Proof by induction 8C

1 $n=1;\ \text{LHS}=\begin{pmatrix}1 & 2\\ 0 & 1\end{pmatrix}^1=\begin{pmatrix}1 & 2\\ 0 & 1\end{pmatrix}$

$$\text{RHS}=\begin{pmatrix}1 & 2(1)\\ 0 & 1\end{pmatrix}=\begin{pmatrix}1 & 2\\ 0 & 1\end{pmatrix}$$

As LHS = RHS, the matrix equation is true for $n=1$.

Assume that the matrix equation is true for $n=k$.

i.e. $\begin{pmatrix}1 & 2\\ 0 & 1\end{pmatrix}^k=\begin{pmatrix}1 & 2k\\ 0 & 1\end{pmatrix}$

With $n=k+1$ the matrix equation becomes

$$\begin{pmatrix}1 & 2\\ 0 & 1\end{pmatrix}^{k+1}=\begin{pmatrix}1 & 2\\ 0 & 1\end{pmatrix}^k\begin{pmatrix}1 & 2\\ 0 & 1\end{pmatrix}$$

$$=\begin{pmatrix}1 & 2k\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & 2\\ 0 & 1\end{pmatrix}$$

$$=\begin{pmatrix}1+0 & 2+2k\\ 0+0 & 0+1\end{pmatrix}.$$

$$=\begin{pmatrix}1 & 2(k+1)\\ 0 & 1\end{pmatrix}$$

Therefore the matrix equation is true when $n=k+1$.

If the matrix equation is true for $n=k$, then it is shown to be true for $n=k+1$. As the matrix equation is true for $n=1$, it is now also true for all $n\geqslant 1$ and $n\in\mathbb{Z}^+$, by mathematical induction.

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2 $n=1;\ \text{LHS}=\begin{pmatrix}3 & -4\\ 1 & -1\end{pmatrix}^1=\begin{pmatrix}3 & -4\\ 1 & -1\end{pmatrix}$

$$\text{RHS}=\begin{pmatrix}2(1)+1 & -4(1)\\ 1 & -2(1)+1\end{pmatrix}=\begin{pmatrix}3 & -4\\ 1 & -1\end{pmatrix}$$

As $\text{LHS}=\text{RHS},$ the matrix equation is true for $n=1.$

Assume that the matrix equation is true for $n=k.$

i.e. $\begin{pmatrix}3 & -4\\ 1 & -1\end{pmatrix}^k=\begin{pmatrix}2k+1 & -4k\\ k & -2k+1\end{pmatrix}.$

With $n=k+1$ the matrix equation becomes

$$\begin{pmatrix}3 & -4\\ 1 & -1\end{pmatrix}^{k+1}=\begin{pmatrix}3 & -4\\ 1 & -1\end{pmatrix}^k\begin{pmatrix}3 & -4\\ 1 & -1\end{pmatrix}$$

$$=\begin{pmatrix}2k+1 & -4k\\ k & -2k+1\end{pmatrix}\begin{pmatrix}3 & -4\\ 1 & -1\end{pmatrix}$$

$$=\begin{pmatrix}6k+3-4k & -8k-4+4k\\ 3k-2k+1 & -4k+2k-1\end{pmatrix}$$

$$=\begin{pmatrix}2k+3 & -4k-4\\ k+1 & -2k-1\end{pmatrix}$$

$$=\begin{pmatrix}2(k+1)+1 & -4(k+1)\\ (k+1) & -2(k+1)+1\end{pmatrix}$$

Therefore the matrix equation is true when $n=k+1.$

If the matrix equation is true for $n=k,$ then it is shown to be true for $n=k+1.$ As the matrix equation is true for $n=1,$ it is now also true for all $n\geqslant 1$ and $n\in\mathbb{Z}^+$ by mathematical induction.

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3 $n=1$; LHS $=\begin{pmatrix}2 & 0\\ 1 & 1\end{pmatrix}^1=\begin{pmatrix}2 & 0\\ 1 & 1\end{pmatrix}$

RHS $=\begin{pmatrix}2^1 & 0\\ 2^1-1 & 1\end{pmatrix}=\begin{pmatrix}2 & 0\\ 1 & 1\end{pmatrix}$

As LHS=RHS, the matrix equation is true for $n = 1$.

Assume that the matrix equation is true for $n = k$.

ie. $\begin{pmatrix}2 & 0\\ 1 & 1\end{pmatrix}^k=\begin{pmatrix}2^k & 0\\ 2^k-1 & 1\end{pmatrix}$

With $n=k+1$ the matrix equation becomes

$$\begin{pmatrix}2 & 0\\ 1 & 1\end{pmatrix}^{k+1}=\begin{pmatrix}2 & 0\\ 1 & 1\end{pmatrix}^k\begin{pmatrix}2 & 0\\ 1 & 1\end{pmatrix}$$

$$=\begin{pmatrix}2^k & 0\\ 2^k-1 & 1\end{pmatrix}\begin{pmatrix}2 & 0\\ 1 & 1\end{pmatrix}$$

$$=\begin{pmatrix}2(2^k)+0 & 0+0\\ 2(2^k)-2+1 & 0+1\end{pmatrix}$$

$$=\begin{pmatrix}2^{1(2^k)} & 0\\ 2^{1(2^k)}-1 & 1\end{pmatrix}$$

$$=\begin{pmatrix}2^{k+1} & 0\\ 2^{k+1}-1 & 1\end{pmatrix}$$

Therefore the matrix equation is true when $n=k+1$.

If the matrix equation is true for $n=k$, then it is shown to be true for $n=k+1$. As the matrix equation is true for $n=1$, it is now also true for all $n \geqslant 1$ and $n \in \mathbb{Z}^+$ by mathematical induction.

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4 a $n=1$; LHS $= \begin{pmatrix} 5 & -8 \\ 2 & -3 \end{pmatrix}^1 = \begin{pmatrix} 5 & -8 \\ 2 & -3 \end{pmatrix}$

$$\text{RHS} = \begin{pmatrix} 4(1)+1 & -8(1) \\ 2(1) & 1-4(1) \end{pmatrix} = \begin{pmatrix} 5 & -8 \\ 2 & -3 \end{pmatrix}$$

As LHS = RHS, the matrix equation is true for $n = 1$.

Assume that the matrix equation is true for $n = k$.

ie. $\begin{pmatrix} 5 & -8 \\ 2 & -3 \end{pmatrix}^k = \begin{pmatrix} 4k+1 & -8k \\ 2k & 1-4k \end{pmatrix}$.

With $n = k + 1$ the matrix equation becomes

$$\begin{pmatrix} 5 & -8 \\ 2 & -3 \end{pmatrix}^{k+1} = \begin{pmatrix} 5 & -8 \\ 2 & -3 \end{pmatrix}^k \begin{pmatrix} 5 & -8 \\ 2 & -3 \end{pmatrix}$$

$$= \begin{pmatrix} 4k+1 & -8k \\ 2k & 1-4k \end{pmatrix} \begin{pmatrix} 5 & -8 \\ 2 & -3 \end{pmatrix}$$

$$= \begin{pmatrix} 20k+5-16k & -32-8+24k \\ 10k+2-8k & -16k-3+12k \end{pmatrix}$$

$$= \begin{pmatrix} 4k+5 & -8k-8 \\ 2k+2 & -4k-3 \end{pmatrix}$$

$$= \begin{pmatrix} 4(k+1)+1 & -8(k+1) \\ 2(k+1) & 1-4(k+1) \end{pmatrix}$$

Therefore the matrix equation is true when $n = k + 1$.

If the matrix equation is true for $n = k$, then it is shown to be true for $n = k + 1$. As the matrix equation is true for $n = 1$, it is now also true for all $n \geqslant 1$ and $n \in \mathbb{Z}^+$ by mathematical induction.

b n = 6

5 a Basis: $n = 1$: LHS = RHS $= \begin{pmatrix} 2 & 5 \\ 0 & 1 \end{pmatrix}$

Assumption: $\mathbf{M}^k = \begin{pmatrix} 2k & 5(2k-1) \\ 0 & 1 \end{pmatrix}$

Induction: With $n = k + 1$ the matrix equation becomes

$$\mathbf{M}^{k+1} = \mathbf{M}^k \begin{pmatrix} 2 & 5 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 2k & 5(2k-1) \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 5 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 2k+1 & 5\times 2k + 5(2k-1) \\ 0 & 1 \end{pmatrix}$$

So if the statement holds for $n = k$, it holds for $n = k + 1$.

Conclusion: The statement holds for all $n \in \mathbb{Z}^+$.

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5 b $\begin{pmatrix} 2-n & 5(2-n-1) \\ 0 & 1 \end{pmatrix}$

Challenge

Basis: $n = 1$: LHS = RHS = $\begin{pmatrix} 3 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 4 \end{pmatrix}$

Assumption: $\begin{pmatrix} 3 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 4 \end{pmatrix} k = \begin{pmatrix} 3k & \frac{3k-1}{2} & 0 \\ 0 & 1 & 0 \\ 0 & \frac{1-4k}{3} & 4k \end{pmatrix}$

Induction:

$$\begin{pmatrix} 3 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 4 \end{pmatrix}^{k+1} = \begin{pmatrix} 3 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 4 \end{pmatrix}^{k} \begin{pmatrix} 3 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 4 \end{pmatrix}$$

$$= \begin{pmatrix} 3k & \frac{3k-1}{2} & 0 \\ 0 & 1 & 0 \\ 0 & \frac{1-4k}{3} & 4k \end{pmatrix} \begin{pmatrix} 3 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 4 \end{pmatrix}$$

$$= \begin{pmatrix} 3k+1 & 3k+\frac{3k-1}{2} & 0 \\ 0 & 1 & 0 \\ 0 & \frac{1-4k}{3}-4k & 4k\times 4 \end{pmatrix} = \begin{pmatrix} 3k+1 & \frac{3k+1-1}{2} & 0 \\ 0 & 1 & 0 \\ 0 & \frac{1-4k+1}{3} & 4k+1 \end{pmatrix}$$

So if the statement holds for $n = k$, it holds for $n = k + 1$.

If the matrix equation is true for $n = k$, then it is shown to be true for $n = k + 1$. As the matrix equation is true for $n = 1$, it is now also true for all $n \geqslant 1$ and $n \in \mathbb{Z}^+$ by mathematical induction.

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