## Exercise 9D

**1 a** Direction vectors are $\mathbf{a} = \begin{pmatrix} 3 \\ -5 \\ -1 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 2 \\ 1 \\ -9 \end{pmatrix}$

$$\cos\theta = \frac{\mathbf{a}.\mathbf{b}}{|\mathbf{a}||\mathbf{b}|}$$

$$\mathbf{a}.\mathbf{b} = \begin{pmatrix} 3 \\ -5 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 1 \\ -9 \end{pmatrix} = 6 - 5 + 9 = 10$$

$$|\mathbf{a}| = \sqrt{3^2 + (-5)^2 + (-1)^2} = \sqrt{35}$$

$$|\mathbf{b}| = \sqrt{2^2 + 1^2 + (-9)^2} = \sqrt{86}$$

$$\cos\theta = \frac{10}{\sqrt{35}\sqrt{86}}$$

$\theta = 79.5^\circ$ (1 d.p.)

The acute angle between the lines is 79.5° (1 d.p.)

**b** Direction vectors are $\mathbf{a} = \begin{pmatrix} -2 \\ -1 \\ 3 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} -4 \\ -2 \\ 1 \end{pmatrix}$

$$\cos\theta = \frac{\mathbf{a}.\mathbf{b}}{|\mathbf{a}||\mathbf{b}|}$$

$$\mathbf{a}.\mathbf{b} = \begin{pmatrix} -2 \\ -1 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} -4 \\ -2 \\ 1 \end{pmatrix} = 8 + 2 + 3 = 13$$

$$|\mathbf{a}| = \sqrt{(-2)^2 + (-1)^2 + 3^2} = \sqrt{14}$$

$$|\mathbf{b}| = \sqrt{(-4)^2 + (-2)^2 + 1^2} = \sqrt{21}$$

$$\cos\theta = \frac{13}{\sqrt{14}\sqrt{21}}$$

$\theta = 40.7^\circ$ (1 d.p.)

The acute angle between the lines is 40.7° (1 d.p.)

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**1 c** Direction vectors are $\mathbf{a} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 2 \\ -7 \\ 3 \end{pmatrix}$

$$\cos\theta = \frac{\mathbf{a}.\mathbf{b}}{|\mathbf{a}||\mathbf{b}|}$$

$$\mathbf{a}.\mathbf{b} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ -7 \\ 3 \end{pmatrix} = 2 - 7 + 3 = -2$$

$$|\mathbf{a}| = \sqrt{1^2 + 1^2 + 1^2} = \sqrt{3}$$

$$|\mathbf{b}| = \sqrt{2^2 + (-7)^2 + 3^2} = \sqrt{62}$$

$$\cos\theta = \frac{-2}{\sqrt{3}\sqrt{62}}$$

$\theta = 98.4°$ (1 d.p.)

This is the angle between the two vectors.

The acute angle between the lines is 180° – 98.4° = 81.6° (1 d.p.).

**d** Direction vectors are $\mathbf{a} = \begin{pmatrix} 8 \\ -1 \\ -2 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 1 \\ 3 \\ -7 \end{pmatrix}$

$$\cos\theta = \frac{\mathbf{a}.\mathbf{b}}{|\mathbf{a}||\mathbf{b}|}$$

$$\mathbf{a}.\mathbf{b} = \begin{pmatrix} 8 \\ -1 \\ -2 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 3 \\ -7 \end{pmatrix} = 8 - 3 + 14 = 19$$

$$|\mathbf{a}| = \sqrt{8^2 + (-1)^2 + (-2)^2} = \sqrt{69}$$

$$|\mathbf{b}| = \sqrt{1^2 + 3^2 + (-7)^2} = \sqrt{59}$$

$$\cos\theta = \frac{19}{\sqrt{69}\sqrt{59}}$$

$\theta = 72.7°$ (1 d.p.)

The acute angle between the lines is 72.7° (1 d.p.)

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**1 e** Direction vectors are $\mathbf{a} = \begin{pmatrix} 11 \\ 5 \\ -3 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} -3 \\ 5 \\ 4 \end{pmatrix}$

$$\cos\theta = \frac{\mathbf{a}.\mathbf{b}}{|\mathbf{a}||\mathbf{b}|}$$

$$\mathbf{a}.\mathbf{b} = \begin{pmatrix} 11 \\ 5 \\ -3 \end{pmatrix} \cdot \begin{pmatrix} -3 \\ 5 \\ 4 \end{pmatrix} = -33 + 25 - 12 = -20$$

$$|\mathbf{a}| = \sqrt{11^2 + 5^2 + (-3)^2} = \sqrt{155}$$

$$|\mathbf{b}| = \sqrt{(-3)^2 + 5^2 + 4^2} = \sqrt{50}$$

$$\cos\theta = \frac{-20}{\sqrt{155}\sqrt{50}}$$

$\theta = 103.1°$ (1 d.p.)

This is the angle between the two vectors.

The acute angle between the lines is 180° – 103.1° = 76.9° (1 d.p.).

**2 a** $\mathbf{r} \cdot (2\mathbf{i} + \mathbf{j} + \mathbf{k}) = (\mathbf{i} - \mathbf{j} - \mathbf{k}) \cdot (2\mathbf{i} + \mathbf{j} + \mathbf{k})$

$= 2 - 1 - 1$

i.e. $\mathbf{r} \cdot (2\mathbf{i} + \mathbf{j} + \mathbf{k}) = 0$

**b** $\mathbf{r} \cdot (5\mathbf{i} - \mathbf{j} - 3\mathbf{k}) = (\mathbf{i} + 2\mathbf{j} + \mathbf{k}) \cdot (5\mathbf{i} - \mathbf{j} - 3\mathbf{k})$

$= 5 - 2 - 3$

i.e. $\mathbf{r} \cdot (5\mathbf{i} - \mathbf{j} - 3\mathbf{k}) = 0$

**c** $\mathbf{r} \cdot (\mathbf{i} + 3\mathbf{j} + 4\mathbf{k}) = (2\mathbf{i} - 3\mathbf{k}) \cdot (\mathbf{i} + 3\mathbf{j} + 4\mathbf{k})$

$= 2 - 12$

i.e. $\mathbf{r} \cdot (\mathbf{i} + 3\mathbf{j} + 4\mathbf{k}) = -10$

**d** $\mathbf{r} \cdot (4\mathbf{i} + \mathbf{j} - 5\mathbf{k}) = (4\mathbf{i} - 2\mathbf{j} + \mathbf{k}) \cdot (4\mathbf{i} + \mathbf{j} - 5\mathbf{k})$

$= 16 - 2 - 5$

i.e. $\mathbf{r} \cdot (4\mathbf{i} + \mathbf{j} - 5\mathbf{k}) = 9$

**3 a** $2x + y + z = 0$

**b** $5x - y - 3z = 0$

**c** $x + 3y + 4z = -10$

**d** $4x + y - 5z = 9$

Replace **r** by $x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ in each equation.

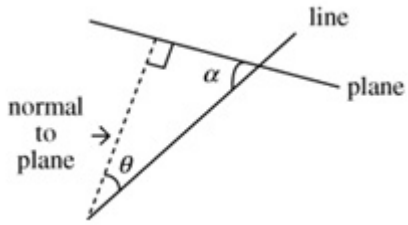
**4** Expanding the scalar product gives

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \cdot \begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix} = k$$

$\Rightarrow n_1x + n_2y + n_3z = k$

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**5**




Find the acute angle between the given line and the normal to the plane, then subtract from 90°.

Let $\theta$ be the acute angle between the line and the normal to the plane.

Then $\cos\theta = \left|\dfrac{(4\mathbf{i}+4\mathbf{j}+7\mathbf{k})\cdot(2\mathbf{i}+\mathbf{j}-2\mathbf{k})}{\sqrt{4^2+4^2+7^2}\sqrt{2^2+1^2+(-2)^2}}\right|$

$= \left|\dfrac{8+4-14}{\sqrt{81}\sqrt{9}}\right|$

$= \left|\dfrac{-2}{27}\right| = \dfrac{2}{27}$

Let $\alpha$ be the angle between the line and the plane.

Then $\theta + \alpha = 90^\circ$

So $\sin\alpha = \cos\theta = \dfrac{2}{27}$

$\therefore \alpha = 4.25^\circ$ (3 s.f.)

**6** Let $\theta$ be the acute angle between the line and the normal to the plane.

Then $\cos\theta = \dfrac{(3\mathbf{i}+4\mathbf{j}-12\mathbf{k})\cdot(4\mathbf{i}-4\mathbf{j}-7\mathbf{k})}{\sqrt{3^2+4^2+(-12)^2}\sqrt{4^2+(-4)^2+(-7)^2}}$

$= \dfrac{12-16+84}{\sqrt{169}\sqrt{81}}$

$= \dfrac{80}{13\times 9}$

$= \dfrac{80}{117}$

Find the acute angle between the given line and the normal to the plane, then subtract from 90°.

Let $\alpha$ be the angle between the line and the plane.

Then $\theta + \alpha = 90^\circ$

So $\sin\alpha = \cos\theta = \dfrac{80}{117}$

$\therefore \alpha = 43.1^\circ$ (3s.f.)

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**7** Then angle $\theta$ between the two normal vectors $\mathbf{i}+2\mathbf{j}-2\mathbf{k}$ and $-4\mathbf{i}+4\mathbf{j}+7\mathbf{k}$ is given by

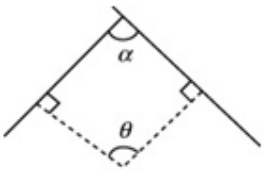
$$\cos\theta = \frac{(\mathbf{i}+2\mathbf{j}-2\mathbf{k})\cdot(-4\mathbf{i}+4\mathbf{j}+7\mathbf{k})}{|\mathbf{i}+2\mathbf{j}-2\mathbf{k}||-4\mathbf{i}+4\mathbf{j}+7\mathbf{k}|}$$

$$= \frac{-4+8-14}{\sqrt{1^2+2^2+(-2)^2}\sqrt{(-4)^2+4^2+7^2}}$$

$$= \frac{-10}{\sqrt{9}\sqrt{81}}$$

$$= -\frac{10}{27}$$

First find the angle between the two normal vectors.




The acute angle, $\alpha$, between the two planes is such that
$\alpha+\theta=180^\circ$
So $\cos\alpha = -\cos\theta$

$$= \frac{10}{27}$$

$$\therefore \alpha = 68.3^\circ \text{ (3 s.f.)}$$

**8** The angle $\theta$ between the two normal vectors $3\mathbf{i}-4\mathbf{j}+12\mathbf{k}$ and $5\mathbf{i}-12\mathbf{k}$ is given by

$$\cos\theta = \frac{(3\mathbf{i}-4\mathbf{j}+12\mathbf{k})\cdot(5\mathbf{i}-12\mathbf{k})}{|3\mathbf{i}-4\mathbf{j}+12\mathbf{k}||5\mathbf{i}-12\mathbf{k}|}$$

$$= \frac{15-144}{\sqrt{3^2+(-4)^2+12^2}\sqrt{5^2+(-12)^2}}$$

$$= \frac{-129}{\sqrt{169}\sqrt{169}}$$

$$= \frac{-129}{169}$$

The acute angle $\alpha$ between the planes is such that $\alpha+\theta=180^\circ$

So $\cos\alpha = -\cos\theta = \frac{129}{169}$

$\therefore \alpha = 40.2^\circ$ (3 s.f.)

**9 a** Line $l_1$: $\mathbf{r} = \begin{pmatrix}1\\4\\2\end{pmatrix} + t\begin{pmatrix}8\\5\\1\end{pmatrix}$

When $t = 1$, $\mathbf{r} = \begin{pmatrix}9\\9\\3\end{pmatrix}$

So the point (9, 9, 3) lies on $l_1$.

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**9 b** Direction vectors are $\mathrm{a} = \begin{pmatrix} 8 \\ 5 \\ 1 \end{pmatrix}$ and $\mathrm{b} = \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix}$

$$\mathbf{a}.\mathbf{b} = \begin{pmatrix} 8 \\ 5 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix} = 24 + 5 + 0 = 29$$

$$|\mathrm{a}| = \sqrt{8^2 + 5^2 + 1^2} = \sqrt{90}$$

$$|\mathrm{b}| = \sqrt{3^2 + 1^2 + 0^2} = \sqrt{10}$$

$$\cos\theta = \frac{29}{\sqrt{90}\sqrt{10}} = \frac{29}{\sqrt{900}} = \frac{29}{30}$$

**c** $\mathbf{PQ} = \sqrt{(9-1)^2 + (9-4)^2 + (3-2)^2}$

$$= \sqrt{8^2 + 5^2 + 1^2} = \sqrt{90}$$

Line $l_2$: $\mathbf{r} = \begin{pmatrix} 1 \\ 4 \\ 2 \end{pmatrix} + s\begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1+3s \\ 4+s \\ 2 \end{pmatrix}$

Let the coordinates of $R$ be $(1+3s,\ 4+s,\ 2)$

$\mathbf{PR} = \sqrt{(1+3s-1)^2 + (4+s-4)^2 + (2-2)^2}$

$$= \sqrt{9s^2 + s^2} = \sqrt{10s^2}$$

$PQ^2 = PR^2 : 90 = 10s^2$

$\Rightarrow 5^2 = 9$

$\Rightarrow s = \pm 3$

When $s = 3$, $\mathbf{r} = \begin{pmatrix} 10 \\ 7 \\ 2 \end{pmatrix}$ $\quad R : (10,\ 7,\ 2)$

When $s = -3$, $\mathbf{r} = \begin{pmatrix} -8 \\ 1 \\ 2 \end{pmatrix}$ $\quad R : (-8,\ 1,\ 2)$

**10 a** $\frac{3-6}{-1} = \frac{3+3}{2} = \frac{7+2}{3} = 3$

and

$\frac{3+5}{2} = \frac{3-15}{-3} = \frac{7-3}{1} = 4$

So $A$ lies on both $l_1$ and $l_2$

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**10 b** $\mathbf{a} = \begin{pmatrix} -1 \\ 2 \\ 3 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix}$

$\therefore \mathbf{a} \cdot \mathbf{b} = -2 - 6 + 3 = -5$

$|\mathbf{a}| = \sqrt{(-1)^2 + 2^2 + 3^2} = \sqrt{14}$

$|\mathbf{b}| = \sqrt{2^2 + (-3)^2 + 1^2} = \sqrt{14}$

$\therefore \cos\theta = \left| \frac{-5}{\sqrt{14}\sqrt{14}} \right| = \frac{5}{14}$

$\Rightarrow \theta = 69.1^o$ (1d.p.)

**11** $\begin{pmatrix} 1 \\ 3 \\ 3 \end{pmatrix} + 3\begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 10 \\ 9 \\ 0 \end{pmatrix} \Rightarrow A = (10, 9, 0)$

$\begin{pmatrix} 3 \\ 5 \\ -2 \end{pmatrix} - 2\begin{pmatrix} -4 \\ -3 \\ 1 \end{pmatrix} = \begin{pmatrix} 11 \\ 11 \\ -4 \end{pmatrix} \Rightarrow B = (11, 11, -4)$

$\therefore \overrightarrow{AB} = \mathbf{v}_1 = \begin{pmatrix} 1 \\ 2 \\ -4 \end{pmatrix}$

$l_1$ has direction vector $\mathbf{v}_2 = \begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix}$

$\mathbf{v}_1 \cdot \mathbf{v}_2 = \begin{pmatrix} 1 \\ 2 \\ -4 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix} = 3 + 4 + 4 = 11$

$|\mathbf{v}_1| = \sqrt{1^2 + 2^2 + (-4)^2} = \sqrt{21}$

$|\mathbf{v}_2| = \sqrt{3^2 + 2^2 + (-1)^2} = \sqrt{14}$

$\therefore \cos\theta = \frac{11}{\sqrt{21}\sqrt{14}} \Rightarrow \theta = 50.1^o$ (1d.p.)

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**12 a** Find the plane containing $A=(3,5,-1)$; $B=(2,-2,4)$; $C=(4,3,0)$ and show that $D=(1,4,-3)$ does not lie in it.

$\overrightarrow{AB}=\begin{pmatrix}-1\\-7\\5\end{pmatrix}$, $\overrightarrow{AC}=\begin{pmatrix}1\\-2\\1\end{pmatrix}$

So an equation for the plane is

$\mathbf{r}=\begin{pmatrix}3\\5\\-1\end{pmatrix}+\lambda\begin{pmatrix}-1\\-7\\5\end{pmatrix}+\mu\begin{pmatrix}1\\-2\\1\end{pmatrix}$

If $D$ lies in the plane, then there exist $\lambda$ and $\mu$ such that

$\begin{pmatrix}3\\5\\-1\end{pmatrix}+\lambda\begin{pmatrix}-1\\-7\\5\end{pmatrix}+\mu\begin{pmatrix}1\\-2\\1\end{pmatrix}=\begin{pmatrix}1\\4\\-3\end{pmatrix}$

**i** component: $3-\lambda+\mu=1\Rightarrow-\lambda+\mu=-2$

**j** component: $5-7\lambda-2\mu=4\Rightarrow7\lambda+2\mu=1$

**k** component: $-1+5\lambda+\mu=-3\Rightarrow5\lambda+\mu=-2$

**i** and **j** components $\Rightarrow\ \lambda=\frac{5}{9},\ \mu=-\frac{13}{9}$

Check **k** component: $5\left(\frac{5}{9}\right)-\frac{13}{9}=\frac{4}{3}\neq-2$

So the equations are not consistent.

Therefore the points are not coplanar.

**b** $\overrightarrow{AD}=\mathbf{b}=\begin{pmatrix}-2\\-1\\-2\end{pmatrix}$

The plane has normal vector $\mathbf{n}=\begin{pmatrix}1\\2\\3\end{pmatrix}$, since this is perpendicular to both $\overrightarrow{AB}$ and $\overrightarrow{AC}$

$\therefore\mathbf{b}\cdot\mathbf{n}=\begin{pmatrix}-2\\-1\\-2\end{pmatrix}\cdot\begin{pmatrix}1\\2\\3\end{pmatrix}=-10$

$|\mathbf{b}|=\sqrt{(-2)^2+(-1)^2+(-2)^2}=3$

$|\mathbf{n}|=\sqrt{1^2+2^2+3^2}=\sqrt{14}$

$\Rightarrow\sin\theta=\left|\frac{-10}{3\sqrt{14}}\right|=\frac{10}{3\sqrt{14}}$

$\Rightarrow\theta=63.0^o$ (1d.p.)

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**13** Let $E$ be the midpoint of $AD$. Then the coordinate of $E$ is $\left(\frac{1}{2}, 0, \frac{1}{2}\right)$

The angle $\theta$ between planes $ABD$ and $ADC$ is the same as the angle between the lines $EB$ and $EC$

$$\overrightarrow{EB} = \overrightarrow{OB} - \overrightarrow{OE} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} - \begin{pmatrix} \frac{1}{2} \\ 0 \\ \frac{1}{2} \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} \\ 1 \\ \frac{1}{2} \end{pmatrix}$$

$$\overrightarrow{EC} = \overrightarrow{OC} - \overrightarrow{OE} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} - \begin{pmatrix} \frac{1}{2} \\ 0 \\ \frac{1}{2} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ 1 \\ -\frac{1}{2} \end{pmatrix}$$

$$\overrightarrow{EB}.\overrightarrow{EC} = \begin{pmatrix} -\frac{1}{2} \\ 1 \\ \frac{1}{2} \end{pmatrix}.\begin{pmatrix} \frac{1}{2} \\ 1 \\ -\frac{1}{2} \end{pmatrix} = -\frac{1}{4} + 1 - \frac{1}{4} = \frac{1}{2} \qquad (1)$$

and $\overrightarrow{EB}.\overrightarrow{EC} = \left|\begin{pmatrix} -\frac{1}{2} \\ 1 \\ \frac{1}{2} \end{pmatrix}\right|\left|\begin{pmatrix} \frac{1}{2} \\ 1 \\ -\frac{1}{2} \end{pmatrix}\right| \cos\theta = \sqrt{\left(-\frac{1}{2}\right)^2 + 1^2 + \left(\frac{1}{2}\right)^2}\sqrt{\left(\frac{1}{2}\right)^2 + 1^2 + \left(-\frac{1}{2}\right)^2}\cos\theta$

$$= \sqrt{\frac{3}{2}}\sqrt{\frac{3}{2}}\cos\theta = \frac{3}{2}\cos\theta \qquad (2)$$

So equating equations (1) and (2) gives:

$$\frac{1}{2} = \frac{3}{2}\cos\theta$$

So therefore $\cos\theta = \frac{1}{3}$

Since the tetrahedron is regular, the angle between any two adjacent faces will be the same,

i.e. $\arccos\left(\frac{1}{3}\right)$

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**14** Let point $D$ be the point of the top of the pole.

So the coordinate of $D$ is $(0,0,20)$

Let the points of the bases of the ropes be $A=(0,8,2)$, $B=(12,-5,3)$ and $C=(-2,6,5)$ respectively.

Then $\overrightarrow{DA}=\begin{pmatrix}0\\8\\-18\end{pmatrix}$, and $\overrightarrow{DC}=\begin{pmatrix}-2\\6\\-15\end{pmatrix}$

Let $\theta$ be the angle between $DA$ and $DC$

$$\overrightarrow{DA}.\overrightarrow{DC}=\begin{pmatrix}0\\8\\-18\end{pmatrix}.\begin{pmatrix}-2\\6\\-15\end{pmatrix}=(0\times-2)+(8\times6)+(-18\times-15)=318$$

$$\left|\begin{pmatrix}0\\8\\-18\end{pmatrix}\right|=\sqrt{0^2+8^2+(-18)^2}=\sqrt{388} \text{ and } \left|\begin{pmatrix}-2\\6\\-15\end{pmatrix}\right|=\sqrt{(-2)^2+6^2+(-15)^2}=\sqrt{265}$$

So $\cos\theta=\dfrac{318}{\sqrt{388}\sqrt{265}}=0.9917$

and $\theta=\arccos(0.9917)=7.4^\circ$

So the angle between at least one pair of guide ropes is less than $15^\circ$. Therefore the flagpole will not be stable.

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