## Matrices 6C

**1 a** $\begin{vmatrix} 3 & 4 \\ -1 & 2 \end{vmatrix} = 3\times 2 - 4\times(-1) = 10$

**b** $\begin{vmatrix} 4 & 2 \\ 1 & 2 \end{vmatrix} = 4\times 2 - 2\times 1 = 6$

**c** $\begin{vmatrix} -2 & 1 \\ 1 & 2 \end{vmatrix} = (-2)\times 0 - 1\times 3 = -3$

**d** $\begin{vmatrix} -4 & -4 \\ 1 & 1 \end{vmatrix} = (-4)\times 1 - (-4)\times 1 = 0$

**e** $\begin{vmatrix} 7 & -4 \\ 0 & 3 \end{vmatrix} = 7\times 3 - (-4)\times 0 = 21$

**f** $\begin{vmatrix} -1 & -1 \\ -6 & -10 \end{vmatrix} = (-1)\times(-10) - (-1)\times(-6) = 4$

**2 a** $\det\begin{vmatrix} a & 1+a \\ 3 & 2 \end{vmatrix} = 2a - 3(1+a)$

$= 2a - 3 - 3a$

$= -3 - a$

Matrix is singular for $a = -3$

**b** Let $\mathbf{A} = \begin{pmatrix} 1+a & 3-a \\ a+2 & 1-a \end{pmatrix}$

$\det \mathbf{A} = (1+a)(1-a) - (3-a)(a+2)$

$= 1 - a^2 - (-a^2 + a + 6)$

$= 1 - a^2 + a^2 - a - 6$

$= -a - 5$

$\det \mathbf{A} = 0 \Rightarrow a = -5$

**c** Let $\mathbf{B} = \begin{pmatrix} 2+a & 1-a \\ 1-a & a \end{pmatrix}$

$\det \mathbf{B} = 2a + a^2 - (1-a)^2$

$= 2a + a^2 - 1 + 2a - a^2$

$= 4a - 1$

$\det \mathbf{B} = 0 \Rightarrow a = \frac{1}{4}$

**3** $\det \mathbf{M} = \begin{vmatrix} -2 & 1-k \\ k-1 & k \end{vmatrix} = (-2)\times k - (1-k)\times(k-1)$

$= k^2 - 4k + 1$

For singular matrix $k^2 - 4k + 1 = 0$

$k = \frac{-(-4) \pm \sqrt{(-4)^2 - 4\times 1\times 1}}{2\times 1}$

$= \frac{4 \pm \sqrt{12}}{2}$

$k = 2 + \sqrt{3},\ k = 2 - \sqrt{3}$

**4** $\det \mathbf{P} = \begin{vmatrix} 3k & 4-k \\ k-2 & -k \end{vmatrix}$

$= 3k\times(-k) - (4-k)\times(k-2)$

$= -2k^2 - 6k + 8$

For singular matrix $2k^2 + 6k - 8 = 0$

$(k+4)(k-1) = 0$

$k = -4$ and $k = 1$

**5 a** $\mathbf{A} = \begin{pmatrix} a & 2a \\ b & 2b \end{pmatrix} \Rightarrow \det \mathbf{A} = 2ab - 2ab = 0$

$\mathbf{B} = \begin{pmatrix} 2b & -2a \\ -b & a \end{pmatrix} \Rightarrow \det \mathbf{B} = 2ab - 2ab = 0$

**b** $\mathbf{AB} = \begin{pmatrix} a & 2a \\ b & 2b \end{pmatrix}\begin{pmatrix} 2b & -2a \\ -b & a \end{pmatrix}$

$= \begin{pmatrix} 2ab - 2ab & -2a^2 + 2a^2 \\ 2b^2 - 2b^2 & -2ab + 2ab \end{pmatrix}$

$= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

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**6 a** $\begin{vmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{vmatrix} = 1\begin{vmatrix} 2 & 0 \\ 0 & 3 \end{vmatrix} - 0\begin{vmatrix} 0 & 0 \\ 0 & 3 \end{vmatrix} + 0\begin{vmatrix} 0 & 2 \\ 0 & 0 \end{vmatrix}$

$= 1(6-0) - 0 + 0 = 6$

**b** $\begin{vmatrix} 0 & 4 & 0 \\ 5 & -2 & 3 \\ 2 & 1 & 4 \end{vmatrix} = 0\begin{vmatrix} -2 & 3 \\ 1 & 4 \end{vmatrix} - 4\begin{vmatrix} 5 & 3 \\ 2 & 4 \end{vmatrix} + 0\begin{vmatrix} 5 & 2 \\ 2 & 1 \end{vmatrix}$

$= 0 - 4(20-6) + 0 = -56$

**c** $\begin{vmatrix} 1 & 0 & 1 \\ 2 & 4 & 1 \\ 3 & 5 & 2 \end{vmatrix} = 1\begin{vmatrix} 4 & 1 \\ 5 & 2 \end{vmatrix} - 0\begin{vmatrix} 2 & 1 \\ 3 & 2 \end{vmatrix} + 1\begin{vmatrix} 2 & 4 \\ 3 & 5 \end{vmatrix}$

$= 1(8-5) - 0 + 1(10-12)$

$= 3 - 2 = 1$

**d** $\begin{vmatrix} 2 & -3 & 4 \\ 2 & 2 & 2 \\ 5 & 5 & 5 \end{vmatrix} = 2\begin{vmatrix} 2 & 2 \\ 5 & 5 \end{vmatrix} - (-3)\begin{vmatrix} 2 & 2 \\ 5 & 5 \end{vmatrix} + 4\begin{vmatrix} 2 & 2 \\ 5 & 5 \end{vmatrix}$

$= 2(10-10) + 3(10-10) + 4(10-10) = 0$

**7 a** $\begin{vmatrix} 4 & 3 & -1 \\ 2 & -2 & 0 \\ 0 & 4 & -2 \end{vmatrix}$

$= 4\begin{vmatrix} -2 & 0 \\ 4 & -2 \end{vmatrix} - 3\begin{vmatrix} 2 & 0 \\ 0 & -2 \end{vmatrix} + (-1)\begin{vmatrix} 2 & -2 \\ 0 & 4 \end{vmatrix}$

$= 4(4-0) - 3(-4-0) - 1(8-0)$

$= 16 + 12 - 8 = 20$

**b** $\begin{vmatrix} 3 & -2 & 1 \\ 4 & 1 & -3 \\ 7 & 2 & -4 \end{vmatrix}$

$= 3\begin{vmatrix} 1 & -3 \\ 2 & -4 \end{vmatrix} - (-2)\begin{vmatrix} 4 & -3 \\ 7 & -4 \end{vmatrix} + 1\begin{vmatrix} 4 & 1 \\ 7 & 2 \end{vmatrix}$

$= 3(-4+6) + 2(-16+21) + 1(8-7)$

$= 6 + 10 + 1 = 17$

**7 c** $\begin{vmatrix} 5 & -2 & -3 \\ 6 & 4 & 2 \\ -2 & -4 & -3 \end{vmatrix}$

$= 5\begin{vmatrix} 4 & 2 \\ -4 & -3 \end{vmatrix} - (-2)\begin{vmatrix} 6 & 2 \\ -2 & -3 \end{vmatrix} + (-3)\begin{vmatrix} 6 & 4 \\ -2 & -4 \end{vmatrix}$

$= 5(-12+8) + 2(-18+4) - 3(-24+8)$

$= 5 \times (-4) + 2 \times (-14) - 3 \times (-16)$

$= -20 - 28 + 48 = 0$

**8** $\det(\mathbf{A}) = \begin{vmatrix} 2 & 1 & -4 \\ 2k+1 & 3 & k \\ 1 & 0 & 1 \end{vmatrix}$

$= 2\begin{vmatrix} 3 & k \\ 0 & 1 \end{vmatrix} - 1\begin{vmatrix} 2k+1 & k \\ 1 & 1 \end{vmatrix} + (-4)\begin{vmatrix} 2k+1 & 3 \\ 1 & 0 \end{vmatrix}$

$= 2(3-0) - 1(2k+1-k) - 4(0-3)$

$= 6 - k - 1 + 12 = 17 - k$

As $\mathbf{A}$ is singular,

$\det(\mathbf{A}) = 0$

$17 - k = 0$

$k = 17$

**9** $\det(\mathbf{A}) = \begin{vmatrix} 2 & -1 & 3 \\ k & 2 & 4 \\ -2 & 1 & k+3 \end{vmatrix}$

$= 2\begin{vmatrix} 2 & 4 \\ 1 & k+3 \end{vmatrix} - (-1)\begin{vmatrix} k & 4 \\ -2 & k+3 \end{vmatrix} + 3\begin{vmatrix} k & 2 \\ -2 & 1 \end{vmatrix}$

$= 2(2k+6-4) + 1(k^2+3k+8) + 3(k+4)$

$= 4k + 4 + k^2 + 3k + 8 + 3k + 12$

$= k^2 + 10k + 24$

As $\det(\mathbf{A}) = 8$

$k^2 + 10k + 24 = 8$

$k^2 + 10k + 16 = 0$

$(k+8)(k+2) = 0$

$k = -8, -2$

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**10 a** $\det(\mathbf{A}) = \begin{vmatrix} 2 & 5 & 3 \\ -2 & 0 & 4 \\ 3 & 10 & 8 \end{vmatrix}$

$= 2\begin{vmatrix} 0 & 4 \\ 10 & 8 \end{vmatrix} - 5\begin{vmatrix} -2 & 4 \\ 3 & 8 \end{vmatrix} + 3\begin{vmatrix} -2 & 0 \\ 3 & 10 \end{vmatrix}$

$= 2(0-40) - 5(-16-12) + 3(-20-0)$

$= -80 + 140 - 60 = 0$

Hence **A** is singular.

**b** $\mathbf{AB} = \begin{pmatrix} 2 & 5 & 3 \\ -2 & 0 & 4 \\ 3 & 10 & 8 \end{pmatrix}\begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 2 \\ 0 & -2 & -1 \end{pmatrix}$

$= \begin{pmatrix} 2+5+0 & 2+10-6 & 0+10-3 \\ -2+0+0 & -2+0-8 & 0+0-4 \\ 3+10+0 & 3+20-16 & 0+20-8 \end{pmatrix}$

$= \begin{pmatrix} 7 & 6 & 7 \\ -2 & -10 & -4 \\ 13 & 7 & 12 \end{pmatrix}$

**c** $\det(\mathbf{AB}) = \begin{vmatrix} 7 & 6 & 7 \\ -2 & -10 & -4 \\ 13 & 7 & 12 \end{vmatrix}$

$= 7\begin{vmatrix} -10 & -4 \\ 7 & 12 \end{vmatrix} - 6\begin{vmatrix} -2 & -4 \\ 13 & 12 \end{vmatrix} + 7\begin{vmatrix} -2 & -10 \\ 13 & 7 \end{vmatrix}$

$= 7(-120+28) - 6(-24+52) + 7(-14+130)$

$= 7\times(-92) - 6\times 28 + 7\times 116$

$= -644 - 168 + 812 = 0$

Hence **AB** is also singular.

**11** $\begin{vmatrix} 0 & a & -b \\ -a & 0 & c \\ b & -c & 0 \end{vmatrix} = 0\begin{vmatrix} 0 & c \\ -c & 0 \end{vmatrix}$

$- a\begin{vmatrix} -a & c \\ b & 0 \end{vmatrix} + (-b)\begin{vmatrix} -a & 0 \\ b & -c \end{vmatrix}$

$= 0 - a(0-cb) - b(ac-0)$

$= abc - abc = 0$

Hence the matrix is singular for all $a$, $b$ and $c$.

**12** $\begin{vmatrix} 2 & -2 & 4 \\ 3 & x & -2 \\ -1 & 3 & x \end{vmatrix} = 2\begin{vmatrix} x & -2 \\ 3 & x \end{vmatrix}$

$-(-2)\begin{vmatrix} 3 & -2 \\ -1 & x \end{vmatrix} + 4\begin{vmatrix} 3 & x \\ -1 & 3 \end{vmatrix}$

$= 2(x^2+6) + 2(3x-2) + 4(9+x)$

$= 2x^2 + 12 + 6x - 4 + 36 + 4x$

$= 2x^2 + 10x + 44$

$= 2(x^2+5)x + 44$

$= 2\left(x^2 + 5x + \left(\frac{5}{2}\right)^2\right) + 44 - 2\times\left(\frac{5}{2}\right)^2$

$= 2\left(x + \frac{5}{2}\right)^2 + 31\frac{1}{2} \geq 31\frac{1}{2}$, for all real $x$

Hence the determinant cannot be zero and the matrix is non-singular for all real $x$.

**13** $\begin{vmatrix} x-3 & -2 & 0 \\ 1 & x & -2 \\ -2 & -1 & x+1 \end{vmatrix}$

$= (x-3)\begin{vmatrix} x & -2 \\ -1 & x+1 \end{vmatrix} - (-2)\begin{vmatrix} 1 & -2 \\ -2 & x+1 \end{vmatrix} + 0\begin{vmatrix} 1 & x \\ -2 & -1 \end{vmatrix}$

$= (x-3)(x^2+x-2) + 2(x+1-4) + 0$

$= x^3 + x^2 - 2x - 3x^2 - 3x + 6 + 2x - 6$

$= x^3 - 2x^2 - 3x$

From the matrix to be singular, the determinant must be zero.

$x^3 - 2x^2 - 3x = x(x^2 - 2x - 3)$

$= x(x-3)(x+1)$

$= 0$

$x = -1, 0, 3$

**14 a** $\det \mathbf{M} = \begin{vmatrix} 1 & -3 \\ 2 & 1 \end{vmatrix} = 1\times 1 - (-3)\times 2 = 7$

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**14 b** $\det \mathbf{N} = \begin{vmatrix} -1 & k \\ 4 & 3 \end{vmatrix} = (-1) \times 1 - k \times 4 = -3 - 4k$

$\det \mathbf{N} = 7$

$-3 - 4k = 7$

$4k = -10 \Rightarrow k = -2.5$

**c** $\mathbf{MN} = \begin{pmatrix} 1 & -3 \\ 2 & 1 \end{pmatrix} \times \begin{pmatrix} -1 & -2.5 \\ 4 & 3 \end{pmatrix}$

$= \begin{pmatrix} -13 & -11.5 \\ 2 & -2 \end{pmatrix}$

**d** $\det \mathbf{MN} = \begin{vmatrix} -13 & -11.5 \\ 2 & -2 \end{vmatrix}$

$= (-13) \times (-2) - (-11.5) \times 2 = 49$

$\det \mathbf{M} = 7$

$\det \mathbf{N} = -3 + (-4) \times (-2.5) = 7$

$\det \mathbf{MN} = 7 \times 7 = 49$

**15 a** $\det \mathbf{A} = \begin{vmatrix} 2 & 1 & -1 \\ 1 & 0 & 4 \\ -4 & 2 & 1 \end{vmatrix}$

$= 2(0 - 8) - 1(1 + 16) - 1(2 - 0)$

$= -16 - 17 - 2$

$= -35$

**b** $\det \mathbf{B} = \begin{vmatrix} 3 & 1 & 2 \\ k & 4 & 5 \\ 0 & 2 & 3 \end{vmatrix}$

$= 3(12 - 10) - 1(3k - 0) + 2(2k - 0)$

$= 6 - 3k + 4k$

$= 6 + k$

$\det \mathbf{B} = 2$

$6 + k = 2 \Rightarrow k = -4$

**c** $\mathbf{AB} = \begin{pmatrix} 2 & 1 & -1 \\ 1 & 0 & 4 \\ -4 & 2 & 1 \end{pmatrix} \times \begin{pmatrix} 3 & 1 & 2 \\ -4 & 4 & 5 \\ 0 & 2 & 3 \end{pmatrix}$

$= \begin{pmatrix} 2 & 4 & 6 \\ 3 & 9 & 14 \\ -20 & 6 & 5 \end{pmatrix}$

**15 d** $\det \mathbf{AB} = \begin{vmatrix} 2 & 4 & 6 \\ 3 & 9 & 14 \\ -20 & 6 & 5 \end{vmatrix}$

$= 2(45 - 84) - 4(15 + 280) + 6(18 + 180)$

$= -78 - 1180 + 1188$

$= -70$

$= (-35) \times 2 = \det \mathbf{A} \times \det \mathbf{B}$

**Challenge**

**a** $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc = 0$ for singular matrix

$ad - bc = 0 \Rightarrow ad = bc$

The possibilities are:

$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}, \begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix},$

$\begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}, \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}, \begin{pmatrix} -1 & -1 \\ -1 & -1 \end{pmatrix}$

**b** $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},$

$\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$

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