{i}+\mathbf{j}+\mathbf{k})$

$= 3\mathbf{i}-4\mathbf{j}+\mathbf{k}$

$\therefore$ shortest distance $= \left| \dfrac{(2\mathbf{i}+2\mathbf{j}+3\mathbf{k})\cdot(3\mathbf{i}-4\mathbf{j}+\mathbf{k})}{\sqrt{3^2+(-4)^2+1^2}} \right|$

$= \dfrac{6-8+3}{\sqrt{26}}$

$= \dfrac{1}{\sqrt{26}}$

$= 0.196$ (3 s.f.)

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4 Let A be the point (4,1, –1) and B be the point $(3 + 2t, –1 –t, 2 –t)$ which lies on the line.

Find the distance between (4, 1, –1) and $(3 + 2t, –1 – t, 2 –t)$ at a point on the line.

Then $\overrightarrow{BA} = \mathbf{a} - \mathbf{b}$

$$= [4-(3+2t), 1-(-1-t), -1-(2-t)]$$
$$= [1-2t, 2+t, -3+t]$$
$$\therefore \left|\overrightarrow{BA}\right|^2 = (1-2t)^2 + (2+t)^2 + (-3+t)^2$$
$$= 6t^2 - 6t + 14$$

$\left|\overrightarrow{BA}\right|$ is a minimum when $\left|\overrightarrow{BA}\right|^2$ is minimum

This minimum value can be found by calculus or completion of the square.

$$\left|\overrightarrow{BA}\right|^2 = 6(t^2 - t) + 14$$
$$= 6\left(t - \frac{1}{2}\right)^2 + 14 - \frac{6}{4}$$

This is a minimum when $t = \frac{1}{2}$ and

$$\left|\overrightarrow{BA}\right|^2 = 14 - 1\frac{1}{2} = 12\frac{1}{2}$$
$$\therefore \left|\overrightarrow{BA}\right| = \sqrt{12\frac{1}{2}} = 3.54 \text{ (3 s.f.)}$$

5 a The distance from the origin to the plane $\mathbf{r} \cdot (6\mathbf{i} + 6\mathbf{j} - 7\mathbf{k}) = 55$ is $\frac{55}{\sqrt{6^2 + 6^2 + (-7)^2}}$

$$= \frac{55}{\sqrt{121}}$$
$$= \frac{55}{11}$$
$$= 5$$

First find the distance from the origin to each plane, then subtract.

The distance from the origin to the plane $\mathbf{r} \cdot (6\mathbf{i} + 6\mathbf{j} - 7\mathbf{k}) = 22$ is $\frac{22}{\sqrt{6^2 + 6^2 + (-7)^2}}$

$$= \frac{22}{11}$$
$$= 2$$

$\therefore$ The distance between the planes is $5 - 2 = 3$

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5 b $\mathbf{r} = 3\mathbf{i} + 4\mathbf{j} + \mathbf{k} + \lambda(4\mathbf{i} + \mathbf{k}) + \mu(8\mathbf{i} + 3\mathbf{j} + 3\mathbf{k})$

Express the equations of the planes in the form $\mathbf{r} \cdot \mathbf{n} = \mathbf{a} \cdot \mathbf{n}$

The normal to the plane is $\mathbf{n}$ where

$\mathbf{n} = (4\mathbf{i} + \mathbf{k}) \times (8\mathbf{i} + 3\mathbf{j} + 3\mathbf{k})$

$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 4 & 0 & 1 \\ 8 & 3 & 3 \end{vmatrix} = -3\mathbf{i} - 4\mathbf{j} + 12\mathbf{k}$$

$\therefore$ Equation of plane may be written

$\mathbf{r} \cdot (-3\mathbf{i} - 4\mathbf{j} + 12\mathbf{k}) = (3\mathbf{i} + 4\mathbf{j} + \mathbf{k}) \cdot (-3\mathbf{i} - 4\mathbf{j} + 12\mathbf{k})$

i.e. $\mathbf{r} \cdot (-3\mathbf{i} - 4\mathbf{j} + 12\mathbf{k}) = -13$

The distance from the origin to this plane is $\dfrac{-13}{\sqrt{(-3)^2 + (-4) + 12^2}} = -1$

The second plane

$\mathbf{r} = 14\mathbf{i} + 2\mathbf{j} + 2\mathbf{k} + \lambda(3\mathbf{j} + \mathbf{k}) + \mu(8\mathbf{i} - 9\mathbf{j} - \mathbf{k})$ has normal $\mathbf{n}$ where

$$\mathbf{n} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 3 & 1 \\ 8 & -9 & -1 \end{vmatrix} = 6\mathbf{i} + 8\mathbf{j} - 24\mathbf{k}$$

This shows it is parallel to the first plane as the normal vectors are parallel.

$\therefore$ Equation of second plane may be written

$\mathbf{r} \cdot (6\mathbf{i} + 8\mathbf{j} - 24\mathbf{k}) = (14\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}) \cdot (6\mathbf{i} + 8\mathbf{j} - 24\mathbf{k})$

i.e. $\mathbf{r} \cdot (6\mathbf{i} + 8\mathbf{j} - 24\mathbf{k}) = 52$ or $\mathbf{r} \cdot (-3\mathbf{i} - 4\mathbf{j} + 12\mathbf{k}) = -26$

This distance from the origin to this plane is $\dfrac{-26}{\sqrt{3^2 + 4^2 + (-12)^2}} = -2$

$\therefore$ The distance between the two planes is $-1 - (-2) = 1$.

6 a The length of the normal vector $10\mathbf{i} + 10\mathbf{j} + 23\mathbf{k}$ is $\sqrt{10^2 + 10^2 + 23^2} = \sqrt{729} = 27$

$\therefore \dfrac{1}{27}(10\mathbf{i} + 10\mathbf{j} + 23\mathbf{k})$ is a unit vector normal to the plane.

The plane has equation

$\mathbf{r} \cdot (10\mathbf{i} + 10\mathbf{j} + 23\mathbf{k}) = 81$

or $\mathbf{r} \cdot \dfrac{1}{27}(10\mathbf{i} + 10\mathbf{j} + 23\mathbf{k}) = \dfrac{81}{27} = 3$

$\therefore$ The perpendicular distance from the origin to the plane is 3.

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6 b A plane parallel to π through the point (–1, –1, 4) has equation

$$\mathbf{r}\cdot\frac{1}{27}(10\mathbf{i}+10\mathbf{j}+23\mathbf{k}) = (-\mathbf{i}-\mathbf{j}+4\mathbf{k})\cdot\frac{1}{27}(10\mathbf{i}+10\mathbf{j}+23\mathbf{k})$$
$$= \frac{-10}{27}-\frac{10}{27}+\frac{92}{27}$$
$$= \frac{72}{27}$$
$$= \frac{8}{3}$$

∴ The perpendicular distance from the origin to this new plane is $2\frac{2}{3}$

This distance between the planes is $3-2\frac{2}{3}=\frac{1}{3}$

∴ The perpendicular distance from the point $(-1,-1,4)$ to the plane π is $\frac{1}{3}$

c A plane parallel to π through the point (2, 1, 3) has equation

$$\mathbf{r}\cdot\frac{1}{24}(10\mathbf{i}+10\mathbf{j}+23\mathbf{k}) = (2\mathbf{i}+\mathbf{j}+3\mathbf{k})\cdot\frac{1}{27}(10\mathbf{i}+10\mathbf{j}+23\mathbf{k})$$
$$= \frac{20}{27}+\frac{10}{27}+\frac{69}{27}$$
$$= \frac{99}{27}$$
$$= \frac{11}{3}$$

∴ The perpendicular distance from the origin to this new plane is $3\frac{2}{3}$

∴ The distances between this plane and π is $3\frac{2}{3}-3=\frac{2}{3}$

∴ The perpendicular distance from $(2,1,3)$ to π is $\frac{2}{3}$

d A plane parallel to π through the point $(6,12,-9)$ has equation

$$\mathbf{r}\cdot\frac{1}{27}(10\mathbf{i}+10\mathbf{j}+23\mathbf{k}) = (6\mathbf{i}+12\mathbf{j}-9\mathbf{k})\cdot\frac{1}{27}(10\mathbf{i}+10\mathbf{j}+23\mathbf{k})$$
$$= \frac{60}{27}+\frac{120}{27}-\frac{207}{27}$$
$$= -\frac{27}{27}$$
$$= -1$$

∴ The perpendicular distance from the origin to this new plane is 1, in the opposite direction.

∴ The distance between this plane and π is $3-(-1)=4$

∴ The perpendicular distance from (2, 1, 3) to π is 4.

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7 Let **M** be the midpoint of the line segment joining P to its reflection in l.

Then, for some λ

$$\overrightarrow{PM} = \begin{pmatrix} -1+\lambda \\ -1+2\lambda \\ 1-\lambda \end{pmatrix}$$

$\overrightarrow{PM}$ must be perpendicular to l

$$\therefore \begin{pmatrix} -1+\lambda \\ -1+2\lambda \\ 1-\lambda \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} = 0$$

$$\Rightarrow \lambda = \frac{2}{3}$$

$$\therefore \overrightarrow{PM} = \begin{pmatrix} -\frac{1}{3} \\ \frac{1}{3} \\ \frac{1}{3} \end{pmatrix}$$

The reflection of P in l has position vector

$$\overrightarrow{OP} + 2\overrightarrow{PM} = \begin{pmatrix} 3 \\ 0 \\ 2 \end{pmatrix} + \begin{pmatrix} -\frac{2}{3} \\ \frac{2}{3} \\ \frac{2}{3} \end{pmatrix} = \begin{pmatrix} \frac{7}{3} \\ \frac{2}{3} \\ \frac{8}{3} \end{pmatrix}$$

$$\therefore \left(\frac{7}{3}, \frac{2}{3}, \frac{8}{3}\right)$$

8 a $$\frac{|1(-2)+0(1)+3(1)-5|}{\sqrt{(-2)^2+1^2+1^2}} = \frac{4}{\sqrt{6}} = \frac{2\sqrt{6}}{3}$$

b A unit normal vector to the plane is $\frac{1}{\sqrt{6}}\begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}$

The line segment along the shortest distance from P to the plane is normal to the plane,

and the distance is $\frac{2\sqrt{6}}{3}$

Therefore Q has position vector

$$\overrightarrow{OQ} = \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix} + 2\left(\frac{2\sqrt{6}}{3}\right)\left(\frac{1}{\sqrt{6}}\right)\begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -\frac{5}{3} \\ \frac{4}{3} \\ \frac{13}{3} \end{pmatrix}$$

$$\therefore Q\left(-\frac{5}{3}, \frac{4}{3}, \frac{13}{3}\right)$$

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9 a The kestrel's path in vector form is

$$\mathbf{r} = \begin{pmatrix} 3 \\ 5 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 9 \\ -5 \\ 1.2 \end{pmatrix},\ 0 \leqslant \lambda \leqslant 1$$

Let **G** be the position of the birdwatcher.

Let **H** be a general point in the path. Then,

$$\overrightarrow{GH} = \begin{pmatrix} -2+9\lambda \\ 1-5\lambda \\ -0.7+1.2\lambda \end{pmatrix}$$

The shortest distance between the kestrel and the birdwatcher will occur when $\overrightarrow{GH}$ is perpendicular to the path.

$$\therefore \begin{pmatrix} -2+9\lambda \\ 1-5\lambda \\ -0.7+1.2\lambda \end{pmatrix} \cdot \begin{pmatrix} 9 \\ -5 \\ 1.2 \end{pmatrix} = 0$$

$$\Rightarrow \lambda = \frac{35}{158}$$

$$\therefore \overrightarrow{GH} = \begin{pmatrix} -2+9\left(\frac{35}{158}\right) \\ 1-5\left(\frac{35}{158}\right) \\ -0.7+1.2\left(\frac{35}{158}\right) \end{pmatrix}$$

$$\therefore |\overrightarrow{GH}| = 0.45 \quad (2 \text{ d.p.})$$

$$\Rightarrow\ |\overrightarrow{GH}| < 0.5$$

So the birdwatcher will be able to spot the kestrel.

b In practice, the bird is very unlikely to fly in a straight line from *A* to *B*

10 a $$\frac{|4(3)-(-2)+8(4)-6|}{\sqrt{3^2+(-2)^2+4^2}} = \frac{40}{\sqrt{29}} = \frac{40\sqrt{29}}{29}$$

b $$\left(\lambda \begin{pmatrix} 2 \\ -2 \\ 3 \end{pmatrix} + \mu \begin{pmatrix} -3 \\ 0 \\ 3 \end{pmatrix}\right) \cdot \begin{pmatrix} 2 \\ 5 \\ 2 \end{pmatrix}$$

$$= \lambda \begin{pmatrix} 2 \\ -2 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 5 \\ 2 \end{pmatrix} + \mu \begin{pmatrix} -3 \\ 0 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 5 \\ 2 \end{pmatrix}$$

$$= \lambda(4-10+6) + \mu(-6+0+6) = 0$$

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10 c $\mathbf{n}_1 = \begin{pmatrix} 3 \\ -2 \\ 4 \end{pmatrix}, \mathbf{n}_2 = \begin{pmatrix} 2 \\ 5 \\ 2 \end{pmatrix}$

$\mathbf{n}_1 \cdot \mathbf{n}_2 = \begin{pmatrix} 3 \\ -2 \\ 4 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 5 \\ 2 \end{pmatrix} = 6 - 10 + 8 = 4$

$|\mathbf{n}_1| = \sqrt{29}, \quad |\mathbf{n}_2| = \sqrt{33}$

$\therefore \cos\theta = \dfrac{4}{\sqrt{29}\sqrt{33}}$

$\Rightarrow \theta = 82.6^o$ (1d.p.)

11 a A vector equation for l_1 is

$\mathbf{r} = \begin{pmatrix} -2 \\ 2 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -1 \\ -2 \end{pmatrix}$

Let M be a general point on l_1

$\overrightarrow{AM} = \begin{pmatrix} -5 + 2\lambda \\ 3 - \lambda \\ -3 - 2\lambda \end{pmatrix}$

$|\overrightarrow{AM}|$ is minimum when $\overrightarrow{AM}$ is perpendicular to l_1

$\therefore \begin{pmatrix} -5 + 2\lambda \\ 3 - \lambda \\ -3 - 2\lambda \end{pmatrix} \cdot \begin{pmatrix} 2 \\ -1 \\ -2 \end{pmatrix} = 0$

$\Rightarrow \lambda = \dfrac{7}{9}$

$\therefore \overrightarrow{AM} = \dfrac{1}{9} \begin{pmatrix} -31 \\ 20 \\ -41 \end{pmatrix}$

$\Rightarrow |\overrightarrow{AM}| = \dfrac{13\sqrt{2}}{3}$

b A vector equation is

$\begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} -31 \\ 20 \\ -41 \end{pmatrix}$

So a cartesian equation is

$\dfrac{x-3}{-31} = \dfrac{y+1}{20} = \dfrac{z-2}{-41}$

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12 First find the intersection between l_1 and the plane.

$$\begin{pmatrix} 2+4\lambda \\ -1-3\lambda \\ 2+4\lambda \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix} = 1+3\lambda = 4$$

$\Rightarrow \lambda = 1$

$\therefore (6, -4, 6)$

Choose any other point along l_1, and reflect it in the plane to obtain e.g. $\begin{pmatrix} 4 \\ -3 \\ 0 \end{pmatrix}$

$$r = \begin{pmatrix} 4 \\ -3 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ -1 \\ 6 \end{pmatrix}$$

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