Exercise 9B

1 a Let

$\mathbf{a} = \mathbf{i} + 2\mathbf{j}, \mathbf{b} = (3\mathbf{i} + \mathbf{j} - \mathbf{k}) - (\mathbf{i} + 2\mathbf{j})$

$= 2\mathbf{i} - \mathbf{j} - \mathbf{k}$

and

$\mathbf{c} = (4\mathbf{i} + 3\mathbf{j} + 3\mathbf{k}) - (\mathbf{i} + 2\mathbf{j})$

$= 3\mathbf{i} + \mathbf{j} + 2\mathbf{k}$

$\therefore \mathbf{r} = \mathbf{i} + 2\mathbf{j} + \lambda(2\mathbf{i} - \mathbf{j} - \mathbf{k}) + \mu(3\mathbf{i} + \mathbf{j} + 2\mathbf{k})$

b Let

$\mathbf{a} = 3\mathbf{i} + 4\mathbf{j} + \mathbf{k}, \mathbf{b}$

$= -\mathbf{i} - 2\mathbf{j} - (3\mathbf{i} + 4\mathbf{j} + \mathbf{k})$

$= -4\mathbf{i} - 6\mathbf{j} - \mathbf{k}$

and

$\mathbf{c} = 2\mathbf{i} + \mathbf{j} + 4\mathbf{k} - (3\mathbf{i} + 4\mathbf{j} + \mathbf{k})$

$= -\mathbf{i} - 3\mathbf{j} + 3\mathbf{k}$

$\therefore \mathbf{r} = 3\mathbf{i} + 4\mathbf{j} + \mathbf{k} + \lambda(-4\mathbf{i} - 6\mathbf{j} - \mathbf{k}) + \mu(-\mathbf{i} - 3\mathbf{j} + 3\mathbf{k})$

or $\mathbf{r} = 3\mathbf{i} + 4\mathbf{j} + \mathbf{k} + \lambda'(4\mathbf{i} + 6\mathbf{j} + \mathbf{k}) + \mu(-\mathbf{i} - 3\mathbf{j} + 3\mathbf{k})$

c Let

$\mathbf{a} = 2\mathbf{i} - \mathbf{j} - \mathbf{k}, \mathbf{b}$

$= 3\mathbf{i} + \mathbf{j} + 2\mathbf{k} - (2\mathbf{i} - \mathbf{j} - \mathbf{k})$

$= \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$

and $\mathbf{c} = 4\mathbf{i} + \mathbf{k} - (2\mathbf{i} - \mathbf{j} - \mathbf{k}) = 2\mathbf{i} + \mathbf{j} + 2\mathbf{k}$

$\therefore \mathbf{r} = 2\mathbf{i} - \mathbf{j} - \mathbf{k} + \lambda(\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}) + \mu(2\mathbf{i} + \mathbf{j} + 2\mathbf{k})$

d Let

$\mathbf{a} = -\mathbf{i} + \mathbf{j} + 3\mathbf{k}, \mathbf{b}$

$= -\mathbf{i} + 2\mathbf{j} + 5\mathbf{k} - (-\mathbf{i} + \mathbf{j} + 3\mathbf{k})$

$= \mathbf{j} + 2\mathbf{k}$

and

$\mathbf{c} = 4\mathbf{j} + 4\mathbf{k} - (-\mathbf{i} + \mathbf{j} + 3\mathbf{k})$

$= \mathbf{i} + 3\mathbf{j} + \mathbf{k}$

$\therefore \mathbf{r} = -\mathbf{i} + \mathbf{j} + 3\mathbf{k} + \lambda(\mathbf{j} + 2\mathbf{k}) + \mu(\mathbf{i} + 3\mathbf{j} + \mathbf{k})$

2 The plane has normal vector $\begin{pmatrix} -1 \\ 3 \\ 2 \end{pmatrix}$ so a cartesian equation of the plane takes takes the form

$-x + 3y + 2z = d$

The plane contains the point $(4, -2, 6)$ so

$-(4) + 3(-2) + 2(6) = d \Rightarrow d = 2$

$\therefore \ -x + 3y + 2z = 2$

3 a $\begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ 2 \\ -2 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix} = \begin{pmatrix} 7 \\ -1 \\ k \end{pmatrix}$

i component: $2 + 3\lambda + \mu$

$= 7 \Rightarrow 3\lambda + \mu = 5$

j component: $-1 + 2\lambda - \mu$

$= -1 \Rightarrow 2\lambda - \mu = 0$

Solving these simultaneous equations:

$\lambda = 1, \ \mu = 2$

k component: $3 + 1(-2) + 2(3)$

$= k \Rightarrow k = 7$

b $\begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ 2 \\ -2 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 \\ k \\ 11 \end{pmatrix}$

i component: $2 + 3\lambda + \mu = 1$

$\Rightarrow 3\lambda + \mu = -1$

k component: $3 - 2\lambda + 3\mu = 11$

$\Rightarrow -2\lambda + 3\mu = 8$

Solving these simultaneous equations:

$\lambda = -1, \ \mu = 2$

j component: $-1 - 2 + 2(-1) = k \Rightarrow k = -5$

c $\begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ 2 \\ -2 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix} = \begin{pmatrix} k \\ -4 \\ 10 \end{pmatrix}$

j component: $-1 + 2\lambda - \mu = -4$

$\Rightarrow 2\lambda - \mu = -3$

k component: $3 - 2\lambda + 3\mu = 10$

$\Rightarrow -2\lambda + 3\mu = 7$

Solving these simultaneous equations:

$\lambda = -\frac{1}{2}, \ \mu = 2$

i component: $2 - \frac{1}{2}(3) + 2(1) = k \Rightarrow k = \frac{5}{2}$

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3 d $\begin{pmatrix}2\\-1\\3\end{pmatrix}+\lambda\begin{pmatrix}3\\2\\-2\end{pmatrix}+\mu\begin{pmatrix}1\\-1\\3\end{pmatrix}=\begin{pmatrix}10\\k\\-k\end{pmatrix}$

j component: $-1+2\lambda-\mu=k$

$\Rightarrow 2\lambda-\mu=k+1$

k component: $3-2\lambda+3\mu=-k$

$\Rightarrow -2\lambda+3\mu=-k-3$

Adding these two equations together:

$2\mu=-2\Rightarrow\mu=-1$

i component: $2+3\lambda+(-1)=10$

$\Rightarrow\lambda=3$

j component: $-1+3(2)+(-1)(-1)=k$

$\Rightarrow k=6$

4 a i $2(1)-3(2)+5(1)=1$

ii $2(2)-3(-4)+5(-3)=1$

b $\mathbf{n}=2\mathbf{i}-3\mathbf{j}+5\mathbf{k}$

5 a l is normal to the plane and passes through $(2,3,-2)$ so an equation for l is

$$\mathbf{r}=\begin{pmatrix}2\\3\\-2\end{pmatrix}+\lambda\begin{pmatrix}5\\-3\\-4\end{pmatrix}$$

b $\mathbf{r}=\begin{pmatrix}2\\3\\-2\end{pmatrix}+\lambda\begin{pmatrix}5\\-3\\-4\end{pmatrix}$

$$\Rightarrow\frac{x-2}{5}=\frac{y-3}{-3}=\frac{z+2}{-4}$$

6 $x=0,\ 0\le y,z\le 3$

$x=3,\ 0\le y,z\le 3$

$y=0,\ 0\le x,z\le 3$

$y=3,\ 0\le x,z\le 3$

$z=0,\ 0\le x,y\le 3$

$z=3,\ 0\le x,y\le 3$

7 Find the plane containing $A(2,2,3)$ $B(1,5,3)$ and $C(4,3,-1)$, then show that $D(3,6,-1)$ lies in this plane.

$$\overrightarrow{AB}=\overrightarrow{OB}-\overrightarrow{OA}=\begin{pmatrix}-1\\3\\0\end{pmatrix}$$

$$\overrightarrow{AC}=\overrightarrow{OC}-\overrightarrow{OA}=\begin{pmatrix}2\\1\\-4\end{pmatrix}$$

So a vector equation of the plane is

$$\mathbf{r}=\begin{pmatrix}2\\2\\3\end{pmatrix}+\lambda\begin{pmatrix}-1\\3\\0\end{pmatrix}+\mu\begin{pmatrix}2\\1\\-4\end{pmatrix}$$

If D also lies in this plane then there exist λ,μ such that

$$\begin{pmatrix}2\\2\\3\end{pmatrix}+\lambda\begin{pmatrix}-1\\3\\0\end{pmatrix}+\mu\begin{pmatrix}2\\1\\-4\end{pmatrix}=\begin{pmatrix}3\\6\\-1\end{pmatrix}$$

k component: $3-4\mu=-1\Rightarrow\mu=1$

j component: $2+3\lambda+1(1)=6\Rightarrow\lambda=1$

Check **i** component: $2+1(-1)+1(2)=3$

Therefore the points are coplanar.

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8 Find the plane containing $A(2,3,4)$ $B(2,-1,3)$ and $C(5,3,-2)$, then show that $D(-1,-9,8)$ lies in this plane.

$$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \begin{pmatrix} 0 \\ -4 \\ -1 \end{pmatrix}$$

$$\overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA} = \begin{pmatrix} 3 \\ 0 \\ -6 \end{pmatrix}$$

So a vector equation of the plane is

$$\mathbf{r} = \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix} + \lambda \begin{pmatrix} 0 \\ -4 \\ -1 \end{pmatrix} + \mu \begin{pmatrix} 3 \\ 0 \\ -6 \end{pmatrix}$$

If D also lies in this plane then there exist λ, μ such that

$$\begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix} + \lambda \begin{pmatrix} 0 \\ -4 \\ -1 \end{pmatrix} + \mu \begin{pmatrix} 3 \\ 0 \\ -6 \end{pmatrix} = \begin{pmatrix} -1 \\ -9 \\ 8 \end{pmatrix}$$

i component: $2 + 3\mu = -1 \Rightarrow \mu = -1$

j component: $3 - 4\lambda = -9 \Rightarrow \lambda = 3$

Check **k** component: $4 + 3(-1) - (-6) = 7 \neq 8$

So the equations are not consistent.

Therefore the points are not coplanar.

9 **a** $\mathbf{a} = \begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix} + 1 \begin{pmatrix} -2 \\ 3 \\ 5 \end{pmatrix} + 2 \begin{pmatrix} 4 \\ 2 \\ -3 \end{pmatrix} = \begin{pmatrix} 9 \\ 5 \\ 0 \end{pmatrix}$

b $\begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} -2 \\ 3 \\ 5 \end{pmatrix} + \mu \begin{pmatrix} 4 \\ 2 \\ -3 \end{pmatrix} = \begin{pmatrix} 1 \\ -7 \\ -1 \end{pmatrix}$

i component: $3 - 2\lambda + 4\mu = 1 \Rightarrow \lambda - 2\mu = 1$

j component: $-2 + 3\lambda + 2\mu = -7 \Rightarrow 3\lambda + 2\mu = -5$

k component: $1 + 5\lambda - 3\mu = -1 \Rightarrow 5\lambda - 3\mu = -2$

$\lambda = \mu = -1$ satisfies all of the equations, so $B(1,-7,-1)$ lies on the plane.

c $A(9,5,0)$ and $B(1,-7,-1)$

So an equation for the line through A and B is $\mathbf{r} = \begin{pmatrix} 9 \\ 5 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} -8 \\ -12 \\ -1 \end{pmatrix}$

d $|\overrightarrow{OA}| = |\overrightarrow{OC}|$

$|\overrightarrow{OA}| = \sqrt{9^2 + 5^2} = \sqrt{106}$

$\therefore |\overrightarrow{OC}| = \sqrt{(9-8\lambda)^2 + (5-12\lambda)^2 + \lambda^2} = \sqrt{106}$

$\Rightarrow 209\lambda^2 - 264\lambda + 106 = 106$

$\Rightarrow \lambda(19\lambda - 24) = 0$

$\lambda \neq 0$ (otherwise that would give A) so $\lambda = \frac{24}{19}$

Therefore C has position vector

$$\begin{pmatrix} 9 \\ 5 \\ 0 \end{pmatrix} + \frac{24}{19} \begin{pmatrix} -8 \\ -12 \\ -1 \end{pmatrix} = \begin{pmatrix} -\frac{21}{19} \\ -\frac{193}{19} \\ -\frac{24}{19} \end{pmatrix}$$

Challenge

We need to show that the plane contains the line. λ and μ can take any real values. So if we restrict the values of λ and μ, those points also lie in the plane. Let λ = 3t - 2 and μ = t + 1. This gives the equation of the line, so it must be contained in the plane.

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