Complex numbers 1A

1 a $\sqrt{9}\sqrt{(-1)} = 3\text{i}$

b $\sqrt{49}\sqrt{(-1)} = 7\text{i}$

c $\sqrt{121}\sqrt{(-1)} = 11\text{i}$

d $\sqrt{1000}\sqrt{(-1)} = 100\text{i}$

e $\sqrt{225}\sqrt{(-1)} = 15\text{i}$

f $\sqrt{5}\sqrt{(-1)} = \text{i}\sqrt{5}$

g $\sqrt{12}\sqrt{(-1)} = \sqrt{4}\sqrt{3}\sqrt{(-1)} = 2\text{i}\sqrt{3}$

h $\sqrt{45}\sqrt{(-1)} = \sqrt{9}\sqrt{5}\sqrt{(-1)} = 3\text{i}\sqrt{5}$

i $\sqrt{200}\sqrt{(-1)} = \sqrt{100}\sqrt{2}\sqrt{(-1)} = 10\text{i}\sqrt{2}$

j $\sqrt{147}\sqrt{(-1)} = \sqrt{49}\sqrt{3}\sqrt{(-1)} = 7\text{i}\sqrt{3}$

2 a $(5+8)+\text{i}(2+9) = 13+11\text{i}$

b $(4+1)+\text{i}(10-8) = 5+2\text{i}$

c $(7-3)+\text{i}(6-5) = 4+\text{i}$

d $\left(\frac{1}{2}+\frac{5}{2}\right)+\text{i}\left(\frac{1}{3}+\frac{5}{3}\right) = 3+2\text{i}$

e $(20-11)+\text{i}(12-3) = 9+9\text{i}$

f $(2--5)+\text{i}(-1-3) = 7-4\text{i}$

g $(-4--8)+\text{i}(-6--8) = 4+2\text{i}$

h $(3\sqrt{2}-\sqrt{2})+\text{i}(1--1) = 2\sqrt{2}+2\text{i}$

i $(-2+1--12)+\text{i}(-7+3-1) = 11-5\text{i}$

j $(18-15-3)+\text{i}(5--2-7) = 0$

3 a $14+4\text{i}$

b $24-12\text{i}$

c $(6+2\text{i})+(6+3\text{i}) = (6+6)+\text{i}(2+3)$
$= 12+5\text{i}$

d $(20+15\text{i})+(4-8\text{i}) = (20+4)+\text{i}(15-8)$
$= 24+7\text{i}$

e $\frac{6-4\text{i}}{2} = \frac{6}{2}-\frac{4}{2}\text{i}$
$= 3-2\text{i}$

f $\frac{15+25\text{i}}{5} = \frac{15}{5}-\frac{25}{5}\text{i}$
$= 3+5\text{i}$

g $\frac{9+11\text{i}}{3} = \frac{9}{3}+\frac{11}{3}\text{i}$
$= 3+\frac{11}{3}\text{i}$

h $\frac{-8+3\text{i}}{4}-\frac{7-2\text{i}}{2} = -2+\frac{3}{4}\text{i}-\frac{7}{2}+\text{i}$
$= \left(-\frac{4}{2}-\frac{7}{2}\right)+\text{i}\left(\frac{3}{4}+\frac{4}{4}\right)$
$= -\frac{11}{2}+\frac{7}{4}\text{i}$

4 a $\frac{4-2\text{i}}{\sqrt{2}} = \frac{4-2\text{i}}{\sqrt{2}}\times\frac{\sqrt{2}}{\sqrt{2}}$
$= \frac{4\sqrt{2}-2\sqrt{2}\text{i}}{2}$
$= 2\sqrt{2}-\text{i}\sqrt{2}$

b $\frac{2-6\text{i}}{1+\sqrt{3}} = \frac{2-6\text{i}}{1+\sqrt{3}}\times\frac{\left(1-\sqrt{3}\right)}{\left(1-\sqrt{3}\right)}$
$= \frac{2-2\sqrt{3}-6\text{i}+6\sqrt{3}\text{i}}{1-3}$
$= \frac{2-2\sqrt{3}}{-2}+\frac{6\sqrt{3}-6}{-2}\text{i}$
$= \left(-1+\sqrt{3}\right)+\left(3-3\sqrt{3}\right)\text{i}$

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5 a $z-w=(7-6\mathrm{i})-(7+6\mathrm{i})$
$=(7-7)+\mathrm{i}(-6-6)$
$=-12\mathrm{i}$

b $w+z=(7+6\mathrm{i})+(7-6\mathrm{i})$
$=(7+7)+\mathrm{i}(6-6)$
$=14$

6 $7+2\mathrm{i}=z_2-z_1$
$=(-3+b\mathrm{i})-(a+9\mathrm{i})$
$=(-3-a)+(b-9)\mathrm{i}$
Equate real parts:
$7=-3-a\Rightarrow a=-10$
Equate imaginary parts:
$2=b-9\Rightarrow b=11$

7 a $z_1-z_2=(4+\mathrm{i})-(7-3\mathrm{i})$
$=(4-7)+\mathrm{i}(1--3)$
$=-3+4\mathrm{i}$

b $4z_2=4(7-3\mathrm{i})$
$=28-12\mathrm{i}$

c $2z_1+5z_2=2(4+\mathrm{i})+5(7-3\mathrm{i})$
$=8+2\mathrm{i}+35-15\mathrm{i}$
$=43-13\mathrm{i}$

8 a $z+w=(a+b\mathrm{i})+a-b\mathrm{i}$
$=2a$
So $z+w$ is always real.

b $z-w=(a+b\mathrm{i})-(a-b\mathrm{i})$
$=2b\mathrm{i}$
So $z-w$ is always imaginary.

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Complex numbers 1B

1 a $z^2 + 121 = 0$

$z^2 = -121$

$z = \pm\sqrt{-121}$

$= \pm\sqrt{121}\sqrt{-1}$

$= \pm 11\text{i}$

b $z^2 + 40 = 0$

$z^2 = -40$

$z = \pm\sqrt{-40}$

$= \pm\sqrt{4}\sqrt{10}\sqrt{-1}$

$= \pm\sqrt{4}\sqrt{10}\text{i}$

$= \pm 2\text{i}\sqrt{10}$

c $2z^2 + 120 = 0$

$2z^2 = -120$

$z^2 = -60$

$z = \pm\sqrt{-60}$

$= \pm\sqrt{4}\sqrt{15}\sqrt{-1}$

$= \pm 2\text{i}\sqrt{15}$

d $3z^2 + 150 = 38 - z^2$

$4z^2 = -112$

$z^2 = -28$

$z = \pm\sqrt{-28}$

$= \pm\sqrt{4}\sqrt{7}\sqrt{-1}$

$= \pm 2\text{i}\sqrt{7}$

e $z^2 + 30 = -3z^2 - 66$

$4z^2 = -96$

$z^2 = -24$

$z = \pm\sqrt{-24}$

$= \pm\sqrt{4}\sqrt{6}\sqrt{-1}$

$= \pm 2\text{i}\sqrt{6}$

1 f

$6z^2 + 1 = 2z^2$

$4z^2 = -1$

$z^2 = -\frac{1}{4}$

$z = \pm\sqrt{\frac{-1}{4}}$

$= \pm\sqrt{\frac{1}{4}}\sqrt{-1}$

$= \pm\frac{1}{2}\text{i}$

2 a $(z-3)^2 - 9 = -16$

$(z-3)^2 = -7$

$z - 3 = \pm\text{i}\sqrt{7}$

$z = 3 \pm \text{i}\sqrt{7}$

b $2(z-7)^2 + 30 = 6$

$2(z-7)^2 = -24$

$(z-7)^2 = -12$

$z - 7 = \pm 2\text{i}\sqrt{3}$

$z = 7 \pm 2\text{i}\sqrt{3}$

c $16(z+1)^2 + 11 = 2$

$16(z+1)^2 = -9$

$(z+1)^2 = -\frac{9}{16}$

$z + 1 = \pm\frac{3}{4}\text{i}$

$z = -1 \pm \frac{3}{4}\text{i}$

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3 [Note that **3a**, **3b** & **3c** use the quadratic formula, and **3d**, **3e** & **3f** use completion of the square]

a $a=1,\ b=2,\ c=5$

$$z=\frac{-2\pm\sqrt{(4-20)}}{2}$$
$$=\frac{-2\pm 4\mathrm{i}}{2}$$
$$=-1\pm 2\mathrm{i}$$

b $a=1,\ b=-2,\ c=10$

$$z=\frac{2\pm\sqrt{(4-40)}}{2}$$
$$=\frac{2\pm 6\mathrm{i}}{2}$$
$$=1\pm 3\mathrm{i}$$

c $a=1,\ b=4,\ c=29$

$$z=\frac{-4\pm\sqrt{(16-116)}}{2}$$
$$=\frac{-4\pm 10\mathrm{i}}{2}$$
$$=-2\pm 5\mathrm{i}$$

d

$$z^2+10z+26=0$$
$$(z+5)^2-25+26=0$$
$$(z+5)^2=-1$$
$$z+5=\pm\mathrm{i}$$
$$z=-5\pm\mathrm{i}$$

e

$$z^2+5z+25=0$$
$$\left(z+\frac{5}{2}\right)^2-\frac{25}{4}+25=0$$
$$\left(z+\frac{5}{2}\right)^2=-\frac{75}{4}$$
$$z+\frac{5}{2}=\pm\frac{\mathrm{i}\sqrt{25\times 3}}{\sqrt{4}}$$
$$z=-\frac{5}{2}\pm\frac{5\mathrm{i}\sqrt{3}}{2}$$

3 f

$$z^2+3z+5=0$$
$$\left(z+\frac{3}{2}\right)^2-\frac{9}{4}+5=0$$
$$\left(z+\frac{3}{2}\right)^2=-\frac{11}{4}$$
$$z+\frac{3}{2}=\pm\frac{\mathrm{i}\sqrt{11}}{2}$$
$$z=-\frac{3}{2}\pm\frac{\mathrm{i}\sqrt{11}}{2}$$

4 [Note that **4a** uses completion of the square and **4b** & **4c** use the quadratic formula]

a

$$2z^2+5z+4=0$$
$$z^2+\frac{5}{2}z+2=0$$
$$\left(z+\frac{5}{4}\right)^2-\frac{25}{16}+2=0$$
$$\left(z+\frac{5}{4}\right)^2=-\frac{7}{16}$$
$$z+\frac{5}{4}=\pm\frac{\mathrm{i}\sqrt{7}}{4}$$
$$z=-\frac{5}{4}\pm\frac{\mathrm{i}\sqrt{7}}{4}$$

b Use the quadratic formula with $a = 7$, $b = -3$ and $c = 3$

$$z=\frac{-(-3)\pm\sqrt{(-3)^2-(4)(7)(3)}}{2(7)}$$
$$=\frac{3\pm\sqrt{9-84}}{14}$$
$$=\frac{3\pm\sqrt{-75}}{14}$$
$$=\frac{3}{14}\pm\frac{5\sqrt{3}}{14}\mathrm{i}$$

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4 c Use the quadratic formula with $a = 5$, $b = -1$ and $c = 3$

$$z = \frac{-(-1) \pm \sqrt{(-1)^2 - 4(5)(3)}}{2(5)}$$
$$= \frac{1 \pm \sqrt{1-60}}{10}$$
$$= \frac{1}{10} \pm \frac{\sqrt{59}}{10}\mathrm{i}$$

5 Method 1: Completing the square

$$(z-4)^2 - 16 + 21 = 0$$
$$(z-4)^2 = -5$$
$$z - 4 = \pm\mathrm{i}\sqrt{5}$$
$$z = 4 \pm \mathrm{i}\sqrt{5}$$

So $z_1 = 4 + \mathrm{i}\sqrt{5}$ and $z_2 = 4 - \mathrm{i}\sqrt{5}$

Method 2: using the quadratic formula

$$z = \frac{-(-8) \pm \sqrt{(-8)^2 - 4(1)(21)}}{2(1)}$$
$$= \frac{8 \pm \sqrt{-20}}{2}$$
$$= \frac{8 \pm \mathrm{i}\sqrt{4}\sqrt{5}}{2}$$
$$= 4 \pm \mathrm{i}\sqrt{5}$$

6 $z^2 + bz + 11 = 0$ has two distinct complex roots when

$$b^2 - 4(1)(11) < 0$$
$$b^2 < 44$$
$$-\sqrt{44} < b < \sqrt{44}$$
or $-2\sqrt{11} < b < 2\sqrt{11}$

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Complex numbers 1C

1 a

$$\begin{aligned}(5+\mathrm{i})(3+4\mathrm{i}) &= 5(3+4\mathrm{i})+\mathrm{i}(3+4\mathrm{i})\\ &= 15+20\mathrm{i}+3\mathrm{i}+4\mathrm{i}^2\\ &= 15+20\mathrm{i}+3\mathrm{i}-4\\ &= (15-4)+\mathrm{i}(20+3)\\ &= 11+23\mathrm{i}\end{aligned}$$

b
$$\begin{aligned}(6+3\mathrm{i})(7+2\mathrm{i}) &= 6(7+2\mathrm{i})+3\mathrm{i}(7+2\mathrm{i})\\ &= 42+12\mathrm{i}+21\mathrm{i}+6\mathrm{i}^2\\ &= 42+12\mathrm{i}+21\mathrm{i}-6\\ &= (42-6)+\mathrm{i}(12+21)\\ &= 36+33\mathrm{i}\end{aligned}$$

c
$$\begin{aligned}(5-2\mathrm{i})(1+5\mathrm{i}) &= 5(1+5\mathrm{i})-2\mathrm{i}(1+5\mathrm{i})\\ &= 5+25\mathrm{i}-2\mathrm{i}-10\mathrm{i}^2\\ &= 5+25\mathrm{i}-2\mathrm{i}+10\\ &= (5+10)+\mathrm{i}(25-2)\\ &= 15+23\mathrm{i}\end{aligned}$$

d
$$\begin{aligned}(13-3\mathrm{i})(2-8\mathrm{i}) &= 13(2-8\mathrm{i})-3\mathrm{i}(2-8\mathrm{i})\\ &= 26-104\mathrm{i}-6\mathrm{i}+24\mathrm{i}^2\\ &= 26-104\mathrm{i}-6\mathrm{i}-24\\ &= (26+24)+\mathrm{i}(104-6)\\ &= 2-110\mathrm{i}\end{aligned}$$

e
$$\begin{aligned}(-3-\mathrm{i})(4+7\mathrm{i}) &= -3(4+7\mathrm{i})-\mathrm{i}(4+7\mathrm{i})\\ &= -12-21\mathrm{i}-4\mathrm{i}-7\mathrm{i}^2\\ &= -12-21\mathrm{i}-4\mathrm{i}+7\\ &= (-12+7)+\mathrm{i}(-21-4)\\ &= -5-25\mathrm{i}\end{aligned}$$

f
$$\begin{aligned}(8+5\mathrm{i})(8+5\mathrm{i}) &= 8(8+5\mathrm{i})+5\mathrm{i}(8+5\mathrm{i})\\ &= 64+40\mathrm{i}+40\mathrm{i}+25\mathrm{i}^2\\ &= 64+40\mathrm{i}+40\mathrm{i}-25\\ &= (64-25)+\mathrm{i}(40+40)\\ &= 39+80\mathrm{i}\end{aligned}$$

1 g
$$\begin{aligned}(2-9\mathrm{i})(2-9\mathrm{i}) &= 2(2-9\mathrm{i})-9\mathrm{i}(2-9\mathrm{i})\\ &= 4-18\mathrm{i}-18\mathrm{i}+81\mathrm{i}^2\\ &= 4-18\mathrm{i}-18\mathrm{i}-81\\ &= (4-81)+\mathrm{i}(-18-18)\\ &= -77-36\mathrm{i}\end{aligned}$$

h
$$\begin{aligned}&(1+\mathrm{i})(2+\mathrm{i})(3+\mathrm{i})\\ &= (1+\mathrm{i})[2(3+\mathrm{i})+\mathrm{i}(3+\mathrm{i})]\\ &= (1+\mathrm{i})[6+2\mathrm{i}+3\mathrm{i}+\mathrm{i}^2]\\ &= (1+\mathrm{i})[6+2\mathrm{i}+3\mathrm{i}-1]\\ &= (1+\mathrm{i})(5+5\mathrm{i})\\ &= 1(5+5\mathrm{i})+\mathrm{i}(5+5\mathrm{i})\\ &= 5+5\mathrm{i}+5\mathrm{i}+5\mathrm{i}^2\\ &= 5+5\mathrm{i}+5\mathrm{i}-5\\ &= 10\mathrm{i}\end{aligned}$$

i
$$\begin{aligned}&(3-2\mathrm{i})(5+\mathrm{i})(4-2\mathrm{i})\\ &= (3-2\mathrm{i})[5(4-2\mathrm{i})+\mathrm{i}(4-2\mathrm{i})]\\ &= (3-2\mathrm{i})[20-10\mathrm{i}+4\mathrm{i}-2\mathrm{i}^2]\\ &= (3-2\mathrm{i})[20-10\mathrm{i}+4\mathrm{i}+2]\\ &= (3-2\mathrm{i})[22-6\mathrm{i}]\\ &= 3(22-6\mathrm{i})-2\mathrm{i}(22-6\mathrm{i})\\ &= 66-18\mathrm{i}-44\mathrm{i}+12\mathrm{i}^2\\ &= 66-18\mathrm{i}-44\mathrm{i}-12\\ &= 54-62\mathrm{i}\end{aligned}$$

1 j
$$\begin{aligned}(2+3i)^3 &= (2+3\mathrm{i})[(2+3\mathrm{i})(2+3\mathrm{i})]\\ &= (2+3\mathrm{i})[2(2+3\mathrm{i})+3\mathrm{i}(2+3\mathrm{i})]\\ &= (2+3\mathrm{i})[4+6\mathrm{i}+6\mathrm{i}+9\mathrm{i}^2]\\ &= (2+3\mathrm{i})[4+6\mathrm{i}+6\mathrm{i}-9]\\ &= (2+3\mathrm{i})(-5+12\mathrm{i})\\ &= 2(-5+12\mathrm{i})+3\mathrm{i}(-5+12\mathrm{i})\\ &= -10+24\mathrm{i}-15\mathrm{i}+36\mathrm{i}^2\\ &= -10+24\mathrm{i}-15\mathrm{i}-36\\ &= -46+9\mathrm{i}\end{aligned}$$

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2 a $(4+5\mathrm{i})(4-5\mathrm{i}) = 16-20\mathrm{i}+20\mathrm{i}-25\mathrm{i}^2$
$= 16-20\mathrm{i}+20\mathrm{i}+25$
$= 41$

b $(7-2\mathrm{i})(7+2\mathrm{i}) = 49+14\mathrm{i}-14\mathrm{i}-4\mathrm{i}^2$
$= 49+14\mathrm{i}-14\mathrm{i}+4$
$= 53$

c The answers to **2a** and **2b** are both real.

d Let a and b be any real numbers.
$(a+b\mathrm{i})(a-b\mathrm{i}) = a^2-ab\mathrm{i}+ab\mathrm{i}-b^2\mathrm{i}^2$
$= a^2-ab\mathrm{i}+ab\mathrm{i}+b^2$
$= a^2+b^2$

For any a and b, the imaginary parts cancel (i.e. they sum to zero), so the answer is always real.

3 $(a+3\mathrm{i})(1+b\mathrm{i}) = 25-39i$
$a+ab\mathrm{i}+3\mathrm{i}+3b\mathrm{i}^2 = 25-39\mathrm{i}$
$a+ab\mathrm{i}+3\mathrm{i}-3b = 25-39\mathrm{i}$
$(a-3b)+(ab+3)\mathrm{i} = 25-39\mathrm{i}$

Equating real parts:
$a-3b = 25$
$a = 3b+25$ (**1**)
Equating imaginary parts:
$ab+3 = -39$ (**2**)

Substituting (**1**) into (**2**):
$(3b+25)b+3 = -39$
$3b^2+25b+3 = -39$
$3b^2+25b+42 = 0$
$(3b+7)(b+6) = 0$
So $b = \frac{-7}{3}$ or $b = -6$

Substituting $b = -6$ into (**1**):
$a(-6)+3 = -39$
$a = 7$
Substituting $b = \frac{-7}{3}$ into (**1**):
$a\left(-\frac{7}{3}\right)+3 = -39$
$a = 18$

3 Hence, the two pairs of values are:
$a = 7,\ b = -6$
$a = 18,\ b = -\frac{7}{3}$

4 a $\mathrm{i}\times\mathrm{i}\times\mathrm{i}\times\mathrm{i}\times\mathrm{i}\times\mathrm{i} = \mathrm{i}^2\times\mathrm{i}^2\times\mathrm{i}^2$
$= -1\times-1\times-1$
$= -1$

b $3\mathrm{i}\times3\mathrm{i}\times3\mathrm{i}\times3\mathrm{i} = 81(\mathrm{i}\times\mathrm{i}\times\mathrm{i}\times\mathrm{i})$
$= 81(\mathrm{i}^2\times\mathrm{i}^2)$
$= 81(-1\times-1)$
$= 81$

c $(\mathrm{i}\times\mathrm{i}\times\mathrm{i}\times\mathrm{i}\times\mathrm{i})+\mathrm{i} = (\mathrm{i}^2\times\mathrm{i}^2\times\mathrm{i})+\mathrm{i}$
$= (-1\times-1\times\mathrm{i})+\mathrm{i}$
$= \mathrm{i}+\mathrm{i}$
$= 2\mathrm{i}$

d $(4\mathrm{i})^3-4\mathrm{i}^3 = (4\mathrm{i}\times4\mathrm{i}\times4\mathrm{i})-4(\mathrm{i}\times\mathrm{i}\times\mathrm{i})$
$= 64(\mathrm{i}\times\mathrm{i}\times\mathrm{i})-4(\mathrm{i}\times\mathrm{i}\times\mathrm{i})$
$= 60(\mathrm{i}\times\mathrm{i}\times\mathrm{i})$
$= 60(-1\times\mathrm{i})$
$= -60\mathrm{i}$

5 To expand $(1+\mathrm{i})^6$, use the binomial expansion of $(a+b)^6$
$(a+b)^6 = a^6+6a^5b+15a^4b^2+20a^3b^2$
$+15a^2b^4+6ab^5+b^6$

Substitute $a = 1$ and $b = \mathrm{i}$:
$(1+\mathrm{i})^6 = (1)^6+6(1)^5(\mathrm{i})+15(1)^4(\mathrm{i})^2+20(1)^3(\mathrm{i})^3$
$+15(1)^2(\mathrm{i})^4+6(1)(\mathrm{i})^5+(\mathrm{i})^6$
$(1+\mathrm{i})^6 = 1+6\mathrm{i}+15\mathrm{i}^2+20\mathrm{i}^3+15\mathrm{i}^4+6\mathrm{i}^5+\mathrm{i}^6$

[Use $\mathrm{i}^2 = -1, \mathrm{i}^3 = -\mathrm{i}, \mathrm{i}^4 = 1, \mathrm{i}^5 = \mathrm{i}$ and $\mathrm{i}^6 = -1$]

$(1+\mathrm{i})^6 = 1+6\mathrm{i}-15-20\mathrm{i}+15+6\mathrm{i}-1$
$= -8\mathrm{i}$

So $a = 0$ and $b = -8$

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6 To expand $(3-2\mathrm{i})^4$, use the binomial expansion of $(a+b)^4$:

$$(a+b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$$

Substitute $a=3$ and $b=2\mathrm{i}$:

$$(3-2\mathrm{i})^4 = (3)^4 + 4(3)^3(-2\mathrm{i}) + 6(3)^2(-2\mathrm{i})^2 + 4(3)(-2\mathrm{i})^3 + (-2\mathrm{i})^4$$

$$(3-2\mathrm{i})^4 = 81 - 216\mathrm{i} + 216\mathrm{i}^2 - 96\mathrm{i}^3 + 16\mathrm{i}^4$$

[Use $\mathrm{i}^2 = -1, \mathrm{i}^3 = -\mathrm{i}$ and $\mathrm{i}^4 = 1$]

$$(3-2\mathrm{i})^4 = 81 - 216\mathrm{i} - 216 + 96\mathrm{i} + 16$$
$$= -119 - 120\mathrm{i}$$

So the real part of $(3-2\mathrm{i})^4$ is -119

7 a $\mathrm{f}(2\mathrm{i}) = 2(2\mathrm{i})^2 - (2\mathrm{i}) + 8$
$= 2(4\mathrm{i}^2) - 2\mathrm{i} + 8$
$= 8(-1) - 2\mathrm{i} + 8$
$= -2\mathrm{i}$

b $\mathrm{f}(3-6\mathrm{i})$
$= 2(3-6\mathrm{i})^2 - (3-6\mathrm{i}) + 8$
$= 2(9 - 36\mathrm{i} + 36\mathrm{i}^2) - 3 + 6\mathrm{i} + 8$
$= 18 - 72\mathrm{i} + 72\mathrm{i}^2 - 3 + 6\mathrm{i} + 8$
$= 18 - 72\mathrm{i} - 72 - 3 + 6\mathrm{i} + 8$
$= -49 - 66\mathrm{i}$

8 $\mathrm{f}(1-4\mathrm{i}) = (1-4\mathrm{i})^2 - 2(1-4\mathrm{i}) + 17$
$= 1 - 8\mathrm{i} + 16\mathrm{i}^2 - 2 + 8\mathrm{i} + 17$
$= 1 - 8\mathrm{i} - 16 - 2 + 8\mathrm{i} + 17$
$= 0$

$\mathrm{f}(1-4\mathrm{i}) = 0 \Rightarrow z = 1-4\mathrm{i}$ is a solution of $\mathrm{f}(z) = 0$

9 a $\mathrm{i}^3 = (\mathrm{i}^2)(\mathrm{i}) = (-1)(\mathrm{i}) = -\mathrm{i}$
$\mathrm{i}^4 = (\mathrm{i}^2)(\mathrm{i}^2) = (-1)(-1) = 1$

9 b $\mathrm{i}^5 = (\mathrm{i}^4)(\mathrm{i}) = (1)(\mathrm{i}) = \mathrm{i}$
$\mathrm{i}^6 = (\mathrm{i}^4)(\mathrm{i}^2) = (1)(-1) = -1$
$\mathrm{i}^7 = (\mathrm{i}^4)(\mathrm{i}^3) = (1)(-\mathrm{i}) = -\mathrm{i}$
$\mathrm{i}^8 = (\mathrm{i}^4)(\mathrm{i}^4) = (1)(1) = 1$

c For $n \in \mathbb{N}$,
$\mathrm{i}^{4n} = 1$, $\mathrm{i}^{4n+1} = \mathrm{i}$, $\mathrm{i}^{4n+2} = -1$, $\mathrm{i}^{4n+3} = -\mathrm{i}$

Hence:

i $100 = 4 \times 25 \Rightarrow \mathrm{i}^{100} = 1$

ii $253 = (4 \times 63) + 1 \Rightarrow \mathrm{i}^{253} = \mathrm{i}$

iii $301 = (4 \times 75) + 1 \Rightarrow \mathrm{i}^{301} = \mathrm{i}$

Challenge

a $(a+b\mathrm{i})^2 = (a+b\mathrm{i})(a+b\mathrm{i})$
$= a^2 + ab\mathrm{i} + ab\mathrm{i} + b^2\mathrm{i}^2$
$= a^2 + 2ab\mathrm{i} + b^2(-1)$
$= (a^2 - b^2) + 2ab\mathrm{i}$

b Let $a^2 - b^2 = 40$ **(1)**
and $2ab = -42$ **(2)**
Then by part **a**, $\sqrt{40-42\mathrm{i}} = a + b\mathrm{i}$
Rearranging **(2)** gives $a = -\frac{21}{b}$ **(3)**
Substituting **(3)** into **(1)** gives:

$$\left(-\frac{21}{b}\right)^2 - b^2 = 40$$
$$\frac{441}{b^2} - b^2 = 40$$
$$b^4 + 40b^2 - 441 = 0$$
$$(b^2 - 9)(b^2 + 49) = 0$$

b is real, so $b^2 \neq -49$
Hence $b^2 - 9 = 0 \Rightarrow b = -3$
Substituting $b = -3$ into **(3)**:

$$a = -\frac{21}{b} = \frac{-21}{-3} = 7$$

So $\sqrt{40-42i} = 7 - 3i$

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Complex numbers 1D

1 a $z^* = 8 - 2\mathrm{i}$

b $z^* = 6 + 5\mathrm{i}$

c $z^* = \frac{2}{3} + \frac{1}{2}\mathrm{i}$

d $z^* = \sqrt{5} - \mathrm{i}\sqrt{10}$

2 a $z + z^* = (6 - 3\mathrm{i}) + (6 + 3\mathrm{i})$
$= 12$

$zz^* = (6 - 3\mathrm{i})(6 + 3\mathrm{i})$
$= 6(6 + 3\mathrm{i}) - 3\mathrm{i}(6 + 3\mathrm{i})$
$= 36 + 18\mathrm{i} - 18\mathrm{i} - 9\mathrm{i}^2$
$= (36 + 9) + \mathrm{i}(18 - 18)$
$= 45$

b $z + z^* = (10 + 5\mathrm{i}) + (10 - 5\mathrm{i})$
$= 20$

$zz^* = (10 + 5\mathrm{i})(10 - 5\mathrm{i})$
$= 10(10 - 5\mathrm{i}) + 5\mathrm{i}(10 - 5\mathrm{i})$
$= 100 - 50\mathrm{i} + 50\mathrm{i} - 25\mathrm{i}^2$
$(100 + 25) + \mathrm{i}(-50 + 50)$
$= 125$

c $z + z^* = \left(\frac{3}{4} + \frac{1}{4}\mathrm{i}\right) + \left(\frac{3}{4} - \frac{1}{4}\mathrm{i}\right)$
$= \frac{3}{2}$

$zz^* = \left(\frac{3}{4} + \frac{1}{4}\mathrm{i}\right)\left(\frac{3}{4} - \frac{1}{4}\mathrm{i}\right)$
$= \frac{3}{4}\left(\frac{3}{4} - \frac{1}{4}\mathrm{i}\right) + \frac{1}{4}\mathrm{i}\left(\frac{3}{4} - \frac{1}{4}\mathrm{i}\right)$
$= \frac{9}{16} - \frac{3}{16}\mathrm{i} + \frac{3}{16}\mathrm{i} - \frac{1}{16}\mathrm{i}^2$
$= \frac{10}{16} = \frac{5}{8}$

2 d $z + z^* = \left(\sqrt{5} - 3\mathrm{i}\sqrt{5}\right) + \left(\sqrt{5} + 3\mathrm{i}\sqrt{5}\right)$
$= 2\sqrt{5}$

$zz^* = \left(\sqrt{5} - 3\mathrm{i}\sqrt{5}\right)\left(\sqrt{5} + 3\mathrm{i}\sqrt{5}\right)$
$= \sqrt{5}\left(\sqrt{5} + 3\mathrm{i}\sqrt{5}\right) - 3\mathrm{i}\sqrt{5}\left(\sqrt{5} + 3\mathrm{i}\sqrt{5}\right)$
$= 5 + 15\mathrm{i} - 15\mathrm{i} - 45\mathrm{i}^2$
$= (5 + 45) + \mathrm{i}(15 - 15)$
$= 50$

3 a $\frac{3 - 5\mathrm{i}}{1 + 3\mathrm{i}} = \frac{(3 - 5\mathrm{i})(1 - 3\mathrm{i})}{(1 + 3\mathrm{i})(1 - 3\mathrm{i})}$
$(3 - 5\mathrm{i})(1 - 3\mathrm{i}) = 3(1 - 3\mathrm{i}) - 5\mathrm{i}(1 - 3\mathrm{i})$
$= 3 - 9\mathrm{i} - 5\mathrm{i} + 15\mathrm{i}^2$
$= -12 - 14\mathrm{i}$
$(1 + 3\mathrm{i})(1 - 3\mathrm{i}) = 1(1 - 3\mathrm{i}) + 3\mathrm{i}(1 - 3\mathrm{i})$
$= 1 - 3\mathrm{i} + 3\mathrm{i} - 9\mathrm{i}^2$
$= 10$
$\frac{3 - 5\mathrm{i}}{1 + 3\mathrm{i}} = \frac{-12 - 14\mathrm{i}}{10} = -\frac{6}{5} - \frac{7}{5}\mathrm{i}$

b $\frac{3 + 5\mathrm{i}}{6 - 8\mathrm{i}} = \frac{(3 + 5\mathrm{i})(6 + 8\mathrm{i})}{(6 - 8\mathrm{i})(6 + 8\mathrm{i})}$
$(3 + 5\mathrm{i})(6 + 8\mathrm{i}) = 3(6 + 8\mathrm{i}) + 5\mathrm{i}(6 + 8\mathrm{i})$
$= 18 + 24\mathrm{i} + 30\mathrm{i} + 40\mathrm{i}^2$
$= -22 + 54\mathrm{i}$
$(6 - 8\mathrm{i})(6 + 8\mathrm{i}) = 6(6 + 8\mathrm{i}) - 8\mathrm{i}(6 + 8\mathrm{i})$
$= 36 + 48\mathrm{i} - 48\mathrm{i} - 64\mathrm{i}^2$
$= 100$
$\frac{3 + 5\mathrm{i}}{6 - 8\mathrm{i}} = \frac{-22 + 54\mathrm{i}}{100} = -\frac{11}{50} + \frac{27}{50}\mathrm{i}$

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3 c

$$\frac{28-3\mathrm{i}}{1-\mathrm{i}}=\frac{(28-3\mathrm{i})(1+\mathrm{i})}{(1-\mathrm{i})(1+\mathrm{i})}$$

$$\begin{aligned}(28-3\mathrm{i})(1+\mathrm{i}) &= 28(1+\mathrm{i})-3\mathrm{i}(1+\mathrm{i})\\ &= 28+28\mathrm{i}-3\mathrm{i}-3\mathrm{i}^2\\ &= 31+25\mathrm{i}\end{aligned}$$

$$\begin{aligned}(1-\mathrm{i})(1+\mathrm{i}) &= 1(1+\mathrm{i})-\mathrm{i}(1+\mathrm{i})\\ &= 1+\mathrm{i}-\mathrm{i}-\mathrm{i}^2\\ &= 2\end{aligned}$$

$$\frac{28-3\mathrm{i}}{1-\mathrm{i}}=\frac{31+25\mathrm{i}}{2}=\frac{31}{2}+\frac{25}{2}\mathrm{i}$$

d

$$\frac{2+\mathrm{i}}{1+4\mathrm{i}}=\frac{(2+\mathrm{i})(1-4\mathrm{i})}{(1+4\mathrm{i})(1-4\mathrm{i})}$$

$$\begin{aligned}(2+\mathrm{i})(1-4\mathrm{i}) &= 2(1-4\mathrm{i})+\mathrm{i}(1-4\mathrm{i})\\ &= 2-8\mathrm{i}+\mathrm{i}-4\mathrm{i}^2\\ &= 6-7\mathrm{i}\end{aligned}$$

$$\begin{aligned}(1+4\mathrm{i})(1-4\mathrm{i}) &= 1(1-4\mathrm{i})+4\mathrm{i}(1-4\mathrm{i})\\ &= 1-4\mathrm{i}+4\mathrm{i}-16\mathrm{i}^2\\ &= 17\end{aligned}$$

$$\frac{2+\mathrm{i}}{1+4\mathrm{i}}=\frac{6-7\mathrm{i}}{17}=\frac{6}{17}-\frac{7}{17}\mathrm{i}$$

4

$$\begin{aligned}\frac{(3-4\mathrm{i})^2}{1+\mathrm{i}} &= \frac{\left(9-24\mathrm{i}+16\mathrm{i}^2\right)}{\left(1+\mathrm{i}\right)}\times\frac{\left(1-\mathrm{i}\right)}{\left(1-\mathrm{i}\right)}\\ &= \frac{9-24\mathrm{i}+16\mathrm{i}^2-9\mathrm{i}+24\mathrm{i}^2-16\mathrm{i}^3}{1-\mathrm{i}+\mathrm{i}-\mathrm{i}^2}\\ &= \frac{9-24\mathrm{i}-16-9\mathrm{i}-24+16\mathrm{i}}{1-\left(-1\right)}\\ &= \frac{-31-17\mathrm{i}}{2}\\ &= -\frac{31}{2}-\frac{17}{2}i\end{aligned}$$

5 a

$$\begin{aligned}z_1z_2 &= (1+\mathrm{i})(2+\mathrm{i})\\ &= 1(2+\mathrm{i})+\mathrm{i}(2+\mathrm{i})\\ &= 2+\mathrm{i}+2\mathrm{i}+\mathrm{i}^2\\ &= 1+3\mathrm{i}\end{aligned}$$

$$\frac{z_1z_2}{z_3}=\frac{1+3\mathrm{i}}{3+\mathrm{i}}=\frac{(1+3\mathrm{i})(3-\mathrm{i})}{(3+\mathrm{i})(3-\mathrm{i})}$$

$$\begin{aligned}(1+3\mathrm{i})(3-\mathrm{i}) &= 1(3-\mathrm{i})+3\mathrm{i}(3-\mathrm{i})\\ &= 3-\mathrm{i}+9\mathrm{i}-3\mathrm{i}^2\\ &= 6+8\mathrm{i}\end{aligned}$$

$$\begin{aligned}(3+\mathrm{i})(3-\mathrm{i}) &= 3(3-\mathrm{i})+\mathrm{i}(3-\mathrm{i})\\ &= 9-3\mathrm{i}+3\mathrm{i}-\mathrm{i}^2\\ &= 10\end{aligned}$$

$$\frac{z_1z_2}{z_3}=\frac{6+8\mathrm{i}}{10}=\frac{3}{5}+\frac{4}{5}\mathrm{i}$$

b

$$\begin{aligned}(z_2)^2 &= (2+\mathrm{i})(2+\mathrm{i})\\ &= 2(2+\mathrm{i})+\mathrm{i}(2+\mathrm{i})\\ &= 4+2\mathrm{i}+2\mathrm{i}+\mathrm{i}^2\\ &= 3+4\mathrm{i}\end{aligned}$$

$$\frac{(z_2)^2}{z_1}=\frac{3+4\mathrm{i}}{1+\mathrm{i}}=\frac{(3+4\mathrm{i})(1-\mathrm{i})}{(1+\mathrm{i})(1-\mathrm{i})}$$

$$\begin{aligned}(3+4\mathrm{i})(1-\mathrm{i}) &= 3(1-\mathrm{i})+4\mathrm{i}(1-\mathrm{i})\\ &= 3-3\mathrm{i}+4\mathrm{i}-4\mathrm{i}^2\\ &= 7+\mathrm{i}\end{aligned}$$

$$\begin{aligned}(1+\mathrm{i})(1-\mathrm{i}) &= 1(1-\mathrm{i})+\mathrm{i}(1-\mathrm{i})\\ &= 1-\mathrm{i}+\mathrm{i}-\mathrm{i}^2\\ &= 2\end{aligned}$$

$$\frac{(z_2)^2}{z_1}=\frac{7+\mathrm{i}}{2}=\frac{7}{2}+\frac{1}{2}\mathrm{i}$$

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5 c

$$2z_1+5z_3 = 2(1+\mathrm{i})+5(3+\mathrm{i})$$
$$= 2+2\mathrm{i}+15+5\mathrm{i}$$
$$= 17+7\mathrm{i}$$
$$\frac{2z_1+5z_3}{z_2} = \frac{17+7\mathrm{i}}{2+\mathrm{i}} = \frac{(17+7\mathrm{i})(2-\mathrm{i})}{(2+\mathrm{i})(2-\mathrm{i})}$$
$$(17+7\mathrm{i})(2-\mathrm{i}) = 17(2-\mathrm{i})+7\mathrm{i}(2-\mathrm{i})$$
$$= 34-17\mathrm{i}+14\mathrm{i}-7\mathrm{i}^2$$
$$= 41-3\mathrm{i}$$
$$(2+\mathrm{i})(2-\mathrm{i}) = 2(2-\mathrm{i})+\mathrm{i}(2-\mathrm{i})$$
$$= 4-2\mathrm{i}+2\mathrm{i}-\mathrm{i}^2$$
$$= 5$$
$$\frac{2z_1+5z_3}{z_2} = \frac{41-3\mathrm{i}}{5} = \frac{41}{5}-\frac{3}{5}\mathrm{i}$$

6

$$\frac{5+2\mathrm{i}}{z} = 2-\mathrm{i}$$
$$z = \frac{5+2\mathrm{i}}{2-\mathrm{i}} = \frac{(5+2\mathrm{i})(2+\mathrm{i})}{(2-\mathrm{i})(2+\mathrm{i})}$$
$$(5+2\mathrm{i})(2+\mathrm{i}) = 5(2+\mathrm{i})+2\mathrm{i}(2+\mathrm{i})$$
$$= 10+5\mathrm{i}+4\mathrm{i}+2\mathrm{i}^2$$
$$= 8+9\mathrm{i}$$
$$(2-\mathrm{i})(2+\mathrm{i}) = 2(2+\mathrm{i})-\mathrm{i}(2+\mathrm{i})$$
$$= 4+2\mathrm{i}-2\mathrm{i}-\mathrm{i}^2$$
$$= 5$$
$$z = \frac{8+9\mathrm{i}}{5} = \frac{8}{5}+\frac{9}{5}\mathrm{i}$$

7

$$\frac{6+8\mathrm{i}}{1+\mathrm{i}}+\frac{6+8\mathrm{i}}{1-\mathrm{i}}$$
$$= \frac{(6+8\mathrm{i})(1-\mathrm{i})+(6+8\mathrm{i})(1+\mathrm{i})}{(1+\mathrm{i})(1-\mathrm{i})}$$
$$= \frac{6(1-\mathrm{i})+8\mathrm{i}(1-\mathrm{i})+6(1+\mathrm{i})+8\mathrm{i}(1+\mathrm{i})}{1(1-\mathrm{i})+\mathrm{i}(1-\mathrm{i})}$$
$$= \frac{6-6\mathrm{i}+8\mathrm{i}-8\mathrm{i}^2+6+6\mathrm{i}+8\mathrm{i}+8\mathrm{i}^2}{1-\mathrm{i}+\mathrm{i}-\mathrm{i}^2}$$
$$= \frac{12+16\mathrm{i}}{2} = 6+8\mathrm{i}$$

8

$$\frac{4}{8-\mathrm{i}\sqrt{2}} = \frac{4}{\left(8-\mathrm{i}\sqrt{2}\right)}\times\frac{\left(8+\mathrm{i}\sqrt{2}\right)}{\left(8+\mathrm{i}\sqrt{2}\right)}$$
$$= \frac{32+4\mathrm{i}\sqrt{2}}{64+8\mathrm{i}\sqrt{2}-8\mathrm{i}\sqrt{2}-\mathrm{i}^2\left(\sqrt{2}\right)^2}$$
$$= \frac{32+4\mathrm{i}\sqrt{2}}{66}$$
$$= \frac{16}{33}+\frac{2\sqrt{2}}{33}\mathrm{i}$$

9

$$\frac{1}{w} = \frac{1}{1-9\mathrm{i}}$$
$$= \frac{1}{\left(1-9\mathrm{i}\right)}\times\frac{\left(1+9\mathrm{i}\right)}{\left(1+9\mathrm{i}\right)}$$
$$= \frac{1+9\mathrm{i}}{1+9\mathrm{i}-9\mathrm{i}-81\mathrm{i}^2}$$
$$= \frac{1+9\mathrm{i}}{82}$$
$$= \frac{1}{82}+\frac{9}{82}\mathrm{i}$$

10

$$\frac{z+4}{z-3} = \frac{4-\mathrm{i}\sqrt{2}+4}{4-\mathrm{i}\sqrt{2}-3}$$
$$= \frac{8-\mathrm{i}\sqrt{2}}{1-\mathrm{i}\sqrt{2}}$$
$$= \frac{\left(8-\mathrm{i}\sqrt{2}\right)}{\left(1-\mathrm{i}\sqrt{2}\right)}\times\frac{\left(1+\mathrm{i}\sqrt{2}\right)}{\left(1+\mathrm{i}\sqrt{2}\right)}$$
$$= \frac{8+8\mathrm{i}\sqrt{2}-\mathrm{i}\sqrt{2}-\mathrm{i}^2\left(\sqrt{2}\right)^2}{1+\mathrm{i}\sqrt{2}-\mathrm{i}\sqrt{2}-\mathrm{i}^2\left(\sqrt{2}\right)^2}$$
$$= \frac{8+7\mathrm{i}\sqrt{2}+2}{1+2}$$
$$= \frac{10}{3}+\frac{7\sqrt{2}}{3}\mathrm{i}$$

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11 $(4+2\mathrm{i})(z-2\mathrm{i})=6-4\mathrm{i}$

$$z-2\mathrm{i}=\frac{6-4\mathrm{i}}{4+2\mathrm{i}}=\frac{(6-4\mathrm{i})}{(4+2\mathrm{i})}\times\frac{(4-2\mathrm{i})}{(4-2\mathrm{i})}$$

$$z-2\mathrm{i}=\frac{24-12\mathrm{i}-16\mathrm{i}+8\mathrm{i}^2}{16-8\mathrm{i}+8\mathrm{i}-(2\mathrm{i})^2}$$

$$z-2\mathrm{i}=\frac{16-28\mathrm{i}}{20}$$

$$z-2\mathrm{i}=\frac{4}{5}-\frac{7}{5}\mathrm{i}$$

$$z=\frac{4}{5}+\frac{3}{5}\mathrm{i}$$

Alternative Method:

Let $z=a+b\mathrm{i}$

Substitute $z=a+b\mathrm{i}$ into the equation $6-4\mathrm{i}=(4+2\mathrm{i})(z-2\mathrm{i})$:

$$6-4\mathrm{i}=(4+2\mathrm{i})(a+(b-2)\mathrm{i})$$

$$=4a+4(b-2)\mathrm{i}+2a\mathrm{i}+2(b-2)\mathrm{i}^2$$

$$=4a+4b\mathrm{i}-8\mathrm{i}+2a\mathrm{i}-2(b-2)$$

$$=4a+4b\mathrm{i}-8\mathrm{i}+2a\mathrm{i}-2b+4$$

$$=(4a-2b+4)+(2a+4b-8)\mathrm{i}$$

Equate real parts:

$6=4a-2b+4\Rightarrow 2a-b=1$ (**1**)

Equate imaginary parts:

$-4=2a+4b-8\Rightarrow a+2b=2$ (**2**)

$(2)+2\times(1)\Rightarrow 5a=4$

$$a=\frac{4}{5}$$

Substitute $a=\frac{4}{5}$ into (**2**)

$$\frac{4}{5}+2b=2$$

$$b=\frac{3}{5}$$

Hence $z=\frac{4}{5}+\frac{3}{5}\mathrm{i}$

12 $\frac{z_1}{z_2}=\frac{p-7\mathrm{i}}{2+5\mathrm{i}}$

$$=\frac{(p-7\mathrm{i})}{(2+5\mathrm{i})}\times\frac{(2-5\mathrm{i})}{(2-5\mathrm{i})}$$

$$=\frac{2p-5p\mathrm{i}-14\mathrm{i}+35\mathrm{i}^2}{4-10\mathrm{i}+10\mathrm{i}-25\mathrm{i}^2}$$

$$=\frac{2p-35}{29}+\frac{-5p-14}{29}\mathrm{i}$$

13 $z=\sqrt{5}+4\mathrm{i}$

$z^*=\sqrt{5}-4\mathrm{i}$

$$\frac{z}{z^*}=\frac{(\sqrt{5}+4\mathrm{i})}{(\sqrt{5}-4\mathrm{i})}\times\frac{(\sqrt{5}+4\mathrm{i})}{(\sqrt{5}+4\mathrm{i})}$$

$$=\frac{5+4\sqrt{5}\mathrm{i}+4\sqrt{5}\mathrm{i}+16\mathrm{i}^2}{5+4\sqrt{5}\mathrm{i}-4\sqrt{5}\mathrm{i}-16\mathrm{i}^2}$$

$$=-\frac{11}{21}+\frac{8\sqrt{5}}{21}\mathrm{i}$$

14 a $\frac{p+5\mathrm{i}}{p-2\mathrm{i}}=\frac{(p+5\mathrm{i})}{(p-2\mathrm{i})}\times\frac{(p+2\mathrm{i})}{(p+2\mathrm{i})}$

$$=\frac{p^2+2p\mathrm{i}+5p\mathrm{i}+10\mathrm{i}^2}{p^2+2p\mathrm{i}-2p\mathrm{i}-4\mathrm{i}^2}$$

$$=\frac{p^2-10}{p^2+4}+\frac{7p}{p^2+4}\mathrm{i}$$ (**1**)

The real part is equal to $\frac{1}{2}$, so

$$\frac{p^2-10}{p^2+4}=\frac{1}{2}$$

$$2p^2-20=p^2+4$$

$$p^2=24$$

$$p=\pm 2\sqrt{6}$$

But since $p>0$, then $p=2\sqrt{6}$

b Substituting $p=2\sqrt{6}$ into (**1**):

$$z=\frac{\left(2\sqrt{6}\right)^2-10}{\left(2\sqrt{6}\right)^2+4}+\frac{7\left(2\sqrt{6}\right)}{\left(2\sqrt{8}\right)^2+4}\mathrm{i}$$

$$=\frac{24-10}{24+4}+\frac{14\sqrt{6}}{24+4}\mathrm{i}$$

$$=\frac{1}{2}+\frac{\sqrt{6}}{2}\mathrm{i}$$

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Complex numbers 1E

1 a $z^2 + 2z + 26 = 0$
Using the quadratic formula gives

$$z = \frac{-2 \pm \sqrt{4 - 4(1)(26)}}{2}$$
$$= \frac{-2 \pm 10\text{i}}{2}$$
$\alpha = -1 + 5\text{i},\ \beta = -1 - 5\text{i}$ **(**or vice versa)

b $\alpha + \beta = (-1 + 5\text{i}) + (-1 - 5\text{i}) = -2$

c $\alpha\beta = (-1 + 5\text{i})(-1 - 5\text{i})$
$= -1(-1 - 5\text{i}) + 5\text{i}(-1 - 5\text{i})$
$= 1 + 5\text{i} - 5\text{i} - 25\text{i}^2$
$= 26$

2 a $z^2 - 8z + 25 = 0$

$$z = \frac{8 \pm \sqrt{(-8)^2 - 4(1)(25)}}{2}$$
$$= \frac{8 \pm 6\text{i}}{2}$$
$\alpha = 4 + 3\text{i},\ \beta = 4 - 3\text{i}$ (or vice versa)

b $\alpha + \beta = (4 + 3\text{i}) + (4 - 3\text{i}) = 8$

c $\alpha\beta = (4 + 3\text{i})(4 - 3\text{i})$
$= 4(4 - 3\text{i}) + 3\text{i}(4 - 3\text{i})$
$= 16 - 12\text{i} + 12\text{i} - 9\text{i}^2$
$= 25$

3 a The other root is the complex conjugate $2 - 3\text{i}$.

b $\left(z - (2 + 3\text{i})\right)\left(z - (2 - 3\text{i})\right)$
$= z^2 - (2 - 3\text{i})z - (2 + 3\text{i})z$
$\quad + (2 + 3\text{i})(2 - 3\text{i})$
$= z^2 - 2z + 3\text{i}z - 2z - 3\text{i}z + 4 - 6\text{i} + 6\text{i} - 9\text{i}^2$
$= z^2 - 4z + 4 + 9$
$= z^2 - 4z + 13$

Equation is $z^2 - 4z + 13 = 0$
where $b = -4,\ c = 13$

4 a The other root is the complex conjugate $5 + \text{i}$.

b $\left(z - (5 - \text{i})\right)\left(z - (5 + \text{i})\right)$
$= z^2 - (5 + \text{i})z - (5 - \text{i})z + (5 - \text{i})(5 + \text{i})$
$= z^2 - 5z - \text{i}z - 5z + \text{i}z + 25 + 5\text{i} - 5\text{i} - \text{i}^2$
$= z^2 - 10z + 25 + 1$
$= z^2 - 10z + 26$

So, $p = -10$ and $q = 26$

5 The second root is the complex conjugate of $z_1 = -5 + 4i$
i.e. $z_2 = -5 - 4i$
$\left(z - (-5 + 4\text{i})\right)\left(z - (-5 - 4\text{i})\right)$
$= z^2 - (-5 - 4\text{i})z - (-5 + 4\text{i})z$
$\quad + (-5 + 4\text{i})(-5 - 4\text{i})$
$= z^2 + 5z + 4\text{i}z + 5z - 4\text{i}z$
$\quad + 25 + 20\text{i} - 20\text{i} - 16\text{i}^2$
$= z^2 + 10z + 25 + 16$
$= z^2 + 10z + 41$

So, $b = 10$ and $c = 41$

6 The other root is $1 - 2\text{i}$.

If the roots are α and β, the equation is $(z - \alpha)(z - \beta) = z^2 - (\alpha + \beta)z + \alpha\beta = 0$

$\alpha + \beta = (1 + 2\text{i}) + (1 - 2\text{i}) = 2$
$\alpha\beta = (1 + 2\text{i})(1 - 2\text{i})$
$= 1(1 - 2\text{i}) + 2\text{i}(1 - 2\text{i})$
$= 1 - 2\text{i} + 2\text{i} - 4\text{i}^2 = 5$
Equation is $z^2 - 2z + 5 = 0$

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7 The other root is 3 + 5i.

If the roots are α and β, the equation is $(z-\alpha)(z-\beta) = z^2 - (\alpha+\beta)z + \alpha\beta = 0$

$$\alpha + \beta = (3-5\text{i}) + (3+5\text{i}) = 6$$
$$\alpha\beta = (3-5\text{i})(3+5\text{i})$$
$$= 3(3+5\text{i}) - 5\text{i}(3+5\text{i})$$
$$= 9 + 15\text{i} - 15\text{i} - 25\text{i}^2 = 34$$

Equation is $z^2 - 6z + 34 = 0$

8 a $$z = \frac{5}{3-\text{i}}$$
$$= \frac{5}{(3-\text{i})} \times \frac{(3+\text{i})}{(3+\text{i})}$$
$$= \frac{15+5\text{i}}{9+3\text{i}-3\text{i}-\text{i}^2}$$
$$= \frac{15}{10} + \frac{5}{10}\text{i}$$
$$= \frac{3}{2} + \frac{1}{2}\text{i}$$

b If $z = \frac{3}{2} + \frac{1}{2}\text{i}$ is a root, then $z = \frac{3}{2} - \frac{1}{2}\text{i}$ is also a root.

$$0 = \left(z - \left(\frac{3}{2} + \frac{1}{2}\text{i}\right)\right)\left(z - \left(\frac{3}{2} - \frac{1}{2}\text{i}\right)\right)$$
$$= z^2 - \left(\frac{3}{2} - \frac{1}{2}\text{i}\right)z - \left(\frac{3}{2} + \frac{1}{2}\text{i}\right)z$$
$$+ \left(\frac{3}{2} + \frac{1}{2}\text{i}\right)\left(\frac{3}{2} - \frac{1}{2}\text{i}\right)$$
$$= z^2 - \frac{3}{2}z + \frac{1}{2}\text{i}z - \frac{3}{2}z - \frac{1}{2}\text{i}z$$
$$+ \frac{9}{4} - \frac{3}{4}\text{i} + \frac{3}{4}\text{i} - \frac{1}{4}\text{i}^2$$
$$= z^2 - 3z + \frac{10}{4}$$
$$= z^2 - 3z + \frac{5}{2}$$

So $p = -3$ and $q = \frac{5}{2}$

9 $z = 5 + q\text{i}$

The second root is the complex conjugate, $z = 5 - q\text{i}$

$$0 = \left(z - (5+q\text{i})\right)\left(z - (5-q\text{i})\right)$$
$$= z^2 - (5-q\text{i})z - (5+q\text{i})z$$
$$+ (5+q\text{i})(5-q\text{i})$$
$$= z^2 - 5z + q\text{i}z - 5z - q\text{i}z$$
$$+ 25 - 5q\text{i} + 5q\text{i} - q^2\text{i}^2$$
$$= z^2 - 10z + 25 + q^2$$

So $z^2 - 10z + 25 + q^2 \equiv z^2 - 4pz + 34$

Equating z terms:

$$4p = 10 \Rightarrow p = \frac{5}{2}$$

Equating constant terms:

$$25 + q^2 = 34 \Rightarrow q^2 = 9$$
$$\Rightarrow q = \pm 3$$

But $q > 0$, and hence $q = 3$

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Complex numbers 1F

1 a $f(z) = z^3 - 6z^2 + 21z - 26$

$f(2) = (2)^3 - 6(2)^2 + 21(2) - 26$

$= 8 - 24 + 42 - 26$

$= 0$

b If $f(2) = 0$ then $z - 2$ is a factor of f(z)

$$z - 2 \overline{)\, z^3 - 6z^2 + 21z - 26}$$

$z^2 - 4z + 13$

$z^3 - 2z^2$

$-4z^2 + 21z$

$-4z^2 + 8z$

$13z - 26$

$13z - 26$

0

So $0 = f(z)$

$= z^3 - 6z^2 + 21z - 26$

$= (z - 2)(z^2 - 4z + 13)$

Either $(z - 2) = 0$ or $(z^2 - 4z + 13) = 0$

Solve $z^2 - 4z + 13 = 0$ by completing the square:

$(z - 2)^2 - 4 + 13 = 0$

$(z - 2)^2 = -9$

$z - 2 = \pm 3\text{i}$

$z = 2 \pm 3\text{i}$

So the roots of $z^3 - 6z^2 + 21z - 26 = 0$ are $2,\ 2 + 3\text{i}$ and $2 - 3\text{i}$

2 a $f(z) = 2z^3 + 5z^2 + 9z - 6$

$f\left(\frac{1}{2}\right) = 2\left(\frac{1}{2}\right)^3 + 5\left(\frac{1}{2}\right)^2 + 9\left(\frac{1}{2}\right) - 6$

$= \frac{2}{8} + \frac{5}{4} + \frac{9}{2} - 6$

$= \frac{1}{4} + \frac{5}{4} + \frac{18}{4} - \frac{24}{4}$

$= 0$

2 b $f\left(\frac{1}{2}\right) = 0 \Rightarrow 2z - 1$ is a factor of $f(z) = 0$

Hence

$f(z) = 2z^3 + 5z^2 + 9z - 6$

$= (2z - 1)(z^2 + bz + c)$

$= 2z^3 + (2b - 1)z^2 + (2c - b)z - c$

Equating z^2 terms:

$2b - 1 = 5$, so $b = 3$

Equating constant terms:

$-c = -6$, so $c = 6$

Hence $f(z) = (2z - 1)(z^2 + 3z + 6)$

c $f(z) = 0 \Rightarrow$ Either $(2z - 1) = 0$

or $(z^2 + 3z + 6) = 0$

Solve $(z^2 + 3z + 6) = 0$ by completing the square:

$\left(z + \frac{3}{2}\right)^2 - \frac{9}{4} + 6 = 0$

$\left(z + \frac{3}{2}\right)^2 = -\frac{15}{4}$

$z + \frac{3}{2} = \pm\frac{\sqrt{15}}{2}\text{i}$

$z = -\frac{3}{2} \pm \frac{\sqrt{15}}{2}\text{i}$

So the roots of $2z^3 + 5z^2 + 9z - 6 = 0$ are $\frac{1}{2}$, $\left(-\frac{3}{2} + \frac{\sqrt{15}}{2}\text{i}\right)$, and $\left(-\frac{3}{2} - \frac{\sqrt{15}}{2}\text{i}\right)$

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3 $g(z) = 2z^3 - 4z^2 - 5z - 3$

$z = 3$ is a root of $g(z) = 0$

$\Rightarrow z - 3$ is a factor of $g(z)$

$$z-3\overline{)\,2z^3-4z^2-5z-3}$$

$$\begin{array}{r} 2z^2+2z+1 \\ 2z^3-6z^2 \\ \hline 2z^2-5z \\ 2z^2-6z \\ \hline z-3 \\ z-3 \\ \hline 0 \end{array}$$

So $g(z) = (z-3)(2z^2+2z+1) = 0$

Solve $2z^2+2z+1=0$ using the quadratic formula:

$$z = \frac{-2 \pm \sqrt{(2)^2 - (4)(2)(1)}}{2(2)}$$

$$z = \frac{-2 \pm \sqrt{4-8}}{4}$$

$$z = \frac{-2 \pm 2\mathrm{i}}{4}$$

$$z = -\frac{1}{2} \pm \frac{1}{2}\mathrm{i}$$

So the roots of $2z^3 - 4z^2 - 5z - 3 = 0$ are 3, $-\frac{1}{2} + \frac{1}{2}\mathrm{i}$ and $-\frac{1}{2} - \frac{1}{2}\mathrm{i}$

4 a $p(z) = z^3 + 4z^2 - 15z - 68$

$z = -4 + \mathrm{i}$ is a solution to $p(z) = 0$, so

$z = -4 - \mathrm{i}$ is also a solution to $p(z) = 0$.

Hence $(z - (-4+\mathrm{i}))(z - (-4-\mathrm{i}))$ is a factor of $p(z)$

$$\begin{aligned} &(z-(-4+\mathrm{i}))(z-(-4-\mathrm{i})) \\ &= z^2 - (-4-\mathrm{i})z - (-4+\mathrm{i})z \\ &\qquad + (-4+\mathrm{i})(-4-\mathrm{i}) \\ &= z^2 + 4z + \mathrm{i}z + 4z - \mathrm{i}z + 16 + 4\mathrm{i} - 4\mathrm{i} - \mathrm{i}^2 \\ &= z^2 + 8z + 17 \end{aligned}$$

$\therefore z^2 + 8z + 17$ is a factor of $p(z)$

4 b $0 = p(z)$

$$\begin{aligned} &= z^3 + 4z^2 - 15z - 68 \\ &= (z^2 + 8z + 17)(az + b) \\ &= az^3 + (b + 8a)z^2 + (17a + 8b)z + 17b \end{aligned}$$

Equate z^3 coefficients:

$a = 1$

Equate constants:

$17b = -68 \Rightarrow b = -4$

Hence $z - 4$ is a factor of $p(z)$ and $z = 4$ is a solution of $p(z) = 0$

So the roots of $z^3 + 4z^2 - 15z - 68 = 0$ are 4, $-4 + \mathrm{i}$ and $-4 - \mathrm{i}$

5 a $z^3 + 9z^2 + 33z + 25 = (z+1)(z^2 + az + b)$

$$= z^3 + (1+a)z^2 + (a+b)z + b$$

Equate z^2 terms:

$a + 1 = 9$, so $a = 8$

Equate constant terms:

$b = 25$

b $f(z) = (z+1)(z^2 + 8z + 25) = 0$

Either $z + 1 = 0 \Rightarrow z = -1$ **or**

$z^2 + 8z + 25 = 0$

Solve $z^2 + 8z + 25 = 0$ by completing the square:

$$(z+4)^2 - 16 + 25 = 0$$

$$(z+4)^2 = -9$$

$$z + 4 = \pm 3i$$

$$z = -4 \pm 3i$$

So the roots of $z^3 + 9z^2 + 33z + 25 = 0$ are -1, $-4 + 3\mathrm{i}$ and $-4 - 3\mathrm{i}$

c $-1 + (-4 + 3\mathrm{i}) + (-4 - 3\mathrm{i}) = -9$

6 a If $3 + \mathrm{i}$ is a root, then $3 - \mathrm{i}$ is also a root.

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6 b $g(z)=(z-6)(z-(3+i))(z-(3-i))$

$=(z-6)(z^2-(3-i)z-(3+i)z+(3+i)(3-i))$

$=(z-6)(z^2-3z+iz-3z-iz+9-3i+3i-i^2)$

$=(z-6)(z^2-6z+10)$

$=z^3-6z^2+10z-6z^2+36z-60$

$=z^3-12z^2+46z-60$

So $c=46$ and $d=-60$.

7 $h(z)=2z^3+3z^2+3z+1$

$=(2z+1)(az^2+bz+c)$

$=2az^3+(a+2b)z^2+(2c+b)z+c$

Equate z^3 coefficient:
$2a=2$, so $a=1$

Equate z^2 coefficients:
$a+2b=3$, so $b=1$

Equate constants:
$c=1$

$0=h(z)$

$=(2z+1)(z^2+z+1)$

Either $2z+1=0\Rightarrow z=-\frac{1}{2}$ or

$z^2+z+1=0$

Solve $z^2+z+1=0$ by completing the square:

$\left(z+\frac{1}{2}\right)^2-\frac{1}{4}+1=0$

$\left(z+\frac{1}{2}\right)^2=-\frac{3}{4}$

$z+\frac{1}{2}=\pm\frac{\sqrt{3}}{2}i$

$z=-\frac{1}{2}\pm\frac{\sqrt{3}}{2}i$

So the roots of $2z^3+3z^2+3z+1=0$ are $-\frac{1}{2}$, $\left(-\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)$ and $\left(-\frac{1}{2}-\frac{\sqrt{3}}{2}i\right)$

8 a $f(z)=z^3-6z^2+28z+k$

$0=f(2)$

$=(2)^3-6(2)^2+28(2)+k$

$=8-24+56+k$

$k=-40$

b $z=2$ is a root $\Rightarrow z-2$ is a factor

$$z-2\overline{)\,z^3-6z^2+28z-40}\quad z^2-4z+20$$

z^3-2z^2

$-4z^2+28z$

$-4z^2+8z$

$20z-40$

$20z-40$

0

$0=f(z)$

$=(z-2)(z^2-4z+20)$

Either $z=2$ or $z^2-4z+20=0$

Solve $z^2-4z+20=0$ by completing the square:

$z^2-4z+20=0$

$(z-2)^2-4+20=0$

$(z-2)^2=-16$

$z-2=\pm4i$

$z=2\pm4i$

So the other roots of $z^3-6z^2+28z+40=0$ are $2+4i$ and $2-4i$.

9 $z^4-16=0$

$(z^2-4)(z^2+4)=0$

$(z-2)(z+2)(z^2+4)=0$

Either $z-2=0\Rightarrow z=2$

or $z+2=0\Rightarrow z=-2$

or $z^2+4=0\Rightarrow z^2=-4$

$\Rightarrow z=\pm2i$

So the roots of $z^4-16=0$ are $2,-2,2i$ and $-2i$

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10 a $z^4 - 12z^3 + 31z^2 + 108z - 360$

$= (z^2 - 9)(z^2 + bz + c)$

$= z^4 + bz^3 + (-9 + c)z^2 - 9bz - 9c$

Equate z^3 coefficients:

$b = -12$

Equate constants:

$-9c = -360$, so $c = 40$

So $f(z) = (z^2 - 9)(z^2 - 12z + 40)$

b $0 = f(z)$

$= (z^2 - 9)(z^2 - 12z + 40)$

Either $z^2 - 9 = 0 \Rightarrow z = \pm 3$

Or $z^2 - 12z + 40 = 0$

$(z - 6)^2 - 36 + 40 = 0$

$(z - 6)^2 = -4$

$z - 6 = \pm 2i$

$z = 6 \pm 2i$

So the solutions to $f(z) = 0$ are

$3,\ -3,\ 6 + 2i$ and $6 - 2i$

11 If $g(2 + 3i) = 0$ then also $g(2 - 3i) = 0$

So $(z - (2 + 3i))(z - (2 - 3i))$ is a factor of $g(z)$

Now $(z - (2 + 3i))(z - (2 - 3i))$

$= z^2 - (2 - 3i)z - (2 + 3i)z + (2 + 3i)(2 - 3i)$

$= z^2 - 2z + 3iz - 2z - 3iz + 4 - 6i + 6i - 9i^2$

$= z^2 - 4z + 13$

So $z^2 - 4z + 13$ is a factor of $g(z)$

Use long division to find another factor:

$$\begin{array}{r} z^2 + 6z + 10 \\ z^2 - 4z + 13 \overline{)\ z^4 + 2z^3 - z^2 + 38z + 130} \\ z^4 - 4z^3 + 13z^2 \\ \hline 6z^3 - 14z^2 + 38z \\ 6z^3 - 24z^2 + 78z \\ \hline 10z^2 - 40z + 130 \\ 10z^2 - 40z + 130 \\ \hline 0 \end{array}$$

11 So $g(z) = (z^2 - 4z + 13)(z^2 + 6z + 10)$

$g(z) = 0 \Rightarrow$

either $(z^2 - 4z + 13) = 0 \Rightarrow z = 2 \pm 3i$

(these were the two solutions stated at the beginning of the question)

or $z^2 + 6z + 10 = 0$

$(z + 3)^2 - 9 + 10 = 0$

$(z + 3)^2 = -1$

$z + 3 = \pm i$

$z = -3 \pm i$

So the roots of $g(z) = 0$ are

$2 + 3i,\ 2 - 3i,\ -3 + i$ and $-3 - i$

12 a If $z = 2 - 3i$ is a root then

$z = 2 + 3i$ is also a root

So $(z - (2 - 3i))(z - (2 + 3i))$ is a factor of $f(z)$

Now $(z - (2 - 3i))(z - (2 + 3i))$

$= z^2 - (2 + 3i)z - (2 - 3i)z + (2 - 3i)(2 + 3i)$

$= z^2 - 2z - 3iz - 2z + 3iz + 4 + 6i - 6i - 9i^2$

$= z^2 - 4z + 13$

So $z^2 - 4z + 13$ is a factor of $f(z)$

$\therefore\ f(z) = z^4 - 10z^3 + 71z^2 + Qz + 442$

$= (z^2 - 4z + 13)(z^2 + bz + c)$

$= z^4 + (4 + b)z^3 + (c - 4b + 13)z^2 + (13b - 4c)z + 13c$

Equate z^3 coefficients:

$-4 + b = -10$, so $b = -6$

Equate constants:

$13c = 442$, so $c = 34$

Hence $(z^2 - 6z + 34)$ is a factor of $f(z)$

12 b To find Q, equate z coefficients:

$13b - 4c = Q$

$13(-6) - 4(34) = Q$

$Q = -214$

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12 c $0 = \mathrm{f}(z)$

$= (z^2 - 4z + 13)(z^2 - 6z + 34)$

either $(z^2 - 4z + 13) = 0 \Rightarrow z = 2 \pm 3\mathrm{i}$

(these were the two solutions stated at the beginning of the question)

or $z^2 - 6z + 34 = 0$

$(z-3)^2 - 9 + 34 = 0$

$(z-3)^2 = -25$

$z - 3 = \pm 5\mathrm{i}$

$z = 3 \pm 5\mathrm{i}$

So the roots of $\mathrm{f}(z) = 0$ are $2 - 3\mathrm{i},\ 2 + 3\mathrm{i},\ 3 + 5\mathrm{i}$ and $3 - 5\mathrm{i}$.

Challenge

$z^5 + bz^4 + cz^3 + dz^2 + ez + f = 0$ (*)

The other two roots of (*) are –2i and 1 – i

For any pair of complex conjugate roots α and β of (*), the quadratic expression

$(z-\alpha)(z-\beta) = z^2 - (\alpha + \beta)z + \alpha\beta$

is a factor of (*)

Using $\alpha = 2\mathrm{i}$ and $\beta = -2\mathrm{i}$,

$\alpha + \beta = 2\mathrm{i} - 2\mathrm{i} = 0$

$\alpha\beta = (2\mathrm{i})(-2\mathrm{i}) = -4\mathrm{i}^2 = 4$

So one quadratic factor of (*) is

$z^2 + 4 = 0$

Using $\alpha = 1 + \mathrm{i}$ and $\beta = 1 - \mathrm{i}$,

$\alpha + \beta = (1+\mathrm{i}) + (1-\mathrm{i}) = 2$

$\alpha\beta = (1+\mathrm{i})(1-\mathrm{i})$

$= 1(1-\mathrm{i}) + \mathrm{i}(1-\mathrm{i})$

$= 1 - \mathrm{i} + \mathrm{i} - \mathrm{i}^2 = 2$

So a second quadratic factor of (*) is

$z^2 - 2z + 2 = 0$

Also, since –2 is the only real root of (*), then $(z+2)$ is a linear factor of (*)

Hence,

$0 = z^5 + bz^4 + cz^3 + dz^2 + ez + f$

$= (z+2)(z^2+4)(z^2-2z+2)$

$= (z^3 + 2z^2 + 4z + 8)(z^2 - 2z + 2)$

$= z^3(z^2 - 2z + 2) + 2z^2(z^2 - 2z + 2)$

$+ 4z(z^2 - 2z + 2) + 8(z^2 - 2z + 2)$

$= z^5 - 2z^4 + 2z^3 + 2z^4 - 4z^3 + 4z^2$

$+ 4z^3 - 8z^2 + 8z + 8z^2 - 16z + 16$

$= z^5 + 2z^3 + 4z^2 - 8z + 16$

$b = 0,\ c = 2,\ d = 4,\ e = -8,\ f = 16$

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Complex numbers Mixed exercise 1

1 a $z_1 + z_2 = (8-3\mathrm{i}) + (-2+4\mathrm{i})$
$= 6+\mathrm{i}$

b $3z_2 = 3(-2+4\mathrm{i})$
$= -6+12\mathrm{i}$

c $6z_1 - z_2 = 6(8-3\mathrm{i}) - (-2+4\mathrm{i})$
$= 48-18\mathrm{i}+2-4\mathrm{i}$
$= 50-22\mathrm{i}$

2 If $z^2+bz+14=0$ has no real roots then the discriminant <0
$b^2 - 4(1)(14) < 0$
$b^2 < 56$
$-\sqrt{56} < b < \sqrt{56}$
$-2\sqrt{14} < b < 2\sqrt{14}$

3 $z^2 - 6z + 12 = 0$
Solve by completing the square:
$(z-3)^2 - 9 + 12 = 0$
$(z-3)^2 = -3$
$z-3 = \pm\mathrm{i}\sqrt{3}$
$z = 3 \pm \mathrm{i}\sqrt{3}$

So $z_1 = 3+\mathrm{i}\sqrt{3}$ and $z_2 = 3-\mathrm{i}\sqrt{3}$

4 To expand $(1+2\mathrm{i})^5$ use the binomial expansion of $(a+b)^5$
$(a+b)^5 = a^5 + 5a^4b + 10a^3b^2$
$+10a^2b^3 + 5ab^4 + b^5$
So
$(1+2\mathrm{i})^5 = (1)^5 + 5(1)^4(2\mathrm{i}) + 10(1)^3(2\mathrm{i})^2$
$+10(1)^2(2\mathrm{i})^3 + 5(1)(2\mathrm{i})^4 + (2\mathrm{i})^5$
$= 1+10\mathrm{i}+40\mathrm{i}^2+80\mathrm{i}^3+80\mathrm{i}^4+32\mathrm{i}^5$
Use the fact that $\mathrm{i}^2 = -1, \mathrm{i}^3 = -\mathrm{i}, \mathrm{i}^4 = 1$ and $\mathrm{i}^5 = \mathrm{i}$ to simplify
$(1+2\mathrm{i})^5 = 1+10\mathrm{i}-40-80\mathrm{i}+80+32\mathrm{i}$
$= 41-38\mathrm{i}$

5 $\mathrm{f}(z) = z^2 - 6z + 10$
$\mathrm{f}(3+\mathrm{i}) = (3+\mathrm{i})^2 - 6(3+\mathrm{i}) + 10$
$= 9+3\mathrm{i}+3\mathrm{i}+\mathrm{i}^2-18-6\mathrm{i}+10$
$= (9-1-18+10) + (3+3-6)\mathrm{i}$
$= 0$

6 a If $z_1 = 4+2\mathrm{i}$ then z_1^* is the complex conjugate. So $z_1^* = 4-2\mathrm{i}$

b $z_1 z_2 = (4+2\mathrm{i})(-3+\mathrm{i})$
$= -12+4\mathrm{i}-6\mathrm{i}+2\mathrm{i}^2$
$= -14-2\mathrm{i}$

c $\frac{z_1}{z_2} = \frac{4+2\mathrm{i}}{-3+\mathrm{i}}$
$= \frac{(4+2\mathrm{i})}{(-3+\mathrm{i})} \times \frac{(-3-\mathrm{i})}{(-3-\mathrm{i})}$
$= \frac{-12-4\mathrm{i}-6\mathrm{i}-2\mathrm{i}^2}{9+3\mathrm{i}-3\mathrm{i}-\mathrm{i}^2}$
$= \frac{-10-10\mathrm{i}}{10}$
$= -1-\mathrm{i}$

7 $(7-2\mathrm{i})^2 = 49-14\mathrm{i}-14\mathrm{i}+4\mathrm{i}^2$
$= 45-28\mathrm{i}$
$\frac{(7-2\mathrm{i})^2}{1+\mathrm{i}\sqrt{3}} = \frac{(45-28\mathrm{i})}{(1+\mathrm{i}\sqrt{3})} \times \frac{(1-\mathrm{i}\sqrt{3})}{(1-\mathrm{i}\sqrt{3})}$
$= \frac{45-45\mathrm{i}\sqrt{3}-28\mathrm{i}+28\mathrm{i}^2\sqrt{3}}{1-\mathrm{i}\sqrt{3}+\mathrm{i}\sqrt{3}-(\sqrt{3})^2\mathrm{i}^2}$
$= \frac{45-28\sqrt{3}}{4} + \left(\frac{-28-45\sqrt{3}}{4}\right)\mathrm{i}$

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8 $\dfrac{4-7\text{i}}{z}=3+\text{i}$

$$z=\frac{4-7\text{i}}{3+\text{i}}$$

$$z=\frac{(4-7\text{i})}{3+\text{i}}\times\frac{(3-\text{i})}{(3-\text{i})}$$

$$=\frac{12-4\text{i}-21\text{i}+7\text{i}^2}{9-3\text{i}+3\text{i}-\text{i}^2}$$

$$=\frac{5-25\text{i}}{10}$$

$$=\frac{1}{2}-\frac{5}{2}\text{i}$$

9 $z=\dfrac{1}{2+\text{i}}$

$$=\frac{1}{(2+\text{i})}\times\frac{(2-\text{i})}{(2-\text{i})}$$

$$=\frac{2-\text{i}}{4-2\text{i}+2\text{i}-\text{i}^2}$$

$$z=\frac{2-\text{i}}{5}=\frac{2}{5}-\frac{1}{5}\text{i}$$

a $z^2=\left(\dfrac{2-\text{i}}{5}\right)\left(\dfrac{2-\text{i}}{5}\right)$

$$=\frac{4-2\text{i}-2\text{i}+\text{i}^2}{25}$$

So $z^2=\dfrac{3}{25}-\dfrac{4}{25}\text{i}$

b $z-\dfrac{1}{z}=\left(\dfrac{2}{5}-\dfrac{1}{5}\text{i}\right)-\dfrac{1}{\left(\frac{1}{2+\text{i}}\right)}$

$$=\left(\frac{2}{5}-\frac{1}{5}\text{i}\right)-(2+\text{i})$$

$$=-\frac{8}{5}-\frac{6}{5}\text{i}$$

10 If $z=a+b\text{i}$, then $z^*=a-b\text{i}$

So $\dfrac{z}{z^*}=\dfrac{a+b\text{i}}{a-b\text{i}}$

$$=\frac{(a+b\text{i})}{(a-b\text{i})}\times\frac{(a+b\text{i})}{(a+b\text{i})}$$

$$=\frac{a^2+ab\text{i}+ab\text{i}+b^2\text{i}^2}{a^2+ab\text{i}-ab\text{i}-b^2\text{i}^2}$$

$$=\frac{a^2-b^2+2ab\text{i}}{a^2+b^2}$$

$$=\left(\frac{a^2-b^2}{a^2+b^2}\right)+\left(\frac{2ab}{a^2+b^2}\right)\text{i}$$

11 a $z=\dfrac{3+q\text{i}}{q-5\text{i}}$

Multiply by the complex conjugate

$$z=\frac{(3+q\text{i})}{(q-5\text{i})}\times\frac{(q+5\text{i})}{(q+5\text{i})}$$

$$=\frac{3q+15\text{i}+q^2\text{i}+5q\text{i}^2}{q^2+5q\text{i}-5q\text{i}-25\text{i}^2}$$

$$=\left(\frac{-2q}{q^2+25}\right)+\left(\frac{q^2+15}{q^2+25}\right)\text{i}$$

The real part of z is $\dfrac{1}{13}$, so

$$\frac{-2q}{q^2+25}=\frac{1}{13}$$

$$-26q=q^2+25$$

$$q^2+26q+25=0$$

$$(q+25)(q+1)=0$$

So $q=-1$ or $q=-25$

b If $q=-1$

$$z=\frac{-2(-1)}{(-1)^2+25}+\frac{(-1)^2+15}{(-1)^2+25}\text{i}$$

$$=\frac{2}{26}+\frac{16}{26}\text{i}$$

$$=\frac{1}{13}+\frac{8}{13}\text{i}$$

If $q=-25$

$$z=\left(\frac{-2(-25)}{(-25)^2+25}\right)+\left(\frac{(-25)^2+15}{(-25)^2+25}\right)\text{i}$$

$$=\frac{50}{650}+\frac{640}{650}\text{i}$$

$$=\frac{1}{13}+\frac{64}{65}\text{i}$$

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12 $z+4\mathrm{i}z^{*}=-3+18\mathrm{i}$ (*)

Substitute $z=x+\mathrm{i}y$ and $z^{*}=x-\mathrm{i}y$ in (*)

$x+\mathrm{i}y+4\mathrm{i}(x-\mathrm{i}y)=-3+18\mathrm{i}$

$x+\mathrm{i}y+4x\mathrm{i}-4y\mathrm{i}^{2}=-3+18\mathrm{i}$

$(x+4y)+(4x+y)\mathrm{i}=-3+18\mathrm{i}$

Equate real parts and imaginary parts:

$x+4y=-3 \quad (\mathbf{1})$

$4x+y=18 \quad (\mathbf{2})$

Multiply $(\mathbf{2})$ by 4:

$16x+4y=72 \quad (\mathbf{3})$

Subtract $(\mathbf{1})$ from $(\mathbf{3})$

$15x=75$, so $x=5$

Substitute $x=5$ into (**2**)

$4(5)+y=18$, so $y=-2$.

13 $\dfrac{z}{w}=\dfrac{9+6\mathrm{i}}{2-3\mathrm{i}}$

Multiply by the complex conjugate

$\dfrac{z}{w}=\dfrac{(9+6\mathrm{i})}{2-3\mathrm{i}}\times\dfrac{(2+3\mathrm{i})}{2+3\mathrm{i}}$

$=\dfrac{18+27\mathrm{i}+12\mathrm{i}+18\mathrm{i}^{2}}{4+6\mathrm{i}-6\mathrm{i}-9\mathrm{i}^{2}}$

$=\dfrac{39}{13}\mathrm{i}$

$=3\mathrm{i}$

14 $z=\dfrac{q+3\mathrm{i}}{4+q\mathrm{i}}$

Multiply by the complex conjugate

$z=\dfrac{(q+3\mathrm{i})}{(4+q\mathrm{i})}\times\dfrac{(4-q\mathrm{i})}{(4-q\mathrm{i})}$

$=\dfrac{4q-q^{2}\mathrm{i}+12\mathrm{i}-3q\mathrm{i}^{2}}{16-4q\mathrm{i}+4q\mathrm{i}-q^{2}\mathrm{i}^{2}}$

$=\dfrac{4q+3q}{16+q^{2}}+\dfrac{12-q^{2}}{16+q}\mathrm{i}$

$=\dfrac{7q}{16+q^{2}}+\dfrac{12-q^{2}}{16+q^{2}}\mathrm{i}$

15 a If $6-2\mathrm{i}$ is a root, then the complex conjugate $6+2\mathrm{i}$ is also a root.

b $(z-(6-2\mathrm{i}))(z-(6+2\mathrm{i}))$

$=z^{2}-(6+2\mathrm{i})z-(6-2\mathrm{i})z$

$+(6-2\mathrm{i})(6+2\mathrm{i})$

$=z^{2}-6z-2\mathrm{i}z-6z+2\mathrm{i}z$

$+36+12\mathrm{i}-12\mathrm{i}-4\mathrm{i}^{2}$

$=z^{2}-12z+36+4$

$=z^{2}-12z+40$

The equation is $z^{2}-12z+40=0$

16 If $z=4-k\mathrm{i}$ is a root, then the complex conjugate $4+k\mathrm{i}$ is also a root.

$(z-(4-k\mathrm{i}))(z-(4+k\mathrm{i}))$

$=z^{2}-(4+k\mathrm{i})z-(4-k\mathrm{i})z$

$+(4-k\mathrm{i})(4+k\mathrm{i})$

$=z^{2}-4z-k\mathrm{i}z-4z+k\mathrm{i}z$

$+16+4k\mathrm{i}-4k\mathrm{i}-k^{2}\mathrm{i}^{2}$

$=z^{2}-8z+16+k^{2}$

So $z^{2}-8z+16k^{2}=z^{2}-2mz+52$

Equate z coefficients:

$-8=2m$, so $m=4$.

Equate constants:

$16+k^{2}=52$, so $k^{2}=36\Rightarrow k=\pm 6$

But $k>0$, so $k=6$.

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17 If $2+\mathrm{i}$ is a root of $\mathrm{h}(z)=0$ then the complex conjugate $2-\mathrm{i}$ is also a root.

$$\begin{aligned}\mathrm{h}(z)&=(z-(2+\mathrm{i}))(z-(2-\mathrm{i}))\\&=z^2-(2-\mathrm{i})z-(2+\mathrm{i})z+(2+\mathrm{i})(2-\mathrm{i})\\&=z^2-2z+\mathrm{i}z-2z-\mathrm{i}z+4-2\mathrm{i}+2\mathrm{i}-\mathrm{i}^2\\&=z^2-4z+5\end{aligned}$$

Divide to find the remaining factor

$$\begin{array}{r}z+4\\ z^2-4z+5\overline{)\,z^3+0z^2-11z+20}\\ z^3-4z^2+5z\\ \hline 4z^2-16z+20\\ 4z^2-16z+20\\ \hline 0\end{array}$$

So $\mathrm{h}(z)=(z^2-4z+5)(z+4)=0$ has roots $2+\mathrm{i}$, $2-\mathrm{i}$ and -4

18 If $\mathrm{f}(1+3\mathrm{i})=0$, then $\mathrm{f}(1-3\mathrm{i})=0$.

So $\mathrm{f}(z)=(z-(1+3\mathrm{i}))(z-(1-3\mathrm{i}))$

$$\begin{aligned}&=z^2-(1-3\mathrm{i})z-(1+3\mathrm{i})z+(1+3\mathrm{i})(1-3\mathrm{i})\\&=z^2-z+3\mathrm{i}z-z-3\mathrm{i}z+1-3\mathrm{i}+3\mathrm{i}-9\mathrm{i}^2\\&=z^2-2z+10\end{aligned}$$

Divide to find the remaining factor

$$\begin{array}{r}z+2\\ z^2-2z+10\overline{)\,z^3+0z^2+6z+20}\\ z^3-2z^2+10z\\ \hline 2z^2-4z+20\\ 2z^2-4z+20\\ \hline 0\end{array}$$

So $z^3+62+20=(z^2-2z+10)(z+2)$

either $(z^2-2z+10)=0\Rightarrow z=1\pm 3\mathrm{i}$

(as these were the roots given at the beginning of the question)

or $z+2=0\Rightarrow z=-2$

So the roots of $z^3+6z+20=0$ are $1+3\mathrm{i}$, $1-3\mathrm{i}$ and -2

19 a $\mathrm{f}(z)=z^3+3z^2+kz+48$

If $\mathrm{f}(4\mathrm{i})=0$, then:

$$\begin{aligned}0&=\mathrm{f}(4\mathrm{i})\\&=(4\mathrm{i})^3+3(4\mathrm{i})^2+k(4\mathrm{i})+48\\&=64\mathrm{i}^3+48\mathrm{i}^2+4k\mathrm{i}+48\\&=-64\mathrm{i}-48+4k\mathrm{i}+48\\&=(4k-64)\mathrm{i}\end{aligned}$$

Hence $4k-64=0\Rightarrow k=16$.

b If $4\mathrm{i}$ is a root then $-4\mathrm{i}$ is also a root and $(z-4\mathrm{i})(z-(-4\mathrm{i}))$ is a quadratic factor of $\mathrm{f}(z)$

Now $(z-4\mathrm{i})(z-(-4\mathrm{i}))=(z-4\mathrm{i})(z+4\mathrm{i})$

$$\begin{aligned}&=z^2+4\mathrm{i}z-4\mathrm{i}z-16\mathrm{i}^2\\&=z^2+16\end{aligned}$$

Divide to find the remaining factor

$$\begin{array}{r}z+3\\ z^2+0z+16\overline{)\,z^3+3z^3+16z+48}\\ z^3+0z^2+16z\\ \hline 3z^2+0z+48\\ 3z^2+0z+48\\ \hline 0\end{array}$$

So $\mathrm{f}(z)=(z^2+16)(z+3)=0$ has two other roots which are $-4\mathrm{i}$ and -3

20 a $z^4-z^3-16z^2-74z-60$

$$=(z^2-5z-6)(z^2+bz+c)$$

Equate z^3 coefficients:

$b-5=-1$, so $b=4$.

Equate constants:

$-6c=-60$, so $c=10$.

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20 b $0 = \mathrm{f}(z)$

$= (z^2 - 5z - 6)(z^2 + 4z + 10)$

$= (z-6)(z+1)(z^2 + 4z + 10)$

Either $z - 6 = 0 \Rightarrow z = 6$

or $z + 1 = 0 \Rightarrow z = -1$

or $z^2 + 4z + 10 = 0$

$(z+2)^2 - 4 + 10 = 0$

$(z+2)^2 = -6$

$z + 2 = \pm \mathrm{i}\sqrt{6}$

$z = -2 \pm \mathrm{i}\sqrt{6}$

So the roots of $\mathrm{f}(z) = 0$ are $6,\ -1,\ -2 + \mathrm{i}\sqrt{6}$ and $-2 - \mathrm{i}\sqrt{6}$.

21 If $\mathrm{g}(3 - 2\mathrm{i}) = 0$ then $\mathrm{g}(3 + 2\mathrm{i}) = 0$ and $(z - (3 - 2\mathrm{i}))(z - (3 + 2\mathrm{i}))$ is a quadratic factor of $\mathrm{g}(z)$

Now $(z - (3 - 2\mathrm{i}))(z - (3 + 2\mathrm{i}))$

$= z^2 - (3 + 2\mathrm{i})z - (3 - 2\mathrm{i})z + (3 - 2\mathrm{i})(3 + 2\mathrm{i})$

$= z^2 - 3z - 2\mathrm{i}z - 3z + 2\mathrm{i}z + 9 + 6\mathrm{i} - 6\mathrm{i} - 4\mathrm{i}^2$

$= z^2 - 6z + 9 + 4$

$= z^2 - 6z + 13$

Divide to find the remaining factor

$$z^2 - 6z + 13 \overline{)\ z^4 - 6z^3 + 19z^2 - 36z + 78}$$

$$z^2 + 6$$

$$z^4 - 6z^3 + 13z^2$$

$$6z^2 - 36z + 78$$

$$6z^2 - 36z + 78$$

$$0$$

$0 = \mathrm{g}(z)$

$= (z^2 + 6)(z^2 - 6z + 13)$

Either $z^2 - 6z + 13 = 0$

$\Rightarrow z = 3 \pm 2\mathrm{i}$

(as these were the two original roots)

or $z^2 + 6 = 0$

$z^2 = -6$

$z = \pm \mathrm{i}\sqrt{6}$

21 (cont.)

So the roots of $\mathrm{g}(z) = 0$ are

$3 + 2\mathrm{i},\ 3 - 2\mathrm{i},\ \mathrm{i}\sqrt{6}$ and $-\mathrm{i}\sqrt{6}$.

22 a $\mathrm{f}(z) = z^4 - 2z^3 - 5z^2 + pz + 24$

$0 = \mathrm{f}(4)$

$= 4^4 - 2(4)^3 - 5(4)^2 + 4p + 24$

$= 256 - 128 - 80 + 4p + 24$

$4p = -72$

$p = -18$

b If $\mathrm{f}(4) = 0$ then $(z - 4)$ is a factor of $\mathrm{f}(z)$

Divide to find the remaining factors

$$z - 4 \overline{)\ z^4 - 2z^3 - 5z^2 - 18z + 24}$$

$$z^3 + 2z^3 + 3z - 6$$

$$z^4 - 4z^3$$

$$2z^3 - 5z^2$$

$$2z^3 - 8z^2$$

$$3z^2 - 18z$$

$$3z^3 - 12z$$

$$-6z + 24$$

$$-6z + 24$$

$$0$$

So $0 = \mathrm{f}(z)$

$= (z - 4)(z^3 + 2z^2 + 3z - 6)$

Either $z - 4 = 0 \Rightarrow z = 4$

or $z^3 + 2z^2 + 3z - 6 = 0$ (*)

Now (*) is a cubic equation, so must have at least one real root (since complex roots always come in pairs).

Try small real values of z to find a factor of (*):

Let $z = 1$: $(1)^3 + 2(1)^2 + 3(1) - 6 = 0$

So $z = 1$ is a root of (*), and hence $z - 1$ is a factor.

So $z^3 + 2z^2 + 3z - 6 = (z - 1)(z^2 + bz + c)$

Equate z^2 terms:

$-1 + b = 2$, so $b = 3$

Equate constants:

$-6 = -c$, so $c = 6$

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22 b (cont.)

Hence

$z^3+2z^2+3z-6=(z-1)(z^2+3z+6)$

Either $z=1$

or

$$(z^2+3z+6)=0$$

$$\left(z+\frac{3}{2}\right)^2-\frac{9}{4}+6=0$$

$$\left(z+\frac{3}{2}\right)^2=-\frac{15}{4}$$

$$z+\frac{3}{2}=\pm\frac{\sqrt{15}}{2}\mathrm{i}$$

$$z=-\frac{3}{2}\pm\frac{\sqrt{15}}{2}\mathrm{i}$$

So the roots of $\mathrm{f}(z)=0$ are

$1,\ 4,\ \left(-\frac{3}{2}+\frac{\sqrt{15}}{2}\mathrm{i}\right)$ and $\left(-\frac{3}{2}-\frac{\sqrt{15}}{2}\mathrm{i}\right)$

Challenge

a Suppose, for a contradiction, that a cubic equation has repeated non-real roots. This repeated root gives us two roots of the cubic.
The complex conjugate of each repeated root would also be a root of the cubic.
This means we have found four roots of the cubic.
However, this is a contradiction, as a cubic has exactly three roots in total.
Therefore, a cubic equation cannot have repeated non-real roots.

b Any answer of the form $(z+a\mathrm{i})^2(z-a\mathrm{i})^2$ will work.

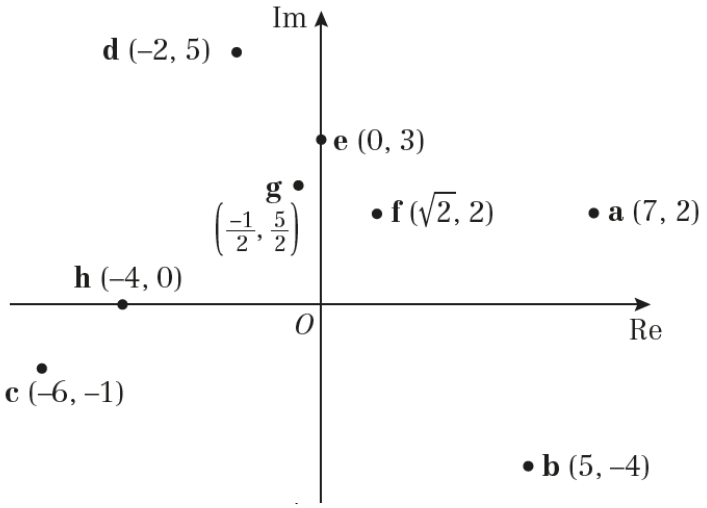
$$(z+a\mathrm{i})^2(z-a\mathrm{i})^2$$
$$=(z^2+2za\mathrm{i}+a^2\mathrm{i}^2)(z^2-2za\mathrm{i}+a^2\mathrm{i}^2)$$
$$=z^4-2a\mathrm{i}z^3+a^2\mathrm{i}^2z^2+2a\mathrm{i}z^3-4a^2\mathrm{i}^2z^2$$
$$+2a^3\mathrm{i}^3z+a^2\mathrm{i}^2z^2-2a^3\mathrm{i}^3z+a^4\mathrm{i}^4$$
$$=z^4-2az^3\mathrm{i}-a^2z^2+2az^3\mathrm{i}+4a^2z^2$$
$$-2a^3z\mathrm{i}-a^2z^2+2a^3z\mathrm{i}+a^4$$
$$=z^4+2a^2z^2+a^4$$

This is a quadratic equation with real coefficients. Its roots are non-real and repeated.

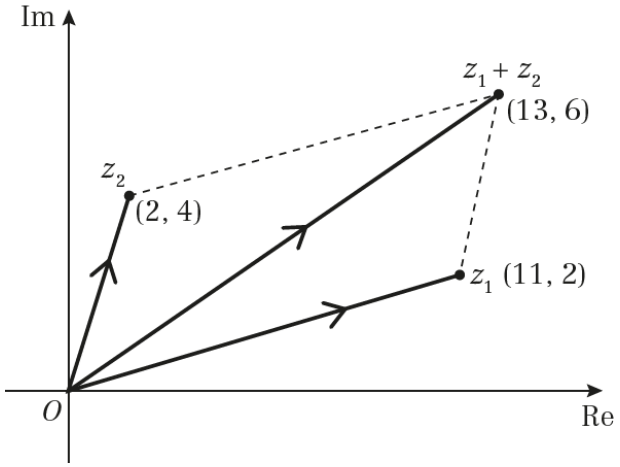
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Argand diagrams 2A

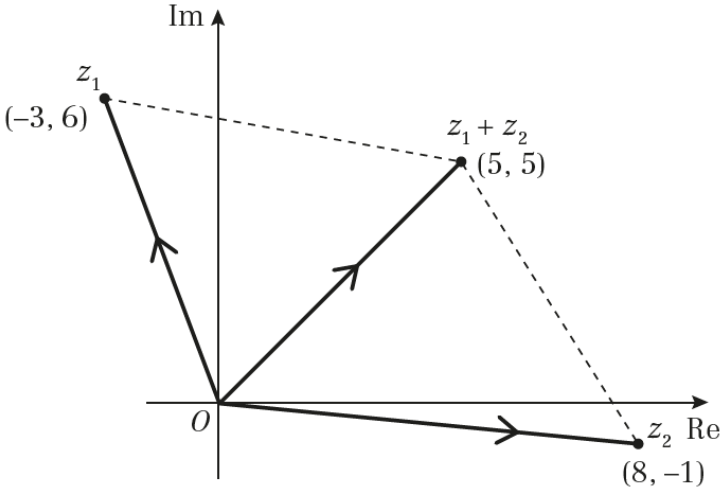
1



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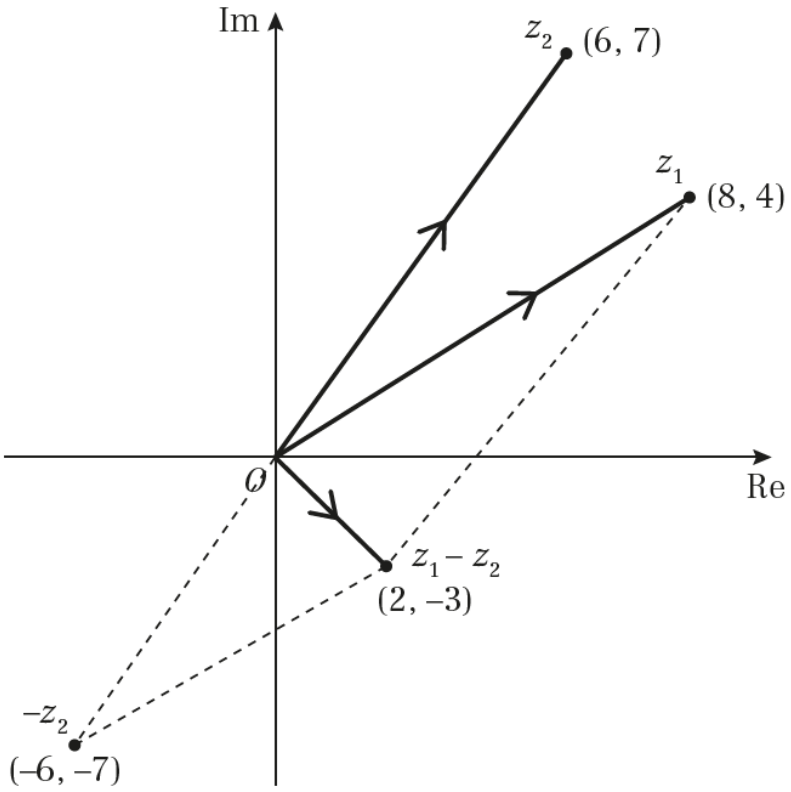


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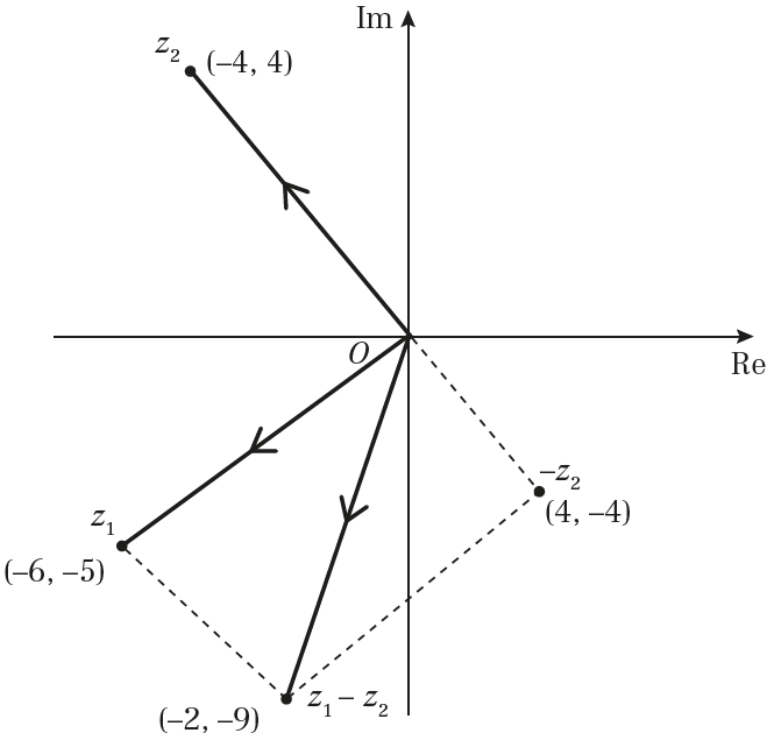


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4



5



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6 a $z_3 = z_1 + z_2$

$-3 + 2\text{i} = (7 - 5\text{i}) + (a + b\text{i})$

$= (7 + a) + (-5 + b)\text{i}$

Equate real coefficients: $-3 = 7 + a$

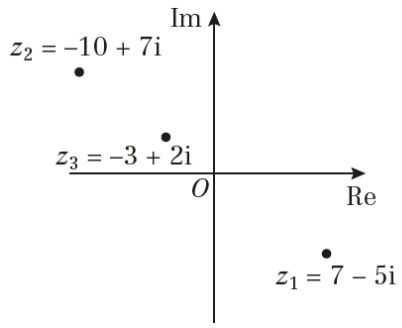
$a = -10$

Equate imaginary coefficients: $2 = -5 + b$

$b = 7$

Hence $z_2 = -10 + 7\text{i}$

b



7 a $z_3 = z_1 + z_2$

$-8 + 5\text{i} = (p + q\text{i}) + (9 - 5\text{i})$

Equate real coefficients: $-8 = p + 9$

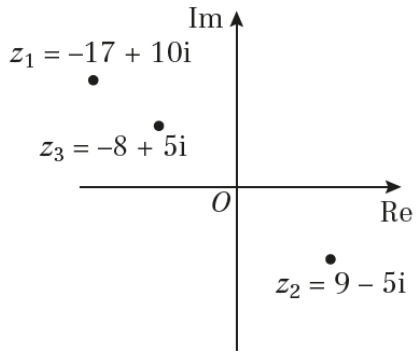
$p = -17$

Equate imaginary coefficients: $5 = -5 + q$

$q = 10$

$z_1 = -17 + 10\text{i}$

b



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8 a $z^2 - 6z + 10 = 0$

Solve by completing the square:

$(z-3)^2 - 9 + 10 = 0$

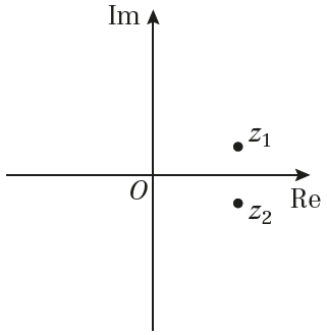
$(z-3)^2 = -1$

$z - 3 = \pm \mathrm{i}$

$z = 3 \pm \mathrm{i}$

So $z_1 = 3 + \mathrm{i}$ and $z_2 = 3 - \mathrm{i}$.

b



9 a $\mathrm{f}(z) = 2z^3 - 19z^2 + 64z - 60$

$$\mathrm{f}\left(\frac{3}{2}\right) = 2\left(\frac{3}{2}\right)^3 - 19\left(\frac{3}{2}\right)^2 + 64\left(\frac{3}{2}\right) - 60$$
$$= 2\left(\frac{27}{8}\right) - 19\left(\frac{9}{4}\right) + 64\left(\frac{3}{2}\right) - 60$$
$$= \frac{54}{8} - \frac{171}{4} + \frac{192}{2} - 60$$
$$= \frac{27}{4} - \frac{171}{4} + \frac{384}{4} - \frac{240}{4}$$
$$= 0$$

b If $\mathrm{f}\left(\frac{3}{2}\right) = 0$, then $2z - 3$ is a factor of $\mathrm{f}(z)$.

Use division to find the other factors:

$$\begin{array}{r} z^2 - 8z + 20 \\ 2z-3 \overline{)\ 2z^3 - 19z^2 + 64z - 60} \\ 2z^3 - 3z^2 \\ -16z^2 + 64z \\ -16z^2 + 24z \\ 40z - 60 \\ 40z - 60 \\ 0 \end{array}$$

So $2z^3 - 19z^2 + 64z - 60 = (2z-3)(z^2 - 8z + 20)$

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9 b Either $2z-3=0 \Rightarrow z=\frac{3}{2}$

or $z^2-8z+20=0$

Solve by completing the square:

$(z-4)^2-16+20=0$

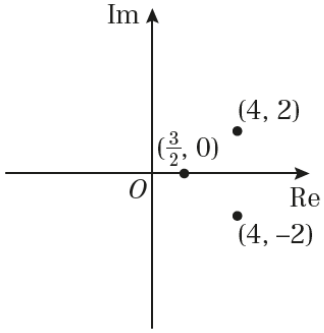
$(z-4)^2=-4$

$z-4=\pm 2\text{i}$

$z=4\pm 2\text{i}$

So the roots of $\text{f}(z)=0$ are $\frac{3}{2}$, $4+2\text{i}$ and $4-2\text{i}$

c



Challenge

a $z^6=1$

$z^6-1=0$

$(z^3-1)(z^3+1)=0$ (*)

Let $\text{f}(z)=z^3-1$.

Since $\text{f}(1)=0$, then $z-1$ is a factor of $\text{f}(z)$.

Hence $\text{f}(z)=(z-1)(z^2+bz+c)$

Equate coefficients of z^2: $0=b-1$

$b=1$

Equate constants: $-1=-c$

$c=1$

So $\text{f}(z)=(z-1)(z^2+z+1)$

Let $\text{g}(z)=z^3+1$.

Since $\text{g}(-1)=0$, then $z+1$ is a factor of $\text{g}(z)$

Hence $\text{g}(z)=(z+1)(z^2+pz+q)$

Equate coefficients of z^2: $0=1+p$

$p=-1$

Equate constants: $q=1$

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Challenge

a So $g(z)=(z+1)(z^2-z+1)$

By (*), $0=f(z)g(z)$

$$0=(z-1)(z^2+z+1)(z+1)(z^2-z+1)$$

Either $z-1=0\Rightarrow z=1$

or

$$z^2+z+1=0$$
$$\left(z+\frac{1}{2}\right)^2-\frac{1}{4}+1=0$$
$$\left(z+\frac{1}{2}\right)^2=-\frac{3}{4}$$
$$z+\frac{1}{2}=\pm\frac{\sqrt{3}}{2}\mathrm{i}$$
$$z=-\frac{1}{2}\pm\frac{\sqrt{3}}{2}\mathrm{i}$$

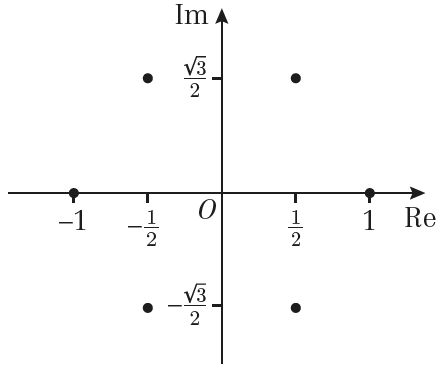
or $z+1=0\Rightarrow z=-1$

or

$$z^2-z+1=0$$
$$\left(z-\frac{1}{2}\right)^2-\frac{1}{4}+1=0$$
$$\left(z-\frac{1}{2}\right)^2=-\frac{3}{4}$$
$$z-\frac{1}{2}=\pm\frac{\sqrt{3}}{2}\mathrm{i}$$
$$z=\frac{1}{2}\pm\frac{\sqrt{3}}{2}\mathrm{i}$$

So the roots of $z^6=1$ are -1, 1, $-\frac{1}{2}+\frac{\sqrt{3}}{2}\mathrm{i}$, $-\frac{1}{2}-\frac{\sqrt{3}}{2}\mathrm{i}$, $\frac{1}{2}+\frac{\sqrt{3}}{2}\mathrm{i}$ and $\frac{1}{2}-\frac{\sqrt{3}}{2}\mathrm{i}$

b



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Challenge

c $(0,1)$ and $(0,-1)$ are on the unit circle.

Use Pythagoras' Theorem to check $\pm\frac{1}{2}\pm\frac{\sqrt{3}}{2}$i also lie on a circle with centre $(0,0)$ and radius 1.

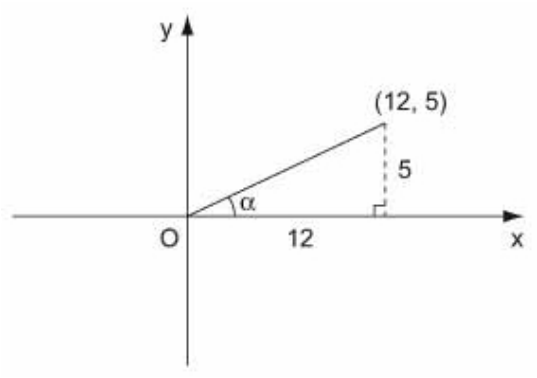
$\left(\pm\frac{1}{2}\right)^2+\left(\pm\frac{\sqrt{3}}{2}\right)^2=\frac{1}{4}+\frac{3}{4}=1$

So all points lie on a circle with centre $(0,0)$ and radius 1.

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Argand diagrams 2B

1 a $z = 12 + 5\text{i}$

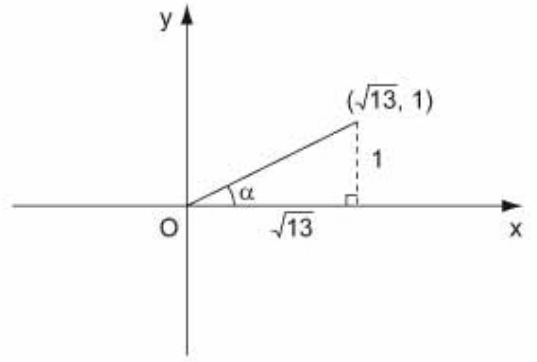


$|z| = \sqrt{(12^2 + 5^2)} = \sqrt{169} = 13$

$\tan\alpha = \frac{5}{12}$ $\alpha = 0.39$ rad.

$\arg z = 0.39$ radians (2 d.p.)

b $z = \sqrt{3} + \text{i}$

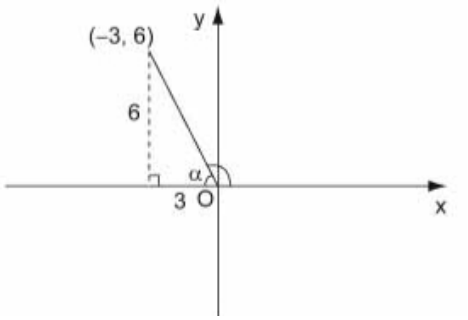


$|z| = \sqrt{\left((\sqrt{3})^2 + 1^2\right)} = \sqrt{4} = 2$

$\tan\alpha = \frac{1}{\sqrt{3}}$ $\alpha = \frac{\pi}{6}$

$\arg z = \frac{\pi}{6}$

c $z = -3 + 6\text{i}$

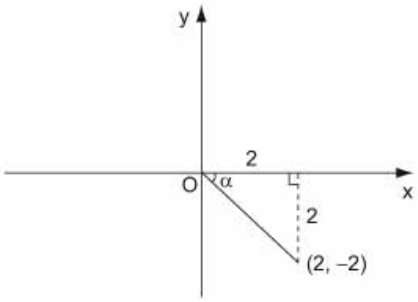


$|z| = \sqrt{\left((-3)^2 + 6^2\right)} = \sqrt{45} = 3\sqrt{5}$

$\tan\alpha = \frac{6}{3}$ $\alpha = 1.107$

$\arg z = \pi - \alpha = 2.03$ radians (2 d.p.)

d $z = 2 - 2\text{i}$

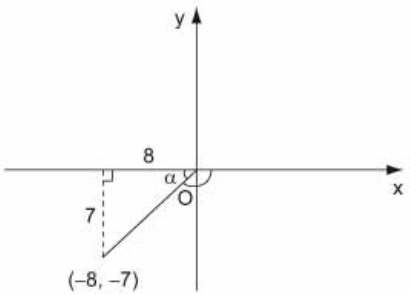


$|z| = \sqrt{\left(2^2 + (-2)^2\right)} = \sqrt{8} = 2\sqrt{2}$

$\tan\alpha = \frac{2}{2}$ $\alpha = \frac{\pi}{4}$

$\arg z = -\alpha = -\frac{\pi}{4}$

e $z = -8 - 7\text{i}$



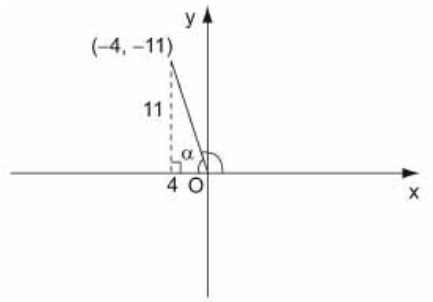
$|z| = \sqrt{\left((-8)^2 + (-7)^2\right)} = \sqrt{113}$

$\tan\alpha = \frac{7}{8}$ $\alpha = 0.7188$

$\arg z = -(\pi - \alpha) = -2.42$ radians (2 d.p.)

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1 f $z = -4 + 1\text{i}$

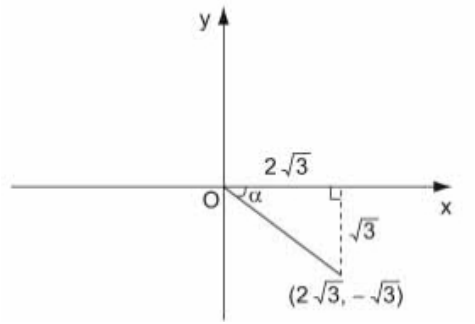


$|z| = \sqrt{\left((-4)^2 + 11^2\right)} = \sqrt{137}$

$\tan\alpha = \frac{11}{4} \quad \alpha = 1.222$

arg $z = \pi - \alpha = 1.92$ radians (2 d.p.)

g $z = 2\sqrt{3} - \text{i}\sqrt{3}$

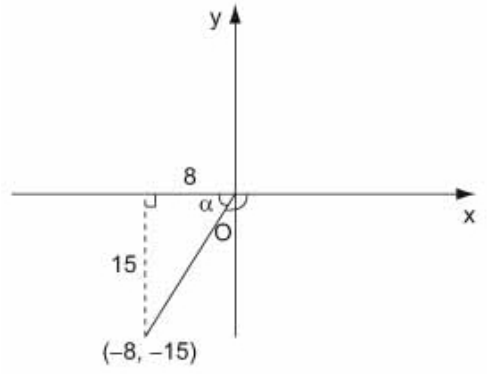


$|z| = \sqrt{\left((2\sqrt{3})^2 + (-\sqrt{3})^2\right)} = \sqrt{15}$

$\tan\alpha = \frac{\sqrt{3}}{2\sqrt{3}} = \frac{1}{2} \quad \alpha = 0.4636$

arg $z = -0.46$ radians (2 d.p.)

h $z = -8 - 15\text{i}$



$|z| = \sqrt{\left((-8)^2 + (-15)^2\right)} = \sqrt{289} = 17$

$\tan\alpha = \frac{15}{8} \quad \alpha = 1.0808$

arg $z = -(\pi - \alpha) = -2.06$ radians (2 d.p.)

2 a i $|2 + 2\text{i}| = \sqrt{(2)^2 + (2)^2}$

$= \sqrt{8}$

$= 2\sqrt{2}$

ii $\tan\alpha = \frac{2}{2}$, so $\alpha = \frac{\pi}{4}$

$\arg z = \frac{\pi}{4}$

b i $|5 + 5\text{i}| = \sqrt{(5)^2 + (5)^2}$

$= \sqrt{50}$

$= 5\sqrt{2}$

ii $\tan\alpha = \frac{5}{5}$, so $\alpha = \frac{\pi}{4}$

$\arg z = \frac{\pi}{4}$

c i $|-6 + 6\text{i}| = \sqrt{(-6)^2 + (6)^2}$

$= \sqrt{72}$

$= 6\sqrt{2}$

ii $\tan\alpha = \frac{6}{6}$, so $\alpha = \frac{\pi}{4}$

Here z is in the second quadrant, so

$\arg z = (\pi - \alpha)$

$\arg z = \pi - \frac{\pi}{4} = \frac{3\pi}{4}$

d i $|-a - a\text{i}| = \sqrt{(-a)^2 + (-a)^2}$

$= \sqrt{2a^2}$

$= a\sqrt{2}$

ii $\tan\alpha = \frac{a}{a}$, so $\alpha = \frac{\pi}{4}$.

Here z is in the third quadrant, so

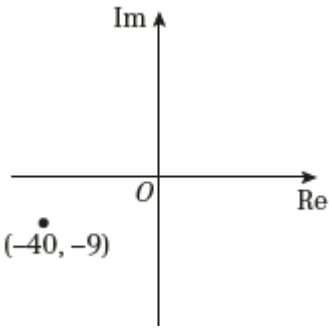
$\arg z = -(\pi - \alpha)$

$\arg z = -\left(\pi - \frac{\pi}{4}\right) = -\frac{3\pi}{4}$

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3 a

a



b $\tan\alpha = \frac{9}{40}$

$\alpha = \arctan\left(\frac{9}{40}\right) = 0.2213...$

Here z is in the third quadrant,

so $\arg z = -(\pi - \alpha)$

$\arg z = -(\pi - 0.2213...) = -2.92$

4 a $z = 3 + 4\text{i}$

$z^2 = (3+4\text{i})^2 = 9 + 12\text{i} + 12\text{i} + 16\text{i}^2$

$= -7 + 24\text{i}$

b $|z^2| = \sqrt{(-7)^2 + (24)^2} = \sqrt{625} = 25$

c $\tan\alpha = \frac{24}{7}$

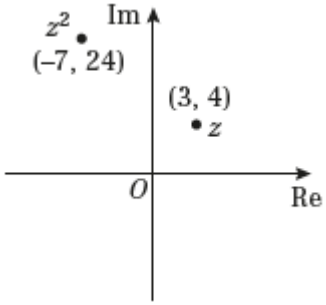
$\alpha = \arctan\left(\frac{24}{7}\right) = 1.2870...$

z^2 is in the second quadrant, so

$\arg z^2 = \pi - \alpha$

$\arg z^2 = \pi - 1.2870... = 1.85$

d



5 a $\frac{z_1}{z_2} = \frac{4+6\text{i}}{1+\text{i}}$

Multiply by the complex conjugate

$\frac{4+6\text{i}}{1+\text{i}} = \frac{(4+6\text{i})}{(1+\text{i})} \times \frac{(1-\text{i})}{(1-\text{i})}$

$= \frac{4 - 4\text{i} + 6\text{i} - 6\text{i}^2}{1 - \text{i} + \text{i} - \text{i}^2}$

$= \frac{10}{2} + \frac{2}{2}\text{i}$

$= 5 + \text{i}$

b $\left|\frac{z_1}{z_2}\right| = \sqrt{(5)^2 + (1)^2}$

$= \sqrt{26}$

c $\tan\alpha = \frac{1}{5}$

$\alpha = \arctan\frac{1}{5} = 0.20$

As z is in the first quadrant

$\arg z = \alpha = 0.20$ radians (2 d.p.)

6 a $\frac{z_1}{z_2} = 1 - \text{i} \Rightarrow z_2 = \frac{z_1}{1-\text{i}}$

$z_2 = \frac{3 + 2p\text{i}}{1 - \text{i}}$

Multiply by the complex conjugate

$z_2 = \frac{(3+2p\text{i})}{(1-\text{i})} \times \frac{(1+\text{i})}{(1+\text{i})}$

$= \frac{3 + 3\text{i} + 2p\text{i} + 2p\text{i}^2}{1 + \text{i} - \text{i} - \text{i}^2}$

$= \left(\frac{3-2p}{2}\right) + \left(\frac{3+2p}{2}\right)\text{i}$

b $\arg z_2 = \arctan\left(\frac{\frac{3+2p}{2}}{\frac{3-2p}{2}}\right)$

$= \arctan\left(\frac{3+2p}{3-2p}\right)$

Since we are told that $\arg z_2 = \arctan 5$ it follows that:

$\frac{3+2p}{3-2p} = 5$

$3 + 2p = 15 - 10p$

$-12 = -12p$

$p = 1$

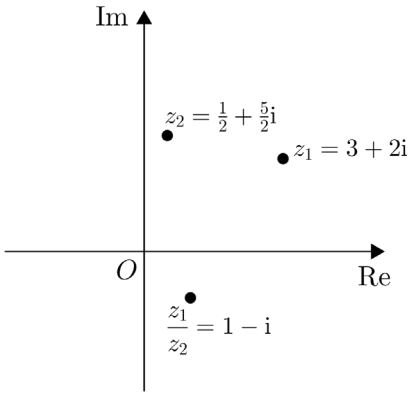
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6 c $z_2 = \dfrac{3-2p}{2} + \dfrac{3+2p}{2}\text{i}$

When $p = 1$, $z_2 = \dfrac{1}{2} + \dfrac{5}{2}\text{i}$

$|z_2| = \sqrt{\left(\dfrac{1}{2}\right)^2 + \left(\dfrac{5}{2}\right)^2} = \sqrt{\dfrac{26}{4}} = \dfrac{\sqrt{26}}{2}$

d



7 a $z = \dfrac{26}{2-3\text{i}}$

Multiply by the complex conjugate

$z = \dfrac{26}{(2-3\text{i})} \times \dfrac{(2+3\text{i})}{(2+3\text{i})}$

$= \dfrac{52+78\text{i}}{4+6\text{i}-6\text{i}-9\text{i}^2}$

$= \dfrac{52}{13} + \dfrac{78}{13}\text{i}$

$= 4 + 6\text{i}$

b $z^2 = (4+6\text{i})^2 = 16 + 24\text{i} + 24\text{i} + 36\text{i}^2$

So $z^2 = -20 + 48\text{i}$

c $|z| = \sqrt{(4)^2 + (6)^2} = \sqrt{52} = 2\sqrt{13}$

d $\tan\alpha = \dfrac{48}{20}$

$\alpha = \tan^{-1}\left(\dfrac{48}{20}\right) = 1.1760...$

Here z^2 is in the second quadrant

so $\arg(z^2) = \pi - \alpha$

$\arg(z^2) = \pi - 1.1760... = 1.97$

radians (2 d.p.)

8 a $z_1 + z_2 = (4+2\text{i}) + (2+4\text{i}) = 6 + 6\text{i}$

$|z_1 + z_2| = \sqrt{(6)^2 + (6)^2} = \sqrt{72} = 6\sqrt{2}$

b $w = \dfrac{z_1 z_3}{z_2} = \dfrac{(4+2\text{i})(a+b\text{i})}{2+4\text{i}}$

$z_1 z_3 = (4+2\text{i})(a+b\text{i})$

$= 4a + 4b\text{i} + 2a\text{i} + 2b\text{i}^2$

$= (4a-2b) + (4b+2a)\text{i}$

$w = \dfrac{(4a-2b)+(4b+2a)\text{i}}{2+4\text{i}}$

Multiply by the complex conjugate

$w = \dfrac{((4a-2b)+(4b+2a)\text{i})}{(2+4\text{i})} \times \dfrac{(2-4\text{i})}{(2-4\text{i})}$

$= \dfrac{8a-4b-(16a-8b)\text{i}+(8b+4a)\text{i}-(16b+8a)\text{i}^2}{4-8\text{i}+8\text{i}-16\text{i}^2}$

$= \dfrac{(16a+12b)+(-12a+16b)\text{i}}{20}$

$= \dfrac{16a+12b}{20} + \dfrac{-12a+16b}{20}\text{i}$

$= \dfrac{4a+3b}{5} + \dfrac{-3a+4b}{5}\text{i}$

c $w = \dfrac{21}{5} - \dfrac{22}{5}\text{i} = \dfrac{4a+3b}{5} + \dfrac{-3a+4b}{5}\text{i}$

Equate real coefficients:

$4a + 3b = 21$ **(1)**

Equate imaginary coefficients:

$-3a + 4b = -22$ **(2)**

$3 \times (1)$: $12a + 9b = 63$ **(3)**

$4 \times (2)$: $-12a + 16b = -88$ **(4)**

$(3) + (4) \Rightarrow 25b = -25$

$b = -1$

Substituting $b = -1$ into (1): $4a - 3 = 21$

$a = 6$

d $\tan\alpha = \dfrac{\frac{22}{5}}{\frac{21}{5}} = \dfrac{22}{21}$

$\alpha = \arctan\left(\dfrac{22}{21}\right) = 0.8086...$

Here z is in the fourth quadrant, so

$\arg z = -\alpha$

$\arg z = -0.81$ radians (2 d.p.)

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9 a $|w| = \sqrt{(6)^2 + (3)^2} = \sqrt{45} = 3\sqrt{5}$

b $\tan\alpha = \frac{3}{6} = \frac{1}{2}$

$\alpha = \tan^{-1}\left(\frac{1}{2}\right) = 0.4636...$

Here w is in the first quadrant so $\arg z = \alpha$

$\arg z = 0.46$ radians (2 d.p.)

c $\arg(\lambda + 5\mathrm{i} + w)$

$= \arg(\lambda + 5\mathrm{i} + 6 + 3\mathrm{i})$

$= \arg((\lambda + 6) + 8\mathrm{i})$

$\tan\alpha = \frac{8}{\lambda + 6}$

As $\tan\frac{\pi}{4} = 1$, it follows that

$\frac{8}{\lambda + 6} = 1$, so $\lambda = 2$.

10 a $|z| = \sqrt{(-1)^2 + \left(\sqrt{3}\right)^2} = \sqrt{4} = 2$

b $z = -1 - \mathrm{i}\sqrt{3}$, so $z^* = -1 + \mathrm{i}\sqrt{3}$

$\frac{z}{z^*} = \frac{-1 - \mathrm{i}\sqrt{3}}{-1 + \mathrm{i}\sqrt{3}}$

Multiply by the complex conjugate

$\frac{z}{z^*} = \frac{\left(-1 - \mathrm{i}\sqrt{3}\right)}{\left(-1 + \mathrm{i}\sqrt{3}\right)} \times \frac{\left(-1 - \mathrm{i}\sqrt{3}\right)}{\left(-1 - \mathrm{i}\sqrt{3}\right)}$

$= \frac{1 + \mathrm{i}\sqrt{3} + \mathrm{i}\sqrt{3} + \mathrm{i}^2\left(\sqrt{3}\right)^2}{1 + \mathrm{i}\sqrt{3} - \mathrm{i}\sqrt{3} - \mathrm{i}^2\left(\sqrt{3}\right)^2}$

$= \frac{-2 + 2\mathrm{i}\sqrt{3}}{4}$

$= -\frac{1}{2} + \frac{\sqrt{3}}{2}\mathrm{i}$

$\left|\frac{z}{z^*}\right| = \sqrt{\left(-\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} = \sqrt{\frac{1}{4} + \frac{3}{4}} = \sqrt{1} = 1$

c For $\arg z$, $\tan\alpha = \frac{\sqrt{3}}{1}$

$\alpha = \arctan\left(\sqrt{3}\right) = \frac{\pi}{3}$

Here z is in the third quadrant, so

$\arg z = -(\pi - \alpha)$

$\arg z = -\left(\pi - \frac{\pi}{3}\right) = -\frac{2\pi}{3}$

For $\arg z^*$, $\tan\alpha = \frac{\sqrt{3}}{1}$

$\alpha = \arctan\left(\sqrt{3}\right) = \frac{\pi}{3}$

Here z^* is in the second quadrant, so $\arg z^* = \pi - \alpha$

$\arg z^* = \pi - \frac{\pi}{3} = \frac{2\pi}{3}$

For $\arg\left(\frac{z}{z^*}\right)$, $\tan\alpha = \frac{\frac{\sqrt{3}}{2}}{\frac{1}{2}} = \frac{2\sqrt{3}}{2} = \sqrt{3}$

$\alpha = \arctan\left(\sqrt{3}\right) = \frac{\pi}{3}$

Here $\frac{z}{z^*}$ is in the second quadrant, so $\arg\left(\frac{z}{z^*}\right) = \pi - \alpha$

$\arg\left(\frac{z}{z^*}\right) = \pi - \frac{\pi}{3} = \frac{2\pi}{3}$

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11 $w+z=(k+\mathrm{i})+(-4+5k\mathrm{i})$

$=(k-4)+(1+5k)\mathrm{i}$

$\arg(w+z)=\frac{2\pi}{3}$, and $\tan\frac{2\pi}{3}=-\sqrt{3}$

$$\frac{1+5k}{k-4}=\frac{-\sqrt{3}}{1}$$

$$1+5k=-k\sqrt{3}+4\sqrt{3}$$

So $5k+k\sqrt{3}=4\sqrt{3}-1$

$$k\left(5+\sqrt{3}\right)=4\sqrt{3}-1$$

$$k=\frac{4\sqrt{3}-1}{5+\sqrt{3}}$$

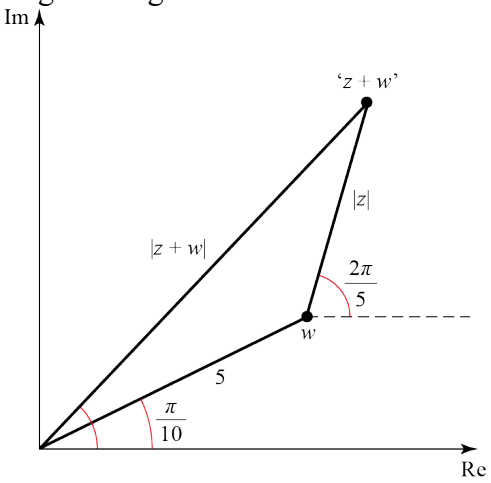
Rationalise the denominator by multiplying by the conjugate

$$k=\frac{\left(4\sqrt{3}-1\right)}{\left(5+\sqrt{3}\right)}\times\frac{\left(5-\sqrt{3}\right)}{\left(5-\sqrt{3}\right)}$$

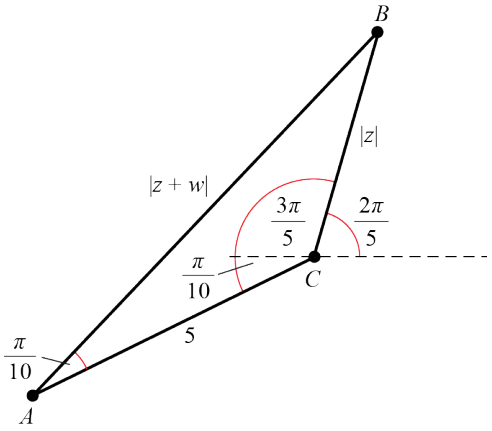
$$=\frac{20\sqrt{3}-12-5+\sqrt{3}}{25-5\sqrt{3}+5\sqrt{3}-3}$$

$$=\frac{21\sqrt{3}-17}{22}$$

12 Represent the given information on an Argand diagram:



Using this information, consider the triangle ΔABC shown below:



$$\angle A=\frac{\pi}{5}-\frac{\pi}{10}=\frac{\pi}{10}$$

$$\angle C=\frac{3\pi}{5}+\frac{\pi}{10}=\frac{7\pi}{10}$$

The angles in a triangle sum to π, so

$$\angle B=\pi-\angle C-\angle A$$

$$=\pi-\frac{7\pi}{10}-\frac{\pi}{10}$$

$$=\frac{\pi}{5}$$

Using the sine rule:

$$\frac{\sin A}{a}=\frac{\sin B}{b}=\frac{\sin C}{c}$$

$$\frac{\sin\frac{\pi}{10}}{|z|}=\frac{\sin\frac{\pi}{5}}{5}=\frac{\sin\frac{7\pi}{10}}{|z+w|}$$

$$5\sin\frac{\pi}{10}=|z|\sin\frac{\pi}{5}$$

$$|z|=\frac{5\sin\frac{\pi}{10}}{\sin\frac{\pi}{5}}\approx 2.63$$

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Argand diagrams 2C

1 a $r = \sqrt{\left(2^2 + 2^2\right)} = \sqrt{8} = 2\sqrt{2}$

$\tan\alpha = \frac{2}{2} = 1. \quad \alpha = \frac{\pi}{4}$

$\theta = \frac{\pi}{4}$

$2 + 2\mathrm{i} = 2\sqrt{2}\left(\cos\frac{\pi}{4} + \mathrm{i}\sin\frac{\pi}{4}\right)$

b $r = \sqrt{\left(0^2 + 3^2\right)} = \sqrt{9} = 3$

$\tan\alpha = \infty \quad \alpha = \frac{\pi}{2}$

$\theta = \frac{\pi}{2}$

$3\mathrm{i} = 3\left(\cos\frac{\pi}{2} + \mathrm{i}\sin\frac{\pi}{2}\right)$

c $r = \sqrt{\left((-3)^2 + 4^2\right)} = \sqrt{25} = 5$

$\tan\alpha = \frac{4}{3}. \quad \alpha = 0.927 \text{ rad.}$

$\theta = \pi - \alpha = 2.21$

$-3 + 4\mathrm{i} = 5(\cos 2.21 + \mathrm{i}\sin 2.21)$

d $r = \sqrt{\left(1^2 + (-\sqrt{3})^2\right)} = \sqrt{4} = 2$

$\tan\alpha = \frac{\sqrt{3}}{1}. \quad \alpha = \frac{\pi}{3}$

$\theta = -\frac{\pi}{3}$

$1 - \sqrt{3}\mathrm{i} = 2\left(\cos\left(\frac{-\pi}{3}\right) + \mathrm{i}\sin\left(\frac{-\pi}{3}\right)\right).$

e $r = \sqrt{\left((-2)^2 + (-5)^2\right)} = \sqrt{29}$

$\tan\alpha = \frac{5}{2}. \quad \alpha = 1.190 \text{ rad}$

$\theta = -(\pi - \alpha) = -1.95$

$-2 - 5\mathrm{i} = \sqrt{29}(\cos(-1.95) + \mathrm{i}\sin(-1.95)).$

f $r = \sqrt{\left((-20)^2 + 0^2\right)} = \sqrt{400} = 20$

$\tan\alpha = 0$

$\theta = \pi$

$-20 = 20(\cos\pi + \mathrm{i}\sin\pi)$

g $r = \sqrt{\left(7^2 + (-24)^2\right)} = \sqrt{625} = 25$

$\tan\alpha = \frac{24}{7}. \quad \alpha = 1.287 \text{ rad}$

$\theta = -1.29$

$7 - 24\mathrm{i} = 25(\cos(-1.29) + \mathrm{i}\sin(-1.29))$

h $r = \sqrt{\left((-5)^2 + 5^2\right)} = \sqrt{50} = 5\sqrt{2}$

$\tan\alpha = \frac{5}{5} = 1. \quad \alpha = \frac{\pi}{4}.$

$\theta = \pi - \alpha = \frac{3\pi}{4}$

$-5 + 5\mathrm{i} = 5\sqrt{2}\left(\cos\frac{3\pi}{4} + \mathrm{i}\sin\frac{3\pi}{4}\right).$

2 a $\frac{3}{1+\mathrm{i}\sqrt{3}} = \frac{3(1-\mathrm{i}\sqrt{3})}{(1+\mathrm{i}\sqrt{3})(1-\mathrm{i}\sqrt{3})}$

$= \frac{3-3\mathrm{i}\sqrt{3}}{1(1-\mathrm{i}\sqrt{3}) + \mathrm{i}\sqrt{3}(1-\mathrm{i}\sqrt{3})}$

$= \frac{3-3\mathrm{i}\sqrt{3}}{1-\mathrm{i}\sqrt{3}+\mathrm{i}\sqrt{3}-3\mathrm{i}^2}$

$= \frac{3-3\mathrm{i}\sqrt{3}}{4}$

$= \frac{3}{4} - \frac{3\sqrt{3}}{4}\mathrm{i}$

$r = \sqrt{\left(\left(\frac{3}{4}\right)^2 + \left(-\frac{3\sqrt{3}}{4}\right)^2\right)} = \sqrt{\left(\frac{9}{16} + \frac{27}{16}\right)}$

$= \sqrt{\left(\frac{36}{16}\right)} = \frac{3}{2}$

$\tan\alpha = \frac{3\sqrt{3}}{4} \div \frac{3}{4} = \sqrt{3} \quad \alpha = \frac{\pi}{3}$

$\theta = -\frac{\pi}{3}$

$\frac{3}{1+\mathrm{i}\sqrt{3}} = \frac{3}{2}\left(\cos\left(\frac{-\pi}{3}\right) + \mathrm{i}\sin\left(\frac{-\pi}{3}\right)\right)$

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2 b $\dfrac{1}{2-\mathrm{i}} = \dfrac{2+\mathrm{i}}{(2-\mathrm{i})(2+\mathrm{i})}$

$= \dfrac{2+\mathrm{i}}{2(2+\mathrm{i})-\mathrm{i}(2+\mathrm{i})}$

$= \dfrac{2+\mathrm{i}}{4+2\mathrm{i}-2\mathrm{i}-\mathrm{i}^2}$

$= \dfrac{2+\mathrm{i}}{5} = \dfrac{2}{5}+\dfrac{1}{5}\mathrm{i}$

$r = \sqrt{\left(\left(\dfrac{2}{5}\right)^2+\left(\dfrac{1}{5}\right)^2\right)} = \sqrt{\left(\dfrac{4}{25}+\dfrac{1}{25}\right)}$

$= \sqrt{\dfrac{5}{25}} = \dfrac{1}{\sqrt{5}} = \dfrac{\sqrt{5}}{5}$

$\tan\alpha = \dfrac{1}{5}\div\dfrac{2}{5} = \dfrac{1}{2}.$ $\quad \alpha = 0.4636$ rad.

$\theta = 0.46$

$\dfrac{1}{2-\mathrm{i}} = \dfrac{\sqrt{5}}{5}(\cos 0.46 + \mathrm{i}\sin 0.46)$

c $\dfrac{1+\mathrm{i}}{1-\mathrm{i}} = \dfrac{(1+\mathrm{i})(1+\mathrm{i})}{(1-\mathrm{i})(1+\mathrm{i})}$

$= \dfrac{1(1+\mathrm{i})+\mathrm{i}(1+\mathrm{i})}{1(1+\mathrm{i})-\mathrm{i}(1+\mathrm{i})} = \dfrac{1+\mathrm{i}+\mathrm{i}+\mathrm{i}^2}{1+\mathrm{i}-\mathrm{i}-\mathrm{i}^2}$

$= \dfrac{2\mathrm{i}}{2} = \mathrm{i}$

$r = \sqrt{\left(0^2+1^2\right)} = 1$

$\tan\alpha = \infty.$ $\quad \alpha = \dfrac{\pi}{2}$

$\theta = \dfrac{\pi}{2}$

$\dfrac{1+\mathrm{i}}{1-\mathrm{i}} = 1\left(\cos\dfrac{\pi}{2}+\mathrm{i}\sin\dfrac{\pi}{2}\right)$

3 a $5\left(\cos\dfrac{\pi}{2}+\mathrm{i}\sin\dfrac{\pi}{2}\right)$

$= 5(0+\mathrm{i})$

$= 5\mathrm{i}$

b $\dfrac{1}{2}\left(\cos\dfrac{\pi}{6}+\mathrm{i}\sin\dfrac{\pi}{6}\right)$

$= \dfrac{1}{2}\left(\dfrac{\sqrt{3}}{2}+\dfrac{1}{2}\mathrm{i}\right)$

$= \dfrac{\sqrt{3}}{4}+\dfrac{1}{4}\mathrm{i}$

c $6\left(\cos\dfrac{5\pi}{6}+\mathrm{i}\sin\dfrac{5\pi}{6}\right)$

$= 6\left(-\dfrac{\sqrt{3}}{2}+\dfrac{1}{2}\mathrm{i}\right)$

$= -3\sqrt{3}+3\mathrm{i}$

d $3\left(\cos\left(-\dfrac{2\pi}{3}\right)+\mathrm{i}\sin\left(-\dfrac{2\pi}{3}\right)\right)$

$= 3\left(-\dfrac{1}{2}-\dfrac{\sqrt{3}}{2}\mathrm{i}\right)$

$= -\dfrac{3}{2}-\dfrac{3\sqrt{3}}{2}\mathrm{i}$

e $2\sqrt{2}\left(\cos\left(-\dfrac{\pi}{4}\right)+\mathrm{i}\sin\left(-\dfrac{\pi}{4}\right)\right)$

$= 2\sqrt{2}\left(\dfrac{1}{\sqrt{2}}-\dfrac{1}{\sqrt{2}}\mathrm{i}\right)$

$= 2-2\mathrm{i}$

f $-4\left(\cos\dfrac{7\pi}{6}+\mathrm{i}\sin\dfrac{7\pi}{6}\right)$

$= -4\left(-\dfrac{\sqrt{3}}{2}-\dfrac{1}{2}\mathrm{i}\right)$

$= 2\sqrt{3}+2\mathrm{i}$

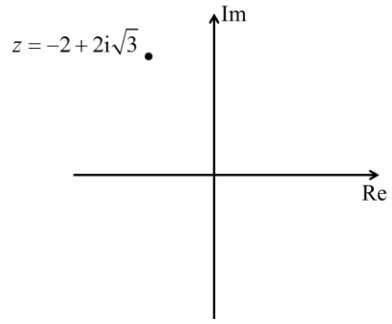
4 a $z = 4\left(\cos\left(\dfrac{2\pi}{3}\right)+\mathrm{i}\sin\left(\dfrac{2\pi}{3}\right)\right)$

$= 4\left(-\dfrac{1}{2}+\mathrm{i}\dfrac{\sqrt{3}}{2}\right)$

$= -2+2\mathrm{i}\sqrt{3}$

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4 b



5 $|z| = 7$ and $\arg z = \dfrac{11\pi}{6}$

$$z = 7\left(\cos\left(\frac{11\pi}{6}\right) + i\sin\left(\frac{11\pi}{6}\right)\right)$$
$$= 7\cos\left(\frac{11\pi}{6}\right) + 7i\sin\left(\frac{11\pi}{6}\right)$$
$$= \frac{7\sqrt{3}}{2} - \frac{7}{2}i$$

So $p = \dfrac{7\sqrt{3}}{2}$ and $q = -\dfrac{7}{2}$

6 $|z| = 5$ and $\arg z = -\dfrac{4\pi}{3}$

$$z = 5\left(\cos\left(-\frac{4\pi}{3}\right) + \sin\left(-\frac{4\pi}{3}\right)\right)$$
$$= 5\cos\left(-\frac{4\pi}{3}\right) + 5i\sin\left(-\frac{4\pi}{3}\right)$$
$$= -\frac{5}{2} + \frac{5i\sqrt{3}}{2}$$

So $a = -\dfrac{5}{2}$ and $b = \dfrac{5\sqrt{3}}{2}$.

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Argand diagrams 2D

1 a i $|z_1 z_2| = |z_1||z_2|$

$|z_1| = 5, |z_2| = 6$

$|z_1 z_2| = 5 \times 6 = 30$

ii $\arg(z_1 z_2) = \arg z_1 + \arg z_2$

$\arg z_1 = \frac{3\pi}{8}, \arg z_2 = \frac{7\pi}{8}$

$\arg(z_1 z_2) = \frac{3\pi}{8} + \frac{7\pi}{8}$

$= \frac{10\pi}{8}$

$= \frac{5\pi}{4}$

iii $z_1 z_2 = r(\cos\theta + i\sin\theta)$

$= 30\left(\cos\frac{5\pi}{4} + i\sin\frac{5\pi}{4}\right)$

b i $|z_1 z_2| = |z_1||z_2|$

$|z_1| = \sqrt{2}, |z_2| = 4\sqrt{2}$

$|z_1 z_2| = \sqrt{2} \times 4\sqrt{2} = 4 \times 2 = 8$

ii $\arg(z_1 z_2) = \arg z_1 + \arg z_2$

$\arg z_1 = \frac{\pi}{3}, \arg z_2 = \frac{3\pi}{4}$

$\arg(z_1 z_2) = \frac{\pi}{3} + \frac{3\pi}{4}$

$= \frac{13\pi}{12}$

iii $z_1 z_2 = r(\cos\theta + i\sin\theta)$

$= 8\left(\cos\frac{13\pi}{12} + i\sin\frac{13\pi}{12}\right)$

2 a $|z_1 z_2| = 8 \times 4 = 32$

$\arg(z_1 z_2) = \frac{8\pi}{5} + \frac{2\pi}{3}$

$= \frac{24\pi}{15} + \frac{10\pi}{15}$

$= \frac{34\pi}{15}$

$\frac{34\pi}{15}$ is not in the range $-\pi < \theta < \pi$

$\frac{34\pi}{15} - 2\pi = \frac{34\pi}{15} - \frac{30\pi}{15} = \frac{4\pi}{15}$

So $\arg(z_1 z_2) = \frac{4\pi}{15}$

b $\left|\frac{z_1}{z_2}\right| = \frac{8}{4} = 2$

$\arg\left(\frac{z_1}{z_2}\right) = \frac{8\pi}{5} - \frac{2\pi}{3}$

$= \frac{24\pi}{15} - \frac{10\pi}{15}$

$= \frac{14\pi}{15}$

c $|z_1^2| = |z_1 z_1| = 8 \times 8 = 64$

$\arg(z_1^2) = \arg(z_1 z_1)$

$= \frac{8\pi}{5} + \frac{8\pi}{5}$

$= \frac{16\pi}{5}$

$\frac{16\pi}{5}$ is not in the range $-\pi < \theta < \pi$

$\frac{16\pi}{5} - 2\pi = \frac{16\pi}{5} - \frac{10\pi}{5} = \frac{6\pi}{5}$

$\frac{6\pi}{5}$ is still not in the range $-\pi < \theta < \pi$,

so we need to subtract 2π again:

$\frac{6\pi}{5} - 2\pi = \frac{6\pi}{5} - \frac{10\pi}{5} = -\frac{4\pi}{5}$

So $\arg(z_1^2) = -\frac{4\pi}{5}$

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3 a $(\cos 2\theta + \mathrm{i}\sin 2\theta)(\cos 3\theta + \mathrm{i}\sin 3\theta)$

$= \cos(2\theta + 3\theta) + \mathrm{i}\sin(2\theta + 3\theta)$

$= \cos 5\theta + \mathrm{i}\sin 5\theta$

b $\left(\cos\frac{3\pi}{11} + \mathrm{i}\sin\frac{3\pi}{11}\right)\left(\cos\frac{8\pi}{11} + \mathrm{i}\sin\frac{8\pi}{11}\right)$

$= \cos\left(\frac{3\pi}{11} + \frac{8\pi}{11}\right) + \mathrm{i}\sin\left(\frac{3\pi}{11} + \frac{8\pi}{11}\right)$

$= \cos\pi + \mathrm{i}\sin\pi$

$= -1 + \mathrm{i}(0)$

$= -1$

c $3\left(\cos\frac{\pi}{4} + \mathrm{i}\sin\frac{\pi}{4}\right) \times 2\left(\cos\frac{\pi}{12} + \mathrm{i}\sin\frac{\pi}{12}\right)$

$= (3\times 2)\left(\cos\left(\frac{\pi}{4} + \frac{\pi}{12}\right) + \mathrm{i}\sin\left(\frac{\pi}{4} + \frac{\pi}{12}\right)\right)$

$= 6\left(\cos\left(\frac{\pi}{3}\right) + \mathrm{i}\sin\left(\frac{\pi}{3}\right)\right)$

$= 6\left(\frac{1}{2} + \frac{\sqrt{3}}{2}\mathrm{i}\right)$

$= 3 + 3\sqrt{3}\mathrm{i}$

d $\sqrt{6}\left(\cos\frac{\pi}{3} - \mathrm{i}\sin\frac{\pi}{3}\right) \times \left(\cos\frac{\pi}{3} + \mathrm{i}\sin\frac{\pi}{3}\right)$

The first factor is not in modulus-argument form. Use

$\cos(-\theta) = \cos\theta$ and $\sin(-\theta) = -\sin\theta$

to write

$\cos\left(-\frac{\pi}{3}\right) = \cos\frac{\pi}{3}$ and $\sin\left(-\frac{\pi}{3}\right) = -\sin\frac{\pi}{3}$

$\sqrt{6}\left(\cos\left(-\frac{\pi}{3}\right) + \mathrm{i}\sin\left(-\frac{\pi}{3}\right)\right)\sqrt{3}\left(\cos\frac{\pi}{3} + \mathrm{i}\sin\frac{\pi}{3}\right)$

$= \sqrt{6}\times\sqrt{3}\left(\cos\left(-\frac{\pi}{3} + \frac{\pi}{3}\right) + \mathrm{i}\sin\left(-\frac{\pi}{3} + \frac{\pi}{3}\right)\right)$

$= \sqrt{18}(\cos 0 + \mathrm{i}\sin 0)$

$= \sqrt{18}(1 + 0\mathrm{i})$

$= 3\sqrt{2}$

e $4\left(\cos\frac{5\pi}{9} - \mathrm{i}\sin\frac{5\pi}{9}\right) \times \frac{1}{2}\left(\cos\frac{5\pi}{18} - \mathrm{i}\sin\frac{5\pi}{18}\right)$

Here, neither factor is in modulus-argument form. Use

$\cos(-\theta) = \cos\theta$ and $\sin(-\theta) = -\sin\theta$

to write:

$\cos\left(-\frac{5\pi}{9}\right) = \cos\frac{5\pi}{9}$ and

$\sin\left(-\frac{5\pi}{9}\right) = -\sin\frac{5\pi}{9}$

Also write:

$\cos\left(-\frac{5\pi}{18}\right) = \cos\frac{5\pi}{18}$ and

$\sin\left(-\frac{5\pi}{18}\right) = -\sin\frac{5\pi}{18}$

$4\left(\cos\left(-\frac{5\pi}{9}\right) + \mathrm{i}\sin\left(-\frac{5\pi}{9}\right)\right) \times \frac{1}{2}\left(\cos\left(-\frac{5\pi}{18}\right) + \mathrm{i}\sin\left(-\frac{5\pi}{18}\right)\right)$

$= 4\times\frac{1}{2}\left(\cos\left(-\frac{5\pi}{9} - \frac{5\pi}{18}\right) + \mathrm{i}\sin\left(-\frac{5\pi}{9} - \frac{5\pi}{18}\right)\right)$

$= 2\left(\cos\left(-\frac{15\pi}{18}\right) + \mathrm{i}\sin\left(-\frac{15\pi}{18}\right)\right)$

$= 2\left(\cos\left(-\frac{5\pi}{6}\right) + \mathrm{i}\sin\left(-\frac{5\pi}{6}\right)\right)$

$= 2\left(-\frac{\sqrt{3}}{2} - \mathrm{i}\frac{1}{2}\right)$

$= -\sqrt{3} - \mathrm{i}$

f $6\left(\cos\frac{\pi}{10} + \mathrm{i}\sin\frac{\pi}{10}\right) \times 5\left(\cos\frac{\pi}{3} + \mathrm{i}\sin\frac{\pi}{3}\right) \times \frac{1}{3}\left(\cos\frac{2\pi}{5} + \mathrm{i}\sin\frac{2\pi}{5}\right)$

$= \left(6\times 5\times\frac{1}{3}\right)\left(\cos\left(\frac{\pi}{10}+\frac{\pi}{3}+\frac{2\pi}{5}\right) + \mathrm{i}\sin\left(\frac{\pi}{10}+\frac{\pi}{3}+\frac{2\pi}{5}\right)\right)$

$= 10\left(\cos\frac{5\pi}{6} + \mathrm{i}\sin\frac{5\pi}{6}\right)$

$= 10\left(-\frac{\sqrt{3}}{2} + \frac{1}{2}\mathrm{i}\right)$

$= -5\sqrt{3} + 5\mathrm{i}$

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3 g First, re-write the second factor so it is in modulus-argument form:

$$(\cos 4\theta + \mathrm{i}\sin 4\theta)(\cos\theta - \mathrm{i}\sin\theta)$$
$$= (\cos 4\theta + \mathrm{i}\sin 4\theta)(\cos(-\theta) + \mathrm{i}\sin(-\theta))$$
$$= \cos\left(4\theta + (-\theta)\right) + \mathrm{i}\sin\left(4\theta + (-\theta)\right)$$
$$= \cos 3\theta + \mathrm{i}\sin 3\theta$$

h

$$3\left(\cos\frac{\pi}{12} + \mathrm{i}\sin\frac{\pi}{12}\right) \times \sqrt{2}\left(\cos\frac{\pi}{3} - \mathrm{i}\sin\frac{\pi}{3}\right)$$
$$= 3\left(\cos\tfrac{\pi}{12} + \mathrm{i}\sin\tfrac{\pi}{12}\right) \times \sqrt{2}\left(\cos\left(-\tfrac{\pi}{3}\right) + \mathrm{i}\sin\left(-\tfrac{\pi}{3}\right)\right)$$
$$= 3(\sqrt{2})\left(\cos\left(\frac{\pi}{12} - \frac{\pi}{3}\right) + \mathrm{i}\sin\left(\frac{\pi}{12} - \frac{\pi}{3}\right)\right)$$
$$= 3\sqrt{2}\left(\cos\left(-\frac{\pi}{4}\right) + \mathrm{i}\sin\left(-\frac{\pi}{4}\right)\right)$$
$$= 3\sqrt{2}\left(\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}\mathrm{i}\right)$$
$$= 3 - 3\mathrm{i}$$

4 a

$$\frac{\cos 5\theta + \mathrm{i}\sin 5\theta}{\cos 2\theta + \mathrm{i}\sin 2\theta}$$
$$= \cos(5\theta - 2\theta) + \mathrm{i}\sin(5\theta - 2\theta)$$
$$= \cos 3\theta + \mathrm{i}\sin 3\theta$$

b

$$\frac{\sqrt{2}\left(\cos\frac{\pi}{2} + \mathrm{i}\sin\frac{\pi}{2}\right)}{\frac{1}{2}\left(\cos\frac{\pi}{4} + \mathrm{i}\sin\frac{\pi}{4}\right)}$$
$$= \frac{\sqrt{2}}{\left(\frac{1}{2}\right)}\left(\cos\left(\frac{\pi}{2} - \frac{\pi}{4}\right) + \mathrm{i}\sin\left(\frac{\pi}{2} - \frac{\pi}{4}\right)\right)$$
$$= 2\sqrt{2}\left(\cos\frac{\pi}{4} + \mathrm{i}\sin\frac{\pi}{4}\right)$$
$$= 2\sqrt{2}\left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\mathrm{i}\right)$$
$$= 2 + 2\mathrm{i}$$

c

$$\frac{3\left(\cos\frac{\pi}{3} + \mathrm{i}\sin\frac{\pi}{3}\right)}{4\left(\cos\frac{5\pi}{6} + \mathrm{i}\sin\frac{5\pi}{6}\right)}$$
$$= \frac{3}{4}\left(\cos\left(\frac{\pi}{3} - \frac{5\pi}{6}\right) + \mathrm{i}\sin\left(\frac{\pi}{3} - \frac{5\pi}{6}\right)\right)$$
$$= \frac{3}{4}\left(\cos\left(-\frac{\pi}{2}\right) + \mathrm{i}\sin\left(-\frac{\pi}{2}\right)\right)$$
$$= \frac{3}{4}(0 - \mathrm{i})$$
$$= -\frac{3}{4}\mathrm{i}$$

d

$$\frac{\cos 2\theta - \mathrm{i}\sin 2\theta}{\cos 3\theta + \mathrm{i}\sin 3\theta}$$
$$= \frac{\cos(-2\theta) + \mathrm{i}\sin(-2\theta)}{\cos 3\theta + \mathrm{i}\sin 3\theta}$$
$$= \cos(-2\theta - 3\theta) + \mathrm{i}\sin(-2\theta - 3\theta)$$
$$= \cos(-5\theta) + \mathrm{i}\sin(-5\theta)$$
$$= \cos 5\theta - \mathrm{i}\sin 5\theta$$

5 a

$$z = -9 + 3\mathrm{i}\sqrt{3}$$
$$|z| = \sqrt{(-9)^2 + \left(3\sqrt{3}\right)^2}$$
$$= \sqrt{81 + 27}$$
$$= \sqrt{108}$$
$$= 6\sqrt{3}$$
$$\tan\alpha = \frac{3\sqrt{3}}{9} = \frac{\sqrt{3}}{3}$$
$$\alpha = \tan^{-1}\left(\frac{\sqrt{3}}{3}\right) = \frac{\pi}{6}$$

As z is in the second quadrant,

$$\arg z = \pi - \alpha$$
$$\arg z = \pi - \frac{\pi}{6} = \frac{5\pi}{6}$$
$$z = 6\sqrt{3}\left(\cos\frac{5\pi}{6} + \mathrm{i}\sin\frac{5\pi}{6}\right)$$

b i

$$w = \sqrt{3}\left(\cos\frac{7\pi}{12} + \mathrm{i}\sin\frac{7\pi}{12}\right)$$

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5 b ii $zw = 6\sqrt{3}\left(\cos\frac{5\pi}{6}+\mathrm{i}\sin\frac{5\pi}{6}\right)\times$
$\sqrt{3}\left(\cos\frac{7\pi}{12}+\mathrm{i}\sin\frac{7\pi}{12}\right)$

$= 6\sqrt{3}\times\sqrt{3}\left(\cos\left(\frac{5\pi}{6}+\frac{7\pi}{12}\right)+\mathrm{i}\sin\left(\frac{5\pi}{6}+\frac{7\pi}{12}\right)\right)$

$= 18\left(\cos\frac{17\pi}{12}+\mathrm{i}\sin\frac{17\pi}{12}\right)$

$\frac{17\pi}{12}$ is not in the range $-\pi<\theta<\pi$.

$\frac{17\pi}{12}-2\pi=\frac{17\pi}{12}-\frac{24\pi}{12}=-\frac{7\pi}{12}$

$zw = 18\left(\cos\left(-\frac{7\pi}{12}\right)+\mathrm{i}\sin\left(-\frac{7\pi}{12}\right)\right)$

$= 18\left(\cos\frac{7\pi}{12}-\mathrm{i}\sin\frac{7\pi}{12}\right)$

iii

$\frac{z}{w}=\frac{6\sqrt{3}\left(\cos\frac{5\pi}{6}+\mathrm{i}\sin\frac{5\pi}{6}\right)}{\sqrt{3}\left(\cos\frac{7\pi}{12}+\mathrm{i}\sin\frac{9\pi}{12}\right)}$

$= \frac{6\sqrt{3}}{\sqrt{3}}\left(\cos\left(\frac{5\pi}{6}-\frac{7\pi}{12}\right)+\mathrm{i}\sin\left(\frac{5\pi}{6}-\frac{7\pi}{12}\right)\right)$

$= 6\left(\cos\frac{3\pi}{12}+\mathrm{i}\sin\frac{3\pi}{12}\right)$

$= 6\left(\cos\frac{\pi}{4}+\mathrm{i}\sin\frac{\pi}{4}\right)$

Challenge

a $z = 1+\mathrm{i}\sqrt{3}$

$|z| = \sqrt{(1)^2+\left(\sqrt{3}\right)^2} = \sqrt{4} = 2$

$\tan\alpha = \frac{\sqrt{3}}{1}$

$\alpha = \tan^{-1}\left(\frac{\sqrt{3}}{1}\right) = \frac{\pi}{3}$

$z = 2\left(\cos\frac{\pi}{3}+\mathrm{i}\sin\frac{\pi}{3}\right)$

$z^7 = 2^7\left(\cos\frac{7\pi}{3}+\mathrm{i}\sin\frac{7\pi}{3}\right)$

$\frac{7\pi}{3}$ is not in the range $-\pi<\theta<\pi$.

$\frac{7\pi}{3}-2\pi=\frac{7\pi}{3}-\frac{6\pi}{3}=\frac{\pi}{3}$

$z^7 = 128\left(\cos\frac{\pi}{3}+\mathrm{i}\sin\frac{\pi}{3}\right)$ (*)

$= 64\left[2\left(\cos\frac{\pi}{3}+\mathrm{i}\sin\frac{\pi}{3}\right)\right]$

$= 64z$

Alternatively, from (*):

$z^7 = 128\left(\frac{1}{2}+\mathrm{i}\left(\frac{\sqrt{3}}{2}\right)\right)$

$z^7 = 64+64\mathrm{i}\sqrt{3}$

So $z^7 = 64\left(1+\mathrm{i}\sqrt{3}\right) = 64z$

Hence $k = 64$.

b $z^4 = 2^4\left(\cos\frac{4\pi}{3}+\mathrm{i}\sin\frac{4\pi}{3}\right)$

$z^4 = 2^4\left(-\cos\frac{\pi}{3}-\mathrm{i}\sin\frac{\pi}{3}\right)$ (**)

$= -2^3\left[2\left(\cos\frac{\pi}{3}+\mathrm{i}\sin\frac{\pi}{3}\right)\right]$

$= -8z$

Alternatively, from (**):

$z^4 = 16\left(-\frac{1}{2}-\frac{\sqrt{3}}{2}\mathrm{i}\right)$

$z^4 = -8-8\mathrm{i}\sqrt{3}$

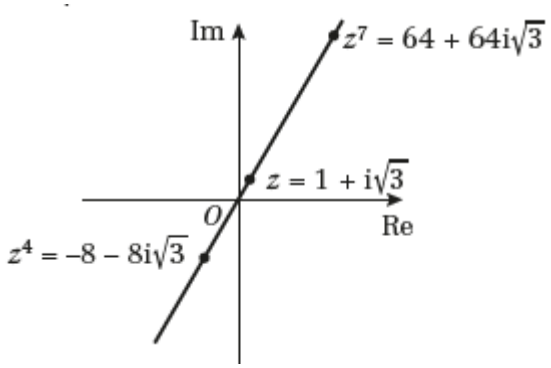
$z^4 = -8\left(1+\mathrm{i}\sqrt{3}\right) = -8z$

So $p = -8$.

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Challenge

c When the points z, z^7 and z^4 are plotted on an Argand diagram, they form a straight line as shown below.



[Note that you can also deduce that z, z^7 and z^4 lie in a straight line by knowing that z^7 and z^4 are scalar multiples of z by factors $k = 64$ and $p = -8$ respectively.]

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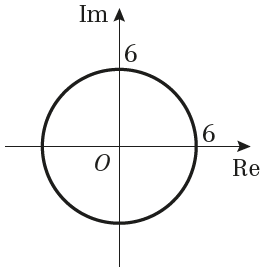
Argand diagrams 2E

1 a $|z| = 6$

circle centre (0, 0), radius 6

equation: $x^2 + y^2 = 6^2$

$x^2 + y^2 = 36$

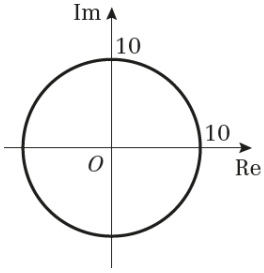


b $|z| = 10$

circle centre (0, 0), radius 10

equation: $x^2 + y^2 = 10^2$

$x^2 + y^2 = 100$

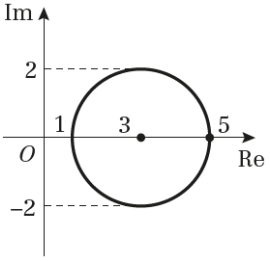


c $|z - 3| = 2$

circle centre (3, 0), radius 2

equation: $(x-3)^2 + y^2 = 2^2$

$(x-3)^2 + y^2 = 4$

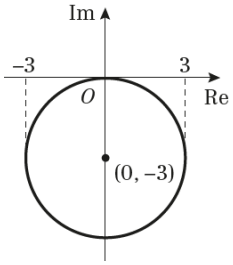


d $|z + 3\mathrm{i}| = 3 \Rightarrow |z - (-3\mathrm{i})| = 3$

circle centre (0, –3), radius 3

equation: $x^2 + (y+3)^2 = 3^2$

$x^2 + (y+3)^2 = 9$

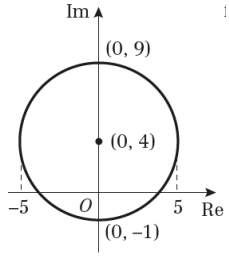


e $|z - 4\mathrm{i}| = 5$

circle centre (0, 4), radius 5

equation: $x^2 + (y-4)^2 = 5^2$

$x^2 + (y-4)^2 = 25$

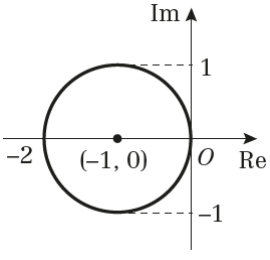


f $|z + 1| = 1 \Rightarrow |z - (-1)| = 1$

circle centre (–1, 0), radius 1

equation: $(x+1)^2 + y^2 = 1^2$

$(x+1)^2 + y^2 = 1$



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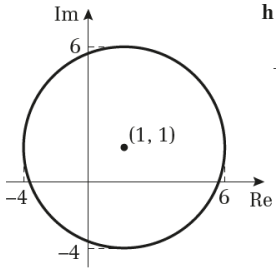
1 g $|z-1-\mathrm{i}|=5 \Rightarrow |z-(1+\mathrm{i})|=5$

circle centre (1, 1), radius 5

equation: $(x-1)^2+(y-1)^2=5^2$

$(x-1)^2+(y-1)^2=25$

h

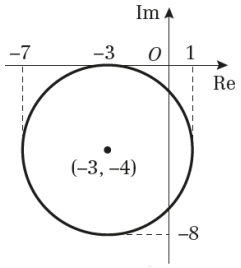


h $|z+3+4\mathrm{i}|=4 \Rightarrow |z-(-3-4\mathrm{i})|=4$

circle centre (–3, –4), radius 4

equation: $(x+3)^2+(y+4)^2=4^2$

$(x+3)^2+(y+4)^2=16$

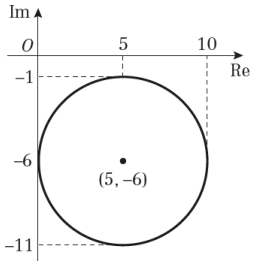


i $|z-5+6\mathrm{i}|=5 \Rightarrow |z-(5-6\mathrm{i})|=4$

circle centre (5, –6), radius 5

equation: $(x-5)^2+(y+6)^2=5^2$

$(x-5)^2+(y+6)^2=25$

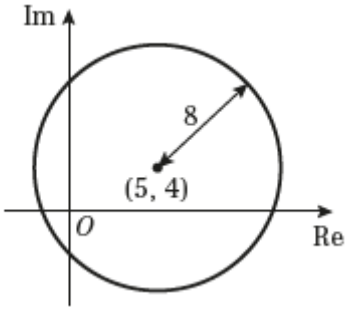


2 a $|z-5-4\mathrm{i}|=8$

$|z-(5+4\mathrm{i})|=8$

The locus of z is a circle centre $(5,4)$ and radius 8.

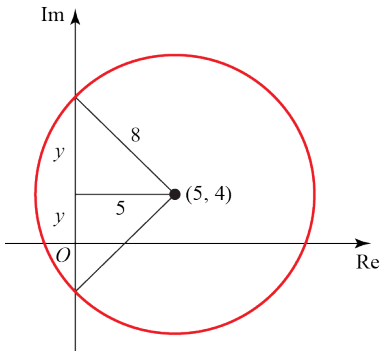
a



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2 b i As $\text{Re}(z) = 0$, this implies that the points lie on the Im axis.

Let y be the vertical distance between the centre of the circle and the points where the circle crosses the Im axis, as shown in the diagram.



Using Pythagoras' Theorem,

$5^2 + y^2 = 8^2$

$25 + y^2 = 64$

$y^2 = 39$

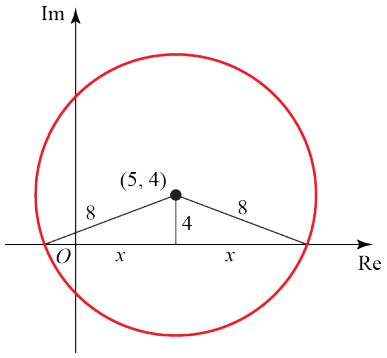
$y = \pm\sqrt{39}$

So $z = (4 + y)\text{i}$ or $z = (4 - y)\text{i}$

So $z = \left(4 + \sqrt{39}\right)\text{i}$ or $z = \left(4 - \sqrt{39}\right)\text{i}$

b ii As $\text{Im}(z) = 0$, this implies that the points lie on the Re axis.

Let x be the horizontal distance between the centre of the circle and the points where the circle crosses the Re axis, as shown in the diagram.



Using Pythagoras' Theorem,

$4^2 + x^2 = 8^2$

$16 + x^2 = 64$

$x^2 = 48$

$x = \pm 4\sqrt{3}$

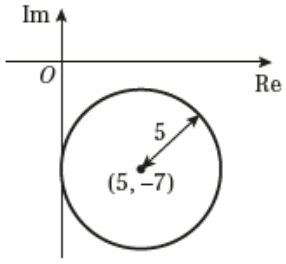
So $z = 5 + x$ or $z = 5 - x$

So $z = 5 + 4\sqrt{3}$ or $z = 5 - 4\sqrt{3}$

3 a $|z - 5 + 7\text{i}| = 5$

$|z - (5 - 7\text{i})| = 5$

The locus of z is a circle with centre $(5, -7)$ and radius 5.



b Let $z = x + \text{i}y$

$|z - 5 + 7\text{i}| = 5$

$|x + \text{i}y - 5 + 7\text{i}| = 5$

$|(x - 5) + \text{i}(y + 7)| = 5$

So, $(x - 5)^2 + (y + 7)^2 = 5^2$

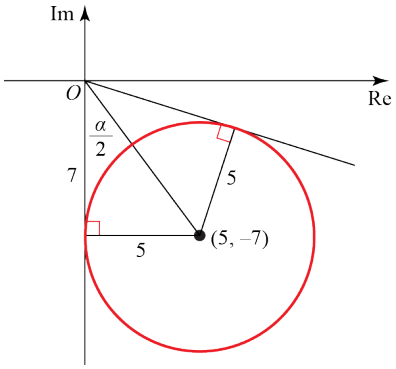
or $(x - 5)^2 + (y + 7)^2 = 25$

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3 c $\tan\frac{\alpha}{2}=\frac{5}{7}$

$$\frac{\alpha}{2}=\arctan\left(\frac{5}{7}\right)$$

$$\alpha=2\arctan\left(\frac{5}{7}\right)$$



The maximum value of $\arg z$ is:

$$\arg z=-\frac{\pi}{2}+2\arctan\left(\frac{5}{7}\right)$$

$$\arg z=-\left(\frac{\pi}{2}-2\arctan\left(\frac{5}{7}\right)\right)$$

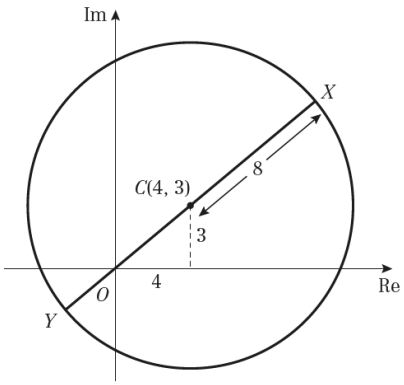
$\arg z=-0.330$ radians (3 s.f.)

4 a $|z-4-3\mathrm{i}|=8\Rightarrow|z-(4+3\mathrm{i})|=8$

This is a circle centre (4, 3), radius 8

Hence the Cartesian equation of the locus of P is $(x-4)^2+(y-3)^2=64$

b



c $|z|$ is the distance from (0, 0) to the locus of points.

$|z|_{\max}$ is the distance OX

$|z|_{\min}$ is the distance OY

Now radius = $CY = CX = 8$ and $OC=\sqrt{4^2+3^2}=\sqrt{25}=5$

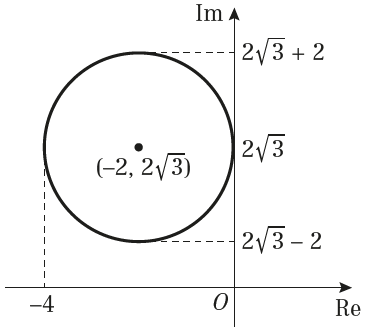
From the diagram, $|z|_{\max} = OC + CX = 5 + 8 = 13$

$|z|_{\min} = CY - OC = 8 - 5 = 3$

The maximum value of $|z|$ is 13 and the minimum value of $|z|$ is 3.

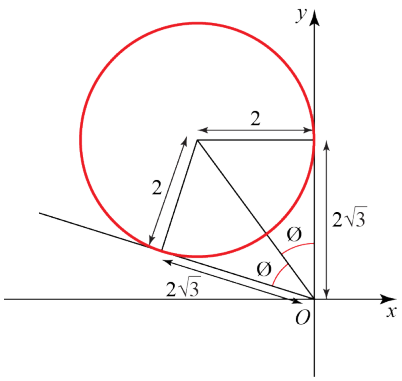
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5 a $|z+2-2\sqrt{3}\mathrm{i}|=2$ is a circle centre $(-2,\ 2\sqrt{3})$, radius 2



b From the diagram, the minimum value of $\arg(z)$ is $\frac{\pi}{2}$.

c From the diagram, the maximum value of $\arg z$ is $\frac{\pi}{2}+2\phi$



Now

$$\tan\phi=\frac{2}{2\sqrt{3}}$$

$$\tan\phi=\frac{1}{\sqrt{3}}$$

$$\phi=\tan^{-1}\left(\frac{1}{\sqrt{3}}\right)=\frac{\pi}{6}$$

Hence $\arg(z)_{\max}=\frac{\pi}{2}+2\left(\frac{\pi}{6}\right)=\frac{5\pi}{6}$.

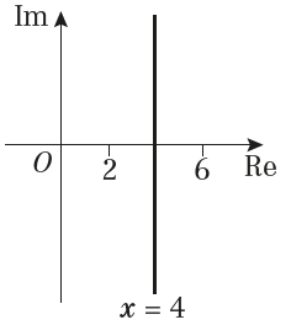
The maximum value of $\arg(z)$ is $\frac{5\pi}{6}$.

6 All parts can be solved using either a geometrical approach, or by substitution of $z=x+\mathrm{i}y$.
Here, we have used a mixture of the two techniques.

a $|z-6|=|z-2|$

Locus is the perpendicular bisector of the line joining (6, 0) and (2, 0).
The midpoint of the line joining (6, 0) and (2, 0) is (4, 0)

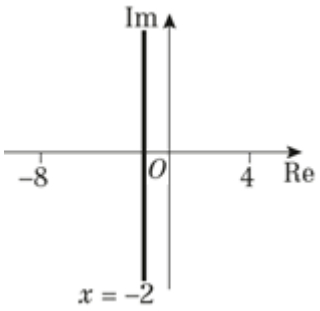
Hece, the locus has equation: $x=4$



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6 b $|z+8|=|z-4|$

$\Rightarrow |z-(-8)|=|z-4|$

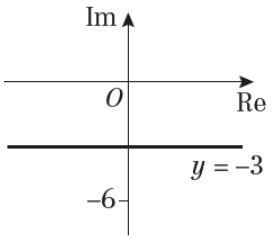


Locus is the perpendicular bisector of the line joining (–8, 0) and (4, 0)
The midpoint of the line joining (–8, 0) and (4, 0) is (–2, 0)

Hence, the locus has equation: $x = -2$

c $|z|=|z+6\mathrm{i}|$

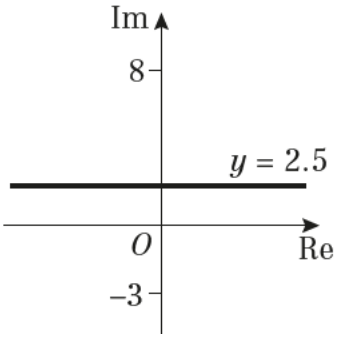
$\Rightarrow |z|=|z-(-6\mathrm{i})|$



Locus is the perpendicular bisector of the line joining (0, 0) to (0, –6).
The midpoint of the line joining (0, 0) to (0, –6) is (0, –3)

Hence, the locus has equation: $y = -3$

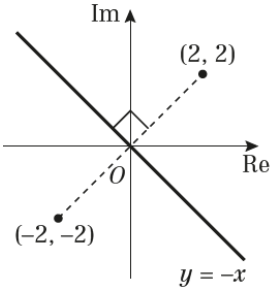
d $|z+3\mathrm{i}|=|z-8\mathrm{i}|$



The locus is the perpendicular bisector of the points $(0,-3)$ and $(0,8)$
As both these points lie on the Im axis, the perpendicular bisector will be a horizontal line.
The midpoint of $(0,-3)$ and $(0,8)$ is $\left(0,\frac{5}{2}\right)$
The equation of the perpendicular bisector is the horizontal line through $\left(0,\frac{5}{2}\right)$.

So the Cartesian equation of the locus is $y=\frac{5}{2}$

e



$|z-2-2\mathrm{i}|=|z+2+2\mathrm{i}|$

Substitute $z = x+\mathrm{i}y$

$|x+\mathrm{i}y-2-2\mathrm{i}|=|x+\mathrm{i}y+2+2\mathrm{i}|$

$\Rightarrow |(x-2)+\mathrm{i}(y-2)|=|(x+2)+\mathrm{i}(y+2)|$

$\Rightarrow (x-2)^2+(y-2)^2=(x+2)^2+(y+2)^2$

$\Rightarrow x^2-4x+4+y^2-4y+4=x^2+4x+4+y^2+4y+4$

$\Rightarrow -4x-4y^2+8=4x+4y+8$

$\Rightarrow 0=8x+8y$

$\Rightarrow -8x=8y$

$\Rightarrow y=-x$

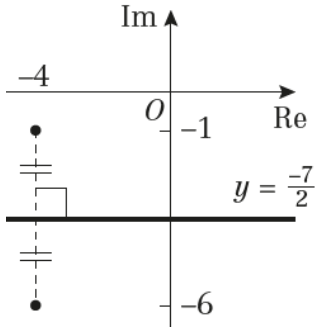
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6 f $|z+4+i|=|z+4+6i|$

$\Rightarrow |z-(-4-i)|=|z+(-4-6i)|$

perpendicular bisector of the line joining (–4, –1) to (–4, –6).

Equation: $y=-\frac{7}{2}$

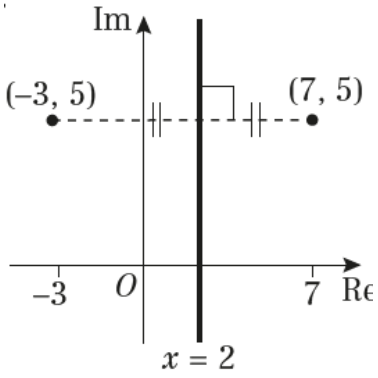


g $|z+3-5i|=|z-7-5i|$

$\Rightarrow |z-(-3+5i)|=|z-(7+5i)|$

perpendicular bisector of the line joining (–3, 5) to (7, 5).

Equation: $x=2$



h $|z+4-2i|=|z-8+2i|$

Substitute $z=x+iy$

$|x+iy+4-2i|=|x+iy-8+2i|$

$\Rightarrow |(x+4)+i(y-2)|=|(x-8)+i(y+2)|$

$\Rightarrow (x+4)^2+(y-2)^2=(x-8)^2+(y+2)^2$

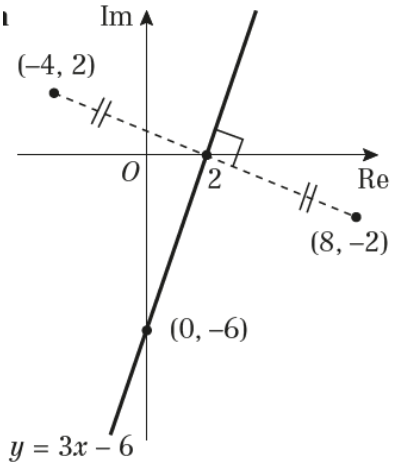
$\Rightarrow x^2+8x+16+y^2-4y+4=x^2-16x+64+y^2+4y+4$

$\Rightarrow 8x-4y+20=-16x+4y+68$

$\Rightarrow 0=-24x+8y+48$

$\Rightarrow 0=-3x+y+6$

$\Rightarrow 3x-6=y$



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6 i

$$\left|\frac{z+3}{z-6\mathrm{i}}\right| = 1$$

$$\Rightarrow |z+3| = |z-6\mathrm{i}|$$

Substitute $z = x + \mathrm{i}y$

$$|x+\mathrm{i}y+3| = |x+\mathrm{i}y-6\mathrm{i}|$$

$$\Rightarrow |(x+3)+\mathrm{i}y| = |x+\mathrm{i}(y-6)|$$

$$\Rightarrow (x+3)^2 + y^2 = x^2 + (y-6)^2$$

$$\Rightarrow \cancel{x^2} + 6x + 9 + \cancel{y^2} = \cancel{x^2} + \cancel{y^2} - 12y + 36$$

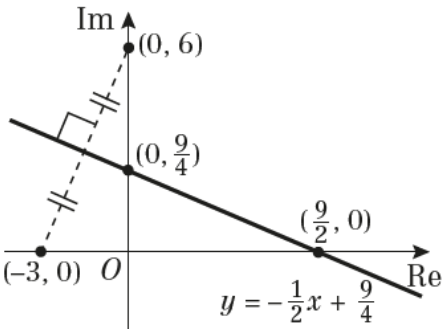
$$\Rightarrow 6x + 12y = 36 - 9$$

$$\Rightarrow 6x + 12y = 27$$

$$\Rightarrow 2x + 4y = 9$$

$$\Rightarrow 4y = 9 - 2x$$

$$\Rightarrow y = -\frac{1}{2}x + \frac{9}{4}$$



j

$$\frac{|z+6-\mathrm{i}|}{|z-10-5\mathrm{i}|} = 1$$

$$|z+6-\mathrm{i}| = |z-10-5\mathrm{i}|$$

$$|z-(-6+\mathrm{i})| = |z-(10+5\mathrm{i})|$$

The locus of z is the perpendicular bisector of the line segment joining the points $(-6,1)$ and $(10,5)$.

The gradient of the line joining $(-6,1)$ and $(10,5)$ is $\frac{1}{4}$.

So, the gradient of the perpendicular bisector is -4.

The midpoint of $(-6,1)$ and $(10,5)$ is $(2,3)$

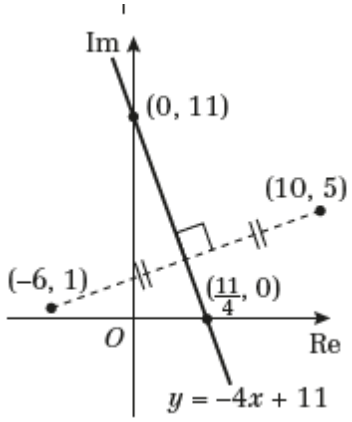
The equation of the perpendicular bisector is found by using $y - y_1 = m(x - x_1)$ with $m = -4$ and $(x_1, y_1) = (2, 3)$

$$y - 3 = -4(x-2)$$

$$y - 3 = -4x + 8$$

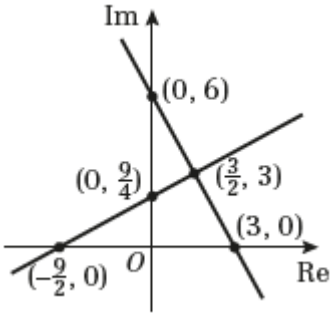
$$y = -4x + 11$$

So the Cartesian equation of the locus is $y = -4x + 11$.



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7 a $|z-3|=|z-6\mathrm{i}|$



The locus of z is the perpendicular bisector of the line segment joining the points $(3,0)$ and $(0,6)$.

The gradient of the line joining $(3,0)$ and $(0,6) = -2$.

So, the gradient of the perpendicular bisector is $\frac{1}{2}$.

The midpoint of $(3,0)$ and $(0,6)$ is $\left(\frac{3}{2},3\right)$.

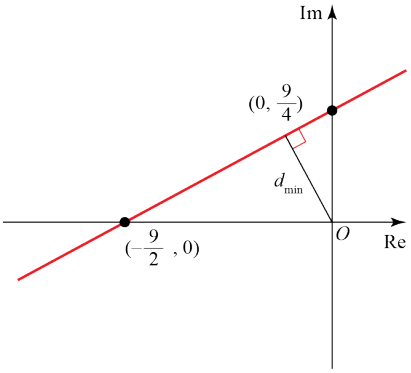
The equation of the perpendicular bisector is found by using $y-y_1=m(x-x_1)$ with $m=\frac{1}{2}$ and $(x_1, y_1)=\left(\frac{3}{2}, 3\right)$

$$y-3=\frac{1}{2}\left(x-\frac{3}{2}\right)$$

$$y-3=\frac{1}{2}x-\frac{3}{4}$$

$$y=\frac{1}{2}x+\frac{9}{4}$$

b $|z|_{\min}$ occurs at the point on the locus where a line from the origin meets the locus of z at right angles (see diagram).



The line labelled $d_{\min}$ is perpendicular to the locus, so has gradient –2
It also passes through the origin, so has equation $y=-2x$

The lines intersect where:

$$-2x=\frac{1}{2}x+\frac{9}{4}$$

$$-\frac{5}{2}x=\frac{9}{4}$$

$$x=-\frac{18}{20}=-\frac{9}{10}$$

If $x=-\frac{9}{10}$, then $y=-2\left(-\frac{9}{10}\right)=\frac{9}{5}$

$$d_{\min}=\sqrt{\left(-\frac{9}{10}\right)^2+\left(\frac{9}{5}\right)^2}$$

$$=\sqrt{\frac{81}{100}+\frac{81}{25}}$$

$$=\sqrt{\frac{405}{100}}$$

$$=\frac{9\sqrt{5}}{10}$$

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8 a&b $|z+3+3\mathrm{i}|=|z-9-5\mathrm{i}|$

$|z-(-3-3\mathrm{i})|=|z-(9+5\mathrm{i})|$

The locus of z is the equation of the perpendicular bisector joining $(-3,-3)$ and $(9,\ 5)$.

The gradient of the line joining $(-3,-3)$ and $(9,5)$ is $\frac{8}{12}=\frac{2}{3}$.

So the gradient of the perpendicular bisector is $-\frac{3}{2}$.

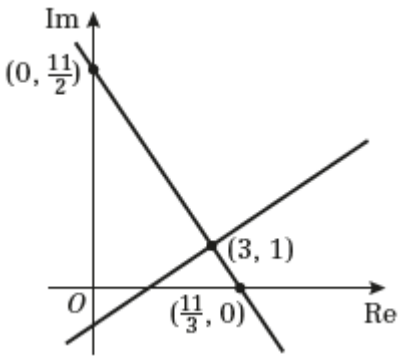
The midpoint of $(-3,-3)$ and $(9,\ 5)$ is $(3,\ 1)$

The equation of the perpendicular bisector is found by using $y-y_1=m(x-x_1)$ with $m=-\frac{3}{2}$ and $(x_1,\ y_1)=(3,\ 1)$

$y-1=-\frac{3}{2}(x-3)$

$y-1=-\frac{3}{2}x+\frac{9}{2}$

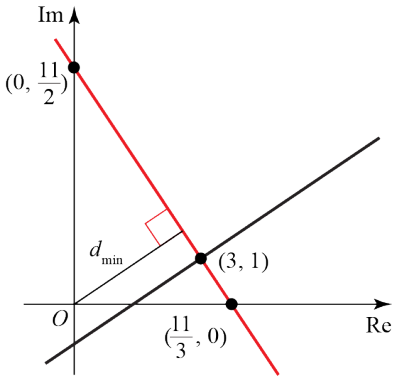
$y=-\frac{3}{2}x+\frac{11}{2}$



c $|z|_{\min}$ occurs at the point on the locus where a line from the origin meets the locus of z at right angles (see diagram).

The line labelled $d_{\min}$ is perpendicular to the locus, so has gradient –2

It also passes through the origin, so has equation $y=\frac{2}{3}x$



The lines intersect where:

$\frac{2}{3}x=-\frac{3}{2}x+\frac{11}{2}$

$\left(\frac{4}{6}+\frac{9}{6}\right)x=\frac{11}{2}$

$\frac{13}{6}x=\frac{11}{2}$

$x=\frac{11\times 6}{2\times 13}=\frac{66}{26}=\frac{33}{13}$

If $x=\frac{33}{13}$, $y=\frac{2}{3}\left(\frac{33}{13}\right)=\frac{22}{13}$

$d_{\min}=\sqrt{\left(\frac{33}{13}\right)^2+\left(\frac{22}{13}\right)^2}$

$=\sqrt{\frac{1573}{169}}$

$=\frac{\sqrt{121}\sqrt{13}}{13}$

$=\frac{11\sqrt{13}}{13}$

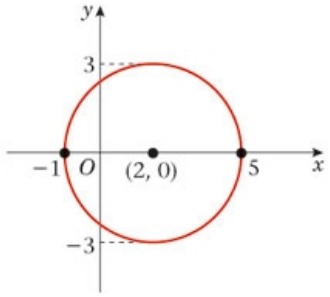
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9 a $|2-z|=3$

$|(-1)(z-2)|=3$

$|-1||(z-2)|=3$ ← $|-1|=1$

$|(z-2)|=3$



circle centre (2, 0), radius, 3

equation: $(x-2)^2+y^2=3^2$

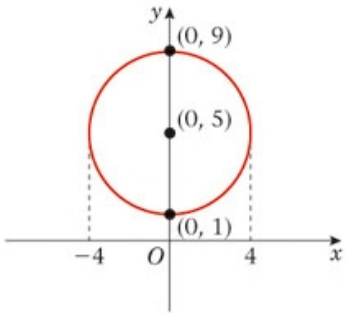
$(x-2)^2+y^2=9$

b $|5\mathrm{i}-z|=4$

$|(-1)(z-5\mathrm{i})|=4$

$|(-1)||(z-5\mathrm{i})|=4$ ← $|-1|=1$

$|(z-5\mathrm{i})|=4$



circle centre (0, 5), radius 4

equation: $x^2+(y-5)^2=4^2$

$x^2+(y-5)^2=16$

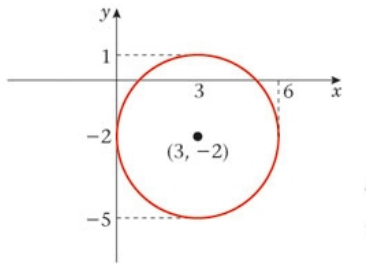
c $|3-2\mathrm{i}-z|=3$

$|(-1)(z-3+2\mathrm{i})|=3$

$|(-1)||(z-3+2\mathrm{i})|=3$ ← $|-1|=1$

$|z-3+2\mathrm{i}|=3$

$|z-(3-2\mathrm{i})|=3$



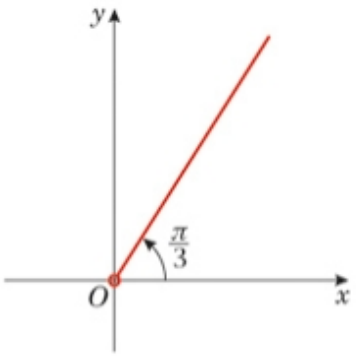
circle centre $(3,-2)$ radius 3

equation: $(x-3)^2+(y+2)^2=3^2$

$(x-3)^2+(y+2)^2=9$

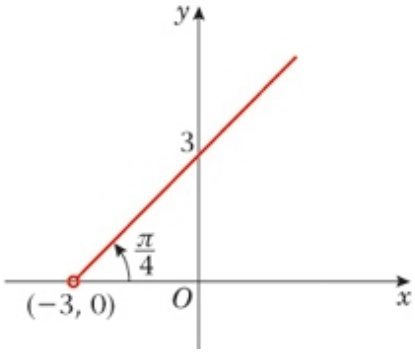
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10 a $\arg z = \frac{\pi}{3}$

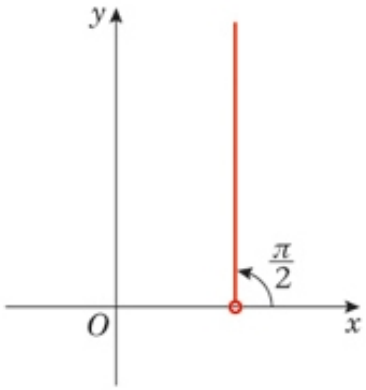


b $\arg(z+3) = \frac{\pi}{4}$

$\Rightarrow \arg(z-(-3)) = \frac{\pi}{4}$

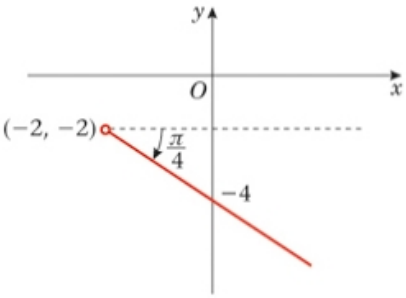


c $\arg(z+2) = \frac{\pi}{2}$



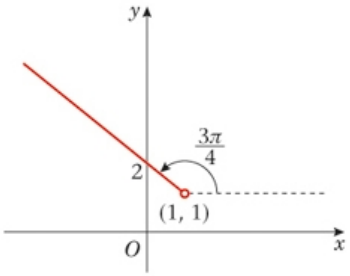
d $\arg(z+2+2\mathrm{i}) = -\frac{\pi}{4}$

$\Rightarrow \arg(z-(-2-2\mathrm{i})) = -\frac{\pi}{4}$



e $\arg(z-1-\mathrm{i}) = \frac{3\pi}{4}$

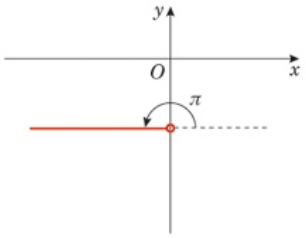
$\Rightarrow \arg(z-(1+\mathrm{i})) = \frac{3\pi}{4}$



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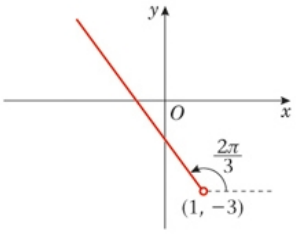
10 f $\arg(z+3\mathrm{i})=\pi$

$\Rightarrow \arg(z-(-3\mathrm{i}))=\pi$



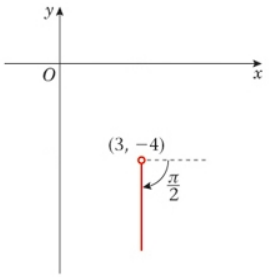
g $\arg(z-1+3\mathrm{i})=\dfrac{2\pi}{3}$

$\Rightarrow \arg(z-(1-3\mathrm{i}))=\dfrac{2\pi}{3}$

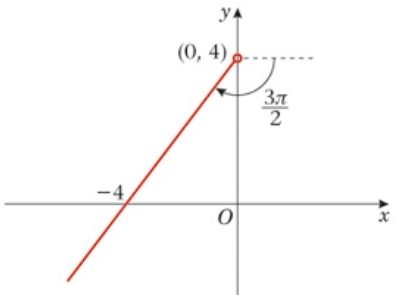


h $\arg(z-3+4\mathrm{i})=-\dfrac{\pi}{2}$

$\Rightarrow \arg(z-(3-4\mathrm{i}))=-\dfrac{\pi}{2}$

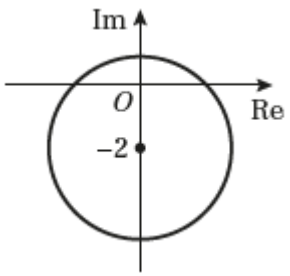


i $\arg(z-4\mathrm{i})=-\dfrac{3\pi}{4}$



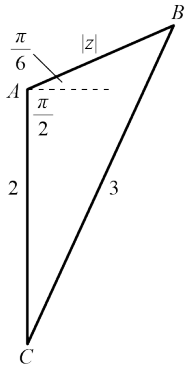
11 a $|z+2\mathrm{i}|=3$

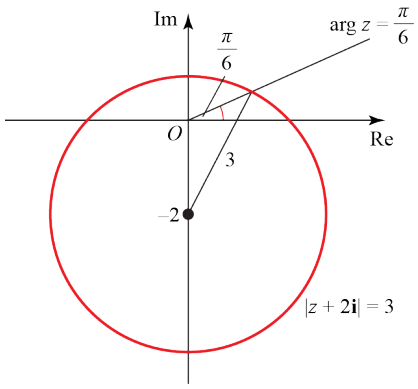
The locus of z is a circle with centre $(0,-2)$ and radius 3.



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11 b Using the information on the diagram to the right, we can create a triangle ABC as follows.





$$\angle A = \frac{\pi}{6} + \frac{\pi}{2} = \frac{2\pi}{3}$$

Use the cosine rule to find $|z|$:

$$a^2 = b^2 + c^2 - 2bc \times \cos A$$

$$3^2 = 2^2 + |z|^2 - 2(2)|z|\cos\frac{2\pi}{3}$$

$$9 = 4 + |z|^2 - 4|z|\left(-\frac{1}{2}\right)$$

$$9 = 4 + |z|^2 + 2|z|$$

$$0 = |z|^2 + 2|z| - 5$$

Solve for $|z|$ by completing the square:

$$(|z| + 1)^2 - 1 - 5 = 0$$

$$(|z| + 1)^2 = 6$$

$$|z| + 1 = \pm\sqrt{6}$$

$$|z| = -1 \pm \sqrt{6}$$

$|z| > 0$, so $|z| = -1 + \sqrt{6}$

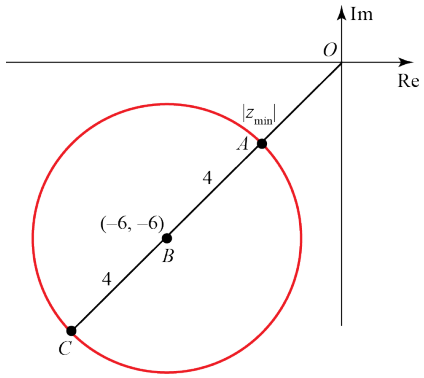
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12 a $|z+6+6\mathrm{i}|=4$

$|z-(-6-6\mathrm{i})|=4$

The locus of z is a circle with centre $(-6,-6)$ and radius 4.

This is shown in the diagram.



First find the distance from OB:

$|OB|=\sqrt{(-6)^2+(-6)^2}=\sqrt{72}=6\sqrt{2}$

Then $|z_{\min}|=|OB|-|AB|$

$=6\sqrt{2}-4$

and $|z_{\max}|=|OB|+|BC|$

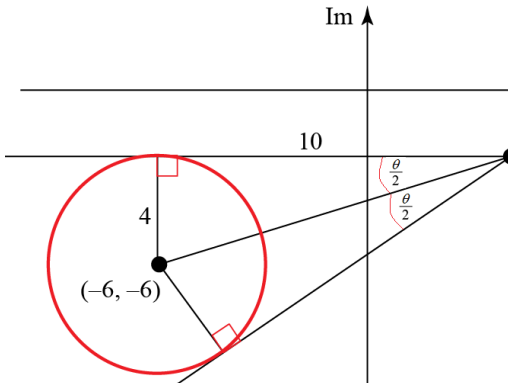
$=6\sqrt{2}+4$

b $\arg(z-4+2\mathrm{i})=\theta$

$\arg(z-(4-2\mathrm{i}))=\theta$

This is the half-line from $(4,-2)$

which makes an angle θ with the positive real axis.



Consider the diagram.

$\tan\left(\frac{\alpha}{2}\right)=\frac{4}{10}=\frac{2}{5}$

$\frac{\alpha}{2}=\arctan\left(\frac{2}{5}\right)$

$\alpha=2\arctan\left(\frac{2}{5}\right)=0.7610..$

The half-line $\arg(z-(4-2\mathrm{i}))=\theta$ intersects the circle $|z-(-6-6\mathrm{i})|=4$ when

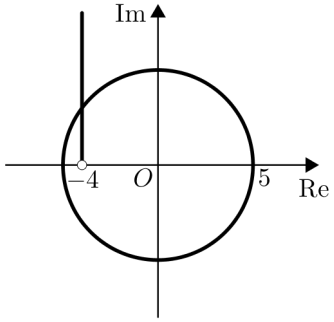
$-\pi\leqslant\theta\leqslant-\pi+0.7610$

$\Rightarrow-\pi\leqslant\theta\leqslant-2.38...$ (*)

So, there will be *no* common solutions for $\arg(z-(4-2\mathrm{i}))=\theta$ and $|z-(-6-6\mathrm{i})|=4$ for values of θ outside of the range given in (*), that is, no common solutions for $-2.38<\theta<\pi$.

13 a Consider P:

$|z|=5\Rightarrow$ The locus of P is a circle centre $(0,0)$ and radius 5.



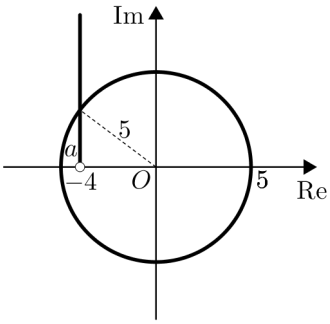
Consider Q:

$\arg(z+4)=\frac{\pi}{2}\Rightarrow\arg(z-(-4))=\frac{\pi}{2}$

The locus of Q is a half-line from $(-4,0)$ making an angle of $\frac{\pi}{2}$ with the positive real axis.

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13 b Consider the diagram shown.



Using Pythagoras, the loci of P and Q intersect when

$(-4)^2 + a^2 = 5^2$

$16 + a^2 = 25$

$a^2 = 9$

$a = \pm 3$

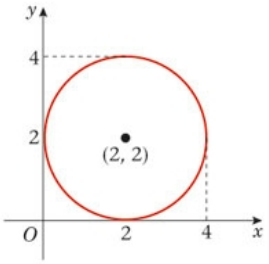
As seen in the diagram, $a > 0$, so $a = 3$.

So $z = -4 + 3\text{i}$.

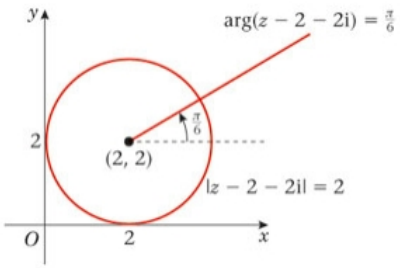
14 a $|z - 2 - 2\text{i}| = 2$

$\Rightarrow |z - (2 + 2\text{i})| = 2$

The locus of z is a circle centre (2, 2), radius 2.



b $\arg(z - 2 - 2\text{i}) = \frac{\pi}{6}$, is a half-line from (2, 2) which makes an angle $\frac{\pi}{6}$ with the positive real axis, as shown in the diagram.



$|z - 2 - 2\text{i}| = 2 \Rightarrow (x-2)^2 + (y-2)^2 = 4$ **(1)**

$\arg(z - 2 - 2\text{i}) = \frac{\pi}{6} \Rightarrow \arg(x + \text{i}y - 2 - 2\text{i}) = \frac{\pi}{6}$

$\Rightarrow \arg((x-2) + \text{i}(y-2)) = \frac{\pi}{6}$

$\Rightarrow \frac{y-2}{x-2} = \tan\left(\frac{\pi}{6}\right) = \frac{1}{\sqrt{3}}$

$\Rightarrow y - 2 = \frac{1}{\sqrt{3}}(x-2)$ (*)

$\Rightarrow (y-2)^2 = \left[\frac{1}{\sqrt{3}}(x-2)\right]^2$

$\Rightarrow (y-2)^2 = \frac{1}{3}(x-2)^2$ **(2)**

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14 b Substituting (**2**) into (**1**) gives $(x-2)^2+\frac{1}{3}(x-2)^2=4$

$$\Rightarrow \frac{4}{3}(x-2)^2=4$$
$$\Rightarrow 4(x-2)^2=12$$
$$\Rightarrow (x-2)^2=3$$
$$\Rightarrow x-2=\pm\sqrt{3}$$
$$\Rightarrow x=2\pm\sqrt{3}$$

From the Argand diagram, $x>2$ so $x=2+\sqrt{3}$ (**3**)

From (*), $y-2=\frac{1}{\sqrt{3}}(x-2)$

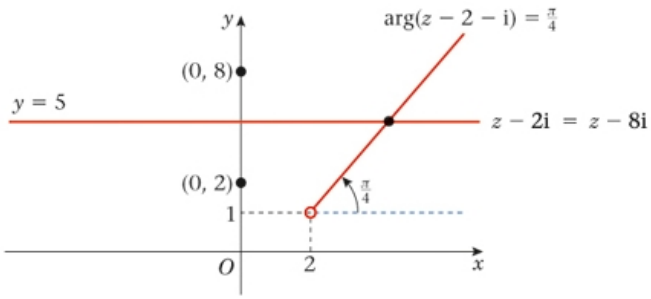
Substituting (**3**) into (*) gives $y-2=\frac{1}{\sqrt{3}}(2+\sqrt{3}-2)$

$$\Rightarrow y-2=\frac{1}{\sqrt{3}}(\sqrt{3})$$
$$\Rightarrow y-2=1$$
$$\Rightarrow y=3$$

Therefore, $z=(2+\sqrt{3})+3\text{i}$

15 a $|z-2\text{i}|=|z-8\text{i}|$

Locus is the perpendicular bisector of the line joining (0, 2) to (0, 8).
This is the line with equation $y=5$.



b $\arg(z-2-\text{i})=\frac{\pi}{4}$

Locus is the half-line from (2, 1) which makes an angle of $\frac{\pi}{4}$ with the positive real axis.

c Find the Cartesian equations of both loci, and then solve them simultaneously to find the point of intersection:

$|z-2\text{i}|=|z-8\text{i}|$ has Cartesian equation $y=5$ (**1**)

$$\arg(z-2-\text{i})=\frac{\pi}{4}\Rightarrow \arg(x+\text{i}y-2-\text{i})=\frac{\pi}{4}$$
$$\Rightarrow \arg((x-2)+\text{i}(y-1))=\frac{\pi}{4}$$
$$\Rightarrow \frac{y-1}{x-2}=\tan\left(\frac{\pi}{4}\right)=1$$
$$\Rightarrow y-1=x-2$$
$$\Rightarrow y=x-1 \quad (\mathbf{2})$$

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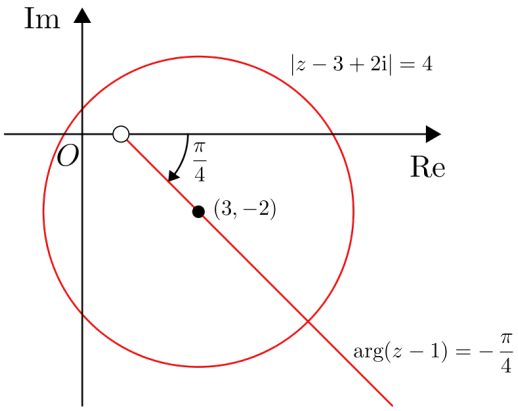
15 c Substituting (**1**) into (**2**) gives $5 = x - 1$

$\Rightarrow 6 = x$

From (**2**), $y = 6 - 1 = 5$

Therefore, $z = 6 + 5\text{i}$

16 a $|z - 3 + 2\text{i}| = 4$ is a circle centre (3, –2) radius 4.



b $\arg(z-1) = -\dfrac{\pi}{4}$ is a half-line from (1, 0) making an angle of $-\dfrac{\pi}{4}$ with the positive real axis.

c Find the Cartesian equations of both loci, and then solve them simultaneously to find the point of intersection:

$|z-3+2\text{i}| = 4 \Rightarrow (x-3)^2 + (y+2)^2 = 16 \quad (\mathbf{1})$

$\arg(z-1) = -\dfrac{\pi}{4} \Rightarrow \arg(x + \text{i}y - 1) = -\dfrac{\pi}{4}$

$\Rightarrow \arg((x-1) + \text{i}y) = -\dfrac{\pi}{4}$

$\Rightarrow \dfrac{y}{x-1} = \tan\left(-\dfrac{\pi}{4}\right)$

$\Rightarrow = \dfrac{y}{x-1} = -1$

$\Rightarrow y = -1(x-1)$

$\Rightarrow y = -x + 1 \quad \text{for } x > 1, y < 0 \quad (\mathbf{2})$

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16 c (cont.)

Substituting **(2)** into **(1)** gives $(x-3)^2+(-x+1+2)^2=16$

$$\Rightarrow (x-3)^2+(-x+3)^2=16$$
$$\Rightarrow x^2-6x+9+x^2-6x+9=16$$
$$\Rightarrow 2x^2-12x+18=16$$
$$\Rightarrow 2x^2-12x+2=0$$
$$\Rightarrow x^2-6x+1=0$$
$$\Rightarrow x=\frac{6\pm\sqrt{36-4(1)(1)}}{2}$$
$$\Rightarrow x=\frac{6\pm\sqrt{32}}{2}$$
$$\Rightarrow x=\frac{6\pm\sqrt{16}\sqrt{2}}{2}$$
$$\Rightarrow x=\frac{6\pm4\sqrt{2}}{2}$$
$$\Rightarrow x=3\pm2\sqrt{2}$$

But $x>1$ so $x=3+2\sqrt{2}$

Substituting $x=3+2\sqrt{2}$ back into **(2)** gives

$$y=-(3+2\sqrt{2})+1$$
$$y=-3-2\sqrt{2}+1$$
$$y=-2-2\sqrt{2}$$

Therefore, $z=(3+2\sqrt{2})+(-2-2\sqrt{2})\text{i}$

So $a=3+2\sqrt{2},\ b=-2-2\sqrt{2}$

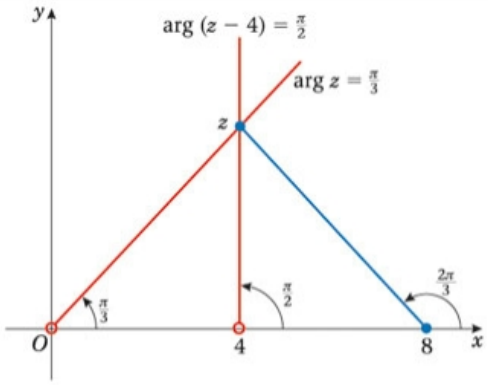
17 a Find the Cartesian equations of both loci, and then solve them simultaneously to find the point of intersection:

$$\arg z=\frac{\pi}{3}\Rightarrow\arg(x+\text{i}y)=\frac{\pi}{3}$$
$$\Rightarrow\frac{y}{x}=\tan\frac{\pi}{3}$$
$$\Rightarrow\frac{y}{x}=\sqrt{3}$$
$$\Rightarrow y=\sqrt{3}x \quad (\text{for } x>0, y>0) \qquad (1)$$

$$\arg(z-4)=\frac{\pi}{2}\Rightarrow x=4\,(\text{for } y>0) \qquad (2)$$

Substituting **(2)** and **(1)** gives $y=\sqrt{3}(4)=4\sqrt{3}$

The value of z satisfying both equations is $y=4+4\sqrt{3}\text{i}$



From part **b** $\arg(z-8)=\frac{2\pi}{3}$.

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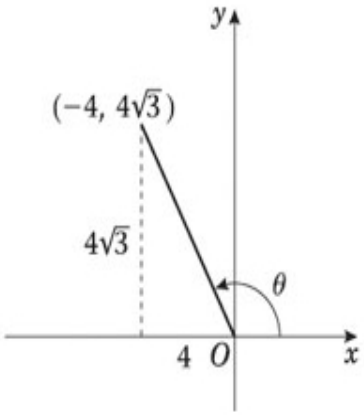
17 b $\arg(z-8\mathrm{i}) = \arg(4+4\sqrt{3}\mathrm{i}-8)$

$= \arg(-4+4\sqrt{3}\mathrm{i}) = \theta$

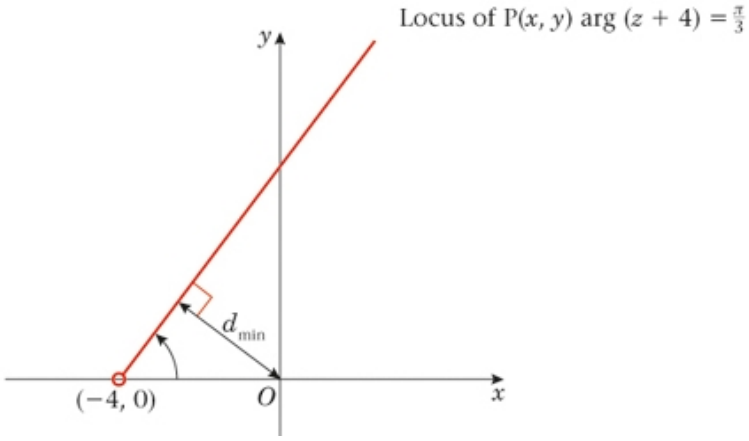
$\therefore \theta = \pi - \tan^{-1}\left(\frac{4\sqrt{3}}{4}\right) = \pi - \frac{\pi}{3}$

$\theta = \frac{2\pi}{3}$

Therefore, $\arg(z-8) = \frac{2\pi}{3}$.

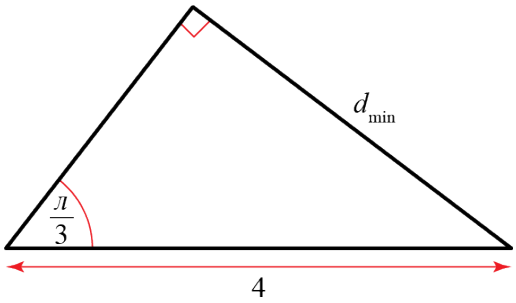


18 a



b $|z|_{\min}$ is the perpendicular distance from (0, 0) to the line $\arg(z+4) = \frac{\pi}{3}$

Consider the triangle shown.



$\frac{d_{\min}}{4} = \sin\left(\frac{\pi}{3}\right)$

$d_{\min} = 4\sin\left(\frac{\pi}{3}\right)$

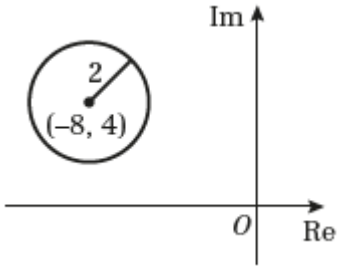
$d_{\min} = \frac{4\sqrt{3}}{2} = 2\sqrt{3}$

Hence $|z|_{\min} = 2\sqrt{3}$.

19 a $|z+8-4\mathrm{i}| = 2$

$|z-(-8+4\mathrm{i})| = 2$

The locus of z is a circle with centre $(-8, 4)$ and radius 2.

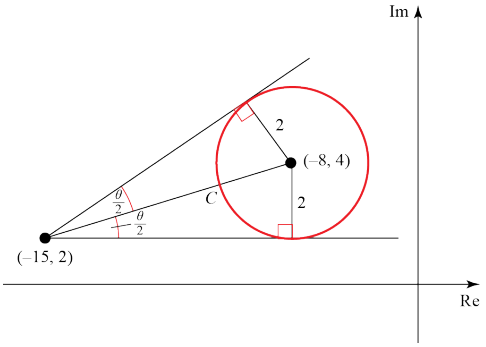


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19 b $\arg(z+15-2\mathrm{i})=\theta$

$\arg(z-(-15+2\mathrm{i}))=\theta$

The locus of z where $\arg(z-(-15+2\mathrm{i}))=\theta$ is a half-line from $(-15,\ 2)$ which makes an angle θ with the positive real axis.



The max value of $\arg(z-(-15+2\mathrm{i}))=\theta$ is found when the half-line is tangent to the top of the circle $|z-(-8+4\mathrm{i})|=2$

Find the length of c shown in the diagram:

$c^2=2^2+7^2$

$c^2=53$

$c=\sqrt{53}$

Then

$\sin\left(\frac{\theta}{2}\right)=\frac{2}{\sqrt{53}}$

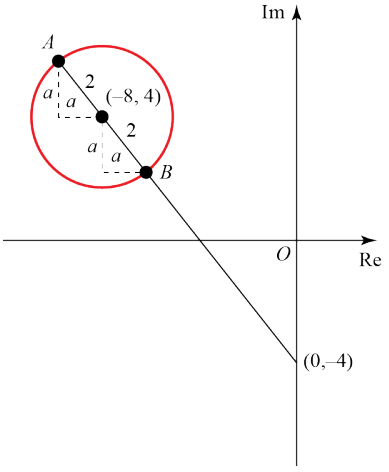
$\frac{\theta}{2}=\arcsin\frac{2}{\sqrt{53}}$

$\theta=2\arcsin\frac{2}{\sqrt{53}}$

c $\arg(z+4\mathrm{i})=\frac{3\pi}{4}$

$\arg\left(z-(-4\mathrm{i})=\frac{3\pi}{4}\right)$

The locus of z is a half-line from $(0,-4)$ with angle $\frac{3\pi}{4}$.



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19 c Find a:

$a^2 + a^2 = 2^2$

$2a^2 = 4$

$a = \pm\sqrt{2}$

a is a distance, so $a > 0 \Rightarrow a = \sqrt{2}$.

Coordinates of A: $\left(-8-\sqrt{2}, 4+\sqrt{2}\right)$

Coordinates of B: $\left(-8+\sqrt{2}, 4-\sqrt{2}\right)$

So the complex numbers satisfying both $|z+8-4\mathrm{i}| = 2$ and $\arg(z+4\mathrm{i}) = \frac{3\pi}{4}$ are

$\left(-8-\sqrt{2}\right)+\left(4+\sqrt{2}\right)\mathrm{i}$ and $\left(-8+\sqrt{2}\right)+\left(4-\sqrt{2}\right)\mathrm{i}$

Challenge

$|z+\mathrm{i}| = 5$

The locus of z is a circle with centre $(0,-1)$ and radius 5.

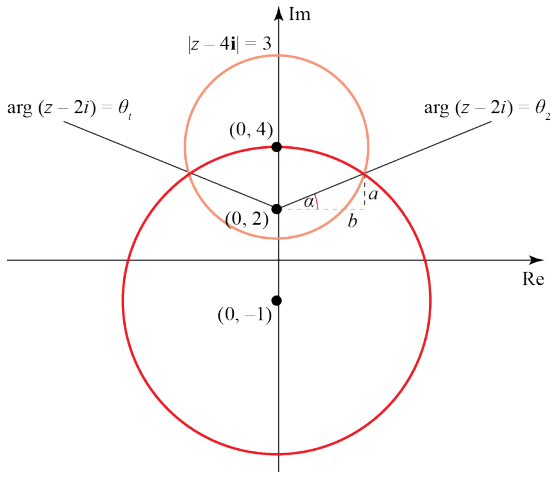
The Cartesian equation of the locus is $x^2 + (y+1)^2 = 5^2$.

$\arg(z-2\mathrm{i}) = \theta$

The locus of z is a half-line from point $(0,2)$ which makes an angle θ with the positive real axis

$|z-4\mathrm{i}| = 3$ is the circle $x^2 + (y-4)^2 = 3^2$

$|z-4\mathrm{i}| < 3$ is the set of points inside the circle $x^2 + (y-4)^2 = 3^2$



Let the two limiting values of θ be θ_1 and θ_2

These values are found when $\arg(z-2\mathrm{i}) = \theta$ meets the points where the circles $x^2 + (y+1)^2 = 5^2$ and $x^2 + (y-4)^2 = 3^2$ intersect.

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Challenge (cont.)

First, find the points where $x^2+(y+1)^2=5^2$ and $x^2+(y-4)^2=3^2$ intersect:

$$25-(y+1)^2=9-(y-4)^2$$
$$25-(y^2+2y+1)=9-(y^2-8y+16)$$
$$25-y^2-2y-1=9-y^2+8y-16$$
$$10y=31$$
$$y=\frac{31}{10}$$

Substitute $y=\frac{31}{10}$ into $x^2+(y+1)^2=25$:

$$x^2+\left(\frac{31}{10}+1\right)^2=25$$
$$x^2+\left(\frac{41}{10}\right)^2=25$$
$$x^2=\frac{819}{100}$$
$$x=\pm\sqrt{\frac{819}{100}}=\pm\frac{3\sqrt{91}}{10}$$

Hence $x^2+(y+1)^2=5^2$ and $x^2+(y-4)^2=3^2$ intersect at $\left(-\frac{3\sqrt{91}}{10},\frac{31}{10}\right)$ and $\left(\frac{3\sqrt{91}}{10},\frac{31}{10}\right)$

Calculate the length a on the diagram:

$$a=\frac{31}{10}-2=\frac{11}{10}$$

Calculate the length b on the diagram:

$$b=\frac{3\sqrt{91}}{10}$$

Then $\tan\alpha=\frac{\frac{11}{10}}{\frac{3\sqrt{91}}{10}}\frac{11}{3\sqrt{91}}$

$$\alpha=\tan^{-1}\frac{11}{3\sqrt{91}}=0.3669...$$

So $\theta_1=\pi-0.3669=2.77$

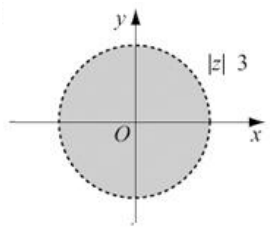
and $\theta_2=0.3669=0.37$

The range of values for θ are $0.37<\theta<2.77$

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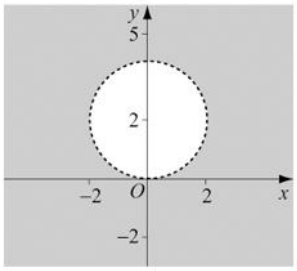
Argand diagrams 2F

1 a

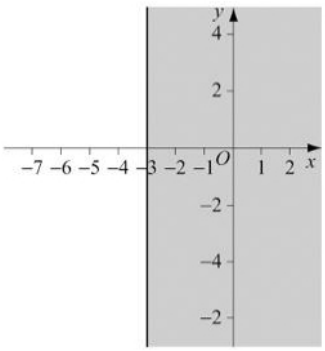


$|z| = 3$ represent a circle centre $(0, 0)$, radius 3

b $|z - (2\mathrm{i})| > 2$

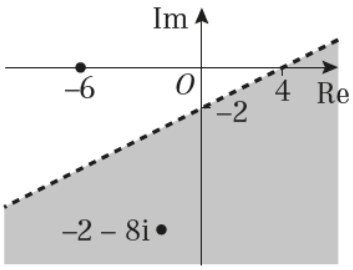


c $|z + 7| \geq |z - 1|$



$|z + 7| = |z - 1|$ represents the perpendicular bisector of the line joining $(-7, 0)$ to $(1, 0)$ which has equation $x = -3$.

d $|z + 6| > |z + 2 + 8\mathrm{i}|$



$|z + 6| = |z + 2 + 8\mathrm{i}|$ represents a perpendicular bisector of the line joining $(-6, 0)$ to $(-2, -8)$.

$|x + \mathrm{i}y + 6| = |x + \mathrm{i}y + 2 + 8\mathrm{i}|$

$\Rightarrow |x + 6 + \mathrm{i}y| = |(x + 2) + \mathrm{i}(y + 8)|$

$\Rightarrow |(x + 6) + \mathrm{i}y|^2 = |(x + 2) + \mathrm{i}(y + 8)|^2$

$\Rightarrow (x + 6)^2 + y^2 = (x + 2)^2 + (y + 8)^2$

$\Rightarrow x^2 + 12x + 36 + y^2 = x^2 + 4x + 4 + y^2 + 16y + 64$

$\Rightarrow 12x + 36 = 4x + 16y + 68$

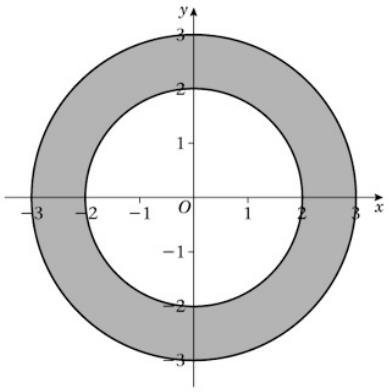
$\Rightarrow 8x + 36 - 68 = 16y$

$\Rightarrow 8x - 32 = 16y$

$\Rightarrow y = \frac{1}{2}x - 2$

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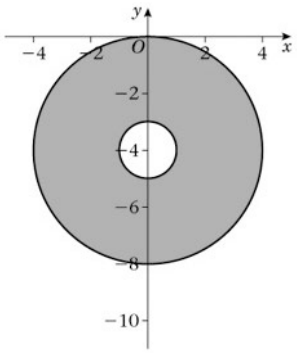
1 e $2 \leqslant |z| \leqslant 3$



$|z| = 2$ represents a circle centre (0, 0), radius 2

$|z| = 3$ represents a circle centre (0, 0), radius 3

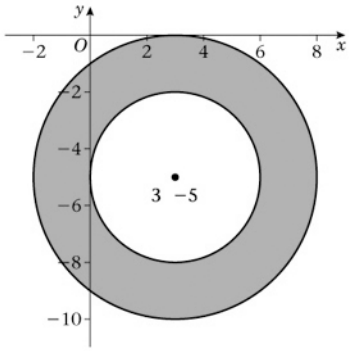
f $1 \leqslant |z + 4i| \leqslant 4$



$|z + 4i| = 1$ represents a circle centre (0, 4), radius 1.

$|z + 4i| = 4$ represents a circle centre (0, –4), radius 4.

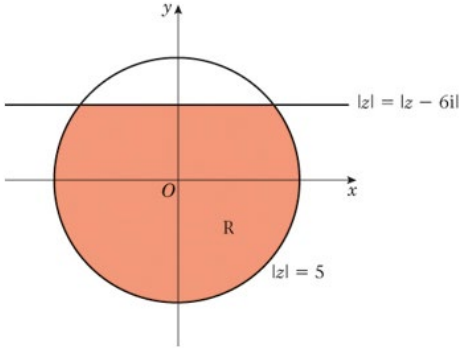
g $3 \leqslant |z - 3 + 5i| \leqslant 5$



$|z - (3 - 5i)| = 3$ represents a circle centre (3, –5), radius 3.

$|z - (3 - 5i)| = 3$ represents a circle centre (3, –5), radius 5.

2 $|z| \leqslant 5$ and $|z| \leqslant |z - 6i|$



$|z| = 5$ represents a circle centre (0, 0), radius 5

$|z| = |z - 6i|$ represents a perpendicular bisector of the line joining (0, 0), to (0, 6) and has the equation $y = 3$.

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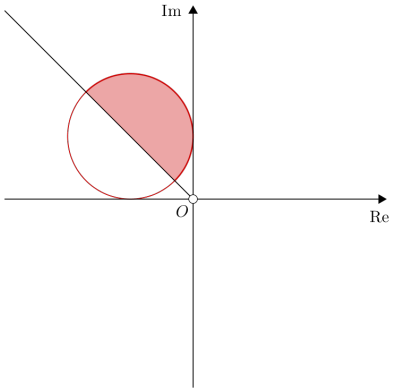
3 $|z+1-\mathrm{i}| \leqslant 1$

$|z-(-1+\mathrm{i})| \leqslant 1$

This represents the region on the boundary and inside the circle with centre $(-1, 1)$ and radius 1.

$0 \leqslant \arg z \leqslant \frac{3\pi}{4}$ This represents the half-line from $(0,0)$ with angle θ,

where $0 \leqslant \theta \leqslant \frac{3\pi}{4}$.



The shaded area shown is where these two regions overlap.

4 $z \in \mathbb{C} : |z| \leqslant 3$

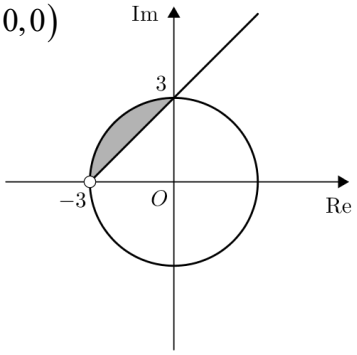
This represents the region on the boundary and inside the circle with centre $(0,0)$ and radius 3.

$z \in \mathbb{C} : \frac{\pi}{4} \leqslant \arg(z+3) \leqslant \pi$

This represents the half-line from $(-3, 0)$ with angle θ,

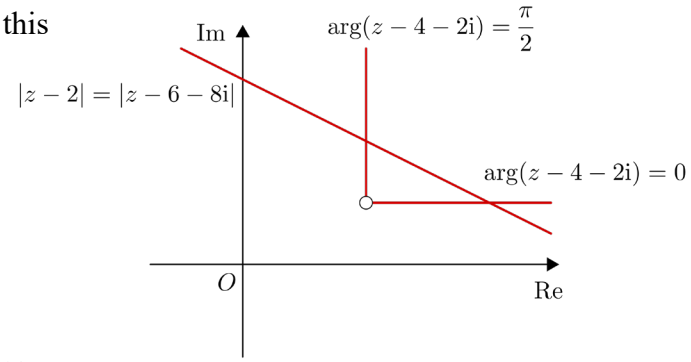
where $\frac{\pi}{4} \leqslant \theta \leqslant \pi$.

The shaded area shown is where these two regions overlap.



5 a i $|z-2| = |z-6-8\mathrm{i}|$ represents the perpendicular bisector of the line joining (2, 0) to (6, 8).

Write $z = x + \mathrm{i}y$ to find a Cartesian form for this perpendicular bisector.



$|x+\mathrm{i}y-2| = |x+\mathrm{i}y-6-8\mathrm{i}|$

$\Rightarrow |(x-2)+\mathrm{i}y| = |(x-6)+\mathrm{i}(y-8)|$

$\Rightarrow |(x-2)+\mathrm{i}y|^2 = |(x-6)+\mathrm{i}(y-8)|^2$

$\Rightarrow (x-2)^2 + y^2 = (x-6)^2 + (y-8)^2$

$\Rightarrow x^2 - 4x + 4 + y^2 = x^2 - 12x + 36 + y^2 - 16y + 64$

$\Rightarrow -4x + 4 = -12x - 16y + 100$

$\Rightarrow 8x + 16y - 96 = 0 \quad (\div 8)$

$\Rightarrow x + 2y - 12 = 0$

$\Rightarrow 2y = -x + 12$

$\Rightarrow y = -\frac{1}{2}x + 6$

ii $\arg(z-4-2\mathrm{i}) = 0$

$\arg(z-(4+2\mathrm{i})) = 0$

This is a half line from (4, 2) making an angle of 0 with the positive Re axis.

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5 a iii $\arg(z-4-2\mathrm{i})=\dfrac{\pi}{2}$

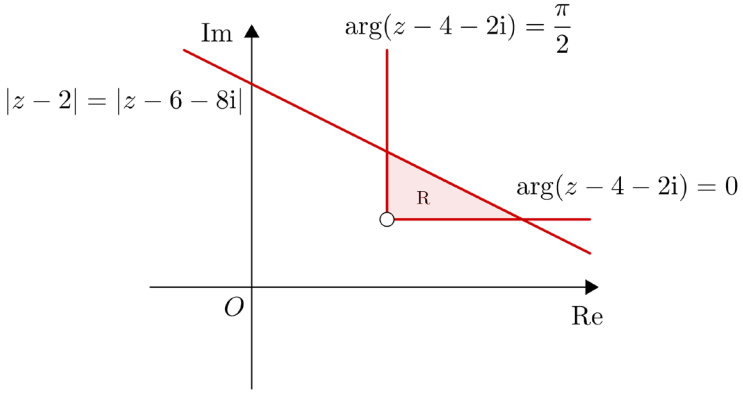
$\arg\left(z-(4+2\mathrm{i})\right)=\dfrac{\pi}{2}$

This is a half line from (4, 2) making an angle of $\dfrac{\pi}{2}$ with the positive Re axis.

b $\{z\in\mathbb{C}:|z-2|\leqslant|z-6-8\mathrm{i}|\}$ represents the region $y\leqslant-\dfrac{1}{2}x+6$

$\left\{z\in\mathbb{C}:0\leqslant\arg(z-4-2\mathrm{i})\leqslant\dfrac{\pi}{2}\right\}$ represented by the region in between and including the two half-lines $\arg\left(z-(4+2\mathrm{i})\right)=0$ and $\arg\left(z-(4+2\mathrm{i})\right)=\dfrac{\pi}{2}$

The region satisfied by both of these is shaded.



6 a i $|x+\mathrm{i}y+10|=|x+\mathrm{i}y-6-4\sqrt{2}\mathrm{i}|$

so $(x+10)^2+y^2=(x-6)^2+(y-4\sqrt{2})^2$

$x^2+20x+100+y^2=x^2-12x+36+y^2-8\sqrt{2}y+32$

$32x=-8\sqrt{2}\,y-32$

$8\sqrt{2}y+32(x+1)=0$

$y+2\sqrt{2}(x+1)=0$

$y=-2\sqrt{2}(x+1)$

ii $(x+1)^2+y^2=9$

$(x^2+2x+y^2=8)$

b Substitute $y=-2\sqrt{2}(x+1)$ into $(x+1)^2+y^2=9$

$(x+1)^2+8(x+1)^2=9$

$9(x+1)^2=9$

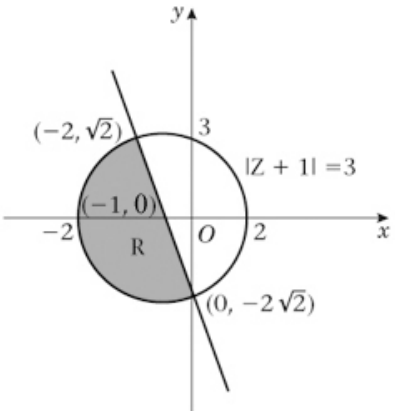
$x+1=\pm1$

$x=0,-2$ $\quad(0,\ -2\sqrt{2})$ and $(-2,\ 2\sqrt{2})$

$z=-2\sqrt{2}\mathrm{i}$ and $z=-2+2\sqrt{2}\mathrm{i}$

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6 c



The region $|z+10| \leqslant |z-6-4i\sqrt{2}|$ is given by $y \leqslant -2\sqrt{2}(x+1)$

$|z+1| \leqslant \sqrt{3}$ is the area inside the circle $(x+1)^2 + y^2 = 9$

The shaded area satisfies both of these inequalities.

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Challenge

$A = \{z \in \mathbb{C} : |z + 5 + 8\mathrm{i}| \leqslant 5\}$

$A = \{z \in \mathbb{C} : |z - (-5 - 8\mathrm{i})| \leqslant 5\}$

This represents the region on the boundary and inside the circle with centre $(-5, -8)$ and radius 5.

$B = \{z \in \mathbb{C} : |z + 8 + 4\mathrm{i}| \leqslant |z + 2 + 12\mathrm{i}|\}$

$B = \{z \in \mathbb{C} : |z - (-8 - 4\mathrm{i})| \leqslant |z - (-2 - 12\mathrm{i})|\}$

This represents the region of all points closer to $(-8, -4)$ than $(-2, -12)$.

To find this region, first find the perpendicular bisector of $(-8, -4)$ than $(-2, -12)$.

The gradient of the line joining $(-8, -4)$ than $(-2, -12)$ is $-\frac{8}{6} = -\frac{4}{3}$

The gradient of the perpendicular bisector is $\frac{3}{4}$

The midpoint of $(-8, -4)$ than $(-2, -12)$ is $(-5, -8)$

The equation of the perpendicular bisector is found by using $y - y_1 = m(x - x_1)$ with $m = \frac{3}{4}$ and $(x_1, y_1) = (-5, -8)$

$y + 8 = \frac{3}{4}(x + 5)$

$y + 8 = \frac{3}{4}x + \frac{15}{4}$

$y = \frac{3}{4}x - \frac{17}{4}$

As $(-8, -4)$ is above this line, the shaded region will also be above the line.

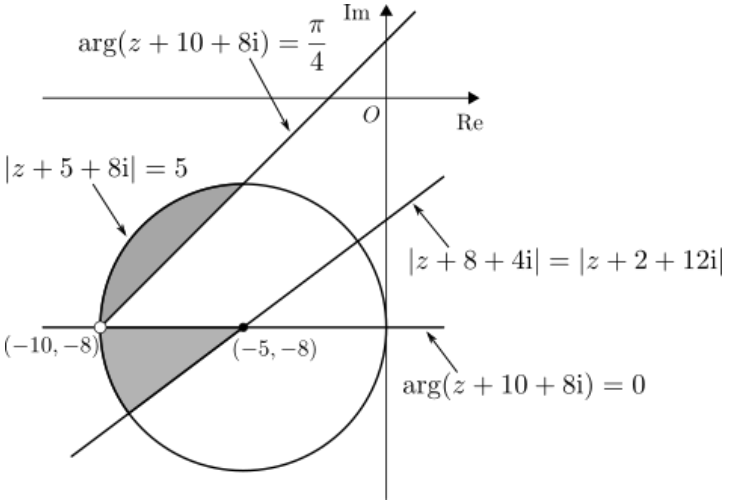
$C = \left\{z \in \mathbb{C} : 0 \leqslant \arg(z + 10 + 8\mathrm{i}) \leqslant \frac{\pi}{4}\right\}$

$C = \left\{z \in \mathbb{C} : 0 \leqslant \arg(z - (-10 - 8\mathrm{i})) \leqslant \frac{\pi}{4}\right\}$

The region is the half-line from $(-10, -8)$ which forms an angle θ with the positive real axis, where $0 \leqslant \theta \leqslant \frac{\pi}{4}$

Remember, we require the region C', so shade the area outside of C.

The diagram shows the correct shaded region together with labels of each locus.



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Argand diagrams Mixed exercise 2

1 a $z^2+5z+10=0$

Solve by completing the square.

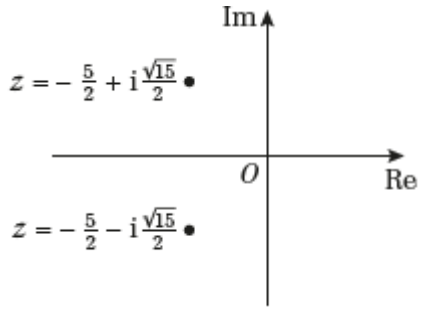
$$\left(z+\frac{5}{2}\right)^2-\frac{25}{4}+10=0$$

$$\left(z+\frac{5}{2}\right)^2=-\frac{15}{4}$$

$$z+\frac{5}{2}=\pm\frac{\sqrt{15}}{2}\mathrm{i}$$

$$z=-\frac{5}{2}\pm\frac{\sqrt{15}}{2}\mathrm{i}$$

b



2 a If $\mathrm{f}(-1+2\mathrm{i})=0$, then $\left(z-(-1+2\mathrm{i})\right)$ and $\left(z-(-1-2\mathrm{i})\right)$ are factors of $\mathrm{f}(z)$.

Hence

$$\left(z-(-1+2\mathrm{i})\right)\left(z-(-1-2\mathrm{i})\right)$$
$$=z^2-(-1-2\mathrm{i})z-(-1+2\mathrm{i})z$$
$$+(-1+2\mathrm{i})(-1-2\mathrm{i})$$
$$=z^2+z+2\mathrm{i}z+z-2\mathrm{i}z+1+2\mathrm{i}-2\mathrm{i}-4\mathrm{i}^2$$
$$=z^2+2z+5 \text{ is a quadratic factor of } \mathrm{f}(z)$$

Use division to find the remaining factor.

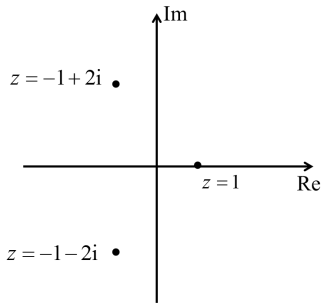
$$\begin{array}{r} z-1 \\ z^2+2z+5\overline{)\,z^3+\ z^2+3z-5} \\ z^3+2z^2+5z \\ -z^2-2z-5 \\ -z^2-2z-5 \\ 0 \end{array}$$

So, $z^3+z^2+3z-5=(z-1)(z^2+2z+5)$

So the solutions of $\mathrm{f}(z)=0$ are

$-1+2\mathrm{i}$, $-1-2\mathrm{i}$ and 1.

b



c The vertices of the triangle are $A(-1,\ 2)$, $B(-1,-2)$ and $C(1,\ 0)$.

Use the distance formula to find the length of each side:

$$AB=\sqrt{(0)^2+(4)^2}=4$$

$$AC=\sqrt{(2)^2+(2)^2}=\sqrt{8}=2\sqrt{2}$$

$$BC=\sqrt{(2)^2+(2)^2}=\sqrt{8}=2\sqrt{2}$$

Then $AC^2+BC^2=\left(2\sqrt{2}\right)^2+\left(2\sqrt{2}\right)^2$

$$=4^2$$
$$=AB^2$$

So by Pythagoras' theorem, ABC is a right-angled triangle which has a right-angle at C.

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3 a $z=-1+4\text{i}$ is a solution implies that the complex conjugate $z=-1-4\text{i}$ is also a solution.

Hence

$$\begin{aligned}&\big(z-(-1+4\text{i})\big)\big(z-(-1-4\text{i})\big)\\&=z^2-(-1-4\text{i})z-(-1+4\text{i})z\\&\quad+(-1+4\text{i})(-1-4\text{i})\\&=z^2+z+4\text{i}z+z-4\text{i}z+1+4\text{i}-4\text{i}-16\text{i}^2\\&=z^2+2z+17 \text{ is a quadratic factor of } \text{f}(z)\end{aligned}$$

Use long division to find the remaining factors.

$$\begin{array}{r} z^2-3z+2 \\ z^2+2z+17\overline{)\,z^4-z^3+13z^2-47z+34} \\ z^4+2z^3+17z^2 \\ -3z^3-4z^2-47z \\ -3z^3-6z^2-51z \\ 2z^2+4z+34 \\ 2z^2+4z+34 \\ 0 \end{array}$$

So $z^4-z^3+13z^2-47z+34$

$$=(z^2+2z+17)(z^2-3z+2)$$

Either $z^2+2z+17=0$

$\Rightarrow z=-1+4\text{i}$ or $z=-1-4\text{i}$

Or $z^2-3z+2=0$

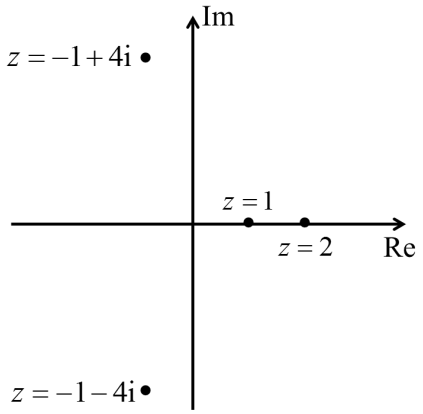
$(z-2)(z-1)=0$

$\Rightarrow z=1$ or $z=2$

So the roots of $\text{f}(z)=0$ are

1, 2, $-1+4\text{i}$ and $-1-4\text{i}$.

b



4 a $(4-3\text{i})x-(1+6\text{i})y-3=0$

$4x-3\text{i}x-y-6\text{i}y-3=0$

Equate real parts:

$4x-y-3=0$ **(1)**

Equate imaginary parts:

$-3x-6y=0$ **(2)**

From **(2)**, $x=-2y$ **(3)**

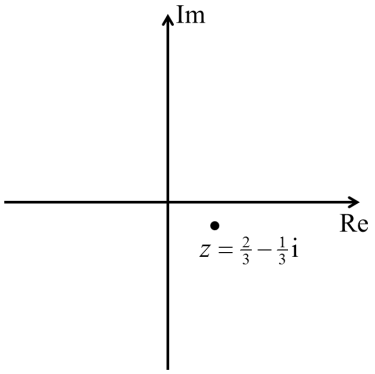
Substitute $x=-2y$ into **(1)**:

$$\begin{aligned}4(-2y)-y-3&=0\\-9y-3&=0\\y&=-\frac{1}{3}\end{aligned}$$

Sub $y=-\frac{1}{3}$ into **(3)**

$$x=-2\left(-\frac{1}{3}\right)=\frac{2}{3}$$

b



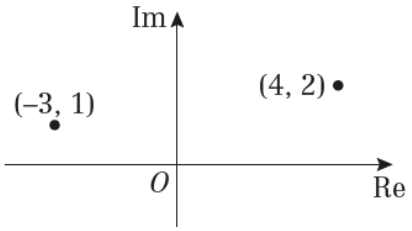
c $|z|=\sqrt{\left(\frac{2}{3}\right)^2+\left(-\frac{1}{3}\right)^2}=\sqrt{\frac{5}{9}}=\frac{\sqrt{5}}{3}$

d $\tan\alpha=\frac{\left(\frac{1}{3}\right)}{\left(\frac{2}{3}\right)}=\frac{1}{2}$

$\alpha=\tan^{-1}\left(\frac{1}{2}\right)=0.4636...$

As z is in the fourth quadrant, $\arg z=-\alpha=-0.46$ radians.

5 a



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5 b $z_1 - z_2 = 4 + 2\text{i} - (-3 + \text{i})$

$= 4 + 2\text{i} + 3 - \text{i}$

$= 7 + \text{i}$

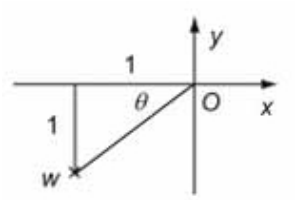
$|z_1 - z_2|^2 = 7^2 + 1^2 = 50$

$|z_1 - z_2| = \sqrt{50} = 5\sqrt{2}$

c $w = \dfrac{4 + 2\text{i}}{-3 + \text{i}} \times \dfrac{-3 - \text{i}}{-3 - \text{i}} = \dfrac{-12 - 4\text{i} - 6\text{i} + 2}{(-3)^2 + 1^2}$

$= \dfrac{-10 - 10\text{i}}{10} = -1 - \text{i}$

d



$\tan\theta = \dfrac{1}{1} = 1 \Rightarrow \theta = \dfrac{\pi}{4}$

w is in the third quadrant.

$\arg w = -\left(\pi - \dfrac{\pi}{4}\right) = -\dfrac{3\pi}{4}$

6 a $z = a + 4\text{i}$

$z^2 = (a + 4\text{i})^2$

$= a^2 + 4a\text{i} + 4a\text{i} + 16\text{i}^2$

$= (a^2 - 16) + 8a\text{i}$

$2z = 2a + 8\text{i}$

$z^2 + 2z = (a^2 - 16) + 8a\text{i} + 2a + 8a\text{i}$

$= (a^2 + 2a - 16) + (8a + 8)\text{i}$

b If $z^2 + 2z$ is real, then $\text{Im}(z^2 + 2z) = 0$

So, $8a + 8 = 0 \Rightarrow a = -1$

c $z = a + 4\text{i}$, so $z = -1 + 4\text{i}$

$|z| = \sqrt{(-1)^2 + (-4)^2} = \sqrt{17} \approx 4.12$

$\tan\alpha = \dfrac{4}{1}$

$\alpha = \tan^{-1} 4 = 1.3258...$

As z is in the second quadrant,

$\arg z = \pi - \alpha$.

$\arg z = \pi - 1.3258... \approx 1.82$

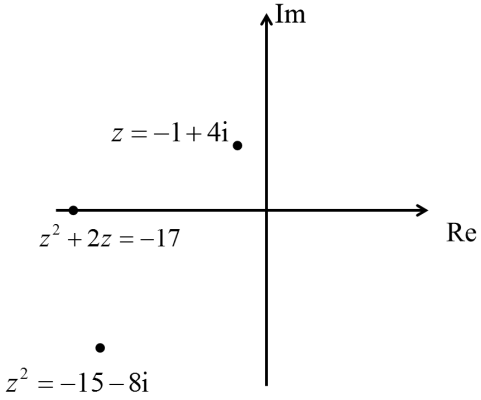
d Use $a = -1$ to find z, z^2 and $z^2 + 2z$

$z = a + 4\text{i} = -1 + 4\text{i}$

$z^2 = (a^2 - 16) + 8a\text{i} = -15 - 8\text{i}$

$z^2 + 2z = (a^2 + 2a - 16) + (8a + 8)\text{i} = -17$

Show these points on an Argand diagram.



7 a $z = \dfrac{3 + 5\text{i}}{2 - \text{i}}$

Multiply by the complex conjugate

$z = \dfrac{(3 + 5\text{i})}{(2 - \text{i})} \times \dfrac{(2 + \text{i})}{(2 + \text{i})}$

$= \dfrac{6 + 3\text{i} + 10\text{i} + 5\text{i}^2}{4 + 2\text{i} - 2\text{i} - \text{i}^2}$

$= \dfrac{1 + 13\text{i}}{5}$

$= \dfrac{1}{5} + \dfrac{13}{5}\text{i}$

$|z| = \sqrt{\left(\dfrac{1}{5}\right)^2 + \left(\dfrac{13}{5}\right)^2} = \sqrt{\dfrac{170}{25}} = \dfrac{1}{5}\sqrt{170}$

b $\tan\alpha = \dfrac{\left(\frac{13}{5}\right)}{\left(\frac{1}{5}\right)} = 13$

$\alpha = \tan^{-1} 13 \approx 1.49$

As z is in the first quadrant,

$\arg z = \alpha \approx 1.49$

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8 a $z = 1 + 2i$

$z^2 = (1+2\text{i})^2$

$= 1 + 2\text{i} + 2\text{i} + 4\text{i}^2$

$= -3 + 4i$

$z^2 - z = (-3+4\text{i}) - (1+2\text{i}) = -4 + 2\text{i}$

$|z^2 - z| = \sqrt{(-4)^2 + (2)^2} = \sqrt{20} = 2\sqrt{5}$

b $\tan\alpha = \frac{2}{4} = \frac{1}{2}$

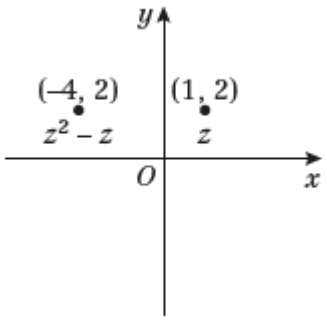
$\alpha = \tan^{-1}\frac{1}{2} = 0.46...$

As $(z^2 - z)$ is in the second quadrant,

$\arg(z^2 - z) = \pi - \alpha$

$\arg(z^2 - z) = \pi - 0.46... \approx 2.68$

c



9 a i $z = \frac{1}{2+\text{i}} \times \frac{2-\text{i}}{2-\text{i}} = \frac{2-\text{i}}{5}$

$= \frac{2}{5} - \frac{1}{5}\text{i}$

$z^2 = \left(\frac{2}{5} - \frac{1}{5}\text{i}\right)^2$

$= \frac{4}{25} - \frac{4}{25}\text{i} + \left(\frac{1}{5}\text{i}\right)^2$

$= \frac{4}{25} - \frac{4}{25}\text{i} - \frac{1}{25}$

$= \frac{3}{25} - \frac{4}{25}\text{i}$

ii $z - \frac{1}{z} = -\frac{2}{5} - \frac{1}{5}\text{i} - (2+\text{i})$

$= \frac{2}{5} - \frac{1}{5}\text{i} - 2 - \text{i}$

$= -\frac{8}{5} - \frac{6}{5}\text{i}$

b $|z^2|^2 = \left(\frac{3}{25}\right)^2 + \left(-\frac{4}{25}\right)^2$

$= \frac{9}{625} + \frac{16}{625} = \frac{25}{625} = \frac{1}{25}$

$|z^2| = \frac{1}{5}$

c $z - \frac{1}{z} = -\frac{8}{5} - \frac{6}{5}\text{i}$

$\tan\alpha = \frac{\left(\frac{6}{5}\right)}{\left(\frac{8}{5}\right)} = \frac{6}{8} = \frac{3}{4}$

$\alpha = \tan^{-1}\frac{3}{4} = 0.6435...$

As z is in the third quadrant,

$\arg z = -(\pi - \alpha)$

$\arg z = -(\pi - 0.6435...) \approx -2.50$

10 a $z = \frac{a+3\text{i}}{2+a\text{i}} = \frac{a+3\text{i}}{2+a\text{i}} \times \frac{2-a\text{i}}{2-a\text{i}}$

$= \frac{2a - a^2\text{i} + 6\text{i} + 3a}{4+a^2}$

$= \frac{5a}{4+a^2} + \frac{6-a^2}{4+a^2}\text{i}$ (*)

Substitute $a = 4$ into (*)

$z = \frac{20}{20} + \frac{-10}{20}\text{i} = 1 - \frac{1}{2}\text{i}$

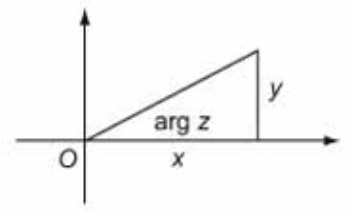
$|z|^2 = 1^2 + \left(-\frac{1}{2}\right)^2 = \frac{5}{4}$

$|z| = \frac{\sqrt{5}}{2}$

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10 b Now $\arg z = \frac{\pi}{4} \Rightarrow \tan(\arg z) = \tan\left(\frac{\pi}{4}\right) = 1$

If $z = x + \mathrm{i}y$, then $\tan(\arg z) = \frac{y}{x}$.



Hence

$$1 = \tan(\arg z)$$
$$= \frac{\left(\frac{6-a^2}{4+a^2}\right)}{\left(\frac{5a}{4+a^2}\right)}$$
$$= \frac{6-a^2}{5a}$$
$$5a = 6 - a^2$$
$$a^2 + 5a - 6 = 0$$
$$(a+6)(a-1) = 0$$

If $a = -6$, substituting into the result (*) in part **a** gives

$$z = \frac{30}{40} - \frac{30}{40}\mathrm{i} = \frac{3}{4} - \frac{3}{4}\mathrm{i}$$

This is in the third quadrant and has a negative argument $\left(-\frac{3\pi}{4}\right)$, so $a = -6$ does not give $\arg z = \frac{\pi}{4}$

If $a = 1$, substituting into the result (*) in part **a** gives

$$z = \frac{5}{5} + \frac{5}{5}\mathrm{i} = 1 + \mathrm{i}$$

This is in the first quadrant and does have an argument $\frac{\pi}{4}$.

Therefore $a = 1$ is the only possible value of a.

11 a $|z_1| = \sqrt{(-1)^2 + (-1)^2} = \sqrt{2}$

$\tan\alpha = \frac{1}{1} \Rightarrow \alpha = \frac{\pi}{4}$

As z_1 is in the third quadrant,

$$\arg z_1 = -\left(\pi - \frac{\pi}{4}\right) - \frac{3\pi}{4}$$
$$z_1 = \sqrt{2}\left(\cos\left(-\frac{3\pi}{4}\right) + \mathrm{i}\sin\left(-\frac{3\pi}{4}\right)\right)$$

$|z_2| = \sqrt{(1)^2 + \left(\sqrt{3}\right)^2} = \sqrt{4} = 2$

$\tan\alpha = \frac{\sqrt{3}}{1} \Rightarrow \alpha = \frac{\pi}{3}$

As z_2 is in the first quadrant

$$\arg z_2 = \frac{\pi}{3}$$
$$z_2 = 2\left(\cos\left(\frac{\pi}{3}\right) + \mathrm{i}\sin\left(\frac{\pi}{3}\right)\right)$$

b i $|z_1 z_2| = |z_1||z_2| = 2\sqrt{2}$

ii $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|} = \frac{\sqrt{2}}{2}$

c i $\arg(z_1 z_2) = \arg(z_1) + \arg(z_2)$
$$= -\frac{3\pi}{4} + \frac{\pi}{3}$$
$$= -\frac{9\pi}{12} + \frac{4\pi}{12}$$
$$= -\frac{5\pi}{12}$$

ii $\arg\left(\frac{z_1}{z_2}\right) = \arg(z_1) - \arg(z_2)$
$$= -\frac{3\pi}{4} - \frac{\pi}{3}$$
$$= -\frac{9\pi}{12} - \frac{4\pi}{12}$$
$$= -\frac{13\pi}{12}$$

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11 c ii $-\dfrac{13\pi}{12}$ is not in the range $-\pi < \theta < \pi$

$$-\frac{13\pi}{12} + 2\pi = -\frac{13\pi}{12} + \frac{24\pi}{12} = \frac{11\pi}{12}$$

$$\arg\left(\frac{z_1}{z_2}\right) = \frac{11\pi}{12}$$

12 a $z = 2 - 2\mathrm{i}\sqrt{3}$

$$|z| = \sqrt{(2)^2 + \left(-2\sqrt{3}\right)^2}$$
$$= \sqrt{4+12}$$
$$= \sqrt{16}$$
$$= 4$$

b $\tan\alpha = \dfrac{2\sqrt{3}}{2} = \sqrt{3}$

$$\alpha = \tan^{-1}\sqrt{3} = \frac{\pi}{3}$$

As z is in the fourth quadrant, $\arg z = -\alpha$

$$\arg z = -\frac{\pi}{3}$$

c $z = 4\left(\cos\left(-\dfrac{\pi}{3}\right) + \mathrm{i}\sin\left(-\dfrac{\pi}{3}\right)\right)$

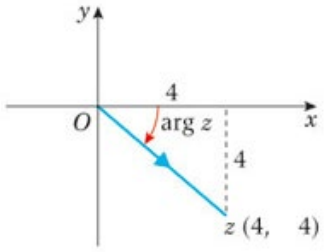
$$w = 4\left(\cos\left(-\frac{\pi}{4}\right) + \mathrm{i}\sin\left(-\frac{\pi}{4}\right)\right)$$

$$\left|\frac{z}{w}\right| = \frac{|z|}{|w|} = \frac{4}{4} = 1$$

d $\arg\left(\dfrac{w}{z}\right) = \arg w - \arg z$

$$= -\frac{\pi}{4} - \left(-\frac{\pi}{3}\right)$$
$$= -\frac{3\pi}{12} + \frac{4\pi}{12}$$
$$= \frac{\pi}{12}$$

13 $4 - 4\mathrm{i}$



modulus $r = \sqrt{4^2 + (-4)^2} = \sqrt{16+16}$

$$= \sqrt{32} = \sqrt{16}\sqrt{2} = 4\sqrt{2}$$

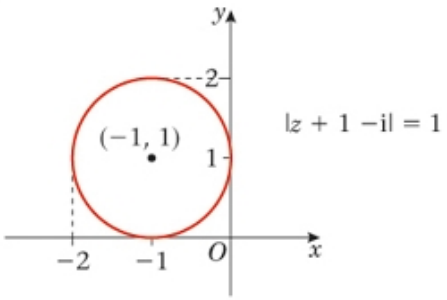
argument $= \theta = -\tan^{-1}\left(\dfrac{4}{4}\right) = -\dfrac{\pi}{4}$

$$4 - 4\mathrm{i} = 4\sqrt{2}\left(\cos\left(-\frac{\pi}{4}\right) + \mathrm{i}\sin\left(-\frac{\pi}{4}\right)\right)$$
$$= 4\sqrt{2}\left(\cos\left(\frac{\pi}{4}\right) - \mathrm{i}\sin\left(-\frac{\pi}{4}\right)\right)$$

14 a

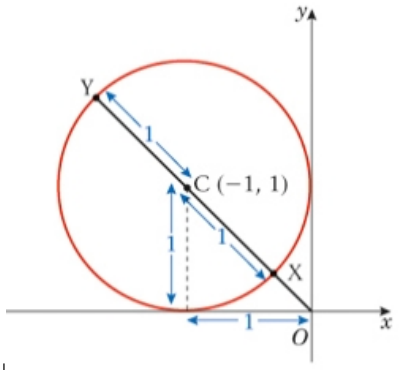
$$|z + 1 - \mathrm{i}| = 1$$
$$|x + \mathrm{i}y + 1 - \mathrm{i}| = 1$$
$$(x+1)^2 + (y-1)^2 = 1$$

b This is a circle, centre $(-1, 1)$, radius 1.



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14 c



$|z|$ is the distance from (0, 0) to the locus of P.

From the Argand diagram,

$|z|_{\max}$ is the distance OY

$|z|_{\min}$ is the distance OX

Note that radius $= CX = CY = 1$

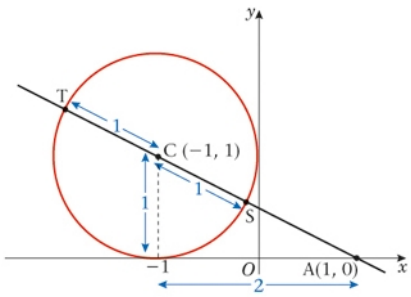
and $OC = \sqrt{1^2 + 1^2} = \sqrt{2}$

$|z|_{\max} = OC + CY = \sqrt{2} + 1$

$|z|_{\min} = OC - CX = \sqrt{2} - 1$

The greatest value of $|z|$ is $\sqrt{2} + 1$ and the least value of $|z|$ is $\sqrt{2} - 1$.

d



$|z-1|$ is the distance from $A(1,0)$ to the locus of points.

From the Argand diagram,

$|z-1|_{\max}$ is the distance AT

$|z-1|_{\min}$ is the distance AS

d Note that radius $= CS = CT = 1$

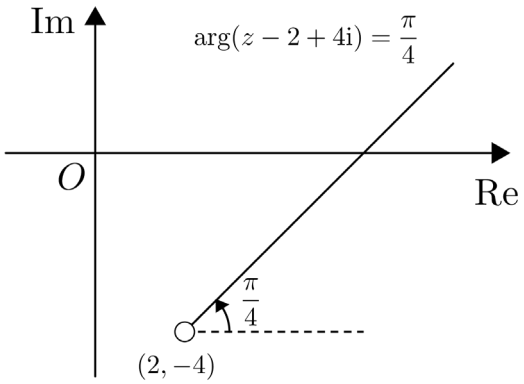
and $AC = \sqrt{1^2 + 2^2} = \sqrt{5}$

$|z-1|_{\max} = AC + CT = \sqrt{5} + 1$

$|z-1|_{\min} = AC - CS = \sqrt{5} - 1$

The greatest value of $|z-1|$ is $\sqrt{5}+1$ and the least value of $|z-1|$ is $\sqrt{5}-1$.

15 a $\arg(z-2+4\mathrm{i}) = \frac{\pi}{4}$ is a half-line from $(2,-4)$ which makes an angle $\frac{\pi}{4}$ with the positive real axis, as shown



b First find a Cartesian form for the locus:

$$\arg(z-2+4\mathrm{i}) = \frac{\pi}{4}$$

$$\Rightarrow \arg(x+\mathrm{i}y-2+4\mathrm{i}) = \frac{\pi}{4}$$

$$\Rightarrow \arg((x-2)+\mathrm{i}(y+4)) = \frac{\pi}{4}$$

$$\Rightarrow \frac{y+4}{x-2} = \tan\frac{\pi}{4} = 1$$

$$\Rightarrow y+4 = x-2$$

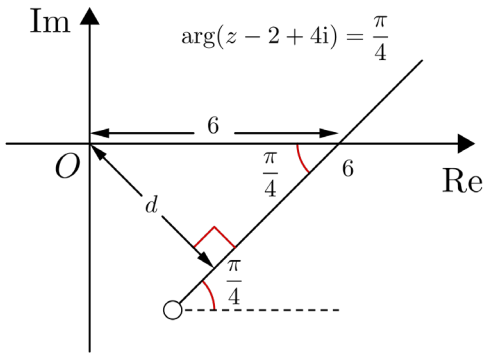
$$\Rightarrow y = x-6, x>2, y>-4$$

$|z|_{\min}$ is a line from the origin which meets the line line $y = x-6$ at right-angles, labelled d in the diagram.

$y = x-6$ cuts x-axis at $0 = x-6 \Rightarrow x = 6$.

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15 b



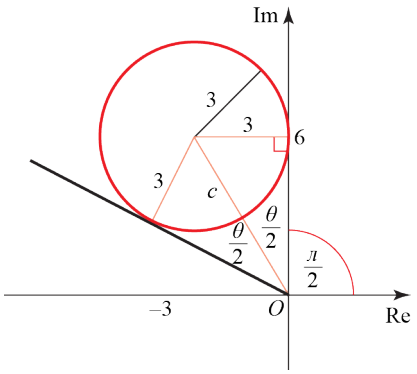
$$|z|_{\min} = d \Rightarrow \frac{d}{6} = \sin\left(\frac{\pi}{4}\right) \Rightarrow d = 6\sin\left(\frac{\pi}{4}\right)$$
$$= 6\left(\frac{1}{\sqrt{2}}\right) = \frac{6\sqrt{2}}{2} = 3\sqrt{2}.$$

Therefore $|z|_{\min} = 3\sqrt{2}$

16 $|z+3-6\mathrm{i}|=3$

$|z-(-3+6\mathrm{i})|=3$

The locus of z is a circle with centre $(-3,6)$ and radius 3.



The maximum value of $\arg z$ occurs at the point A.

16 Find the length of c using Pythagoras' Theorem.

$c^2 = 3^2 + 6^2$

$c^2 = 45$

$c = 3\sqrt{5}$

$$\sin\left(\frac{\theta}{2}\right) = \frac{3}{3\sqrt{5}} = \frac{1}{\sqrt{5}}$$

$$\frac{\theta}{2} = \arcsin\left(\frac{1}{\sqrt{5}}\right)$$

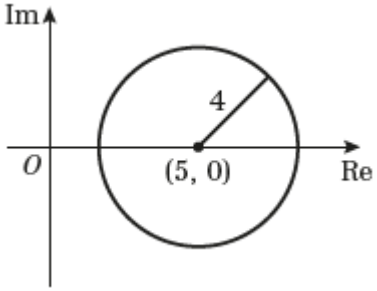
$$\theta = 2\arcsin\left(\frac{1}{\sqrt{5}}\right)$$

The maximum value of $\arg z$ is

$$\theta = \frac{\pi}{2} + 2\arcsin\left(\frac{1}{\sqrt{5}}\right)$$

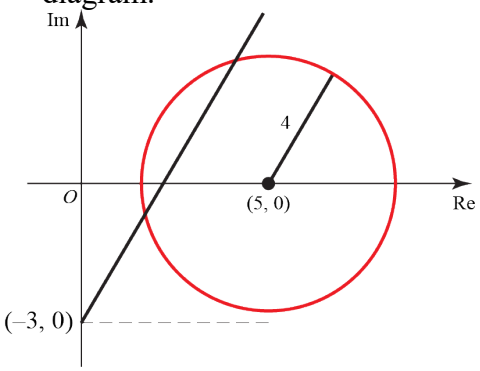
17 a $|z-5|=4$

The locus of z is a circle with centre $(5,0)$ and radius 4. The Cartesian equation of the circle is $(x-5)^2 + y^2 = 4^2$



b $\arg(z+3\mathrm{i}) = \frac{\pi}{3}$

The locus of z is the half-line beginning at $(0,-3)$ which forms an angle $\theta = \frac{\pi}{3}$ with the positive real axis. The half-line will intersect the circle as shown in the diagram.



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17 b As $\theta = \frac{\pi}{3}$, the gradient of the half-line is $\tan\frac{\pi}{3} = \sqrt{3}$.

The equation of the half-line is $y = \left(\sqrt{3}\right)x - 3$

Substitute $y = \left(\sqrt{3}\right)x - 3$ into $(x-5)^2 + y^2 = 4^2$:

$$(x-5)^2 + \left(\left(\sqrt{3}x\right) - 3\right)^2 = 4^2$$

$$x^2 - 10x + 25 + 3x^2 - 6\sqrt{3}x + 9 = 16$$

$$4x^2 + \left(-10 - 6\sqrt{3}\right)x + 18 = 0$$

$$2x^2 + \left(-5 - 3\sqrt{3}\right)x + 9 = 0$$

Use the quadratic formula to solve:

$$x = \frac{5 + 3\sqrt{3} \pm \sqrt{\left(5 + 3\sqrt{3}\right)^2 - (4)(2)(9)}}{2(2)}$$

$$x = \frac{5 + 3\sqrt{3} \pm 5.653...}{4}$$

$x \approx 3.96$ or $x \approx 1.14$

If $x \approx 3.96$, then $y = \left(\sqrt{3}\right)(3.96) - 3 \approx 3.86$

If $x \approx 1.14$, then $y = \left(\sqrt{3}\right)(1.14) - 3 \approx -1.03$

So the coordinates are $(1.14, -1.03)$ and $(3.96, 3.86)$.

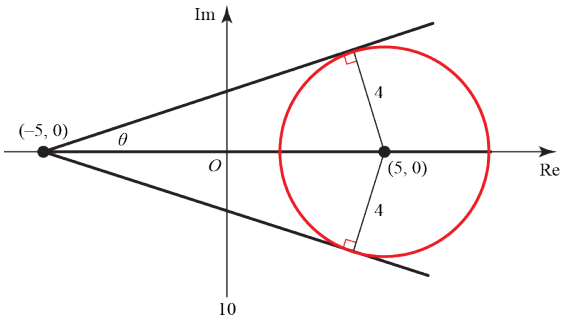
c $|z-5| = 4$

The locus of z is a circle with centre $(5,0)$ and radius 4.

$\arg(z+5) = \theta$

The locus of z is the half-line starting at $(-5,0)$ which forms an angle θ with the positive real axis.

c



$\sin\theta = \frac{4}{10} = \frac{2}{5}$

$\theta = \arcsin\frac{2}{5} = 0.4115...$

So the half-line and the circle will have common solutions in the interval $(-0.41, 0.41)$.

They will have no common solutions in the interval $-\pi < \theta < -0.41$ and $0.41 < \theta < \pi$.

18 a $|z+5-5\mathrm{i}| = |z-6-3\mathrm{i}|$

$|z-(-5+5\mathrm{i})| = |z-(6+3\mathrm{i})|$

The locus of z is the perpendicular bisector of $(-5,5)$ and $(6,3)$.

The gradient of the line joining $(-5,5)$ and $(6,3)$ is $-\frac{2}{11}$.

The gradient of the perpendicular bisector is $\frac{11}{2}$.

The midpoint of $(-5,5)$ and $(6,3)$ is $\left(\frac{1}{2}, 4\right)$.

Use $y - y_1 = m(x - x_1)$ with $m = \frac{11}{2}$ and $(x_1, y_1) = \left(\frac{1}{2}, 4\right)$:

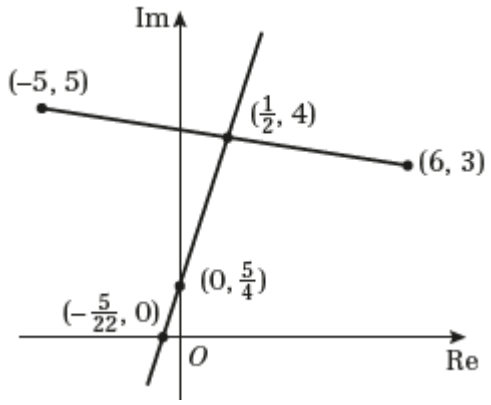
$$y - 4 = \frac{11}{2}\left(x - \frac{1}{2}\right)$$

$$y - 4 = \frac{11}{2}x - \frac{11}{4}$$

$$y = \frac{11}{2}x + \frac{5}{4}$$

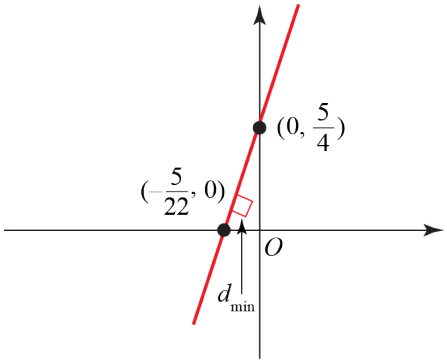
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18 a



b From part **a**, the Cartesian equation of the locus is $y = \frac{11}{2}x + \frac{5}{4}$

c



The equation of the line labelled $d_{\min}$ is $y = -\frac{2}{11}x$ as it is parallel to the line joining $(-5,5)$ and $(6,3)$ and passes through the origin.

The lines $y = -\frac{2}{11}x$ and $y = \frac{11}{2}x + \frac{5}{4}$ intersect where:

$$-\frac{2}{11}x = \frac{11}{2}x + \frac{5}{4}$$
$$\left(-\frac{2}{11} - \frac{11}{2}\right)x = \frac{5}{4}$$
$$\left(-\frac{4}{22} - \frac{121}{22}\right)x = \frac{5}{4}$$
$$-\frac{125}{22}x = \frac{5}{4}$$
$$x = \frac{5}{4} \times -\frac{22}{125}$$
$$x = -\frac{11}{50}$$

c If $x = -\frac{11}{50}$, $y = -\frac{2}{11}\left(-\frac{11}{50}\right) = \frac{1}{25}$

$$d_{\min} = \sqrt{\left(-\frac{11}{50}\right)^2 + \left(\frac{1}{25}\right)^2}$$
$$= \sqrt{\frac{121}{2500} + \frac{1}{625}}$$
$$= \sqrt{\frac{125}{2500}}$$
$$= \sqrt{\frac{1}{20}}$$
$$= \frac{1}{2\sqrt{5}}$$
$$= \frac{\sqrt{5}}{10}$$

So the least possible value of $|z|$ is $\frac{\sqrt{5}}{10}$

19 a $|z-4| = |z-8\mathrm{i}|$

The locus of z is the perpendicular bisector of $(4,0)$ and $(0,8)$.

The gradient of the line joining $(4,0)$ and $(0,8)$ is $-\frac{8}{4} = -2$.

The gradient of the perpendicular bisector is $\frac{1}{2}$.

The midpoint of $(4,0)$ and $(0,8)$ is $(2,4)$

Use $y - y_1 = m(x - x_1)$ with $m = \frac{1}{2}$ and $(x_1, y_1) = (2,4)$:

$$y - 4 = \frac{1}{2}(x-2)$$
$$y - 4 = \frac{1}{2}x - 1$$
$$y = \frac{1}{2}x + 3$$

b $\arg z = \frac{\pi}{4}$

The locus of z is the half-line from $(0,0)$ which forms an angle of $\frac{\pi}{4}$ with the positive real axis.

For a half-line to have angle $\frac{\pi}{4}$, its gradient must be 1. As the half-line begins at the origin, the Cartesian equation of the locus is $y = x$.

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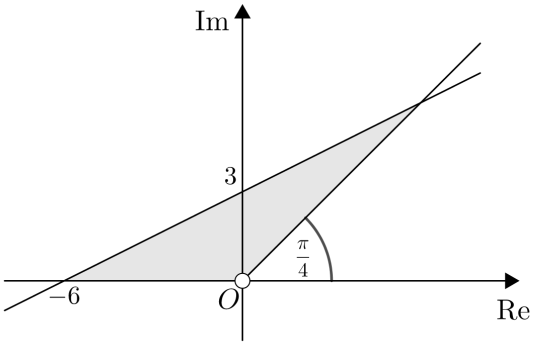
19 b Equate $y = x$ and $y = \frac{1}{2}x + 3$:

$$x = \frac{1}{2}x + 3$$
$$\frac{1}{2}x = 3$$
$$x = 6$$

If $x = 6$, $y = 6$
So $z = 6 + 6\text{i}$

c $|z - 4| \leqslant |z - 8\text{i}|$
Shade the region that is below the line found in part **a** as $(4, 0)$ lies below the line.

$\frac{\pi}{4} \leqslant \arg z \leqslant \pi$
Shade the region between the half-lines with starting point $(0, 0)$ and angles $\frac{\pi}{4}$ and π.



20 a i Method 1: Substitution
Let $|z - 3 + \text{i}| = |z - 1 - \text{i}|$
$\Rightarrow |x + \text{i}y - 3 + \text{i}| = |x + \text{i}y - 1 - \text{i}|$
$\Rightarrow |(x - 3) + \text{i}(y + 1)| = |(x - 1) + \text{i}(y - 1)|$
$\Rightarrow |(x - 3) + \text{i}(y + 1)|^2 = |(x - 1) + \text{i}(y - 1)|^2$
$\Rightarrow (x - 3)^2 + (y + 1)^2 = (x - 1)^2 + (y - 1)^2$
$\Rightarrow x^2 - 6x + 9 + y^2 + 2y + 1$
$\quad = x^2 - 2x + 1 + y^2 - 2y + 1$
$\Rightarrow -6x + 2y + 10 = -2x - 2y + 2$
$\Rightarrow -4x + 4y + 8 = 0$
$\Rightarrow 4y = 4x - 8$
$\Rightarrow y = x - 2$

a i The Cartesian equation of the locus of points representing $|z - 3 + \text{i}| = |z - 1 - \text{i}|$ is $y = x - 2$.

Method 2: Geometrical approach
$|z - 3 + \text{i}| = |z - 1 - \text{i}|$
As $|z - 3 + \text{i}| = |z - 1 - \text{i}|$ is a perpendicular bisector of the line joining
$A(3, -1)$ to $B(1, 1)$,
then $m_{AB} = \frac{1 - -1}{1 - 3} = \frac{2}{-2} = -1$
and perpendicular gradient $= \frac{-1}{-1} = 1$
mid-point of AB is
$\left(\frac{3+1}{2}, \frac{-1+1}{2}\right) = (2, 0)$
$\Rightarrow y - 0 = 1(x - 2)$
$y = x - 2$

a ii $|z - 2| = 2\sqrt{2}$
$\Rightarrow$ circle centre (2, 0), radius $2\sqrt{2}$.
$\Rightarrow$ equation of circle is
$(x - 2)^2 + y^2 = (2\sqrt{2})^2$
$\Rightarrow (x - 2)^2 + y^2 = 8$

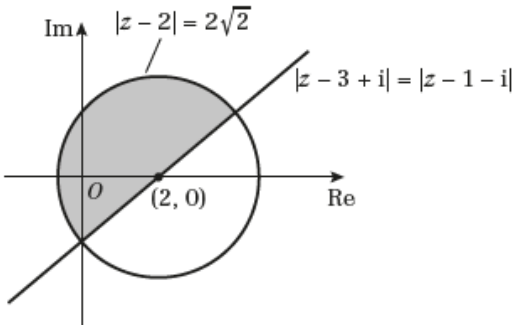
b $|z - 3 + \text{i}| = |z - 1 + \text{i}| \Rightarrow y = x - 2 \quad (\mathbf{1})$
$|z - 2| = 2\sqrt{2} \Rightarrow (x - 2)^2 + y^2 = 8 \quad (\mathbf{2})$
Sub (**1**) in (**2**):
$\Rightarrow (x - 2)^2 + (x - 2)^2 = 8$
$\Rightarrow 2(x - 2)^2 = 8$
$\Rightarrow (x - 2)^2 = 4$
$\Rightarrow x - 2 = \pm\sqrt{4}$
$\Rightarrow x - 2 = \pm 2$
$\Rightarrow x = 2 \pm 2$
$\Rightarrow x = 0, 4$
when $x = 0, y = 0 - 2 = -2 \Rightarrow z = 0 - 2\text{i}$
when $x = 4, y = 4 - 2 = 2 \Rightarrow z = 4 + 2\text{i}$
The values of z are -2i and $4 + 2\text{i}$

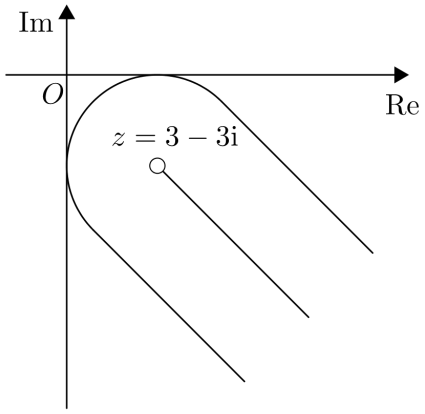
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c $|z-2| \leqslant 2\sqrt{2}$ is the region inside and on the circumference of the circle $|z-2| = 2\sqrt{2}$

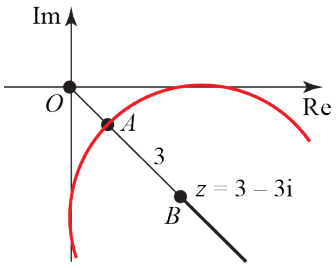
$|z-3+\mathrm{i}| \geqslant |z-1-\mathrm{i}|$ is the set of points which are closer to $(1,\ 1)$ than to $(3,\ -1)$. The boundary is the line $y = x - 2$.

The region which satisfies both inequalities is shaded below





b



The distance from $(0,0)$ to $(3,-3)$ is

$OB = \sqrt{(3)^2 + (-3)^2} = \sqrt{18} = 3\sqrt{2}$

The distance $AB = 3$, since it is a radius of the circle $(x-3)^2 + (y+3)^2 = 3^2$

$OA = OB - AB = 3\sqrt{2} - 3$

Therefore, the minimum distance of $|w|$ is $3\sqrt{2} - 3$

Challenge

a $\arg(z-3+3\mathrm{i}) = -\dfrac{\pi}{4}$

$\arg(z-(3-3\mathrm{i})) = -\dfrac{\pi}{4}$

The locus of z is the half-line starting at $(3,-3)$ which makes an angle $-\dfrac{\pi}{4}$ with the positive real axis.

$|w-z| = 3$

The distance from w to z is always 3. This will be formed by two lines parallel to $\arg(z-(3-3\mathrm{i})) = -\dfrac{\pi}{4}$ at a perpendicular distance of 3, and a semi-circular curve of radius 3 around the point 3 – 3i.

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Series 3A

1 a $\sum_{r=0}^{3}(2r+1) = 1+3+5+7$
$= 16$

b $\sum_{r=1}^{40} r = \frac{1}{2}\times 40\times 41$
$= 820$

c $\sum_{r=1}^{20} r = \frac{1}{2}\times 20\times 21$
$= 210$

d $\sum_{r=1}^{99} r = \frac{1}{2}\times 99\times 100$
$= 4950$

e $\sum_{r=10}^{40} r = \sum_{r=1}^{40} r - \sum_{r=1}^{9} r$
$= \frac{1}{2}\times 40\times 41 - \frac{1}{2}\times 9\times 10$
$= 820 - 45 = 775$

f $\sum_{r=100}^{200} r = \sum_{r=1}^{200} r - \sum_{r=1}^{99} r$
$= \frac{1}{2}\times 200\times 201 - \frac{1}{2}\times 99\times 100$
$= 20\,100 - 4950 = 15\,150$

g $\sum_{r=21}^{40} r = \sum_{r=1}^{40} r - \sum_{r=1}^{20} r$
$= \frac{1}{2}\times 40\times 41 - \frac{1}{2}\times 20\times 21$
$= 820 - 210 = 610$

h $\sum_{r=1}^{k} r + \sum_{r=k+1}^{80} r = \sum_{r=1}^{k} r + \sum_{r=1}^{80} r - \sum_{r=1}^{k} r$
$= \sum_{r=1}^{80} r$
$= \frac{1}{2}\times 80\times 81$
$= 3240$

2 $\frac{1}{2}\times n\times(n+1) = 528$
$n^2 + n - 1056 = 0$
$(n+33)(n-32) = 0$
$n = 32$

3 $\frac{1}{2}k(k+1) = \frac{1}{2}\times\frac{1}{2}\times 20\times 21$
$k^2 + k - 210 = 0$
$(k+15)(k-14) = 0$
$k = 14$

4 a $\sum_{r=1}^{2n-1} r = \frac{1}{2}\times(2n-1)\times(2n-1+1)$
$= \frac{1}{2}\times(2n-1)\times 2n$
$= n(2n-1)$

b $\sum_{r=n+1}^{2n-1} r = \sum_{r=1}^{2n-1} r - \sum_{r=1}^{n} r = n(2n-1) - \frac{1}{2}n(n+1)$
$= \frac{1}{2}n(4n-2-n-1)$
$= \frac{1}{2}n(3n-3) = \frac{3}{2}n(n-1)$

5 $\sum_{r=n-1}^{2n} r = \sum_{r=1}^{2n} r - \sum_{r=1}^{n-2} r$
$= \frac{1}{2}(2n)(2n+1) - \frac{1}{2}(n-2)(n-1)$
$= \frac{1}{2}(2n(2n+1) - (n-2)(n-1))$
$= \frac{1}{2}(3n^2 + 5n - 2)$
$= \frac{1}{2}(n+2)(3n-1)$

6 a $\sum_{r=1}^{n^2} r - \sum_{r=1}^{n} r$
$= \frac{1}{2}n^2(n^2+1) - \frac{1}{2}n(n+1)$
$= \frac{1}{2}n(n(n^2+1) - (n+1))$
$= \frac{1}{2}n(n^3 + n - n - 1)$
$= \frac{1}{2}n(n^3 - 1)$

b $\sum_{r=10}^{81} r = \sum_{r=1}^{9^2} r - \sum_{r=1}^{9} r$
$= \frac{1}{2}n(n^3 - 1)$
$= \frac{1}{2}\times 9\times(9^3 - 1)$
$= 3276$

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7 a $\sum_{r=1}^{55}(3r-1)=3\sum_{r=1}^{55}r-\sum_{r=1}^{55}1$

$=3\times\frac{1}{2}\times55\times56-55$

$=4565$

b $\sum_{r=1}^{90}(2-7r)=2\sum_{r=1}^{90}1-7\sum_{r=1}^{90}r$

$=2\times90-7\times\frac{1}{2}\times90\times91$

$=-28\,485$

c $\sum_{r=1}^{46}(9+2r)=9\sum_{r=1}^{46}1+2\sum_{r=1}^{46}r$

$=9\times46+2\times\frac{1}{2}\times46\times47$

$=2576$

8 a $\sum_{r=1}^{n}(3r+2)=3\sum_{r=1}^{n}r+2\sum_{r=1}^{n}1$

$=\frac{3}{2}n(n+1)+2n$

$=\frac{1}{2}n(3(n+1)+4)=\frac{1}{2}n(3n+7)$

b $\sum_{r=1}^{2n}(5r-4)=5\sum_{r=1}^{2n}r-4\sum_{r=1}^{2n}1$

$=\frac{5}{2}(2n)(2n+1)-4(2n)$

$=5n(2n+1)-8n$

$=10n^2-3n=n(10n-3)$

c $\sum_{r=1}^{n+2}(2r+3)=2\sum_{r=1}^{n+2}r+3\sum_{r=1}^{n+2}1$

$=(n+2)(n+3)+3(n+2)$

$=(n+2)((n+3)+3)$

$=(n+2)(n+6)$

8 d $\sum_{r=3}^{n}(4r+5)=4\sum_{r=3}^{n}4r+5\sum_{r=3}^{n}1$

$=4\left(\sum_{r=1}^{n}r-5\sum_{r=1}^{2}r\right)+5\left(\sum_{r=1}^{n}1-\sum_{r=1}^{2}1\right)$

$=4(\frac{1}{2}n(n+1)-3)+5(n-2)$

$=2n^2+7n-22=(2n+11)(n-2)$

9 a $\sum_{r=1}^{k}(4r-5)=4\sum_{r=1}^{k}r-5\sum_{r=1}^{k}1$

$=\frac{4}{2}k(k+1)-5k$

$=2k^2-2k-5k$

$=2k^2-3k$

b $2k^2-3k>4850\Rightarrow2k^2-3k-4850>0$

$\Rightarrow(2k+97)(k-50)>0$

so $k>50$ since k is positive

$\Rightarrow k=51$

10 $\sum_{r=1}^{n}ar+b=a\sum_{r=1}^{n}r+\sum_{r=1}^{n}b$

$=a\times\frac{1}{2}n(n+1)+bn$

$=\frac{1}{2}an^2+\left(\frac{1}{2}a+b\right)n$

$=\frac{1}{2}n(an+(a+2b))$

Since $\sum_{r=1}^{n}\mathrm{f}(r)=\frac{1}{2}n(7n+1)$

$\Rightarrow a=7$

$\Rightarrow a+2b=1$

$\Rightarrow7+2b=1\Rightarrow b=-3$

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11 a $\sum_{r=1}^{4n-1}(3r+1) = 3\sum_{r=1}^{4n-1} r + \sum_{r=1}^{4n-1} 1$

$= \frac{3}{2}(4n-1)(4n) + (4n-1)$

$= 6n(4n-1) + (4n-1)$

$= 24n^2 - 6n + 4n - 1$

$= 24n^2 - 2n - 1$

b Substituting $n = 25$ into the result from part **a**:

$24 \times 25^2 - 2 \times 25 - 1 = 14\,949$

12 a $\sum_{r=1}^{2k+1}(4-5r) = 4\sum_{r=1}^{2k+1} 1 - 5\sum_{r=1}^{2k+1} r$

$= 4(2k+1) - \frac{5}{2}(2k+1)(2k+2)$

$= (2k+1)(4-5(k+1))$

$= (2k+1)(-1-5k)$

$= -(2k+1)(5k+1)$

b Substituting $k = 12$ into the result from part **a**:

$-25 \times 61 = -1525$

c $\sum_{r=1}^{15}(5r-4) = -\sum_{r=1}^{15}(4-5r)$

Substituting $k = 7$ into the result from part **a**:

$-(-15 \times 36) = 540$

13 Let $f(r) = ar + b$

$\Rightarrow a \times \frac{1}{2}n(n+1) + bn$

$= \frac{1}{2}an^2 + \left(\frac{1}{2}a + b\right)n$

Since $\sum_{r=1}^{n} f(r) = n^2 + 4n$

$\Rightarrow \frac{1}{2}a = 1 \Rightarrow a = 2$

and $\frac{1}{2}a + b = 4 \Rightarrow 1 = b = 4 \Rightarrow b = 3$

$\Rightarrow f(r) = 2r + 3$

14 a $\sum_{r=1}^{n} ar + b = \frac{1}{2}an^2 + \left(\frac{1}{2}a + b\right)n$

$\sum_{r=1}^{4} f(r) = 36 \Rightarrow 8a + 2a + 4b = 36$

$\Rightarrow 10a + 4b = 36 \Rightarrow 5a + 2b = 18$

$\sum_{r=1}^{6} f(r) = 78 \Rightarrow 18a + 3a + 6b = 78$

$\Rightarrow 21a + 6b = 78 \Rightarrow 7a + 2b = 26$

Solving simultaneously:

$2a = 8 \Rightarrow a = 4$

$5 \times 4 + 2b = 18 \Rightarrow b = -1$

$\Rightarrow f(r) = 4r - 1$

$\sum_{r=1}^{n} f(r) = 2n^2 + n$

b $\sum_{r=1}^{n} f(r) = 2n^2 + n$

$\sum_{r=1}^{10} f(r) = 2(10)^2 + 10$

$= 200 + 10$

$= 210$

Challenge

$\sum_{r=n}^{2n}(12 - 2r) = 0$

$\Rightarrow \sum_{r=n}^{2n} 12 - \sum_{r=n}^{2n} 2r = 0$

$\sum_{r=n}^{2n} 12 = \sum_{r=n}^{2n} 2r$

$\sum_{r=n}^{2n} 12 = \sum_{r=1}^{2n} 12 - \sum_{r=1}^{n-1} 12 = 12(n+1)$

$\sum_{r=n}^{2n} 2r = \sum_{r=1}^{2n} 2r - \sum_{r=1}^{n-1} 2r$

$= 2 \times \frac{1}{2} \times 2n(2n+1) - 2 \times \frac{1}{2}(n-1)n$

$= 3n^2 + 3n$

$\Rightarrow 3n^2 + 3n = 12n + 12$

$\Rightarrow n^2 - 3n - 4 = 0$

$\Rightarrow (n-4)(n+1) = 0$

$\Rightarrow n = 4$

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Series 3B

1 a $\sum_{r=1}^{4} r^2 = \frac{1}{6} \times 4 \times 5 \times 9 = 30$

b $\sum_{r=1}^{40} r^2 = \frac{1}{6} \times 40 \times 41 \times 81 = 22\,140$

c $\sum_{r=21}^{40} r^2 = \sum_{r=1}^{40} r^2 - \sum_{r=1}^{20} r^2$

$= \frac{1}{6} \times 40 \times 41 \times 81 - \frac{1}{6} \times 20 \times 21 \times 41$

$= 19270$

d $\sum_{r=1}^{99} r^3 = \frac{1}{4} \times 99^2 \times 100^2 = 24\,502\,500$

e $\sum_{r=1}^{100} r^3 = \frac{1}{4} \times 100^2 \times 101^2 = 25\,502\,500$

f $\sum_{r=100}^{200} r^3 = \sum_{r=1}^{200} r^3 - \sum_{r=1}^{99} r^3$

$= \frac{1}{4} \times 200^2 \times 201^2 - \frac{1}{4} \times 99^2 \times 100^2$

$= 379\,507\,500$

g $\sum_{r=1}^{k} r^2 + \sum_{r=k+1}^{80} r^2 = \sum_{r=1}^{80} r^2$

$= \frac{1}{6} \times 80 \times 81 \times 161$

$= 173\,880$

2 a $\sum_{r=1}^{2n} r^2 = \frac{1}{6}(2n)(2n+1)(4n+1)$

$= \frac{1}{3} n(2n+1)(4n+1)$

b $\sum_{r=1}^{2n-1} r^2 = \frac{1}{6}(2n-1)(2n)(2(2n-1)+1)$

$= \frac{1}{6} \times 2n(2n-1)(4n-1)$

$= \frac{1}{3} n(2n-1)(4n-1)$

2 c $\sum_{r=n}^{2n} r^2 = \sum_{r=1}^{2n} r^2 - \sum_{r=1}^{n-1} r^2$

$= \frac{1}{3} n(2n+1)(4n+1) - \frac{1}{6}(n-1)(n)(2(n-1)+1)$

$= \frac{1}{6} n(2(2n+1)(4n+1) - (n-1)(2n-1))$

$= \frac{1}{6} n(16n^2 + 12n + 2 - 2n^2 + 3n - 1)$

$= \frac{1}{6} n(14n^2 + 15n + 1)$

$= \frac{1}{6} n(n+1)(14n+1)$

3 Let $n + k = m$

$\Rightarrow \sum_{r=1}^{m} r^3 = \frac{1}{4} m^2 (m+1)^2$

and so $\sum_{r=1}^{n+k} r^3 = \frac{1}{4}(n+k)^2 (n+k+1)^2$

4 a $\sum_{r=n+1}^{3n} r^3 = \sum_{r=1}^{3n} r^3 - \sum_{r=1}^{n} r^3$

$= \frac{1}{4}(3n)^2 (3n+1)^2 - \frac{1}{4} n^2 (n+1)^2$

$= \frac{1}{4} n^2 (9(3n+1)^2 - (n+1)^2)$

$= \frac{1}{4} n^2 (81n^2 - n^2 + 54n - 2n + 9 - 1)$

$= n^2 (20n^2 + 13n + 2)$

$= n^2 (4n+1)(5n+2)$

b $\sum_{r=11}^{30} r^3 = 10^2 \times 41 \times 52 = 213\,200$

5 a $\sum_{r=n}^{2n} r^3 = \sum_{r=1}^{2n} r^3 - \sum_{r=1}^{n-1} r^3$

$= \frac{1}{4}(2n)^2 (2n+1)^2 - \frac{1}{4}(n-1)^2 n^2$

$= \frac{1}{4} n^2 (4(2n+1)^2 - (n-1)^2)$

$= \frac{1}{4} n^2 (16n^2 + 16n + 4 - n^2 + 2n - 1)$

$= \frac{1}{4} n^2 (15n^2 + 18n + 3)$

$= \frac{3}{4} n^2 (5n^2 + 6n + 1)$

$= \frac{3}{4} n^2 (n+1)(5n+1)$

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5 b $\sum_{r=30}^{60} r^3 = \frac{3}{4}\times 30^2 \times 31 \times 151 = 3159\,675$

6 a $\sum_{m=1}^{30}(m^2-1) = \sum_{m=1}^{30} m^2 - \sum_{m=1}^{30} 1$

$= \sum_{m=1}^{30} m^2 - 30$

$= \frac{1}{6}\times 30\times 31\times 61 - 30 = 9425$

b $\sum_{r=1}^{40} r(r+4) = \sum_{r=1}^{40} r^2 + 4r = \sum_{r=1}^{40} r^2 + 4\sum_{r=1}^{40} r$

$= \frac{1}{6}\times 40\times 41\times 81 + 4\times\frac{1}{2}\times 40\times 41$

$= 22\,140 + 3280 = 25\,420$

c $\sum_{r=1}^{80} r(r^2+3) = \sum_{r=1}^{80}(r^3+3r) = \sum_{r=1}^{80} r^3 + 3\sum_{r=1}^{80} r$

$= \frac{1}{4}\times 80^2 \times 81^2 + 3\times\frac{1}{2}\times 80\times 81$

$= 10\,497\,600 + 9720 = 10\,507\,320$

d $\sum_{r=11}^{35}(r^3-2) = \sum_{r=1}^{35}(r^3-2) - \sum_{r=1}^{10}(r^3-2)$

$= \sum_{r=1}^{35} r^3 - \sum_{r=1}^{10} r^3 - 2\sum_{r=1}^{35} 1 - 2\sum_{r=1}^{10} 1$

$= \frac{1}{4}\times 35^2\times 36^2 - \frac{1}{4}\times 10^2\times 11^2 - 2(35-10)$

$= 396\,900 - 3025 - 50 = 393\,825$

7 a $\sum_{r=1}^{n}(r+2)(r+5) = \sum_{r=1}^{n}(r^2+7r+10)$

$= \sum_{r=1}^{n} r^2 + 7\sum_{r=1}^{n} r + 10\sum_{r=1}^{n} 1$

$= \frac{1}{6}n(n+1)(2n+1) + \frac{7}{2}n(n+1) + 10n$

$= \frac{1}{6}n(2n^2+3n+1+21n+21+60)$

$= \frac{1}{3}n(n^2+12n+41)$

7 b $\sum_{r=10}^{50}(r+2)(r+5)$

$= \sum_{r=1}^{50}(r+2)(r+5) - \sum_{r=1}^{9}(r+2)(r+5)$

$= \frac{1}{3}\times 50\times(50^2+600+41)$

$\quad -\frac{1}{3}\times 9\times(9^2+108+41)$

$= 52\,350 - 690 = 51\,660$

8 a $\sum_{r=1}^{n}(r^2+3r+1) = \sum_{r=1}^{n} r^2 + 3\sum_{r=1}^{n} r + \sum_{r=1}^{n} 1$

$= \frac{1}{6}n(n+1)(2n+1) + \frac{3}{2}n(n+1) + n$

$= \frac{1}{6}n(2n^2+3n+1+9n+9+6)$

$= \frac{1}{3}n(n^2+6n+8) = \frac{1}{3}n(n+2)(n+4)$

$a = 2,\ b = 4$

b $\sum_{r=19}^{40}(r^2+3r+1)$

$= \sum_{r=1}^{40}(r^2+3r+1) - \sum_{r=1}^{18}(r^2+3r+1)$

$= \frac{1}{3}\times 40\times 42\times 44 - \frac{1}{3}\times 18\times 20\times 22$

$= 22\,000$

9 a $\sum_{r=1}^{n} r^2(r-1) = \sum_{r=1}^{n} r^3 - \sum_{r=1}^{n} r^2$

$= \frac{1}{4}n^2(n+1)^2 - \frac{1}{6}n(n+1)(2n+1)$

$= \frac{1}{12}n(n+1)(3n(n+1) - 2(2n+1))$

$= \frac{1}{12}n(n+1)(3n^2-n-2)$

b $\sum_{r-1}^{2n-1} r^2(r-1)$

$= \frac{1}{12}(2n-1)((2n-1)+1)(3(2n-1)^2$

$\quad -(2n-1)-2)$

$= \frac{1}{12}\times 2n(2n-1)(12n^2-14n+2)$

$= \frac{1}{3}n(2n-1)(6n^2-7n+1)$

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10 a $\sum_{r=1}^{n}(r+1)(r+3)=\sum_{r=1}^{n}(r^2+4r+3)$

$=\sum_{r=1}^{n}r^2+4\sum_{r=1}^{n}r+3\sum_{r=1}^{n}1$

$=\frac{1}{6}n(n+1)(2n+1)+2n(n+1)+3n$

$=\frac{1}{6}n(2n^2+3n+1+12n+12+18)$

$=\frac{1}{6}n(2n^2+15n+31)$

b $\sum_{r=n+1}^{2n}(r+1)(r+3)$

$=\sum_{r=1}^{2n}(r+1)(r+3)-\sum_{r=1}^{n}(r+1)(r+3)$

$=\frac{1}{6}\times 2n(8n^2+30n+31)$

$-\frac{1}{6}\times n\times(2n^2+15n+31)$

$=\frac{1}{6}n(16n^2+60n+62-2n^2-15n-31)$

$=\frac{1}{6}n(14n^2+45n+31)$

11 a $\sum_{r=1}^{n}(r+3)(r+4)=\sum_{r=1}^{n}r^2+7r+12$

$=\sum_{r=1}^{n}r^2+7\sum_{r=1}^{n}r+12\sum_{r=1}^{n}1$

$=\frac{1}{6}n(n+1)(2n+1)+\frac{7}{2}n(n+1)+12n$

$=\frac{1}{6}n(2n^2+3n+1+21n+21+72)$

$=\frac{1}{3}n(n^2+12n+47)$

b $\sum_{r=n+1}^{3n}(r+3)(r+4)$

$=\sum_{r=1}^{3n}(r+3)(r+4)-\sum_{r=1}^{n}(r+3)(r+4)$

$=\frac{1}{3}\times 3n(9n^2+36n+47)$

$-\frac{1}{3}n(n^2+12n+47)$

$=\frac{1}{3}n(27n^2+108n+141-n^2-12n-47)$

$=\frac{1}{3}n(26n^2+96n+94)$

$=\frac{2}{3}n(13n^2+48n+47)$

12 a $\sum_{r=1}^{n}r(r+3)^2=\sum_{r=1}^{n}r^3+6r^2+9r$

$=\sum_{r=1}^{n}r^3+6\sum_{r=1}^{n}r^2+9\sum_{r=1}^{n}r$

$=\frac{1}{4}n^2(n+1)^2+n(n+1)(2n+1)+\frac{9}{2}n(n+1)$

$=\frac{1}{4}n(n+1)(n(n+1)+4(2n+1)+18)$

$=\frac{1}{4}n(n+1)(n^2+9n+22)$

b $\sum_{r=10}^{20}r(r+3)^2$

$=\sum_{r=1}^{20}r(r+3)^2-\sum_{r=1}^{9}r(r+3)^2$

$=\frac{1}{4}\times 20\times 21\times(20^2+180+22)$

$-\frac{1}{4}\times 9\times 10\times(9^2+81+22)$

$=59\,070$

13 a $\sum_{r=1}^{kn}(2r-1)=2\sum_{r=1}^{kn}r-\sum_{r=1}^{kn}1$

$=kn(kn+1)-kn=k^2n^2$

b $\sum_{r=1}^{5n}(2r-1)=25n^2\Rightarrow\sum_{r=1}^{n}r^3=25n^2$

$\Rightarrow\frac{1}{4}n^2(n+1)^2=25n^2$

$\Rightarrow(n+1)^2=100$

$\Rightarrow n=9$

14 a $\sum_{r=1}^{n}(r^3-r^2)=\sum_{r=1}^{n}r^3-\sum_{r-1}^{n}r^2$

$=\frac{1}{4}n^2(n+1)^2-\frac{1}{6}n(n+1)(2n+1)$

$=\frac{1}{12}n(n+1)(3n(n+1)-2(2n+1))$

$=\frac{1}{12}n(n+1)(3n^2-n-2)$

$=\frac{1}{12}n(n+1)(n-1)(3n+2)$

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14 b $\sum_{r=1}^{n} 7r = \frac{7}{2}n(n+1)$

$\Rightarrow \frac{7}{2}n(n+1) = \frac{1}{12}n(n+1)(n-1)(3n+2)$

$\Rightarrow \frac{7}{2} = \frac{1}{12}(n-1)(3n+2)$

$\Rightarrow 42 = (n-1)(3n+2)$

$\Rightarrow 3n^2 - n - 44 = 0$

$\Rightarrow (3n+11)(n-4) = 0$

$\Rightarrow n = 4$

Challenge

a Let $f_2(x) = ax + b$

$\sum_{r=1}^{n} f_2(r) = \frac{1}{2}an(n+1) + bn$

$\Rightarrow \frac{1}{2}an(n+1) + bn = n^2$

$\Rightarrow \frac{1}{2}an^2 + \left(\frac{1}{2}a + b\right)n$

$a = 2, b = -1$

$\Rightarrow f_2(x) = 2x - 1$

Let $f_3(x) = ax^2 + bx + c$

$\sum_{r=1}^{n} f_3(r) = \frac{1}{6}an(n+1)(2n+1) + \frac{1}{2}bn(n+1) + cn$

$\Rightarrow \frac{1}{6}an(n+1)(2n+1) + \frac{1}{2}bn(n+1) + cn = n^3$

$\Rightarrow \frac{1}{3}an^3 + \left(\frac{1}{2}a + \frac{1}{2}b\right)n^2 + \left(\frac{1}{6}a + \frac{1}{2}b + c\right)n = n^3$

$a = 3, b = -3, c = 1$

$\Rightarrow f_3(x) = 3x^2 - 3x + 1$

Let $f_4(x) = ax^3 + bx^2 + cx + d$

$\sum_{r=1}^{n} f_4(r) = \frac{1}{4}an^2(n+1)^2 + \frac{1}{6}bn(n+1)(2n+1)$

$+ \frac{1}{2}cn(n+1) + dn$

$\Rightarrow \frac{1}{4}a = 1$

$\frac{1}{2}a + \frac{1}{3}b = 0$

$\frac{1}{4}a + \frac{1}{2}b + \frac{1}{2}c = 0$

$\frac{1}{6}b + \frac{1}{2}c + d = 0$

$a = 4, b = -6, c = 4, d = -1$

$\Rightarrow f_4(x) = 4x^3 - 6x^2 + 4x - 1$

b Given $h(x) = ax^3 + bx^2 + cx + d$,

$nh(n) = an^4 + bn^3 + cn^2 + dn$

$= \sum_{r=1}^{n} (af_4(r) + bf_3(r) + cf_2(r) + df_1(r))$

$= \sum_{r=1}^{n} g(r)$

for $g(r) = af_4(r) + bf_3(r) + cf_2(r) + df_1(r)$

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Series ME 3

1 a $\sum_{r=1}^{10} r = \frac{1}{2}\times 10\times 11 = 55$

b $\sum_{r=10}^{50} r = \sum_{r=1}^{50} r - \sum_{r=1}^{9} r$

$= \frac{1}{2}\times 50\times 51 - \frac{1}{2}\times 9\times 10$

$= 1230$

c $\sum_{r=1}^{10} r^2 = \frac{1}{6}\times 10\times 11\times 21 = 385$

d $\sum_{r=1}^{10} r^3 = \frac{1}{4}\times 10^2\times 11^2 = 3025$

e $\sum_{r=26}^{50} r^2 = \sum_{r=1}^{50} r^2 - \sum_{r=1}^{25} r^2$

$= \frac{1}{6}\times 50\times 51\times 101 - \frac{1}{6}\times 25\times 26\times 51$

$= 37\,400$

f $\sum_{r=50}^{100} r^3 = \sum_{r=1}^{100} r^3 - \sum_{r=1}^{49} r^3$

$= \frac{1}{4}\times 100^2\times 101^2 - \frac{1}{4}\times 49^2\times 50^2$

$= 24\,001\,875$

g $\sum_{r=1}^{60} r + \sum_{r=1}^{60} r^2$

$= \frac{1}{2}\times 60\times 61 + \frac{1}{6}\times 60\times 61\times 121$

$= 75\,640$

2 a $\sum_{r=1}^{n} (3r-5) = 3\sum_{r=1}^{n} r - \sum_{r=1}^{n} 5$

$= \frac{3}{2}n(n+1) - 5n$

$= \frac{3}{2}n^2 + \frac{3}{2}n - 5n$

$= \frac{3}{2}n^2 - \frac{7}{2}n$

2 b $\sum_{r=1}^{n} (r^2 + r) = \sum_{r=1}^{n} r^2 + \sum_{r=1}^{n} r$

$= \frac{1}{6}n(n+1)(2n+1) + \frac{1}{2}n(n+1)$

$= \frac{1}{6}n(n+1)(2n+1+3)$

$= \frac{1}{6}n(n+1)(2n+4)$

$= \frac{1}{3}n(n+1)(n+2)$

c $\sum_{r=1}^{n} (3r^2 + 7r) = 3\sum_{r=1}^{n} r^2 + 7\sum_{r=1}^{n} r$

$= \frac{3}{6}n(n+1)(2n+1) + \frac{7}{2}n(n+1)$

$= \frac{1}{6}n(n+1)(6n+3+21)$

$= \frac{1}{6}n(n+1)(6n+24)$

$= n(n+1)(n+4)$

d $\sum_{r=1}^{n} (4r^3 + 6r^2) = 4\sum_{r=1}^{n} r^3 + 6\sum_{r=1}^{n} r^2$

$= n^2(n+1)^2 + n(n+1)(2n+1)$

$= n(n+1)(n^2 + n + 2n + 1)$

$= n(n+1)(n^2 + 3n + 1)$

e $\sum_{r=1}^{n} (r^2 - 2r) = \sum_{r=1}^{n} r^2 + 2\sum_{r=1}^{n} r$

$= \frac{1}{6}n(n+1)(2n+1) - n(n+1)$

$= \frac{1}{6}n(n+1)(2n+1-6)$

$= \frac{1}{6}n(n+1)(2n-5)$

f $\sum_{r=1}^{n} (r^2 - 3r) = \sum_{r=1}^{n} r^2 + 3\sum_{r=1}^{n} r$

$= \frac{1}{6}n(n+1)(2n+1) - \frac{3}{2}n(n+1)$

$= \frac{1}{6}n(n+1)(2n+1-9)$

$= \frac{1}{6}n(n+1)(2n-8)$

$= \frac{1}{3}n(n+1)(n-4)$

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2 g $\sum_{r=1}^{n}(r^2-5)=\sum_{r=1}^{n}r^2-5\sum_{r=1}^{n}1$

$=\frac{1}{6}n(n+1)(2n+1)-5n$

$=\frac{1}{6}n(2n^2+3n+1-30)$

$=\frac{1}{6}n(2n^2+3n-29)$

h $\sum_{r=1}^{n}(2r^3+3r^2+r+4)$

$=2\sum_{r=1}^{n}r^3+3\sum_{r=1}^{n}r^2+\sum_{r=1}^{n}r+4\sum_{r=1}^{n}1$

$=\frac{1}{2}n^2(n+1)^2+\frac{1}{2}n(n+1)(2n+1)$

$+\frac{1}{2}n(n+1)+4n$

$=\frac{1}{2}n(n^3+2n^2+n+n^2+3n+1+n+1+8)$

$=\frac{1}{2}n(n^3+3n^2+5n+10)$

3 $\sum_{r=1}^{30}r(3r-1)=\sum_{r=1}^{30}3r^2-r$

$=3\sum_{r=1}^{30}r^2-\sum_{r=1}^{30}r$

$=\frac{1}{2}\times30\times31\times61-\frac{1}{2}\times30\times31$

$=27\,900$

4 a $\sum_{r=1}^{n}r^2(r-3)=\sum_{r=1}^{n}r^3-3r^2$

$=\sum_{r=1}^{n}r^3-3\sum_{r=1}^{n}r^2$

$=\frac{1}{4}n^2(n+1)^2-\frac{1}{2}n(n+1)(2n+1)$

$=\frac{1}{4}n(n+1)(n(n+1)-2(2n+1))$

$=\frac{1}{4}n(n+1)(n^2-3n-2)$

so $a=-3,\ b=-2$.

b $\sum_{r=1}^{20}r^2(r-3)=\frac{1}{4}\times20\times21\times(20^2-60-2)$

$=35490$

5 a $\sum_{r=1}^{n}(2r-1)^2=\sum_{r=1}^{n}4r^2-4r+1$

$=4\sum_{r=1}^{n}r^2-4\sum_{r=1}^{n}r+\sum_{r=1}^{n}1$

$=\frac{2}{3}n(n+1)(2n+1)-2n(n+1)+n$

$=\frac{1}{3}n(n+1)(2(2n+1)-6)+n$

$=\frac{1}{3}n(n+1)(4n-4)+n$

$=\frac{1}{3}n(4n^2-4+3)$

$=\frac{1}{3}n(2n-1)(2n+1)$

b $\sum_{r=1}^{2n}(2r-1)^2=\frac{1}{3}\times2n(4n-1)(4n+1)$

$=\frac{2}{3}n(4n-1)(4n+1)$

6 a $\sum_{r=1}^{n}r(r+2)=\sum_{r=1}^{n}r^2+2r$

$=\sum_{r=1}^{n}r^2+2\sum_{r=1}^{n}r$

$=\frac{1}{6}n(n+1)(2n+1)+n(n+1)$

$=\frac{1}{6}n(n+1)(2n+7)$

b $\sum_{r=15}^{30}r(r+2)=\sum_{r=1}^{30}r(r+2)-\sum_{r=1}^{14}r(r+2)$

$=\frac{1}{6}\times30\times31\times67-\frac{1}{6}\times14\times15\times35$

$=9160$

7 a $\sum_{r=n+1}^{2n}r^2=\sum_{r=1}^{2n}r^2-\sum_{r=1}^{n}r^2$

$=\frac{1}{6}(2n)(2n+1)(2(2n)+1)-\frac{1}{6}n(n+1)(2n+1)$

$=\frac{1}{6}n(2n+1)(2(4n+1)-n-1)$

$=\frac{1}{6}n(2n+1)(7n+1)$

b $\sum_{r=16}^{30}r^2=\frac{1}{6}\times15\times31\times106=8215$

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8 a $\sum_{r=1}^{n}(r^2-r-1)=\sum_{r=1}^{n}r^2-\sum_{r=1}^{n}r-\sum_{r=1}^{n}1$

$=\frac{1}{6}n(n+1)(2n+1)-\frac{1}{2}n(n+1)-n$

$=\frac{1}{3}n(n^2-4)$

b $\sum_{r=10}^{40}(r^2-r-1)$

$=\sum_{r=1}^{40}(r^2-r-1)-\sum_{r=1}^{9}(r^2-r-1)$

$=\frac{1}{3}\times 40\times(40^2-4)-\frac{1}{3}\times 9\times(9^2-4)$

$=21\,049$

c $\sum_{r=1}^{n}(r^2-r-1)=\sum_{r=1}^{2n}r$

$\Rightarrow \frac{1}{3}n(n^2-4)=\frac{1}{2}\times 2n\times(2n+1)$

$\Rightarrow \frac{1}{3}n(n^2-4)=n(2n+1)$

$\Rightarrow \frac{1}{3}(n^2-4)=2n+1$

$\Rightarrow n^2-4=6n+3$

$\Rightarrow n^2-6n-7=0$

$\Rightarrow (n-7)(n+1)=0$

$\Rightarrow n=7$

9 a $\sum_{r=1}^{n}r(2r^2+1)=\sum_{r=1}^{n}2r^3+r$

$=2\sum_{r=1}^{n}r^3+\sum_{r=1}^{n}r$

$=\frac{1}{2}n^2(n+1)^2+\frac{1}{2}n(n+1)$

$=\frac{1}{2}n(n+1)(n(n+1)+1)$

$=\frac{1}{2}n(n+1)(n^2+n+1)$

9 b $\sum_{r=1}^{n}(100r^2-r)=100\sum_{r=1}^{n}r^2-\sum_{r=1}^{n}r$

$=\frac{1}{6}n(n+1)(200n+97)$

Now if $\sum_{r=1}^{n}r(2r^2+1)=\sum_{r=1}^{n}(100r^2-r)$, then

$\frac{1}{2}n(n+1)(n^2+n+1)=\frac{1}{6}n(n+1)(200n+97)$

$\frac{1}{6}n(n+1)(3(n^2+n+1)-(200n+97))=0$

$\frac{1}{6}n(n+1)(3n^2-197n-94)=0$

But $n\neq 0$, $n\neq -1$ and $3n^2-197n-94$ has discriminant 39 937 which is not square.

10 a $\sum_{r=1}^{n}r(r+1)^2=\sum_{r=1}^{n}r^3+2r^2+r$

$=\sum_{r=1}^{n}r^3+2\sum_{r=1}^{n}r^2+\sum_{r=1}^{n}r$

$=\frac{1}{4}n^2(n+1)^2+\frac{1}{3}n(n+1)(2n+1)+\frac{1}{2}n(n+1)$

$=\frac{1}{12}n(n+1)(n+2)(3n+5)$

b $\sum_{r=1}^{n}70r=35n(n+1)$

$\Rightarrow \frac{1}{12}n(n+1)(n+2)(3n+5)=35n(n+1)$

$\Rightarrow \frac{1}{12}(n+2)(3n+5)=35$

$\Rightarrow 3n^2+11n+10=420$

$\Rightarrow 3n^2+11n-410=0$

$\Rightarrow (3n+41)(n-10)=0$

$\Rightarrow n=10$

11 $\frac{1}{6}n(n+1)(2n+1)=\frac{9}{2}(n+1)(n+2)+n+1$

$\Rightarrow n(n+1)(2n+1)=27(n+1)(n+2)+6(n+1)$

$\Rightarrow (n+1)(2n^2+n)=(n+1)(27n+54+6)$

$\Rightarrow 2n^2+n=27n+60$

$\Rightarrow 2n^2-26n-60=0$

$\Rightarrow n^2-13n-30=0$

$\Rightarrow (n-15)(n+2)=0$

$\Rightarrow n=15$

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Challenge

a $\sum_{i=1}^{n}\left(\sum_{r=1}^{i} r^2\right) = \sum_{i=1}^{n} \frac{1}{6} i(i+1)(2i+1)$

$= \frac{1}{6}\sum_{i=1}^{n} 2i^3 + 3i^2 + i$

$= \frac{1}{6}\left(\frac{1}{2}n^2(n+1)^2 + \frac{1}{2}n(n+1)(2n+1) + \frac{1}{2}n(n+1)\right)$

$= \frac{1}{12}\left(n(n+1)(n^2+n+2n+1+1)\right)$

$= \frac{1}{12}n(n+1)(n^2+3n+2)$

$= \frac{1}{12}n(n+1)(n+2)(n+1)$

$= \frac{1}{12}n(n+1)^2(n+2)$

b $\sum_{j=1}^{n}\left(\sum_{i=1}^{j}\left(\sum_{r=1}^{i} r\right)\right) = \sum_{j=1}^{n}\left(\frac{1}{2}\sum_{i=1}^{j} i(i+1)\right)$

$= \frac{1}{2}\sum_{j=1}^{n}\left(\sum_{i=1}^{j} i^2\right) + \frac{1}{2}\sum_{j=1}^{n}\left(\sum_{i=1}^{j} i\right)$

$= \frac{1}{24}n(n+1)^2(n+2) + \frac{1}{4}\sum_{j=1}^{n} j^2 + \frac{1}{4}\sum_{j=1}^{n} j$

$= \frac{1}{24}n(n+1)^2(n+2) + \frac{1}{24}n(n+1)(2n+1)$

$\quad + \frac{1}{8}n(n+1)$

$= \frac{1}{24}n(n+1)(n^2+5n+6)$

$= \frac{1}{24}n(n+1)(n+2)(n+3)$

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Roots of polynomials 4A

1 a $3x^2+7x-4=0$
$a=3, b=7, c=-4$
$\alpha+\beta=-\frac{b}{a}=-\frac{7}{3}$

b $\alpha\beta=\frac{c}{a}=-\frac{4}{3}$

c $\frac{1}{\alpha}+\frac{1}{\beta}=\frac{\alpha+\beta}{\alpha\beta}$
$=\left(\frac{-\frac{7}{3}}{-\frac{4}{3}}\right)$
$=\frac{7}{4}$

d $\alpha^2+\beta^2=(\alpha+\beta)^2-2\alpha\beta$
$=\left(-\frac{7}{3}\right)^2-2\left(-\frac{4}{3}\right)$
$=\frac{49}{9}+\frac{8}{3}$
$=\frac{73}{9}$

2 a $7x^2-3x+1=0$
$a=7, b=-3, c=1$
$\alpha+\beta=-\frac{b}{a}=\frac{3}{7}$

b $\alpha\beta=\frac{c}{a}=\frac{1}{7}$

c $\frac{1}{\alpha}+\frac{1}{\beta}=\frac{\alpha+\beta}{\alpha\beta}$
$=\frac{\frac{3}{7}}{\frac{1}{7}}$
$=3$

d $\alpha^2+\beta^2=(\alpha+\beta)^2-2\alpha\beta$
$=\left(\frac{3}{7}\right)^2-2\left(\frac{1}{7}\right)$
$=\frac{9}{49}-\frac{2}{7}$
$=\frac{9}{49}-\frac{14}{49}$
$=-\frac{5}{49}$

3 a $6x^2-9x+2=0$
$a=6, b=-9, c=2$
$\alpha+\beta=-\frac{b}{a}=-\frac{-9}{6}=\frac{3}{2}$

b $\alpha^2\times\beta^2=(\alpha\beta)^2=\left(\frac{c}{a}\right)^2=\left(\frac{2}{6}\right)^2=\frac{1}{9}$

c $\frac{1}{\alpha}+\frac{1}{\beta}=\frac{\alpha+\beta}{\alpha\beta}=\frac{\frac{3}{2}}{\frac{1}{3}}=\frac{9}{2}$

d $(\alpha+\beta)^3=\alpha^3+3\alpha^2\beta+3\alpha\beta^2+\beta^3$
So $\alpha^3+\beta^3=(\alpha+\beta)^3-3\alpha\beta(\alpha+\beta)$
$=\left(\frac{3}{2}\right)^3-3\left(\frac{1}{3}\right)\left(\frac{3}{2}\right)$
$=\frac{27}{8}-\frac{3}{2}$
$=\frac{27}{8}-\frac{12}{8}$
$=\frac{15}{8}$

4 $ax^2+bx+c=0$
$\alpha=2$ and $\beta=-3$
$\alpha+\beta=2+(-3)=-1=-\frac{b}{a}$
$\alpha\beta=(2)(-3)=-6=\frac{c}{a}$
$ax^2+bx+c=0\Rightarrow x^2+\frac{b}{a}x+\frac{c}{a}=0$
$-\frac{b}{a}=-1\Rightarrow\frac{b}{a}=1$
$\frac{c}{a}=-6$
So equation is $x^2+x-6=0$,
with $a=1, b=1$ and $c=-6$

5 $ax^2+bx+c=0$
$\alpha=-\frac{1}{2}$ and $\beta=-\frac{1}{3}$
$\alpha+\beta=-\frac{1}{2}-\frac{1}{3}=-\frac{5}{6}=-\frac{b}{a}$
$\alpha\beta=\left(-\frac{1}{2}\right)\left(-\frac{1}{3}\right)=\frac{1}{6}=\frac{c}{a}$
$ax^2+bx+c=0\Rightarrow x^2+\frac{b}{a}x+\frac{c}{a}=0$
$-\frac{b}{a}=-\frac{5}{6}$, so $\frac{b}{a}=\frac{5}{6}$
$\frac{c}{a}=\frac{1}{6}$

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5 So equation is $x^2+\frac{5}{6}x+\frac{1}{6}=0$
or $6x^2+5x+1=0$. So
$a=6, b=5$ and $c=1$.

6 $ax^2+bx+c=0$
$\alpha=\frac{-1+\mathrm{i}}{2}$ and $\beta=\frac{-1-\mathrm{i}}{2}$
$\alpha+\beta=-1=-\frac{b}{a}$
$\alpha\beta=\left(\frac{-1+\mathrm{i}}{2}\right)\left(\frac{-1-\mathrm{i}}{2}\right)=\frac{1-\mathrm{i}^2}{2}=\frac{2}{4}=\frac{1}{2}=\frac{c}{a}$
$ax^2+bx+c=0\Rightarrow x^2+\frac{b}{a}x+\frac{c}{a}=0$
$-\frac{b}{a}=-1\Rightarrow\frac{b}{a}=1$
$\frac{c}{a}=\frac{1}{2}$
So equation is $x^2+x+\frac{1}{2}=0$
or $2x^2+2x+1=0$. So
$a=2, b=2$ and $c=1$.

7 a The other root, β, is the complex conjugate. So $\beta=-1+4\mathrm{i}$

b $\alpha=-1-4\mathrm{i}$ and $\beta=-1+4\mathrm{i}$
$\alpha+\beta=-2=-\frac{b}{a}$
$\alpha\beta=(-1-4\mathrm{i})(-1+4\mathrm{i})=1-16\mathrm{i}^2=17=\frac{c}{a}$
$ax^2+bx+c=0\Rightarrow x^2+\frac{b}{a}x+\frac{c}{a}=0$
$-\frac{b}{a}=-2\Rightarrow\frac{b}{a}=2$
$\frac{c}{a}=17$
So equation is $x^2+2x+17=0$,
with $a=1, b=2$ and $c=17$.

8 The sum of the roots is 4,
so $\alpha+\beta=4$
Also $-\frac{b}{a}=4$, but $-\frac{b}{a}=\frac{-(k-3)}{k}=\frac{3-k}{k}$
So $\frac{3-k}{k}=4$
$3-k=4k$
$3=5k$
$k=\frac{3}{5}$

9 The roots are α and $-\alpha$.
$\alpha+-\alpha=0=-\frac{b}{a}$
If $-\frac{b}{a}=0$, then $b=0$
$b=0\Rightarrow 16+n=0\Rightarrow n=-16$

10 The roots are reciprocals, so the roots are α and $\frac{1}{\alpha}$.
$\alpha\left(\frac{1}{\alpha}\right)=1=\frac{c}{a}$
But $\frac{c}{a}=\frac{k}{6}$
So $\frac{k}{6}=1\Rightarrow k=6$

11 The roots are k and $2k$.
$k+2k=3k=-\frac{b}{a}$
But $-\frac{b}{a}=-\frac{4}{m}$, so $3k=-\frac{4}{m}$
$(k)(2k)=2k^2=\frac{c}{a}$
But $\frac{c}{a}=\frac{4m}{m}=4$, so $2k^2=4$
If $2k^2=4, k^2=2$ and $k=\pm\sqrt{2}$.
$3k=-\frac{4}{m}$, so $m=-\frac{4}{3k}$
If $k=\sqrt{2}, m=-\frac{4}{3\sqrt{2}}=-\frac{2\sqrt{2}}{3}$
If $k=-\sqrt{2}, m=\frac{4}{3\sqrt{2}}=\frac{2\sqrt{2}}{3}$

12 a Let $\alpha=p+q\mathrm{i}$ and $\alpha^*=p-q\mathrm{i}$
$\alpha+\alpha^*=(p+q\mathrm{i})+(p-q\mathrm{i})=2p$
$\alpha\alpha^*=(p+q\mathrm{i})(p-q\mathrm{i})=p^2+q^2$
Also $2p=-\frac{b}{a}=-\frac{8}{a}$
and $p^2+q^2=\frac{c}{a}$
If $\mathrm{Re}(\alpha)=2$ then $p=2$, so $2p=4$.
$2p=-\frac{8}{a}\Rightarrow 4=-\frac{8}{a}\Rightarrow a=-2$

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12 b If $\text{Im}(\alpha) = 3i$, then $q = 3$

$p^2 + q^2 = \frac{c}{a}$,

So $(2)^2 + (3)^2 = \frac{c}{-2}$

$13 = \frac{c}{-2}$

$c = -26$

13 a Let $\alpha = m + n\text{i}$ and $\alpha^* = m - n\text{i}$

$\alpha + \alpha^* = (m + n\text{i}) + (m - n\text{i}) = 2m$

$\alpha\alpha^* = (m + n\text{i})(m - n\text{i}) = m^2 + n^2$

Also $2m = -\frac{b}{a} = -\frac{p}{4}$

and $m^2 + n^2 = \frac{c}{a} = \frac{q}{4}$

If $\text{Re}(\alpha) = -3$, then $m = -3$, so $2m = -6$

$2m = -\frac{p}{4} \Rightarrow -6 = -\frac{p}{4} \Rightarrow p = 24$

b If $\text{Im}(\alpha) \neq 0$, then $n \neq 0$

$m^2 + n^2 = \frac{c}{a} \Rightarrow (-3)^2 + n^2 = \frac{q}{4}$

So $q = 36 + 4n^2$

As $n \neq 0$, $4n^2 > 0$, so $q > 36$

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Roots of polynomials 4B

1 a $2x^3+5x^2-2x+3=0$
$a=2, b=5, c=-2$ and $d=3$.
$\alpha+\beta+\gamma=-\frac{b}{a}=-\frac{5}{2}$

b $\alpha\beta\gamma=-\frac{d}{a}=-\frac{3}{2}$

c $\alpha\beta+\beta\gamma+\gamma\alpha=\frac{c}{a}=-\frac{2}{2}=-1$

d $\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma}=\frac{\alpha\beta+\beta\gamma+\gamma\alpha}{\alpha\beta\gamma}$
$=\frac{-1}{\left(-\frac{-3}{2}\right)}=\frac{2}{3}$

2 a $x^3+5x^2+17x+13=0$
$a=1, b=5, c=17$ and $d=13$
$\alpha+\beta+\gamma=-\frac{b}{a}=-5$

b $\alpha\beta\gamma=-\frac{d}{a}=-13$

c $\alpha\beta+\beta\gamma+\gamma\alpha=\frac{c}{a}=17$

d $\alpha^2\beta^2\gamma^2=(\alpha\beta\gamma)^2=(-13)^2=169$

3 a $7x^3-4x^2-x+6=0$
$a=7, b=-4, c=-1$ and $d=6$.
$\alpha+\beta+\gamma=-\frac{b}{a}=-\frac{-4}{7}=\frac{4}{7}$

b $\alpha\beta\gamma=-\frac{d}{9}=-\frac{6}{7}$

c $\alpha^3\beta^3\gamma^3=(\alpha\beta\gamma)^3=\left(-\frac{6}{7}\right)^3=-\frac{216}{343}$

d $\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma}=\frac{\alpha\beta+\beta\gamma+\gamma\alpha}{\alpha\beta\gamma}$
$=\frac{\left(\frac{c}{9}\right)}{\left(-\frac{6}{7}\right)}$
$=\frac{\left(-\frac{1}{7}\right)}{\left(-\frac{6}{7}\right)}$
$=\frac{1}{6}$

4 $ax^3+bx^2+cx+d=0$
$\alpha=\frac{3}{2}, \beta=\frac{1}{2}$. and $\gamma=1$
$\alpha+\beta+\gamma=\frac{3}{2}+\frac{1}{2}+1=3=-\frac{b}{a}$
$\alpha\beta+\beta\gamma+\gamma\alpha=\left(\frac{3}{2}\right)\left(\frac{1}{2}\right)+\left(\frac{1}{2}\right)(1)+$
$(1)\left(\frac{3}{2}\right)=\frac{11}{4}=\frac{c}{a}$
$\alpha\beta\gamma=\left(\frac{3}{2}\right)\left(\frac{1}{2}\right)(1)=\frac{3}{4}=-\frac{d}{a}$
$ax^3+bx^2+cx+d=0\Rightarrow$
$x^3+\frac{b}{a}x^2+\frac{c}{a}x+\frac{d}{a}=0$
$-\frac{b}{a}=3\Rightarrow\frac{b}{a}=-3$
$\frac{c}{a}=\frac{11}{4}$
$-\frac{d}{a}=\frac{3}{4}\Rightarrow\frac{d}{a}=-\frac{3}{4}$
So equation is $x^3-3x^2+\frac{11}{4}x-\frac{3}{4}=0$.
Multiply by 4 for integer coefficients gives
$4x^3-12x^2+11x-3=0$,
with $a=4, b=-12, c=11$ and $d=-3$.

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5 $ax_3 + bx^2 + cx + d = 0$

$\alpha = 1+3\text{i}, \beta = 1-3\text{i}$ and $\gamma = \frac{1}{2}$

$\alpha + \beta + \gamma = (1+3\text{i}) + (1-3\text{i}) + \frac{1}{2} = \frac{5}{2} = -\frac{b}{a}$

$\alpha\beta + \beta\gamma + \gamma\alpha$

$= (1+3\text{i})(1-3\text{i}) + (1-3\text{i})\frac{1}{2} + \frac{1}{2}(1+3\text{i})$

$= 1 - 3i + 3i - 9i^2 + \frac{1}{2} - \frac{3}{2}i + \frac{1}{2} + \frac{3}{2}i$

$= 11 = \frac{c}{a}$

$\alpha\beta\gamma = (1+3\text{i})(1-3\text{i})\left(\frac{1}{2}\right)$

$= \left(1 - 3\text{i} + 3\text{i} + 9\text{i}^2\right)\left(\frac{1}{2}\right)$

$= 5 = -\frac{d}{a}$

$ax^3 + bx^2 + cx + d = 0 \Rightarrow$

$x^3 + \frac{b}{a}x^2 + \frac{c}{a}x + \frac{d}{a} = 0$

$-\frac{b}{a} = \frac{5}{2} \Rightarrow \frac{b}{a} = -\frac{5}{2}$

$\frac{c}{a} = 11$

$-\frac{d}{a} = 5 \Rightarrow \frac{d}{a} = -5$

So equation is $x^3 - \frac{5}{2}x^2 + 11x - 5 = 0.$

Multiplying by 2 for integer coefficients gives $2x^3 - 5x^2 + 22x - 10 = 0,$ with $a = 2, b = -5, c = 22$ and $d = -10.$

6 $ax^3 + bx^2 + cx + d = 0$

$\alpha = \frac{5}{4}, \beta = -\frac{3}{2}$ and $\gamma = \frac{1}{2}$

$\alpha + \beta + \gamma = \frac{5}{4} + \left(-\frac{3}{2}\right) + \frac{1}{2} = \frac{1}{4} = -\frac{b}{a}$

$\alpha\beta + \beta\gamma + \gamma\alpha = \left(\frac{5}{4}\right)\left(-\frac{3}{2}\right) + \left(-\frac{3}{2}\right)\left(\frac{1}{2}\right) + \left(\frac{1}{2}\right)\left(\frac{5}{4}\right)$

$= -\frac{15}{8} - \frac{3}{4} + \frac{5}{8}$

$= -\frac{15}{8} - \frac{6}{8} + \frac{5}{8}$

$= -\frac{16}{8} = -2 = \frac{c}{a}$

$\alpha\beta\gamma = \left(\frac{5}{4}\right)\left(-\frac{3}{2}\right)\left(\frac{1}{2}\right) = -\frac{15}{16} = -\frac{d}{a}$

$ax^3 + bx^2 + cx + d = 0 \Rightarrow$

$x^3 + \frac{b}{a}x^2 + \frac{c}{a}x + \frac{d}{a} = 0$

$-\frac{b}{a} = \frac{1}{4} \Rightarrow \frac{b}{a} = -\frac{1}{4}$

$\frac{c}{a} = -2$

$-\frac{d}{a} = -\frac{15}{16} = \frac{d}{a} = \frac{15}{16}$

So equation is $x^3 - \frac{1}{4}x^2 - 2x + \frac{15}{16} = 0$

Multiply by 16 for integer coefficients gives $16x^2 - 4x^2 - 32x + 15 = 0,$ with $a = 16, b = -4, c = -32$ and $d = 15.$

7 a $16x^3 - kx^2 + 1 = 0$

Here $a = 16, b = -k, c = 0$ and $d = 1.$

$\alpha\beta + \beta\gamma + \gamma\alpha = \frac{c}{a} = 0$

$\alpha\beta\gamma = -\frac{d}{a} = -\frac{1}{16}$

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7 b i $\alpha = \beta \Rightarrow \alpha\beta + \beta\gamma + \gamma\alpha = \alpha^2 + 2\alpha\gamma$

$\alpha = \beta \Rightarrow a\beta\gamma = \alpha^2\gamma$

So $\alpha^2 + 2\alpha\gamma = 0$ and $a^2\gamma = -\frac{1}{16}$

So $\gamma = -\frac{1}{16\alpha^2}$

Substitute $\gamma = -\frac{1}{16\alpha^2}$ into $\alpha^2 + 2\alpha\gamma = 0$ giving:

$\alpha^2 + 2\alpha\left(-\frac{1}{16\alpha^2}\right) = 0$

$\alpha^2 - \frac{1}{8\alpha} = 0$

$8\alpha^3 - 1 = 0$

$\alpha^3 = \frac{1}{8}$

$\alpha = \frac{1}{2}$

If $\alpha = \frac{1}{2}, \gamma = -\frac{1}{16\left(\frac{1}{2}\right)^2} = -\frac{1}{4}$

If $\alpha = \frac{1}{2}, \gamma = -\frac{1}{4}, \alpha\beta\gamma = -\frac{1}{16} \Rightarrow \beta = \frac{1}{2}$.

ii $\alpha + \beta + \gamma = \frac{1}{2} + \frac{1}{2} - \frac{1}{4} = \frac{3}{4} = -\frac{b}{a}$

$a = 16$ and $b = -k \Rightarrow \frac{k}{16} = \frac{3}{4}$, so

$k = 12$.

8 a $2x^3 - kx^2 + 30x - 13 = 0$

Here $a = 2, b = -k, c = 30$ and $d = -13$.

$\alpha\beta + \beta\gamma + \gamma\alpha = \frac{c}{a} = 15$

$\alpha\beta\gamma = -\frac{d}{a} = \frac{13}{2}$

$\alpha + \beta + \gamma = -\frac{b}{a} \Rightarrow \alpha + \beta + \gamma = \frac{k}{2},$

so $k = 2(\alpha + \beta + \gamma)$

b $\alpha = 2 - 3\text{i} \Rightarrow \beta = 2 + 3\text{i}$

$\alpha\beta\gamma = (2 - 3\text{i})(2 + 3\text{i})\gamma$

$= (4 + 6\text{i} - 6\text{i} - 9\text{i}^2)\gamma$

$= 13\gamma$

But $\alpha\beta\gamma = \frac{13}{2}$, so $\gamma = \frac{1}{2}$.

b We know $k = 2(\alpha + \beta + \gamma)$

$k = 2\left((2 - 3\text{i}) + (2 + 3\text{i}) + \frac{1}{2}\right)$

$k = 2\left(\frac{9}{2}\right) = 9.$

9 a $x^3 - mx + n = 0$

Here $a = 1, b = 0, c = -m$ and $d = n$.

Also $\alpha = \alpha, \beta = 1$ and $\gamma = -4$.

α must be real as if α was complex then α^* would also be a complex root and there are already 2 other roots. A cubic has 3 roots.

b $\alpha + \beta + \gamma = \alpha + 1 - 4 = \alpha - 3$

But, $\alpha - 3 = -\frac{b}{a} \Rightarrow \alpha - 3 = 0 \Rightarrow \alpha = 3$

$\alpha\beta + \beta\gamma + \gamma\alpha = (3)(1) + (1)(-4) + (-4)(3)$

$= -13$

But, $\alpha\beta + \beta\gamma + \gamma\alpha = \frac{c}{a} = -\frac{m}{1} = -m$

So, $-m = -13 \Rightarrow m = 13$.

$\alpha\beta\gamma = (3)(1)(-4) = -12$

But $\alpha\beta\gamma = -\frac{a}{a} = -\frac{n}{1} = -n$

So, $-n = -12 \Rightarrow n = 12$

Therefore $m = 13, n = 12$ and $\alpha = 3$.

10 a $2x^3 - 10x^2 + 8x - k = 0$

Here $a = 2, b = -10, c = 8$ and $d = -k$.

One root is $x = 3 - \text{i}$, so another root is $3 + i$.

$\alpha + \beta + \gamma = (3 - \text{i}) + (3 + \text{i}) + \gamma = 6 + \gamma$

But $\alpha + \beta + \gamma = -\frac{b}{a} = \frac{10}{2} = 5$

So $6 + \gamma = 5 \Rightarrow \gamma = -1$

Therefore the roots are $3 - \text{i}, 3 + \text{i}$ and -1.

b $\alpha\beta\gamma = (3 - \text{i})(3 + \text{i})(-1)$

$= (9 + 3\text{i} - 3\text{i} - \text{i}^2)(-1)$

$= (10)(-1)$

$= -10$

But $\alpha\beta\gamma = -\frac{d}{a} = \frac{k}{2}$

So $\frac{k}{2} = -10 \Rightarrow k = -20$.

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11 $x^3 - 14x^2 + 56x - 64 = 0$

Here $a = 1, b = -14, c = 56$ and $d = -64$

The roots are $\alpha = \alpha, \beta = k\alpha$ and $\gamma = k^2\alpha$

$\alpha + \beta + \gamma = \alpha + k\alpha + k^2\alpha$

$= \alpha\left(k^2 + k + 1\right)$

But $\alpha + \beta + \gamma = -\frac{b}{a} = \frac{14}{1} = 14$

So $\alpha\left(k^2 + k + 1\right) = 14$

$\alpha\beta + \beta\gamma + \gamma\alpha$

$= (\alpha)(k\alpha) + (k\alpha)\left(k^2\alpha\right) + \left(k^2\alpha\right)(\alpha)$

$= k\alpha^2 + k^3\alpha^2 + k^2\alpha^2$

$= \alpha^2 k\left(1 + k + k^2\right)$

But $\alpha\beta + \beta\gamma + \gamma\alpha = \frac{c}{a} = 56$

So $\alpha^2 k\left(1 + k + k^2\right) = 56$

$\alpha\beta\gamma = (\alpha)(k\alpha)\left(k^2\alpha\right) = k^3\alpha^3$

But $\alpha\beta\gamma = -\frac{d}{a} = 64$

So $k^3\alpha^3 = 64 = 4^3$

To satisfy both equations $\alpha k = 4$

The possible solutions are therefore:

$\alpha = 2, k = 2$

or

$\alpha = 8, k = \frac{1}{2}$

12 $8x^3 + 12x^2 - cx + d = 0$

Here $a = 8, b = 12, c = -c$ and $d = d$.

The roots are $\alpha = \alpha, \beta = \frac{\alpha}{2}$ and $\gamma = \alpha - 4$.

$\alpha + \beta + \gamma = \alpha + \frac{\alpha}{2} + \alpha - 4 = \frac{5\alpha}{2} - 4$

But $\alpha + \beta + \gamma = -\frac{b}{a} = -\frac{12}{8} = -\frac{3}{2}$

So $\frac{5\alpha}{2} - 4 = -\frac{3}{2}$

$\frac{5\alpha}{2} = \frac{5}{2}$

$\alpha = 1$

Therefore the roots are $\alpha = 1, \beta = \frac{1}{2}$

and $\gamma = -3$.

$\alpha\beta + \beta\gamma + \gamma\alpha = (1)\left(\frac{1}{2}\right) + \left(\frac{1}{2}\right)(-3) + (-3)(1)$

$= \frac{1}{2} - \frac{3}{2} - 3 = -4$

12 But $\alpha\beta + \beta\gamma + \gamma\alpha = \frac{c}{a} = -\frac{c}{8}$

So $-\frac{c}{8} = -4 \Rightarrow c = 32.$

$\alpha\beta\gamma = (1)\left(\frac{1}{2}\right)(-3) = -\frac{3}{2}$

But $\alpha\beta\gamma = -\frac{d}{a} = -\frac{d}{8}$

So $-\frac{d}{8} = -\frac{3}{2} \Rightarrow d = 12.$

13 $2x^3 + 48x^2 + cx + d = 0$

Here $a = 2, b = 48, c = c$ and $d = d$.

The roots are $\alpha = \alpha, \beta = 2\alpha$ and $\gamma = 3\alpha$.

$\alpha + \beta + \gamma = \alpha + 2\alpha + 3\alpha = 6\alpha$

But $\alpha + \beta + \gamma = -\frac{b}{a} = -\frac{48}{2} = -24$

So $6\alpha = -24 \Rightarrow \alpha = -4.$

Therefore the roots are $\alpha = -4, \beta = -8$

and $\gamma = -12$.

$\alpha\beta + \beta\gamma + \gamma\alpha$

$= (-4)(-8) + (-8)(-12) + (-12)(-4)$

$= 32 + 96 + 48$

$= 176$

But $\alpha\beta + \beta\gamma + \gamma\alpha = \frac{c}{a} = \frac{c}{2}$

So $\frac{c}{2} = 176 \Rightarrow c = 352.$

$\alpha\beta\gamma = (-4)(-8)(-12) = -384$

But $\alpha\beta\gamma = -\frac{d}{a} = -\frac{d}{2}$

So $-\frac{d}{2} = -384 \Rightarrow d = 768.$

Challenge

$ax^2 + bx^2 + cx + d = 0$

Roots are α, β and γ.

If α, β and γ are all real, then

$a\beta + \beta\gamma + \gamma\alpha$ and $\alpha\beta\gamma$

must clearly all be real.

So assume $\alpha = a + b\text{i}$

Let $\beta = a - b\text{i},$ the complex conjugate

γ must be real as there can be a maximum of 3 roots.

Let $\alpha = a + b\text{i}, \beta = a - b\text{i}$ and $\gamma = r,$ where $r \in R$

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Challenge

a $\alpha+\beta+\gamma=(a+b\mathrm{i})+(a-b\mathrm{i})+r$
$=2a+r\in IR$

b $\alpha\beta+\beta\gamma+\gamma\alpha$
$=(a+b\mathrm{i})(a-b\mathrm{i})+(a-b\mathrm{i})r+r(a+b\mathrm{i})$
$=a^2-ab\mathrm{i}+ab\mathrm{i}-b^2\mathrm{i}^2+ra-rb\mathrm{i}+ra+rb\mathrm{i}$
$=a^2+b^2+2ar\in R$

c $\alpha\beta\gamma=(a+b\mathrm{i})(a-b\mathrm{i})r$
$=(a^2+b^2)r\in R.$

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Roots of polynomials 4C

1 a $4x^4+3x^3+2x^2-5x+4=0$
$a=4, b=3, c=2, d=-5$ and $e=4$
$\alpha+\beta+\gamma+\delta=-\frac{b}{a}=-\frac{3}{4}$

b $\alpha\beta+\alpha\gamma+\alpha\delta+\beta\gamma+\beta\delta+\gamma\delta=\frac{c}{a}=\frac{2}{4}=\frac{1}{2}$

c $\alpha\beta\gamma+\alpha\beta\delta+\alpha\gamma\delta+\beta\gamma\delta=-\frac{d}{a}=\frac{5}{4}$

d $\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma}+\frac{1}{\delta}=\frac{\alpha\beta\gamma+\alpha\beta\delta+\alpha\gamma\delta+\beta\gamma\delta}{\alpha\beta\gamma\delta}$
$=\frac{\left(\frac{5}{4}\right)}{\left(\frac{e}{a}\right)}$
$=\frac{\left(\frac{5}{4}\right)}{\left(\frac{4}{4}\right)}$
$=\frac{5}{4}$

2 a $2x^4+4x^3-3x^2-x+2=0$
Here $a=2, b=4, c=-3. d=-1$ and $e=2$
$\alpha+\beta+\gamma+\delta=-\frac{b}{a}=-\frac{4}{2}=-2$

b $\alpha\beta+\alpha\gamma+\alpha\delta+\beta\gamma+\beta\delta+\gamma\delta=\frac{c}{a}=-\frac{3}{2}$

c $\alpha\beta\gamma+\alpha\beta\delta+\alpha\gamma\delta+\beta\gamma\delta=-\frac{d}{a}=\frac{1}{2}$

d $\alpha\beta\gamma\delta=\frac{e}{a}=\frac{2}{2}=1$

e $\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma}+\frac{1}{\delta}=\frac{\alpha\beta\gamma+\alpha\beta\delta+\alpha\gamma\delta+\beta\gamma\delta}{\alpha\beta\gamma\delta}$
$=\frac{\frac{1}{2}}{1}$
$=\frac{1}{2}$

3 a $x^4+3x^3+2x^2-x+4=0$
Here $a=1, b=3, c=2, d=-1$ and $e=4$
$\alpha+\beta+\gamma+\delta=-\frac{b}{a}=-3$

b $\alpha\beta+\alpha\gamma+\alpha\delta+\beta\gamma+\beta\delta+\gamma\delta=\frac{c}{a}=\frac{2}{1}=2$

c $\alpha\beta\gamma+\alpha\beta\delta+\alpha\gamma\delta+\beta\gamma\delta=-\frac{d}{a}=\frac{1}{1}=1$

d $\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma}+\frac{1}{\delta}=\frac{\alpha\beta\gamma+\alpha\beta\delta+\alpha\gamma\delta+\beta\gamma\delta}{\alpha\beta\gamma\delta}$
$=\frac{1}{\left(\frac{e}{1}\right)}$
$=\frac{1}{\left(\frac{4}{1}\right)}$
$=\frac{1}{4}$

e $a^2\beta^2\gamma^2\delta^2=(\alpha\beta\gamma\delta)^2=\left(\frac{e}{a}\right)^2=4^2=16$

4 a $7x^4+6x^3-5x^2+4x+3=0$
Here $a=7, b=6, c=-5, d=4$ and $e=3$.
$\alpha+\beta+\gamma+\delta=-\frac{b}{a}=-\frac{6}{7}$

b $\alpha\beta+\alpha\gamma+\alpha\delta+\beta\gamma+\beta\delta+\gamma\delta=\frac{c}{a}=-\frac{5}{7}$

c $\alpha\beta\gamma+\alpha\beta\delta+\alpha\gamma\delta+\beta\gamma\delta=-\frac{d}{a}=-\frac{4}{7}$

d $\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma}+\frac{1}{\delta}=\frac{\alpha\beta\gamma+\alpha\beta\delta+\alpha\gamma\delta+\beta\gamma\delta}{\alpha\beta\gamma\delta}$
$=\frac{\left(-\frac{4}{7}\right)}{\left(\frac{e}{a}\right)}$
$=\frac{\left(-\frac{4}{7}\right)}{\left(\frac{3}{7}\right)}$
$=-\frac{4}{3}$

e $\alpha^3\beta^3\gamma^3\delta^3=(\alpha\beta\gamma\delta)^3=\left(\frac{3}{7}\right)^3=\frac{27}{343}$

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5 $ax^4+bx^3+cx^2+dx+e=0$

The roots are $\alpha=-\frac{3}{2},\beta=-\frac{1}{2},\gamma=-2$ and $\delta=\frac{2}{3}$.

$\alpha+\beta+\gamma+\delta=-\frac{3}{2}-\frac{1}{2}-2+\frac{2}{3}=-\frac{10}{3}=-\frac{b}{a}$

$\alpha\beta+\alpha\gamma+\alpha\delta+\beta\gamma+\beta\delta+\gamma\delta$

$=\left(-\frac{3}{2}\right)\left(-\frac{1}{2}\right)+\left(-\frac{3}{2}\right)(-2)+\left(-\frac{3}{2}\right)\left(\frac{2}{3}\right)$

$+\left(-\frac{1}{2}\right)(-2)+\left(-\frac{1}{2}\right)\left(\frac{2}{3}\right)+(-2)\left(\frac{2}{3}\right)$

$=\frac{3}{4}+3-1+1-\frac{1}{3}-\frac{4}{3}$

$=\frac{25}{12}=\frac{c}{a}$

$\alpha\beta\gamma+\alpha\beta\delta+\alpha\gamma\delta+\beta\gamma\delta$

$=\left(-\frac{3}{2}\right)\left(-\frac{1}{2}\right)(-2)+\left(-\frac{3}{2}\right)\left(-\frac{1}{2}\right)\left(\frac{2}{3}\right)$

$+\left(-\frac{3}{2}\right)(-2)\left(\frac{2}{3}\right)+\left(-\frac{1}{2}\right)(-2)\left(\frac{2}{3}\right)$

$=-\frac{3}{2}+\frac{1}{2}+2+\frac{2}{3}$

$=\frac{5}{3}=-\frac{d}{a}$

$\alpha\beta\gamma\delta=\left(-\frac{3}{2}\right)\left(-\frac{1}{2}\right)(-2)\left(\frac{2}{3}\right)=-1=\frac{e}{a}$

If $ax^4+bx^3+cx^2+dx+e=0,$ then

$x^4+\frac{b}{a}x^3+\frac{c}{a}x^2+\frac{d}{a}x+e=0.$

$-\frac{b}{a}=-\frac{10}{3}\Rightarrow\frac{b}{a}=\frac{10}{3}$

$\frac{c}{a}=\frac{25}{12}$

$-\frac{d}{a}=\frac{5}{3}\Rightarrow\frac{d}{a}=-\frac{5}{3}$

$\frac{e}{a}=-1$

So $x^4+\frac{10}{3}x^3+\frac{25}{12}x^2-\frac{5}{3}x-1=0.$

Multiply by 12 for integer coefficients

$12x^4+40x^3+25x^2-20x-12=0$

So $a=12,b=40,c=25,d=-20$ and $e=-12.$

6 $ax^4+bx^3+cx^2+dx+e=0$

The roots are $\alpha=-\frac{1}{2},\beta=\frac{1}{3},\gamma=1+\mathrm{i}$ and $\delta=1-\mathrm{i}.$

$\alpha+\beta+\gamma+\delta$

$=-\frac{1}{2}+\frac{1}{3}+(1+\mathrm{i})+(1-\mathrm{i})$

$=\frac{11}{6}=-\frac{b}{a}$

$\alpha\beta+\alpha1\gamma+\alpha\delta+\beta\gamma+\beta\delta+\gamma\delta$

$=\left(-\frac{1}{2}\right)\left(\frac{1}{2}\right)+\left(-\frac{1}{2}\right)(1+\mathrm{i})+\left(-\frac{1}{2}\right)(1-\mathrm{i})$

$+\left(\frac{1}{3}\right)(1+\mathrm{i})+\frac{1}{3}(1-\mathrm{i})+(1+\mathrm{i})(1-\mathrm{i})$

$=-\frac{1}{6}-\frac{1}{2}-\frac{1}{2}\mathrm{i}-\frac{1}{2}+\frac{1}{2}\mathrm{i}+\frac{1}{3}$

$+\frac{1}{3}\mathrm{i}+\frac{1}{3}-\frac{1}{3}\mathrm{i}+1-\mathrm{i}+\mathrm{i}-\mathrm{i}^2$

$=-\frac{1}{6}-\frac{1}{2}-\frac{1}{2}+\frac{1}{3}+\frac{1}{3}+1+1=\frac{3}{2}=\frac{c}{a}$

$\alpha\beta\gamma+\alpha\beta\delta+\alpha\gamma\delta+\beta\gamma\delta$

$=\left(-\frac{1}{2}\right)\left(\frac{1}{3}\right)(1+\mathrm{i})+\left(-\frac{1}{2}\right)\left(\frac{1}{3}\right)(1-\mathrm{i})$

$+\left(-\frac{1}{2}\right)(1+\mathrm{i})(1-\mathrm{i})+\left(\frac{1}{3}\right)(1+\mathrm{i})(1-\mathrm{i})$

$=-\frac{1}{6}-\frac{1}{6}\mathrm{i}-\frac{1}{6}+\frac{1}{6}\mathrm{i}-\frac{1}{2}(2)+\frac{1}{3}(2)$

$=-\frac{1}{3}-1+\frac{2}{3}=-\frac{2}{3}=-\frac{d}{a}$

$\alpha\beta\gamma\delta=\left(-\frac{1}{2}\right)\left(\frac{1}{3}\right)(1+\mathrm{i})(1-\mathrm{i})$

$=\left(-\frac{1}{6}\right)(2)$

$=-\frac{1}{3}=\frac{e}{a}$

If $ax^4+bx^3+cx^2+dx+e=0,$

the $x^4+\frac{b}{a}x^3+\frac{c}{a}x^2+\frac{d}{a}x+\frac{e}{a}=0$

$-\frac{b}{a}=\frac{11}{6}\Rightarrow\frac{b}{a}=-\frac{11}{6}$

$\frac{c}{a}=\frac{3}{2}$

$-\frac{d}{a}=-\frac{2}{3}\Rightarrow\frac{d}{a}=\frac{2}{3}$

$\frac{e}{a}=-\frac{1}{3}$

So $x^4-\frac{11}{6}x^3+\frac{3}{2}x^2+\frac{2}{3}x-\frac{1}{3}=0$

Multiply by 6 for integer coefficients

$6x^4-11x^3+9x^2+4x-2=0$

So $a=6,b=-11,c=9,d=4$ and $e=-2.$

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7 $ax^4+bx^3+cx^2+dx+e=0$

$\sum\alpha=\frac{17}{12}\Rightarrow-\frac{b}{a}=\frac{17}{12}\Rightarrow\frac{b}{a}=-\frac{17}{12}$

$\sum\alpha\beta=-\frac{25}{72}\Rightarrow\frac{c}{a}=-\frac{25}{72}$

$\sum\alpha\beta\gamma=-\frac{53}{72}\Rightarrow-\frac{d}{a}=-\frac{53}{72}\Rightarrow\frac{d}{a}=\frac{53}{72}$

$\alpha\beta\gamma\delta=-\frac{1}{6}\Rightarrow\frac{e}{a}=-\frac{1}{6}.$

If $ax^4+bx^3+cx^2+dx+e=0,$

then $x^4+\frac{b}{a}x^3+\frac{c}{a}x^2+\frac{d}{a}x+e=0$

So $x^4-\frac{17}{12}x^3-\frac{25}{72}x^2+\frac{53}{72}x-\frac{1}{6}=0$

Multiply by 72 for integer coefficients:

$72x^4-102x^3-25x^2+53x-12=0.$

So $a=72, b=-102, c=-25, d=53$ and $e=-12.$

8 $x^4-16x^3+86x^2-176x+105=0$

$a=1, b=-16, c=86, d=-176$ and $e=105.$

The roots are $\alpha=\alpha, \beta=\alpha+k, \gamma=\alpha+2k$ and $\delta=\alpha+3k.$

$\alpha+\beta+\gamma+\delta$

$=\alpha+(\alpha+k)+(\alpha+2k)+(\alpha+3k)$

$=4\alpha+6k$

But $\alpha+\beta+\gamma+\delta=-\frac{b}{a}=16$

So $4\alpha+6k=16\Rightarrow 2\alpha+3k=8.$

$\alpha\beta+\alpha\gamma+\alpha\delta+\beta\gamma+\beta\delta+\gamma\delta$

$=\alpha(\alpha+k)+\alpha(\alpha+2k)+\alpha(\alpha+3k)$

$+(\alpha+k)(\alpha+2k)+(\alpha+k)(\alpha+3k)$

$+(\alpha+2k)(\alpha+3k)$

$=\alpha^2+k\alpha+\alpha^2+2k\alpha+\alpha^2+3k\alpha+\alpha^2$

$+3k\alpha+2k^2+\alpha^2+4k\alpha$

$+3k^2+\alpha^2+5k\alpha+6k^2$

$=6\alpha^2+18k\alpha+11k^2$

But $\alpha\beta+\alpha\gamma+\alpha\delta+\beta\gamma+\beta\delta+\gamma\delta=\frac{c}{a}=86$

So $6\alpha^2+18k\alpha+11k^2=86$

If $2\alpha+3k=8\Rightarrow\alpha=\frac{8}{2}-\frac{3}{2}k=4-\frac{3}{2}k.$

Substitute $\alpha=4-\frac{3}{2}k$ into

$6\alpha^2+18k\alpha+11k^2=86.$

$6\left(4-\frac{3}{2}k\right)^2+18k\left(4-\frac{3}{2}k\right)+11k^2=86$

$6\left(16-12k+\frac{9}{4}k^2\right)+72k-27k^2+11k^2=86$

$96-72k+\frac{27}{2}k^2+72k-27k^2+11k^2=86$

$-\frac{5}{2}k^2=-10$

$k^2=4$

$k=\pm2$

If $k=2,$ then $\alpha=4-\frac{3}{2}k\Rightarrow\alpha=1.$

So the roots are $1, 1+2, 1+2(2)$ and $1+3(2)$

Hence the roots are $1, 3, 5$ and $7.$

Note if $k=-2,$ then $\alpha=7,$ which would yield the same roots.

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9 $3072x^4 - 2880x^3 + 840x^2 - 90x + 3 = 0$

Here $a = 3072, b = -2880, c = 840, d = -90$ and $e = 3$.

The roots are $\alpha = \alpha, \beta = r\alpha, \gamma = r^2\alpha$ and $\delta = r^3\alpha$

$\alpha + \beta + \gamma + \delta = \alpha + r\alpha + r^2\alpha + r^3\alpha$

$= \alpha\left(1 + r + r^2 + r^3\right)$

But $\alpha + \beta + \gamma + \delta = -\frac{b}{a} = \frac{2880}{3072} = \frac{15}{16}$

So $\alpha\left(1 + r + r^2 + r^3\right) = \frac{15}{16}$

$\alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta$

$= \alpha(r\alpha) + \alpha\left(r^2\alpha\right) + \alpha\left(r^3\alpha\right) +$

$(r\alpha)\left(r^2\alpha\right) + (r\alpha)\left(r^3\alpha\right) + \left(r^2\alpha\right)\left(r^3\alpha\right)$

$= \alpha^2\left(r + r^2 + r^3 + r^3 + r^4 + r^5\right)$

$= \alpha^2 r\left(1 + r + 2r^2 + r^3 + r^4\right)$

But $\alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta =$

$\frac{c}{a} = \frac{840}{3072} = \frac{35}{128}$

So $\alpha^2 r\left(1 + r + 2r^2 + r^3 + r^4\right) = \frac{35}{128}$

$\alpha\beta\gamma + \alpha\beta\delta + \alpha\gamma\delta + \beta\gamma\delta$

$= (r\alpha)\left(r^2\alpha\right) + \alpha(r\alpha)\left(r^3\alpha\right) +$

$\alpha\left(r^2\alpha\right)\left(r^3\alpha\right) + (r\alpha)\left(r^2\alpha\right)\left(r^3\alpha\right)$

$= r^3\alpha^3 + r^4\alpha^3 + r^5\alpha^3 + r^6\alpha^3$

$\alpha^3 r^3\left(1 + r + r^2 + r^3\right)$

But

$\alpha\beta\gamma + \alpha\beta\delta + \alpha\gamma\delta + \beta\gamma\delta = -\frac{d}{a} = \frac{90}{3072} = \frac{15}{512}$

So $\alpha^3 r^3\left(1 + r + r^2 + r^3\right) = \frac{15}{512}$

$\alpha\beta\gamma\delta = \alpha(r\alpha)\left(r^2\alpha\right)\left(r^3\alpha\right) = \alpha^4 r^6$

But $\alpha\beta\gamma\delta = \frac{e}{a} = \frac{3}{3072} = \frac{1}{1024}$

So $\alpha^4 r^6 = \frac{1}{1024}$

Taking square root of both sides

$\alpha^2 r^3 = \frac{1}{32} = \frac{1}{2^5}$

So $\alpha = \frac{1}{2}$ and $r = \frac{1}{2}$ are possible roots.

9 Check by substituting into $\alpha\left(1 + r + r^2 + r^3\right) = \frac{15}{16}$

$\frac{1}{2}\left(1 + \frac{1}{2} + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^3\right)$

$= \frac{1}{2}\left(1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8}\right)$

$= \frac{1}{2}\left(1 + \frac{4}{8} + \frac{2}{8} + \frac{1}{8}\right)$

$= \frac{1}{2}\left(\frac{15}{8}\right) = \frac{15}{16}$

So $\alpha = \frac{1}{2}$ and $r = \frac{1}{2}$

Substituting these values in the root expressions then gives $x = \frac{1}{2}, \frac{1}{4}, \frac{1}{8}$ and $\frac{1}{16}$.

10 a $40x^3 + 90x^3 - 115x^2 + mx + n = 0$

So $a = 40, b = 90, c = -115, d = m$ and $e = n$.

The roots are $\alpha = 1, \beta = -3, \gamma = \frac{1}{2}$ and $\delta = \delta$

$\alpha + \beta + \gamma + \delta = 1 - 3 + \frac{1}{2} + \delta = -\frac{3}{2} + \delta$

But $\alpha + \beta + \gamma + \delta = -\frac{b}{a} = -\frac{90}{40} = -\frac{9}{4}$

So $-\frac{3}{2} + \delta = -\frac{9}{4}$

$\delta = -\frac{9}{4} + \frac{6}{4}$

$\delta = -\frac{3}{4}$

b $\alpha\beta\gamma + \alpha\beta\delta + \alpha\gamma\delta + \beta\gamma\delta$

$= (1)(-3)\left(\frac{1}{2}\right) + (1)(-3)\left(-\frac{3}{4}\right)$

$+ (1)\left(\frac{1}{2}\right)\left(-\frac{3}{4}\right) + (-3)\left(\frac{1}{2}\right)\left(-\frac{3}{4}\right)$

$= -\frac{3}{2} + \frac{9}{4} - \frac{2}{8} + \frac{9}{8}$

$= -\frac{12}{8} + \frac{18}{8} - \frac{3}{8} + \frac{9}{8}$

$= \frac{12}{8} = \frac{3}{2}$

But $\alpha\beta\gamma + \alpha\beta\delta + \alpha\gamma\delta + \beta\gamma\delta = -\frac{d}{a} = -\frac{m}{40}$

So $-\frac{m}{40} = \frac{3}{2} \Rightarrow m = -60$.

$\alpha\beta\gamma\delta = (1)(-3)\left(\frac{1}{2}\right)\left(-\frac{3}{4}\right) = \frac{9}{8}$

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10 b But $\alpha\beta\gamma\delta = \frac{e}{a} = \frac{n}{40}$

So $\frac{n}{40} = \frac{9}{8} \Rightarrow n = 45$

11 a $2x^4 - 34x^3 + 202x^2 + dx + e = 0$

So $a = 2, b = -34, c = 202, d = d$ and $e = e$.

The roots are $\alpha = \alpha, \beta = \alpha + 1, \gamma = 2\alpha + 1$ and $\delta = 3\alpha + 1$.

$\alpha + \beta + \gamma + \delta$

$= \alpha + (\alpha + 1) + (2\alpha + 1) + (3\alpha + 1)$

$= 7\alpha + 3$

But $\alpha + \beta + \gamma + \delta = -\frac{b}{a} = \frac{34}{2} = 17$

So $7\alpha + 3 = 17$

$7\alpha = 14$

$\alpha = 2$

b So the roots are $\alpha = 2, \beta = 3, \gamma = 5$ and $\delta = 7$.

$\alpha\beta\gamma + \alpha\beta\delta + \alpha\gamma\delta + \beta\gamma\delta$

$= (2)(3)(5) + (2)(3)(7)$

$+ (2)(5)(7) + (3)(5)(7)$

$= 30 + 42 + 70 + 105 = 247.$

But $\alpha\beta\gamma + \alpha\beta\delta + \alpha\gamma\delta + \beta\gamma\delta = -\frac{d}{a} = -\frac{d}{2}$

So $-\frac{d}{2} = 247 \Rightarrow d = -494$

$\alpha\beta\gamma\delta = (2)(3)(5)(7) = 210$

But $\alpha\beta\gamma\delta = \frac{e}{a} = \frac{e}{2}$.

So $\frac{e}{2} = 210 \Rightarrow e = 420.$

12 a $4x^4 - 19x^3 + px^2 + qx + 10 = 0$

So $a = 4, b = -19, c = p, d = q$ and $e = 10$.

If $\gamma = 3 + \mathrm{i}$ is a root then the complex conjugate is also a root. So $\delta = 3 - \mathrm{i}$

The roots are $\alpha, \beta, \gamma = 3 + \mathrm{i}$ and $\delta = 3 - \mathrm{i}$

$\alpha + \beta + \gamma + \delta$

$= \alpha + \beta + (3 + \mathrm{i}) + (3 - \mathrm{i})$

$= \alpha + \beta + 6.$

But $\alpha + \beta + \gamma + \delta = -\frac{b}{a} = \frac{19}{4}$

So, $\alpha + \beta + 6 = \frac{19}{4}$

$4\alpha + 4\beta + 24 = 19$

$4\alpha + 4\beta + 5 = 0$

12 a $\alpha\beta\gamma\delta = \alpha\beta(3 + \mathrm{i})(3 - \mathrm{i}) = 10\alpha\beta$

But $\alpha\beta\gamma\delta = \frac{e}{a} = \frac{10}{4} = \frac{5}{2}$

So $10\alpha\beta = \frac{5}{2}$

$20\alpha\beta - 5 = 0$

$4\alpha\beta - 1 = 0$

b If $4\alpha\beta - 1 = 0$, then $4\alpha\beta = 1$

and $\alpha = \frac{1}{4\beta}$

Substitute $\alpha = \frac{1}{4\beta}$ into $4\alpha + 4\beta + 5 = 0$

$4\left(\frac{1}{4\beta}\right) + 4\beta + 5 = 0$

$\frac{1}{\beta} + 4\beta + 5 = 0$

Multiply by β

$1 + 4\beta^2 + 5\beta = 0$

So $4\beta^2 + 5\beta + 1 = 0$

$(4\beta + 1)(\beta + 1) = 0$

$\beta = -\frac{1}{4}$ or $\beta = -1$

If $\beta = -\frac{1}{4}, \alpha = \frac{1}{4\left(-\frac{1}{4}\right)} = -1$

If $\beta = -1, \alpha = \frac{1}{4(-1)} = -\frac{1}{4}.$

So the roots are $-\frac{1}{4}, -1, 3 + \mathrm{i}$ and $3 - \mathrm{i}$

$\alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta$

$= \left(-\frac{1}{4}\right)(-1) + \left(-\frac{1}{4}\right)(3 + \mathrm{i}) + \left(-\frac{1}{4}\right)(3 - \mathrm{i})$

$+ (-1)(3 + \mathrm{i}) + (-1)(3 - \mathrm{i}) + (3 + \mathrm{i})(3 - \mathrm{i})$

$= \frac{1}{4} - \frac{3}{4} - \frac{3}{4} - 3 - 3 + 10$

$= \frac{11}{4}$

But $\alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta = \frac{c}{a} = \frac{p}{4}$

So, $\frac{p}{4} = \frac{11}{4} \Rightarrow p = 11$

$\alpha\beta\gamma + \alpha\beta\delta + \alpha\gamma\delta + \beta\gamma\delta$

$= \left(-\frac{1}{4}\right)(-1)(3 + \mathrm{i}) + \left(-\frac{1}{4}\right)(-1)(3 - \mathrm{i})$

$+ \left(-\frac{1}{4}\right)(3 + \mathrm{i})(3 - \mathrm{i}) + (-1)(3 + \mathrm{i})(3 - \mathrm{i})$

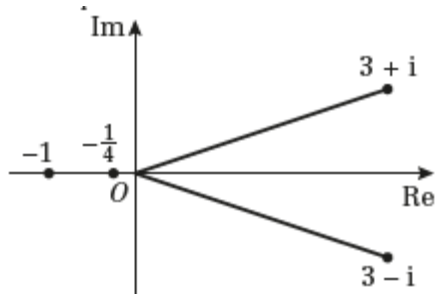
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12 b $= \frac{3}{4} + \frac{3}{4} + \left(-\frac{1}{4}\right)(10) + (-1)(10) = -11$

But $\alpha\beta\gamma + \alpha\beta\delta + \alpha\gamma\delta + \beta\gamma\delta = -\frac{d}{a} = -\frac{q}{4}$

$-\frac{q}{4} = -11 \Rightarrow q = 44$

c



13 a $6x^4 - 10x^3 + 3x^2 + 6x - -40 = 0$

So, $a = 6, b = -10, c = 3, d = 6$ and $e = -40$.

Substitute $x = \frac{1-3\text{i}}{2}$ into the equation

$$6\left(\frac{1-3\text{i}}{2}\right)^4 - 10\left(\frac{1-3\text{i}}{2}\right)^3 + 3\left(\frac{1-3\text{i}}{2}\right)^2 + 6\left(\frac{1-3\text{i}}{2}\right) - 40 = 0$$

$$\frac{6}{2^4}(1-3\text{i})^4 - \frac{10}{2^3}(1-3\text{i})^3 + \frac{3}{2^2}(1-3\text{i})^2 + 3(1-3\text{i}) - 40 = 0$$

$$\frac{3}{8}(1-3\text{i})^4 - \frac{5}{4}(1-3\text{i})^3 + \frac{3}{4}(1-3\text{i})^2 + 3 - 9\text{i} - 40 = 0$$

Expand $(1-3i)^4$ using $(a+b)^4$ with $a = 1$ and $b = -3\text{i}$

$(a+b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$

$(1-3\text{i})^4 = 1 + 4(-3\text{i}) + 6(-3\text{i})^2 + 4(-3\text{i})^3 + (-3\text{i})^4$

$(1-3\text{i})^4 = 1 - 12\text{i} + 54\text{i}^2 - 108\text{i}^3 + 81\text{i}^4$

Use $\text{i} = \text{i}, \text{i}^2 = -1, \text{i}^3 = -\text{i}$ and $\text{i}^4 = 1$ to simplify

$(1-3\text{i})^4 = 1 - 12\text{i} - 54 + 108\text{i} + 81$

$= 28 + 96\text{i}$

Expand $(1-3\text{i})^3$ using $(a+b)^3$ with $a = 1$ and $b = 3\text{i}$

13 a $(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$

$(1-3\text{i})^3 = 1 + 3(-3\text{i}) + 3(-3\text{i})^2 + (-3\text{i})^3$

$= 1 - 9\text{i} + 27\text{i}^2 - 27\text{i}^3$

$= 1 - 9\text{i} - 27 + 27\text{i}$

$= -26 + 18\text{i}$

$(1-3\text{i})^2 = 1 - 3\text{i} - 3\text{i} + 9\text{i}^2$

$= -8 - 6\text{i}$

Substituting

$$\frac{3}{8}(28+96\text{i}) - \frac{5}{4}(-26+18\text{i}) + \frac{3}{4}(-8-6\text{i}) + 3 - 9\text{i} - 40$$

$$= \frac{21}{2} + 36\text{i} + \frac{65}{2} - \frac{45}{2}\text{i} - 6 - \frac{9}{2}\text{i} + 3 - 9\text{i} - 40$$

$$= \left(\frac{21}{2} + \frac{65}{2} - 6 + 3 - 40\right) + \left(36 - \frac{45}{2} - \frac{9}{2} - 9\right)\text{i}$$

$= 0$

b If $x = \frac{1-3\text{i}}{2}$ is a root, then $x = \frac{1+3\text{i}}{2}$ is also a root.

$$\left(x - \left(\frac{1+3\text{i}}{2}\right)\right)\left(x - \left(\frac{1-3\text{i}}{2}\right)\right)$$

$$= x^2 - x\left(\frac{1-3\text{i}}{2}\right) - x\left(\frac{1+3\text{i}}{2}\right) + \left(\frac{1+3\text{i}}{2}\right)\left(\frac{1-3\text{i}}{2}\right)$$

$$= x^2 - \frac{1}{2}x - \frac{3}{2}x\text{i} - \frac{1}{2}x + \frac{3}{2}x\text{i} + \frac{1}{4} - \frac{3}{4}\text{i} + \frac{3}{4}\text{i} - \frac{9}{4}\text{i}^2$$

$= x^2 - x + \frac{5}{2}$

Multiply by 4:

$= 2x^2 - 2x + 5$

Find the other factors using division:

$$\begin{array}{r} 3x^2 - 2x - 8 \\ 2x^2 - 2x + 5 \overline{)\ 6x^4 - 10x^3 + 3x^2 + 6x - 40} \\ 6x^4 - 6x^3 + 15x^2 \\ -4x^3 - 12x^2 + 6x \\ -4x^3 + 4x^2 - 10x \\ -16x^2 + 16x - 40 \\ -16x^2 + 16x - 40 \\ 0 \end{array}$$

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13 b continued

So

$6x^4 - 10x^3 + 3x^2 + 6x - 40$

$= \left(2x^2 - 2x + 5\right)\left(3x^2 - 2x - 8\right) = 0$

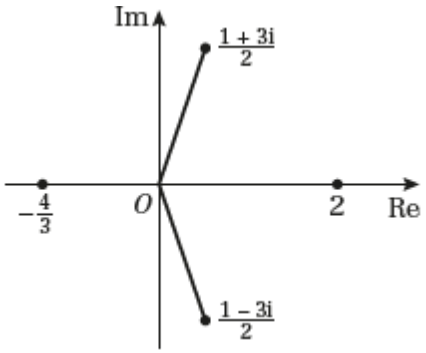
Let $\left(3x^2 - 2x - 8\right) = 0$

$\left(3x + 4\right)\left(x - 2\right) = 0$

$x = -\frac{4}{3}$ or $x = 2$

So the other roots are $x = -\frac{4}{3}$, $x = 2$ and $x = \frac{1+3\mathrm{i}}{2}$.

c



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Roots of polynomials 4D

1 a $\alpha+\beta=4,\ \alpha\beta=3$

$$\frac{1}{\alpha}+\frac{1}{\beta}=\frac{\alpha+\beta}{\alpha\beta}$$
$$=\frac{4}{3}$$

b $\alpha^2\beta^2=(\alpha\beta)^2=3^2=9$

c $\alpha^2+\beta^2=(\alpha+\beta)^2-2\alpha\beta$
$$=4^2-2(3)$$
$$=10$$

d $\alpha^3+\beta^3=(\alpha+\beta)^3-3\alpha\beta(\alpha+\beta)$
$$=4^3-3(3)(4)$$
$$=64-36$$
$$=28$$

2 a $\alpha+\beta=-\frac{2}{3},\alpha\beta=\frac{3}{4}$

$$\frac{1}{\alpha}+\frac{1}{\beta}=\frac{\alpha+\beta}{\alpha\beta}=\frac{-\frac{2}{3}}{\frac{3}{4}}=-\frac{8}{9}$$

b $\alpha^2\beta^2=(\alpha\beta)^2=\left(\frac{3}{4}\right)^2=\frac{9}{16}$

c $\alpha^2+\beta^2=(\alpha+\beta)^2-2\alpha\beta$
$$=\left(-\frac{2}{3}\right)^2-2\left(\frac{3}{4}\right)$$
$$=\frac{4}{9}-\frac{6}{4}=-\frac{38}{36}=-\frac{19}{18}$$

d $\alpha^3+\beta^3=(\alpha+\beta)^3-3\alpha\beta(\alpha+\beta)$
$$=\left(-\frac{2}{3}\right)^2-3\left(\frac{3}{4}\right)\left(-\frac{2}{3}\right)$$
$$=-\frac{8}{27}+\frac{3}{2}=\frac{65}{54}$$

3 a $\alpha+\beta=\frac{5}{4},\alpha\beta=-\frac{1}{3}$

$$(\alpha+2)(\beta+2)=\alpha\beta+2(\alpha+\beta)+4$$
$$=-\frac{1}{3}+2\left(\frac{5}{4}\right)+4$$
$$=\frac{37}{6}$$

3 b $(\alpha-4)(\beta-4)=\alpha\beta-4(\alpha+\beta)+16$
$$=-\frac{1}{3}-4\left(\frac{5}{4}\right)+16$$
$$=\frac{32}{3}$$

c $(\alpha^2+1)(\beta^2+1)=\alpha^2\beta^2+\alpha^2+\beta^2+1$

$\alpha^2\beta^2=(\alpha\beta)^2=\left(-\frac{1}{3}\right)^2=\frac{1}{9}$

$\alpha^2+\beta^2=(\alpha+\beta)^2-2\alpha\beta$
$$=\left(\frac{5}{4}\right)^2-2\left(-\frac{1}{3}\right)$$
$$=\frac{25}{16}+\frac{2}{3}=\frac{107}{48}$$

So

$$\alpha^2\beta^2+(\alpha^2+\beta^2)+1=\frac{1}{9}+\frac{107}{48}+1=\frac{481}{144}$$

4 a $\alpha+\beta+\gamma=2,\alpha\beta+\beta\gamma+\gamma\alpha=-3$ and $\alpha\beta\gamma=4$

$$\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma}=\frac{\alpha\beta+\beta\gamma+\gamma\alpha}{\alpha\beta\gamma}=-\frac{3}{4}$$

b $\alpha^2+\beta^2+\gamma^2$
$$=(\alpha+\beta+\gamma)^2-2(\alpha\beta+\beta\gamma-\gamma\alpha)$$
$$=2^2-2(-3)$$
$$=10$$

c $\alpha^3+\beta^3+\gamma^3$
$$=(\alpha+\beta+\gamma)^3$$
$$-3(\alpha+\beta+\gamma)(\alpha\beta+\beta\gamma+\gamma\alpha)$$
$$+3\alpha\beta\gamma$$
$$=2^3-3(2)(-3)+3(4)$$
$$=8+18+12=38$$

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4 d $(\alpha\beta+\beta\gamma+\gamma\alpha)^2$

$=(\alpha\beta+\beta\gamma+\gamma\alpha)(\alpha\beta+\beta\gamma+\gamma\alpha)$

$=(\alpha\beta)^2+\alpha\beta^2\gamma+\alpha^2\beta\gamma+\alpha\beta^2\gamma+(\beta\gamma)^2$

$+\alpha\beta\gamma^2+\alpha^2\beta\gamma+\alpha\beta\gamma^2+(\gamma\alpha)^2$

$=(\alpha\beta)^2+(\beta\gamma)^2+(\gamma\alpha)^2$

$+2(\alpha^2\beta\gamma+\alpha\beta^2\gamma+\alpha\beta\gamma^2)$

$=(\alpha\beta)^2+(\beta\gamma)^2+(\gamma\alpha)^2$

$+2\alpha\beta\gamma(\alpha+\beta+\gamma)$

Hence:

$(\alpha\beta+\beta\gamma+\gamma\alpha)^2$

$=(\alpha\beta)^2+(\beta\gamma)^2+(\gamma\alpha)^2$

$+2\alpha\beta\gamma(\alpha+\beta+\gamma)$

So:

$(\alpha\beta)^2+(\beta\gamma)^2+(\gamma\alpha)^2$

$=(\alpha\beta+\beta\gamma+\gamma\alpha)^2$

$-2\alpha\beta\gamma(\alpha+\beta+\gamma)$

$=(-3)^2-2(4)(2)$

$=9-16$

$=-7$

5 a $\sum\alpha=\frac{3}{2},\sum\alpha\beta=-\frac{4}{3}$ and $\alpha\beta\gamma=\frac{1}{2}$

$\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma}=\frac{\sum\alpha\beta}{\alpha\beta\gamma}=\frac{\left(-\frac{4}{3}\right)}{\left(\frac{1}{2}\right)}=-\frac{8}{3}$

b

$\alpha^2+\beta^2+\gamma^2=(\alpha+\beta+\gamma)^2-$

$2(\alpha\beta+\beta\gamma+\gamma\alpha)$

$=\left(\sum\alpha\right)^2-2\left(\sum\alpha\beta\right)$

$=\left(\frac{3}{2}\right)^2-2\left(-\frac{4}{3}\right)$

$=\frac{9}{4}+\frac{8}{3}=\frac{59}{12}$

5 c $\alpha^3+\beta^3+\gamma^3$

$=(\alpha+\beta+\gamma)^3$

$-3(\alpha+\beta+\gamma)(\alpha\beta+\beta\gamma+\gamma\alpha)$

$+3\alpha\beta\gamma$

$=\left(\sum\alpha\right)^3-3\left(\sum\alpha\right)\left(\sum\alpha\beta\right)+3\alpha\beta\gamma$

$=\left(\frac{3}{2}\right)^3-3\left(\frac{3}{2}\right)\left(-\frac{4}{3}\right)+3\left(\frac{1}{2}\right)$

$=\frac{27}{8}+6+\frac{3}{2}=\frac{87}{8}$

d $\alpha^3\beta^3\gamma^3=(\alpha\beta\gamma)^3=\left(\frac{1}{2}\right)^2=\frac{1}{8}$

6 a $\alpha+\beta+\gamma=-\frac{1}{2},\alpha\beta+\beta\gamma+\gamma\alpha=\frac{3}{4}$

and $\alpha\beta\gamma=-\frac{2}{5}$

$(\alpha+2)(\beta+2)(\gamma+2)$

$=\alpha\beta\gamma+2(\alpha\beta+\alpha\gamma+\beta\gamma)$

$+4(\alpha+\beta+\gamma)+8$

$=\left(-\frac{2}{5}\right)+2\left(\frac{3}{4}\right)+4\left(-\frac{1}{2}\right)+8$

$=-\frac{2}{5}+\frac{3}{2}-2+8=\frac{71}{10}.$

b $(\alpha-3)(\beta-3)(\gamma-3)$

$=\alpha\beta\gamma-3(\alpha\beta+\alpha\gamma+\beta\gamma)$

$+9(\alpha+\beta+\gamma)-27$

$=\left(-\frac{2}{5}\right)-3\left(\frac{3}{4}\right)+9\left(-\frac{1}{2}\right)-27$

$=-\frac{2}{5}-\frac{9}{4}-\frac{9}{2}-27=-\frac{683}{20}$

c $(1-\alpha)(1-\beta)(1-\gamma)$

$=1-(\alpha+\beta+\gamma)+(\alpha\beta+\alpha\gamma+\beta\gamma)-\alpha\beta\gamma$

$=1-\left(-\frac{1}{2}\right)+\left(\frac{3}{4}\right)-\left(-\frac{2}{5}\right)$

$=1+\frac{1}{2}+\frac{3}{4}+\frac{2}{5}=\frac{53}{20}$

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6 d $(\alpha\beta+\beta\gamma+\gamma\alpha)^2$

$=(\alpha\beta+\beta\gamma+\gamma\alpha)(\alpha\beta+\beta\gamma+\gamma\alpha)$

$=(\alpha\beta)^2+\alpha\beta^2\gamma+\alpha^2\beta\gamma+\alpha\beta^2\gamma+(\beta\gamma)^2$
$+\alpha\beta\gamma^2+\alpha^2\beta\gamma+\alpha\beta\gamma^2+(\gamma\alpha)^2$

$=(\alpha\beta)^2+(\beta\gamma)^2+(\gamma\alpha)^2$
$+2(\alpha^2\beta\gamma+\alpha\beta^2\gamma+\alpha\beta\gamma^2)$

$=(\alpha\beta)^2+(\beta\gamma)^2+(\gamma\alpha)^2$
$+2\alpha\beta\gamma(\alpha+\beta+\gamma)$

Hence:

$(\alpha\beta+\beta\gamma+\gamma\alpha)^2$

$=(\alpha\beta)^2+(\beta\gamma)^2+(\gamma\alpha)^2$
$+2\alpha\beta\gamma(\alpha+\beta+\gamma)$

So:

$(\alpha\beta)^2+(\beta\gamma)^2+(\gamma\alpha)^2$

$=(\alpha\beta+\beta\gamma+\gamma\alpha)^2-2\alpha\beta\gamma(\alpha+\beta+\gamma)$

$=\left(\frac{3}{4}\right)^2-2\left(-\frac{2}{5}\right)\left(-\frac{1}{2}\right)$

$=\frac{9}{16}-\frac{2}{5}$

$=\frac{45}{80}-\frac{32}{80}$

$=\frac{13}{80}$

e $(\alpha\beta+\beta\gamma+\gamma\alpha)^3$

$=(\alpha\beta+\beta\gamma+\gamma\alpha)(\alpha\beta+\beta\gamma+\gamma\alpha)^2$

$=(\alpha\beta+\beta\gamma+\gamma\alpha)\begin{pmatrix}(\alpha\beta)^2+(\beta\gamma)^2\\+(\gamma\alpha)^2\\+2\alpha\beta\gamma(\alpha+\beta+\gamma)\end{pmatrix}$

$=(\alpha\beta)^3+(\beta\gamma)^3+(\gamma\alpha)^3$
$+\alpha\beta^3\gamma^2+\alpha^3\beta\gamma^2+2\alpha^2\beta^2\gamma(\alpha+\beta+\gamma)$
$+\alpha^2\beta^3\gamma+\alpha^2\beta\gamma^3+2\alpha\beta^2\gamma^2(\alpha+\beta+\gamma)$
$+\alpha^3\beta^2\gamma+\alpha\beta^2\gamma^3+2\alpha^2\beta\gamma^2(\alpha+\beta+\gamma)$

$=(\alpha\beta)^3+(\beta\gamma)^2+(\gamma\alpha)^3$
$+2\alpha^3\beta\gamma^2+2\alpha^2\beta^3\gamma+2\alpha^2\beta^2\gamma^2$
$+2\alpha^2\beta^2\gamma^2+2\alpha\beta^3\gamma^2+2\alpha\beta^2\gamma^3$
$+2\alpha^3\beta\gamma^2+2\alpha^2\beta^2\gamma^2+2\alpha^2\beta\gamma^3$

$=(\alpha\beta)^3+(\beta\gamma)^3+(\gamma\alpha)^3+6(\alpha\beta\gamma)^2$
$+3\alpha^2\beta^3\gamma+3\alpha\beta^3\gamma^2+3\alpha\beta^2\gamma^3$
$+3\alpha^3\beta^2\gamma+3\alpha^3\beta\gamma^2+3\alpha^2\beta\gamma^3$

$=(\alpha\beta)^3+(\beta\gamma)^3+(y\alpha)^3+6(\alpha\beta\gamma)^2$
$+3\alpha\beta\gamma\begin{pmatrix}\alpha\beta^2+\beta^2\gamma+\beta\gamma^2\\+\alpha^2\beta+\alpha^2\gamma+\alpha\gamma^2\end{pmatrix}$

$=(\alpha\beta)^3+(\beta\gamma)^3+(y\alpha)^3+6(\alpha\beta\gamma)^2$
$+3\alpha\beta\gamma\begin{pmatrix}\alpha(\beta^2+\gamma^2)\\+\beta(\alpha^2+\gamma^2)\\+\gamma(\alpha^2+\beta^2)\end{pmatrix}$

However:

$(\alpha+\beta+\gamma)(\alpha\beta+\beta\gamma+\gamma\alpha)$

$=\alpha^2\beta+\alpha^2\gamma+\alpha\beta^2$
$+\beta^2\gamma+\beta\gamma^2+\alpha\gamma^2+3\alpha\beta\gamma$

$=\alpha(\beta^2+\gamma^2)+\beta(\alpha^2+\gamma^2)$
$+\gamma(\alpha^2+\beta^2)+3\alpha\beta\gamma$

So

$3\alpha\beta\gamma\begin{pmatrix}\alpha(\beta^2+\gamma^2)\\+\beta(\alpha^2+\gamma^2)\\+\gamma(\alpha^2+\beta^2)\end{pmatrix}$

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6 e $= 3\alpha\beta\gamma\begin{pmatrix}(\alpha+\beta+\gamma)\times\\(\alpha\beta+\beta\gamma+\gamma\alpha)\\-3\alpha\beta\gamma\end{pmatrix}$

Therefore

$(\alpha\beta)^3+(\beta\gamma)^3+(\gamma\alpha)^3$

$=(\alpha\beta+\beta\gamma+\gamma\alpha)^3-6(\alpha\beta\gamma)^2$

$-3\alpha\beta\gamma\begin{pmatrix}(\alpha+\beta+\gamma)\times\\(\alpha\beta+\beta\gamma+\gamma\alpha)\\-3\alpha\beta\gamma\end{pmatrix}$

$=\left(\frac{3}{4}\right)^3-6\left(-\frac{2}{5}\right)^2$

$-3\left(-\frac{2}{5}\right)\left(\left(-\frac{1}{2}\right)\left(\frac{3}{4}\right)-3\left(-\frac{2}{5}\right)\right)$

$=\frac{27}{64}-\frac{24}{25}+\frac{6}{5}\left(\frac{33}{40}\right)$

$=\frac{27}{64}-\frac{24}{25}+\frac{99}{100}$

$=\frac{675}{1600}-\frac{1536}{1600}+\frac{1584}{1600}$

$=\frac{723}{1600}$

7 a $\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma}+\frac{1}{\delta}$

$=\frac{\alpha\beta\gamma+\beta\gamma\delta+\gamma\delta\alpha+\delta\alpha\beta}{\alpha\beta\gamma\delta}$

$=\frac{-4}{-2}=2$

b $\alpha^2+\beta^2+\gamma^2+\delta^2$

$=(\alpha+\beta+\gamma+\delta)^2$

$-2(\alpha\beta+\alpha\gamma+\alpha\delta+\beta\gamma+\beta\delta+\gamma\delta)$

$=(3)^2-2(5)=9-10=-1.$

c $\alpha^4\beta^4\gamma^4\delta^4=(\alpha\beta\gamma\delta)^4=(-2)^4=16.$

8 a $\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma}+\frac{1}{\delta}=\frac{\sum\alpha\beta\gamma}{\alpha\beta\gamma\delta}=\frac{\left(-\frac{1}{5}\right)}{\left(\frac{4}{3}\right)}=-\frac{3}{20}$

8 b $\alpha^2+\beta^2+\gamma^2+\delta^2=\left(\sum\alpha\right)^2-2\left(\sum\alpha\beta\right)$

$=\left(\frac{1}{2}\right)^2-2\left(-\frac{3}{4}\right)$

$=\frac{1}{4}+\frac{6}{4}=\frac{7}{4}$

c $\alpha^3\beta^3\gamma^3\delta^3=(\alpha\beta\gamma\delta)^3=\left(\frac{4}{3}\right)^3=\frac{64}{27}$

d $(\alpha\beta+\beta\gamma+\gamma\alpha+\gamma\delta+\alpha\delta+\beta\delta)^2$

$=(\alpha\beta+\beta\gamma+\gamma\alpha+\gamma\delta+\alpha\delta+\beta\delta)\times$

$(\alpha\beta+\beta\gamma+\gamma\alpha+\gamma\delta+\alpha\delta+\beta\delta)$

$=\alpha^2\beta^2+\alpha\beta^2\gamma+\alpha^2\beta\gamma$

$+\alpha\beta\gamma\delta+\alpha^2\beta\delta+\alpha\beta^2\delta$

$+\alpha\beta^2\gamma+\beta^2\gamma^2+\alpha\beta\gamma^2$

$+\beta\delta\gamma^2+\alpha\beta\gamma\delta+\beta^2\gamma\delta$

$+\alpha^2\beta\gamma+\alpha\beta\gamma^2+\alpha^2\gamma^2$

$+\alpha\gamma^2\delta+\alpha^2\gamma\delta+\alpha\beta\gamma\delta$

$+\alpha\beta\gamma\delta+\beta\gamma^2\delta+\alpha\gamma^2\delta$

$+\gamma^2\delta^2+\alpha\gamma\delta^2+\beta\gamma\delta^2$

$+\alpha^2\beta\delta+\alpha\beta\gamma\delta+\alpha^2\gamma\delta$

$+\alpha\gamma\delta^2+\alpha^2\delta^2+\alpha\beta\delta^2$

$+\alpha\beta^2\delta+\beta^2\gamma\delta+\alpha\beta\gamma\delta$

$+\beta\gamma\delta^2+\alpha\beta\delta^2+\beta^2\delta^2$

$=(\alpha\beta)^2+(\beta\gamma)^2+(\alpha\gamma)^2+(\gamma\delta)^2$

$+(\alpha\delta)^2+(\beta\delta)^2+6\alpha\beta\gamma\delta$

$+\alpha^2\beta\gamma+\alpha^2\beta\delta+\alpha^2\beta\gamma$

$+\alpha^2\gamma\delta+\alpha^2\beta\delta+\alpha^2\gamma\delta$

$+\alpha\beta^2\gamma+\alpha\beta^2\delta+\alpha\beta^2\gamma$

$+\beta^2\gamma\delta+\alpha\beta^2\delta+\beta^2\gamma\delta$

$+\alpha\beta\gamma^2+\alpha\beta\gamma^2+\alpha\gamma^2\delta$

$+\beta\gamma^2\delta+\alpha\gamma^2\delta+\beta\delta\gamma^2$

$+\alpha\beta\delta^2+\beta\gamma\delta^2+\alpha\gamma\delta^2$

$+\alpha\beta\delta^2+\beta\gamma\delta^2+\alpha\gamma\delta^2$

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8 d $=(\alpha\beta)^2+(\beta\gamma)^2+(\alpha\gamma)^2+(\gamma\delta)^2$
$+(\alpha\delta)^2+(\beta\delta)^2+6\alpha\beta\gamma\delta$
$+2\alpha^2\beta\gamma+2\alpha^2\beta\delta+2\alpha^2\gamma\delta$
$+2\alpha\beta^2\gamma+2\alpha\beta^2\delta+2\beta^2\gamma\delta$
$+2\alpha\beta\gamma^2+2\alpha\gamma^2\delta+2\beta\gamma^2\delta$
$+2\alpha\beta\delta^2+2\alpha\gamma\delta^2+2\beta\gamma\delta^2$

However:

$(\alpha+\beta+\gamma+\delta)(\alpha\beta\gamma+\alpha\beta\delta+\alpha\gamma\delta+\beta\gamma\delta)$
$=\alpha^2\beta\gamma+\alpha^2\beta\delta+\alpha^2\gamma\delta$
$+\alpha\beta^2\gamma+\alpha\beta^2\delta+\beta^2\gamma\delta$
$+\alpha\beta\gamma^2+\alpha\gamma^2\delta+\beta\gamma^2\delta$
$+\alpha\beta\delta^2+\alpha\gamma\delta^2+\beta\gamma\delta^2+4\alpha\beta\gamma\delta$

So

$(\alpha\beta+\beta\gamma+\gamma\alpha+\gamma\delta+\alpha\delta+\beta\delta)^2$
$=(\alpha\beta)^2+(\beta\gamma)^2+(\alpha\gamma)^2+(\gamma\delta)^2$
$+(\alpha\delta)^2+(\beta\delta)^2+6\alpha\beta\gamma\delta$
$+2\begin{pmatrix}\alpha+\beta\\+\gamma+\delta\end{pmatrix}\begin{pmatrix}\alpha\beta\gamma+\alpha\beta\delta\\+\alpha\gamma\delta+\beta\gamma\delta\end{pmatrix}$
$-8\alpha\beta\delta\delta$

So:

$(\alpha\beta)^2+(\beta\gamma)^2+(\alpha\gamma)^2$
$+(\gamma\delta)^2+(\alpha\delta)^2+(\beta\delta)^2$
$=(\alpha\beta+\beta\gamma+\gamma\alpha+\gamma\delta+\alpha\delta+\beta\delta)^2$
$+2\alpha\beta\gamma\delta$
$-2\begin{pmatrix}\alpha+\beta\\+\gamma+\delta\end{pmatrix}\begin{pmatrix}\alpha\beta\gamma+\alpha\beta\delta\\+\alpha\gamma\delta+\beta\gamma\delta\end{pmatrix}$
$=\left(-\frac{3}{4}\right)^2+2\left(\frac{4}{3}\right)-2\left(\frac{1}{2}\right)\left(-\frac{1}{5}\right)$
$=\frac{9}{16}+\frac{8}{3}+\frac{1}{5}$
$=\frac{135}{240}+\frac{640}{240}+\frac{48}{240}$
$=\frac{823}{240}$

e $(\alpha\beta\gamma+\alpha\beta\delta+\alpha\gamma\delta+\beta\gamma\delta)^2$
$=(\alpha\beta\gamma+\alpha\beta\delta+\alpha\gamma\delta+\beta\gamma\delta)$
$\times(\alpha\beta\gamma+\alpha\beta\delta+\alpha\gamma\delta+\beta\gamma\delta)$
$=(\alpha\beta\gamma)^2+\alpha^2\beta^2\gamma\delta+\alpha^2\beta\gamma^2\delta+\alpha\beta^2\gamma^2\delta$
$+\alpha^2\beta^2\gamma\delta+(\alpha\beta\delta)^2+\alpha^2\beta\gamma\delta^2+\alpha\beta^2\gamma\delta^2$
$+\alpha^2\beta\gamma^2\delta+\alpha^2\beta\gamma\delta^2+(\alpha\gamma\delta)^2+\alpha\beta\gamma^2\delta^2$
$+\alpha\beta^2\gamma^2\delta+\alpha\beta^2\gamma\delta^2+\alpha\beta\gamma^2\delta^2+(\beta\gamma\delta)^2$
$=(\alpha\beta\gamma)^2+(\alpha\beta\delta)^2+(\alpha\gamma\delta)^2+(\beta\gamma\delta)^2$
$+\alpha\beta\gamma\delta\begin{pmatrix}\alpha\beta+\alpha\gamma+\beta\gamma+\alpha\beta+\alpha\delta+\beta\delta\\+\alpha\gamma+\alpha\delta+\gamma\delta+\beta\gamma+\beta\delta+\gamma\delta\end{pmatrix}$

So $(\alpha\beta\gamma+\alpha\beta\delta+\alpha\gamma\delta+\beta\gamma\delta)^2$
$=(\alpha\beta\gamma)^2+(\alpha\beta\delta)^2+(\alpha\gamma\delta)^2+(\beta\gamma\delta)^2$
$+2\alpha\beta\gamma\delta(\alpha\beta+\alpha\gamma+\alpha\delta+\beta\gamma+\beta\delta+\gamma\delta)$

Hence

$(\alpha\beta\gamma)^2+(\alpha\beta\delta)^2+(\alpha\gamma\delta)^2+(\beta\gamma\delta)^2$
$=(\alpha\beta\gamma+\alpha\beta\delta+\alpha\gamma\delta+\beta\gamma\delta)^2$
$-2\alpha\beta\gamma\delta(\alpha\beta+\alpha\gamma+\alpha\delta+\beta\gamma+\beta\delta+\gamma\delta)$
$=\left(-\frac{1}{5}\right)^2-2\left(\frac{4}{3}\right)\left(-\frac{3}{4}\right)$
$=\frac{1}{25}+2=\frac{51}{25}$

9 a $(\alpha+1)(\beta+1)(\gamma+1)(\delta+1)$
$=\alpha\beta\gamma\delta+(\alpha+\beta+\gamma+\delta)$
$+(\alpha\beta+\alpha\gamma+\alpha\delta+\beta\gamma+\beta\delta+\gamma\delta)$
$+(\alpha\beta\gamma+\alpha\beta\delta+\alpha\gamma\delta+\beta\gamma\delta)+1$
$=\frac{3}{2}+\left(-\frac{1}{2}\right)+\left(-\frac{1}{3}\right)+\left(\frac{1}{4}\right)+1$
$=\frac{23}{12}$

b $(2-\alpha)(2-\beta)(2-\gamma)(2-\delta)$
$=16-8(\alpha+\beta+\gamma+\delta)$
$+4(\alpha\beta+\alpha\gamma+\alpha\delta+\beta\gamma+\beta\delta+\gamma\delta)$
$-2(\alpha\beta\gamma+\alpha\beta\delta+\alpha\gamma\delta+\beta\gamma\delta)+\alpha\beta\gamma\delta$
$=16-8\left(-\frac{1}{2}\right)+4\left(-\frac{1}{3}\right)-2\left(\frac{1}{4}\right)+\frac{3}{2}$
$=16+4-\frac{4}{3}-\frac{1}{2}+\frac{3}{2}=\frac{59}{3}$

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10 a $x^3-6x^2+9x-15=0$

So $a=1, b=-6, c=9$ and $d=-15$.

$\alpha+\beta+\gamma=-\frac{b}{a}=6$

$\alpha\beta+\beta\gamma+\gamma\alpha=\frac{c}{a}=9$

$\alpha\beta\gamma=-\frac{d}{a}=15$

b i $\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma}=\frac{\alpha\beta+\beta\gamma+\gamma\alpha}{\alpha\beta\gamma}=\frac{9}{15}=\frac{3}{5}$

ii

$\alpha^2+\beta^2+\gamma^2=(\alpha+\beta+\gamma)^2-$

$2(\alpha\beta+\beta\gamma+\gamma\alpha)$

$=(6)^2-2(9)=18$

iii $(\alpha-1)(\beta-1)(\gamma-1)$

$=\alpha\beta\gamma-(\alpha\beta+\beta\gamma+\gamma\alpha)$

$+(\alpha+\beta+\gamma)-1$

$=15-9+6-1=11.$

11 a $2x^3+4x^2+7=0$

So $a=2, b=4, c=0$ and $d=7$.

$\alpha+\beta+\gamma=-\frac{b}{a}=-2$

$\alpha\beta+\beta\gamma+\gamma\alpha=\frac{c}{a}=0$

$\alpha\beta\gamma=-\frac{d}{a}=-\frac{7}{2}$

b i $\alpha^2+\beta^2+\gamma^2$

$=(\alpha+\beta+\gamma)^2-2(\alpha\beta+\beta\gamma+\gamma\alpha)$

$=(-2)^2-2(0)$

$=4$

ii $\alpha^3\beta^3\gamma^3=(\alpha\beta\gamma)^3=\left(-\frac{7}{2}\right)^3=-\frac{343}{8}$

iii $(\alpha+2)(\beta+2)(\gamma+2)$

$=\alpha\beta\gamma+4(\alpha+\beta+\gamma)$

$+2(\alpha\beta+\beta\gamma+\gamma\alpha)+8$

$=-\frac{7}{2}+4(-2)+2(0)+8=-\frac{7}{2}$

12 $(\alpha+\beta+\gamma)^3$

$\equiv(\alpha+\beta+\gamma)\begin{pmatrix}\alpha^2+\beta^2+\gamma^2\\+2(\alpha\beta+\beta\gamma+\gamma\alpha)\end{pmatrix}$

$\equiv\alpha^3+\beta^3+\gamma^3$

$+\alpha(\beta^2+\gamma^2)+\beta(\alpha^2+\gamma^2)+\gamma(\alpha^2+\beta^2)$

$+2(\alpha+\beta+\gamma)(\alpha\beta+\beta\gamma+\gamma\alpha)$

Calculate $(\alpha+\beta+\gamma)(\alpha\beta+\beta\gamma+\gamma\alpha)$:

$(\alpha+\beta+\gamma)(\alpha\beta+\beta\gamma+\gamma\alpha)$

$\equiv\alpha^2\beta+\beta^2\alpha+\alpha^2\gamma$

$+\gamma^2\alpha+\beta^2\gamma+\gamma^2\beta+3\alpha\beta\gamma$

$\equiv\alpha(\beta^2+\gamma^2)+\beta(\alpha^2+\gamma^2)$

$+\gamma(\alpha^2+\beta^2)+3\alpha\beta\gamma$

So:

$(\alpha+\beta+\gamma)^3$

$\equiv\alpha^3+\beta^3+\gamma^3$

$+3(\alpha+\beta+\gamma)(\alpha\beta+\beta\gamma+\gamma\alpha)-3\alpha\beta\gamma$

Therefore

$\alpha^3+\beta^3+\gamma^3$

$\equiv(\alpha+\beta+\gamma)^3$

$-3(\alpha+\beta+\gamma)(\alpha\beta+\beta\gamma+\gamma\alpha)+3\alpha\beta\gamma$

13 a $3x^3-px+11=0$

So $a=3, b=0, c=-p$ and $d=11$.

$\alpha\beta+\beta\gamma+\gamma\alpha=\frac{c}{a}=-\frac{p}{3}$

But $\alpha\beta+\beta\gamma+\gamma\alpha=4$

So $-\frac{p}{3}=4\Rightarrow p=-12$

b $\alpha+\beta+\gamma=-\frac{b}{a}=0$

$\alpha\beta\gamma=-\frac{d}{a}=-\frac{11}{3}$

c $(3-a)(3-\beta)(3-\gamma)$

$=27-9(\alpha+\beta+\gamma)+3(\alpha\beta+\beta\gamma+\gamma\alpha)$

$-\alpha\beta\gamma$

$=27-9(0)+3(4)-\left(-\frac{11}{3}\right)$

$=27+12+\frac{11}{3}=\frac{128}{3}$

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14 a $x^4 + 2x^2 - x + 3 = 0$

So $a = 1, b = 0, c = 2, d = -1$ and $e = 3$.

$$\sum \alpha = -\frac{b}{a} = 0$$

$$\sum \alpha\beta = \frac{c}{a} = 2$$

$$\sum \alpha\beta\gamma = -\frac{d}{a} = 1$$

$$\alpha\beta\gamma\delta = \frac{e}{a} = 3$$

b i $\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} + \frac{1}{\delta} = \frac{\sum \alpha\beta\gamma}{\alpha\beta\gamma\delta} = \frac{1}{3}$

ii

$$\alpha^2 + \beta^2 + \gamma^2 + \delta^2 =$$

$$\left(\sum \alpha\right)^2 - 2\left(\sum \alpha\beta\right)$$

$$= 0^2 - 2(2) = -4$$

iii $(\alpha+1)(\beta+1)(\gamma+1)(\delta+1)$

$$= \alpha\beta\gamma\delta + (\alpha\beta\gamma + \alpha\beta\delta + \alpha\gamma\delta + \beta\gamma\delta)$$
$$+ (\alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta)$$
$$+ (\alpha + \beta + \gamma + \delta) + 1$$
$$= 3 + 1 + 2 + 0 + 1 = 7$$

15 a $ax^4 + 3x^3 + 2x^2 + x - 6 = 0$

So $a = a, b = 3, c = 2, d = 1$ and $e = -6$.

$\alpha\beta\gamma\delta = -3$

But $\alpha\beta\gamma\delta = \frac{e}{a} = -\frac{6}{a}$

So $-\frac{6}{a} = -3 \Rightarrow a = 2$

b $\sum \alpha = -\frac{b}{a} = -\frac{3}{2}$

$$\sum \alpha\beta = \frac{c}{a} = \frac{3}{2} = 1$$

$$\sum \alpha\beta\gamma = -\frac{d}{a} = -\frac{1}{2}$$

c $\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} + \frac{1}{\delta} = \frac{\sum \alpha\beta\gamma}{\alpha\beta\gamma\delta} = \frac{\left(-\frac{1}{2}\right)}{-3} = \frac{1}{6}$

16 $\left(\sum \alpha\right)^2 \equiv (\alpha + \beta + \gamma + \delta)^2$

$$\equiv \alpha^2 + \beta^2 + \gamma^2 + \delta^2$$
$$+ 2(\alpha\beta + \beta\gamma + \gamma\delta + \alpha\gamma + \beta\delta + \alpha\delta)$$

Therefore:

$$\alpha^2 + \beta^2 + \gamma^2 + \delta^2$$
$$\equiv \left(\sum \alpha\right)^2 - 2(\alpha\beta + \beta\gamma + \gamma\delta + \alpha\gamma + \beta\delta + \alpha\delta)$$

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Roots of polynomials 4E

1 a $x^3 - 7x^2 + 6x + 5 = 0$

$a = 1, b = -7, c = 6$ and $d = 5$

Method 1:

$\alpha + \beta + \gamma = -\frac{b}{a} = 7$

$\alpha\beta + \beta\gamma + \gamma\alpha = \frac{c}{a} = 6$

$\alpha\beta\gamma = -\frac{d}{a} = -5$

Sum $= (\alpha + 1) + (\beta + 1) + (\gamma + 1)$

$= \alpha + \beta + \gamma + 3$

$= 10$

Pair sum $= (\alpha + 1)(\beta + 1) + (\beta + 1)(\gamma + 1)$

$+ (\gamma + 1)(\alpha + 1)$

$= \alpha\beta + \beta\gamma + \gamma\alpha + 2(\alpha + \beta + \gamma) + 3$

$= 6 + 2(7) + 3$

$= 23$

Product $= (\alpha + 1)(\beta + 1)(\gamma + 1)$

$= \alpha\beta\gamma + (\alpha\beta + \beta\gamma + \gamma\alpha)$

$+ (\alpha + \beta + \gamma) + 1$

$= -5 + 6 + 7 + 1$

$= 9$

Hence the new equation is:

$w^3 - 10w^2 + 23w - 9 = 0$

Method 2:

Let $w = x + 1$, so $x = w - 1$

Substitute $x = w - 1$ into

$x^3 - 7x^2 + 6x + 5 = 0$

$(w-1)^3 - 7(w-1)^2 + 6(w-1) + 5 = 0$

$w^3 - 3w^2 + 3w - 1 - 7(w^2 - 2w + 1)$

$+ 6w - 6 + 5 = 0$

$w^3 - 3w^2 + 3w - 1 - 7w^2 + 14w - 7$

$+ 6w - 6 + 5 = 0$

Simplify to obtain:

$w^3 - 10w^2 + 23w - 9 = 0$

b Remember:

$\alpha + \beta + \gamma = 7$

$\alpha\beta + \beta\gamma + \gamma\alpha = 6$

$\alpha\beta\gamma = -5$

Method 1:

Sum $= 2\alpha + 2\beta + 2\gamma$

$= 2(\alpha + \beta + \gamma)$

$= 14$

Pair sum $= (2\alpha)(2\beta) + (2\beta)(2\gamma)$

$+ (2\gamma)(2\alpha)$

$= 4(\alpha\beta + \beta\gamma + \gamma\alpha)$

$= 24$

Product $= (2\alpha)(2\beta)(2\gamma) = 8\alpha\beta\gamma = -40$

Hence the new equation is:

$w^3 - 14w^2 + 24w + 40 = 0$

Method 2:

$w = 2x$, so $x = \frac{w}{2}$

Substitute $x = \frac{w}{2}$ into

$x^3 - 7x^2 + 6x + 5 = 0$

$\left(\frac{w}{2}\right)^3 - 7\left(\frac{w}{2}\right)^2 + 6\left(\frac{w}{2}\right) + 5 = 0$

$\frac{w^3}{8} - \frac{7}{4}w^2 + 3w + 5 = 0$

Multiply by 8:

$w^3 - 14w^2 + 24w + 40 = 0$

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2 a $3x^3-4x^2-5x+1=0$

$a=3, b=-4, c=-5$ and $d=1$

$\alpha+\beta+\gamma=-\frac{b}{a}=\frac{4}{3}$

$\alpha\beta+\beta\gamma+\gamma\alpha=\frac{c}{a}=-\frac{5}{3}$

$\alpha\beta\gamma=-\frac{d}{a}=-\frac{1}{3}$

Sum $=(\alpha-3)+(\beta-3)+(\gamma-3)$

$=\alpha+\beta+\gamma-9$

$=-\frac{23}{3}$

Pair sum $=(\alpha-3)(\beta-3)+(\beta-3)(\gamma-3)$

$+(\gamma-3)(\alpha-3)$

$=\alpha\beta+\beta\gamma+\gamma\alpha$

$-6(\alpha+\beta+\gamma)+27$

$=-\frac{5}{3}-6\left(\frac{4}{3}\right)+27$

$=-\frac{5}{3}-\frac{24}{3}+\frac{81}{3}$

$=\frac{52}{3}$

Product $=(\alpha-3)(\beta-3)(\gamma-3)$

$=\alpha\beta\gamma-3(\alpha\beta+\beta\gamma+\gamma\alpha)$

$+9(\alpha+\beta+\gamma)-27$

$=-\frac{1}{3}-3\left(-\frac{5}{3}\right)+9\left(\frac{4}{3}\right)-27$

$=-\frac{1}{3}+5+12-27$

$=-\frac{31}{3}$

Hence the new equation is:

$w^3+\frac{23}{3}w^2+\frac{52}{3}w+\frac{31}{3}=0$

Multiply by 3:

$3w^3+23w^2+52w+31=0$

b $w=\frac{x}{2}$, so $x=2w$

Substitute $x=2w$ into

$3x^3-4x^2-5x+1=0$

$3(2w)^3-4(2w)^2-5(2w)+1=0$

$3(8w^3)-4(4w^2)-10w+1=0$

$24w^3-16w^2-10w+1=0$

3 $x^3-3x^2+4x-7=0$

Let $w=2x+1$, so $x=\frac{w-1}{2}$

Substitute $x=\frac{w-1}{2}$ into $x^3-3x^2+4x-7=0$

$\left(\frac{w-1}{2}\right)^3-3\left(\frac{w-1}{2}\right)^2+4\left(\frac{w-1}{2}\right)-7=0$

Multiply through by $2^3=8$ to remove fractions:

$(w-1)^3-6(w-1)^2+16(w-1)-56=0$

$w^3-3w^2+3w-1-6(w^2-2w+1)$

$+16w-16-56=0$

$w^3-3w^2+3w-1-6w^2+12w-6$

$+16w-16-56=0$

$w^3-9w^2+31w-79=0$

4 $x^3+4x^2-4x+2=0$

Let $w=2x-1$, so $x=\frac{w+1}{2}$

Substitute $x=\frac{w+1}{2}$ into $x^3+4x^2-4x+2=0$

$\left(\frac{w+1}{2}\right)^3+4\left(\frac{w+1}{2}\right)^2-4\left(\frac{w+1}{2}\right)+2=0$

Multiply through by $2^3=8$ to remove fractions:

$(w+1)^3+8(w+1)^2-16(w+1)+16=0$

$w^3+3w^2+3w+1+8(w^2+2w+1)$

$-16w-16+16=0$

$w^3+3w^2+3w+1+8w^2+16w+8-16w=0$

$w^3+11w^2+3w+9=0$

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5 $3x^3 - x^2 + 2x - 5 = 0$

Let $w = 3x + 1$, so $x = \dfrac{w-1}{3}$

Substitute $x = \dfrac{w-1}{3}$ into $3x^3 - x^2 + 2x - 5 = 0$

$$3\left(\frac{w-1}{3}\right)^3 - \left(\frac{w-1}{3}\right)^2 + 2\left(\frac{w-1}{3}\right) - 5 = 0$$

Multiply through by $3^3 = 27$ to remove fractions:

$$3(w-1)^3 - 3(w-1)^2 + 18(w-1) - 135 = 0$$
$$3(w^3 - 3w^2 + 3w - 1) - 3(w^2 - 2w + 1)$$
$$+18w - 18 - 135 = 0$$
$$3w^3 - 9w^2 + 9w - 3 - 3w^2 + 6w - 3$$
$$+18w - 18 - 135 = 0$$
$$3w^3 - 12w^2 + 33w - 159 = 0$$

So $w^3 - 4w^2 + 11w - 53 = 0$ as all terms are divisible by 3.

6 a $2x^4 + 4x^3 - 5x^2 + 2x - 1 = 0$

$w = 3x$, so $x = \dfrac{w}{3}$

Substitute $x = \dfrac{w}{3}$ into

$$2x^4 + 4x^3 - 5x^2 + 2x - 1 = 0$$
$$2\left(\frac{w}{3}\right)^4 + 4\left(\frac{w}{3}\right)^3 - 5\left(\frac{w}{3}\right)^2 + 2\left(\frac{w}{3}\right) - 1 = 0$$
$$\frac{2w^4}{81} + \frac{4w^3}{27} - \frac{5w^2}{9} + \frac{2w}{3} - 1 = 0$$

Multiply through by 81 to remove fractions:

$$2w^4 + 12w^3 - 45w^2 + 54w - 81 = 0$$

b $w = x - 1$, so $x = w + 1$

Substitute $x = w + 1$ into

$$2x^4 + 4x^3 - 5x^2 + 2x - 1 = 0$$
$$2(w+1)^4 + 4(w+1)^3 - 5(w+1)^2$$
$$+2(w+1) - 1 = 0$$
$$2(w^4 + 4w^3 + 6w^2 + 4w + 1)$$
$$+4(w^3 + 3w^2 + 3w + 1) - 5(w^2 + 2w + 1)$$
$$+2w + 2 - 1 = 0$$
$$2w^4 + 8w^3 + 12w^2 + 8w + 2$$
$$+4w^3 + 12w^2 + 12w + 4 - 5w^2 - 10w - 5$$
$$+2w + 2 - 1 = 0$$
$$2w^4 + 12w^3 + 19w^2 + 12w + 2 = 0$$

7 a $x^4 + 2x^3 - 3x^2 + 4x + 5 = 0$

$w = 2x$, so $x = \dfrac{w}{2}$

Substitute $x = \dfrac{w}{2}$ into

$$x^4 + 2x^3 - 3x^2 + 4x + 5 = 0$$
$$\left(\frac{w}{2}\right)^4 + 2\left(\frac{w}{2}\right)^3 - 3\left(\frac{w}{2}\right)^2$$
$$+4\left(\frac{w}{2}\right) + 5 = 0$$
$$\frac{w^4}{16} + \frac{2w^3}{8} - \frac{3w^2}{4} + 2w + 5 = 0$$

Multiply through by 16 to remove fractions:

$$w^4 + 4w^3 - 12w^2 + 32w + 80 = 0$$

b $w = x - 2$, so $x = w + 2$

Substitute $x = w + 2$ into

$$x^4 + 2x^3 - 3x^2 + 4x + 5 = 0$$
$$(w+2)^4 + 2(w+2)^3 - 3(w+2)^2$$
$$+4(w+2) + 5 = 0$$
$$w^4 + 8w^3 + 24w^2 + 32w + 16$$
$$+2(w^3 + 6w^2 + 12w + 8)$$
$$-3(w^2 + 4w + 4) + 4w + 8 + 5 = 0$$
$$w^4 + 8w^3 + 24w^2 + 32w + 16$$
$$+2w^3 + 12w^2 + 24w + 16$$
$$-3w^2 - 12w - 12 + 4w + 8 + 5 = 0$$
$$w^4 + 10w^3 + 33w^2 + 48w + 33$$

8 a $3x^4 + 5x^3 - 4x^2 - 3x + 1 = 0$

$w = 3x$, so $x = \dfrac{w}{3}$

Substitute $x = \dfrac{w}{3}$ into

$$3x^4 + 5x^3 - 4x^2 - 3x + 1 = 0$$
$$3\left(\frac{w}{3}\right)^4 + 5\left(\frac{w}{3}\right)^3 - 4\left(\frac{w}{3}\right)^2 - 3\left(\frac{w}{3}\right) + 1 = 0$$
$$\frac{3w^4}{81} + \frac{5w^3}{27} - \frac{4w^2}{9} - w + 1 = 0$$

Multiply through by 81 to remove fractions:

$$3w^4 + 15w^3 - 36w^2 - 81w + 81 = 0$$

All terms are divisible by 3:

$$w^4 + 5w^3 - 12w^2 - 27w + 27 = 0$$

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8 b $w = x+1$, so $x = w-1$

Substitute $x = w-1$ into

$3x^4+5x^3-4x^2-3x+1=0$

$3(w-1)^4+5(w-1)^3-4(w-1)^2$

$-3(w-1)+1=0$

$3(w^4-4w^3+6w^2-4w+1)$

$+5(w^3-3w^2+3w-1)$

$-4(w^2-2w+1)-3w+3+1=0$

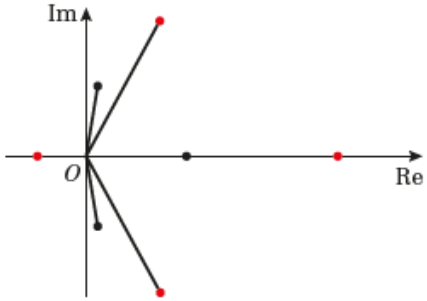
$3w^4-12w^3+18w^2-12w+3$

$+5w^3-15w^2+15w-5$

$-4w^2+8w-4-3w+3+1=0$

$3w^4-7w^3-w^2+8w-2=0$

b



Challenge

a $2x^4-3x^3+x^2-2x-6=0$

$w = 2x+1$, so $x = \frac{w-1}{2}$

Substitute $x = \frac{w-1}{2}$ into

$2\left(\frac{w-1}{2}\right)^4-3\left(\frac{w-1}{2}\right)^3+\left(\frac{w-1}{2}\right)^2$

$-2\left(\frac{w-1}{2}\right)-6=0$

Multiply through by 8 to remove fractions:

$(w-1)^4-3(w-1)^3+2(w-1)^2$

$-8(w-1)-48=0$

$w^4-4w^3+6w^2-4w+1$

$-3(w^3-3w^2+3w-1)$

$+2(w^2-2w+1)-8w+8-48=0$

$w^4-4w^3+6w^2-4w+1$

$-3w^3+9w^2-9w+3$

$+2w^2-4w+2-8w+8-48=0$

$w^4-7w^3+17w^2-25w-34=0$

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Roots of polynomials Mixed exercise 4

1 $ax^4+bx^3+cx^2+dx+e=0$

Roots are $\alpha=\frac{1}{5},\beta=-\frac{2}{5},\gamma=-\frac{3}{5}$ and $\delta=-\frac{1}{2}$.

$\alpha+\beta+\gamma+\delta=\frac{1}{5}-\frac{2}{5}-\frac{3}{5}-\frac{1}{2}=-\frac{13}{10}$

$\alpha\beta+\alpha\gamma+\alpha\delta+\beta\gamma+\beta\delta+\gamma\delta$

$=\left(\frac{1}{5}\right)\left(-\frac{2}{5}\right)+\left(\frac{1}{5}\right)\left(-\frac{3}{5}\right)+\left(\frac{1}{5}\right)\left(-\frac{1}{2}\right)$

$+\left(-\frac{2}{5}\right)\left(-\frac{3}{5}\right)+\left(-\frac{2}{5}\right)\left(-\frac{1}{2}\right)+\left(-\frac{3}{5}\right)\left(-\frac{1}{2}\right)$

$=-\frac{2}{25}-\frac{3}{25}-\frac{1}{10}+\frac{6}{25}+\frac{2}{10}+\frac{3}{10}$

$=\frac{11}{25}$

$\alpha\beta\gamma+\alpha\beta\delta+\alpha\gamma\delta+\beta\gamma\delta$

$=\left(\frac{1}{5}\right)\left(-\frac{2}{5}\right)\left(-\frac{3}{5}\right)+\left(\frac{1}{5}\right)\left(-\frac{2}{5}\right)\left(-\frac{1}{2}\right)$

$+\left(\frac{1}{5}\right)\left(-\frac{3}{5}\right)\left(-\frac{1}{2}\right)+\left(-\frac{2}{5}\right)\left(-\frac{3}{5}\right)\left(-\frac{1}{2}\right)$

$=\frac{6}{125}+\frac{2}{50}+\frac{3}{50}-\frac{6}{50}$

$=\frac{7}{250}$

$\alpha\beta\gamma\delta=\left(\frac{1}{5}\right)\left(-\frac{2}{5}\right)\left(-\frac{3}{5}\right)\left(-\frac{1}{2}\right)=-\frac{6}{250}$

So $-\frac{13}{10}=-\frac{b}{a}\Rightarrow\frac{b}{a}=\frac{13}{10}$

$\frac{11}{25}=\frac{c}{a}\Rightarrow\frac{c}{a}=\frac{11}{25}$

$\frac{7}{250}=-\frac{d}{a}\Rightarrow\frac{d}{a}=-\frac{7}{250}$

$-\frac{6}{250}=\frac{e}{a}\Rightarrow\frac{e}{a}=-\frac{6}{250}$

$ax^4+bx^3+cx^2+dx+e=0$

$\Rightarrow x^4+\frac{b}{a}x^3+\frac{c}{a}x^2+\frac{d}{a}x+\frac{e}{a}=0$

So the equation is:

$x^4+\frac{13}{10}x^3+\frac{11}{25}x^2-\frac{7}{250}x-\frac{6}{250}=0$

Multiply by 250 for integer coefficients:

$250x^4+325x^3+110x^2-7x-6=0$

2 a $x^3+px^2+37x-52=0$

So $a=1,b=p,c=37$ and $d=-52$.

$\alpha\beta+\beta\gamma+\gamma\alpha=\frac{c}{a}=37$

$\alpha\beta\gamma=-\frac{d}{a}=52$

$\alpha+\beta+\gamma=-\frac{b}{a}=-p$, so

$p=-(\alpha+\beta+\gamma)$

b If $\alpha=3-2\mathrm{i}$, then $\beta=3+2\mathrm{i}$

Substitute $\alpha=3-2\mathrm{i}$ and $\beta=3+2\mathrm{i}$ into

$\alpha\beta\gamma=52$

$(3-2\mathrm{i})(3+2\mathrm{i})\gamma=52$

$(9+6\mathrm{i}-6\mathrm{i}-4\mathrm{i}^2)\gamma=52$

$13\gamma=52$

$\gamma=4$

So the roots are $3-2\mathrm{i},3+2\mathrm{i}$ and 4.

$p=-(\alpha+\beta+\gamma)$

$=-((3-2\mathrm{i})+(3+2\mathrm{i})+4)=-10$

3 a $2x^3+5x^2-2x+q=0$

So $a=2,b=5,c=-2$ and $d=q$.

If $x=-2+\mathrm{i}$ is a root then $x=-2-\mathrm{i}$ is also a root.

$\alpha+\beta+\gamma=-\frac{b}{a}=-\frac{5}{2}$

$\alpha\beta+\beta\gamma+\gamma\alpha=\frac{c}{a}=-\frac{2}{2}=-1$

$\alpha\beta\gamma=-\frac{d}{a}=-\frac{q}{2}$

Substitute $\alpha=-2+\mathrm{i}$ and $\beta=-2-\mathrm{i}$ into

$\alpha+\beta+\gamma=-\frac{5}{2}$.

$(-2+\mathrm{i})+(-2-\mathrm{i})+\gamma=-\frac{5}{2}$

$-4+\gamma=-\frac{5}{2}$

$\gamma=\frac{3}{2}$

The other two roots are $-2-\mathrm{i}$ and $\frac{3}{2}$.

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3 b $\alpha\beta\gamma = -\frac{q}{2}$, so

$q = -2\alpha\beta\gamma$

$= -2(-2+\mathrm{i})(-2-\mathrm{i})\left(\frac{3}{2}\right)$

$= (-3)(4+2\mathrm{i}-2\mathrm{i}-\mathrm{i}^2)$

$= (-3)(5)$

$= -15$

So $q = -15$.

4 $x^4 - 40x^3 + 510x^2 - 220x + 1729 = 0$

So $a = 1, b = -40, c = 510, d = -2200$ and $e = 1729$.

Let $\alpha = \alpha$, $\beta = \alpha + 2k$, $\gamma = \alpha + 4k$ and $\delta = \alpha + 6k$.

$\alpha + \beta + \gamma + \delta$

$= \alpha + (\alpha + 2k) + (\alpha + 4k) + (\alpha + 6k)$

$= 4\alpha + 12k$

But $\alpha + \beta + \gamma + \delta = -\frac{b}{a} = 40$

So $4\alpha + 12k = 40$ or $\alpha + 3k = 10$

$\alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta$

$= \alpha(\alpha + 2k) + \alpha(\alpha + 4k) + \alpha(\alpha + 6k)$

$+ (\alpha + 2k)(\alpha + 4k) + (\alpha + 2k)(\alpha + 6k)$

$+ (\alpha + 4k)(\alpha + 6k)$

$= \alpha^2 + 2\alpha k + \alpha^2 + 4\alpha k + \alpha^2 + 6\alpha k$

$+ \alpha^2 + 6\alpha k + 8k^2 + \alpha^2 + 8\alpha k + 12k^2$

$+ \alpha^2 + 10\alpha k + 24k^2$

$= 6\alpha^2 + 36\alpha k + 44k^2$

But $\alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta = \frac{c}{a} = 510$

So $6\alpha^2 + 36\alpha k + 44k^2 = 510$

or $3\alpha^2 + 18\alpha k + 22k^2 = 255$

If $\alpha + 3k = 10$, then $\alpha = 10 - 3k$

Substitute $\alpha = 10 - 3k$ into

$3\alpha^2 + 18\alpha k + 22k^2 = 255$.

$3(10-3k)^2 + 18(10-3k)k + 22k^2 = 255$

$3(100 - 60k + 9k^2) + 18k(10-3k) + 22k^2 = 255$

$300 - 180k + 27k^2 + 180k - 54k^2 + 22k^2 = 255$

$-5k^2 = -45$

$k^2 = 9$

$k = \pm 3$

If $k = 3$, $\alpha = 10 - 3(3) = 1$

If $k = -3$, $\alpha = 10 - 3(-3) = 19$

Either way the roots are $1, 7, 13$ and 19.

5 a $24x^4 - 58x^3 + 17x^2 + dx + e = 0$

So $a = 24, b = -58, c = 17, d = d$ and $e = e$

$\alpha = \frac{1}{2}, \beta = -\frac{1}{3}, \gamma = 2$ and $\delta = \delta$.

$\alpha + \beta + \gamma + \delta = \frac{1}{2} + \left(-\frac{1}{3}\right) + 2 + \delta$

$= \frac{13}{6} + \delta$

But $\alpha + \beta + \gamma + \delta = -\frac{b}{a} = \frac{58}{24} = \frac{29}{12}$

So $\frac{13}{6} + \delta = \frac{29}{12}$, hence $\delta = \frac{3}{12} = \frac{1}{4}$

The fourth root is $\frac{1}{4}$.

b $\alpha\beta\gamma + \alpha\beta\delta + \alpha\gamma\delta + \beta\gamma\delta$

$= \left(\frac{1}{2}\right)\left(-\frac{1}{3}\right)(2) + \left(\frac{1}{2}\right)\left(-\frac{1}{3}\right)\left(\frac{1}{4}\right)$

$+ \left(\frac{1}{2}\right)(2)\left(\frac{1}{4}\right) + \left(-\frac{1}{3}\right)(2)\left(\frac{1}{4}\right)$

$= -\frac{1}{3} - \frac{1}{24} + \frac{1}{4} - \frac{1}{6}$

$= -\frac{8}{24} - \frac{1}{24} + \frac{6}{24} - \frac{4}{24}$

$= -\frac{7}{24}$

But $\alpha\beta\gamma + \alpha\beta\delta + \alpha\gamma\delta + \beta\gamma\delta = -\frac{d}{a} = -\frac{d}{24}$

So $-\frac{d}{24} = -\frac{7}{24}$, hence $d = 7$

$\alpha\beta\gamma\delta = \left(\frac{1}{2}\right)\left(-\frac{1}{3}\right)(2)\left(\frac{1}{4}\right) = -\frac{1}{12}$

But $\alpha\beta\gamma\delta = \frac{e}{a} = \frac{e}{24}$

So $\frac{e}{24} = -\frac{1}{12}$, hence $e = -2$.

6 a $x^4 + 2x^3 + mx^2 + nx + 85 = 0$

So $a = 1, b = 2, c = m, d = n$ and $e = 85$.

Given $\alpha = -2 + \mathrm{i}$ and $\beta = \alpha^* = -2 - \mathrm{i}$

$\alpha + \beta + \gamma + \delta = (-2+\mathrm{i}) + (-2+\mathrm{i}) + \gamma + \delta$

$= -4 + \gamma + \delta$

But $\alpha + \beta + \gamma + \delta = -\frac{b}{a} = -2$

So $-4 + \gamma + \delta = -2$ hence $\gamma + \delta - 2 = 0$

$\alpha\beta\gamma\delta = (-2+\mathrm{i})(-2+\mathrm{i})\gamma\delta$

$= 5\gamma\delta$

But $\alpha\beta\gamma\delta = \frac{e}{a} = 85$

So $5\gamma\delta = 85$ hence $\gamma\delta - 17 = 0$

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6 b If $\gamma\delta - 17 = 0$, then $\gamma = \frac{17}{\delta}$

Substitute $\gamma = \frac{17}{\delta}$ into $\gamma + \delta - 2 = 0$

$\frac{17}{\delta} + \delta - 2 = 0$

Multiply by δ :

$17 + \delta^2 - 2\delta = 0$

$\delta^2 - 2\delta + 17 = 0$

Solve by completing the square:

$(\delta - 1)^2 - 1 + 17 = 0$

$(\delta - 1)^2 = -16$

$\delta - 1 = \pm 4\text{i}$

$\delta = 1 \pm 4\text{i}$

So the roots are $-2 \pm \text{i}$ and $1 \pm 4\text{i}$

But $\alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta = \frac{c}{a} = m$

$= (-2+i)(-2-i) + (-2+i)(1+4i) + (-2+i)(1-4i)$

$+ (-2-i)(1+4i) + (-2-i)(1-4i) + (1+4i)(1-4i)$

and cancelling the imaginary terms gives us

$= 5 - 2 - 2 - 2 + 17 = 14 = m$

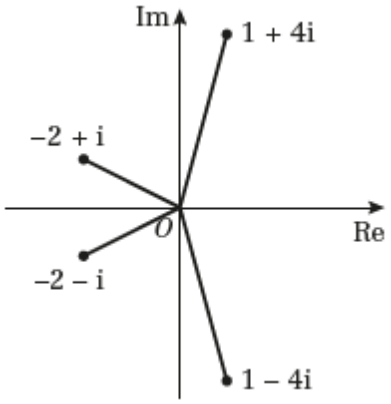
Now $\alpha\beta\gamma + \alpha\beta\delta + \alpha\gamma\delta + \beta\gamma\delta = \frac{-d}{a} = n$

$= 5(\gamma + \delta) + 17(\alpha + \beta) = (5)(2) + (17)(-4) = -58 = -n$

So

$n = 58$

c



7 a $4x^4 - 16x^3 + 115x^2 + 4x - 29 = 0$

So $a = 4, b = -16, c = 115, d = 4$ and $e = -29$

Show $\alpha = 2 - 5\text{i}$ is a root:

$4(2-5\text{i})^4 - 16(2-5\text{i})^3 + 115(2-5\text{i})^2$

$+4(2-5\text{i}) - 29 = 0$

7 a Use

$(a+b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$

with $a = 2$ and $b = -5\text{i}$

$(2-5\text{i})^4 = (2)^4 + 4(2)^3(-5\text{i}) + 6(2)^2(-5\text{i})^2$

$+4(2)(-5\text{i})^3 + (-5\text{i})^4$

$(2-5\text{i})^4 = 16 + (32)(-5\text{i}) + (24)(-5\text{i})^2$

$+(8)(-5\text{i})^3 + (-5\text{i})^4$

$(2-5\text{i})^4 = 16 - 160\text{i} + 600\text{i}^2 - 1000\text{i}^3 + 625\text{i}^4$

Use the fact that $\text{i}^2 = -1$, $\text{i}^3 = -\text{i}$ and $\text{i}^4 = 1$ to simplify:

$(2-5\text{i})^4 = 16 - 160\text{i} - 600 + 1000\text{i} + 625$

$= 41 + 840\text{i}$

Use $(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$ with $a = 2$ and $b = -5\text{i}$

$(2-5\text{i})^3 = (2)^3 + 3(2)^2(-5\text{i})$

$+3(2)(-5\text{i})^2 + (-5\text{i})^3$

$= 8 + (12)(-5\text{i})$

$+(6)(-5\text{i})^2 + (-5\text{i})^3$

$= 8 + -60\text{i} + 150\text{i}^2 - 125\text{i}^3$

Use the fact that $\text{i}^2 = -1$ and $\text{i}^3 = -\text{i}$ to simplify:

$(2-5\text{i})^3 = 8 + -60\text{i} - 150 + 125\text{i}$

$= -142 + 65\text{i}$

$(2-5\text{i})^2 = 4 - 10\text{i} - 10\text{i} + 25\text{i}^2$

$= -21 - 20\text{i}$

So $4(2-5\text{i})^4 - 16(2-5\text{i})^3 + 115(2-5\text{i})^2$

$+4(2-5\text{i}) - 29 = 0$ becomes:

$4(41 + 840\text{i}) - 16(-142 + 65\text{i})$

$+115(-21 - 20\text{i}) + 4(2-5\text{i}) - 29 = 0$

$164 + 3360\text{i} + 2272 - 1040\text{i}$

$-2415 - 2300\text{i} + 8 - 20\text{i} - 29 = 0$

$(164 + 2272 - 2514 + 8 - 29)$

$+(3360 - 1040 - 2300 - 20)\text{i} = 0$

So $\alpha = 2 - 5\text{i}$ is a root.

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7 b If $\alpha = 2-5\text{i}$ is a root, then $\beta = 2+5\text{i}$ is also a root. Let the other two roots be γ and δ.

$\alpha+\beta+\gamma+\delta = (2-5\text{i})+(2+5\text{i})+\gamma+\delta$

$= 4+\gamma+\delta$

But $\alpha+\beta+\gamma+\delta = -\frac{b}{a} = 4$

So $4+\gamma+\delta = 4$ hence $\gamma+\delta = 0$ and $\gamma = -\delta$.

$\alpha\beta\gamma\delta = (2-5\text{i})(2+5\text{i})\gamma\delta$

$= 29\gamma\delta$

But $\alpha\beta\gamma\delta = \frac{e}{a} = -\frac{29}{4}$

So $29\gamma\delta = -\frac{29}{4}$, hence $\gamma\delta = -\frac{1}{4}$.

Substitute $\gamma = -\delta$ into $\gamma\delta = -\frac{1}{4}$.

$-\delta^2 = -\frac{1}{4}$

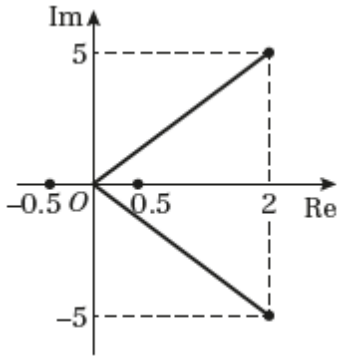
$\delta^2 = \frac{1}{4}$

$\delta = \pm\frac{1}{2}$

So $\gamma = \mp\frac{1}{2}$

So the roots are $2-5\text{i}, 2+5\text{i}, \frac{1}{2}$ and $-\frac{1}{2}$.

c



8 a $2x^3-5x^2+11x-9=0$

So $a=2,\ b=-5,\ c=11$ and $d=-9$

$\alpha+\beta+\gamma = -\frac{b}{a} = \frac{5}{2}$

$\alpha\beta+\beta\gamma+\gamma\alpha = \frac{c}{a} = \frac{11}{2}$

$\alpha\beta\gamma = -\frac{d}{a} = \frac{9}{2}$

b i $\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma} = \frac{\alpha\beta+\beta\gamma+\gamma\alpha}{\alpha\beta\gamma} = \frac{\left(\frac{11}{2}\right)}{\left(\frac{9}{2}\right)} = \frac{11}{9}$

ii $\alpha^2+\beta^2+\gamma^2$

$= (\alpha+\beta+\gamma)^2 - 2(\alpha\beta+\beta\gamma+\gamma\alpha)$

$= \left(\frac{5}{2}\right)^2 - 2\left(\frac{11}{2}\right)$

$= \frac{25}{4} - 11$

$= -\frac{19}{4}$

iii $(\alpha-1)(\beta-1)(\gamma-1)$

$= \alpha\beta\gamma - (\alpha\beta+\beta\gamma+\gamma\alpha)$

$+(\alpha+\beta+\gamma)-1$

$= \frac{9}{2} - \frac{11}{2} + \frac{5}{2} - 1$

$= \frac{1}{2}$

9 a $px^4+12x^3+6x^2+5x-7=0$

So $a=p, b=12, c=6, d=5$ and $e=-7$

$\alpha\beta\gamma\delta = \frac{e}{a} = -\frac{7}{p}$

But $\alpha\beta\gamma\delta = -1$

So $-\frac{7}{p} = -1$, hence $p=7$.

b $\sum\alpha = -\frac{b}{a} = -\frac{12}{7}$

$\sum\alpha\beta = \frac{c}{a} = \frac{6}{7}$

$\sum\alpha\beta\gamma = -\frac{d}{a} = -\frac{5}{7}$

c $\alpha^2+\beta^2+\gamma^2+\delta^2$

$= (\alpha+\beta+\gamma+\delta)^2$

$-2(\alpha\beta+\alpha\gamma+\alpha\delta+\beta\gamma+\beta\delta+\gamma\delta)$

$= \left(-\frac{12}{7}\right)^2 - 2\left(\frac{6}{7}\right)$

$= \frac{144}{49} - \frac{12}{7}$

$= \frac{144}{49} - \frac{84}{49}$

$= \frac{60}{49}$

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10 a $5x^3 + cx + 21 = 0$

So $a = 5,\ b = 0,\ c = c,$ and $d = 21$.

$\alpha\beta + \beta\gamma + \gamma\alpha = \frac{c}{a} = \frac{c}{5}$

But $\alpha\beta + \beta\gamma + \gamma\alpha = -6$

So $\frac{c}{5} = -6$, hence $c = -30$

b $\alpha + \beta + \gamma = -\frac{b}{a} = 0$

$\alpha\beta\gamma = -\frac{d}{a} = -\frac{21}{5}$

c $(1-\alpha)(1-\beta)(1-\gamma)$

$= 1 - (\alpha + \beta + \gamma) + (\alpha\beta + \beta\gamma + \gamma\alpha) - \alpha\beta\gamma$

$= 1 - 0 + (-6) - \left(-\frac{21}{5}\right)$

$= -5 + \frac{21}{5}$

$= -\frac{4}{5}$

11 $2x^3 + 5x^2 + 7x - 2 = 0$

$w = 3x + 1$, so $x = \frac{w-1}{3}$

Substitute $x = \frac{w-1}{3}$ into

$2x^3 + 5x^2 + 7x - 2 = 0$

$2\left(\frac{w-1}{3}\right)^3 + 5\left(\frac{w-1}{3}\right)^2 + 7\left(\frac{w-1}{3}\right) - 2 = 0$

Multiply by 27 to remove fractions:

$2(w-1)^3 + 15(w-1)^2 + 63(w-1) - 54 = 0$

$2(w^3 - 3w^2 + 3w - 1) + 15(w^2 - 2w + 1)$

$+63(w-1) - 54 = 0$

$2w^3 - 6w^2 + 6w - 2 + 15w^2 - 30w + 15$

$+63w - 63 - 54 = 0$

$2w^3 + 9w^2 + 39w - 104 = 0$

12 a $6x^4 - 2x^3 - 5x^2 + 7x + 8 = 0$

$w = 2x$, so $x = \frac{w}{2}$

Substitute $x = \frac{w}{2}$ into

$6x^4 - 2x^3 - 5x^2 + 7x + 8 = 0$

$6\left(\frac{w}{2}\right)^4 - 2\left(\frac{w}{2}\right)^3 - 5\left(\frac{w}{2}\right)^2 + 7\left(\frac{w}{2}\right) + 8 = 0$

12 a Multiply by 16 to eliminate fractions:

$6w^4 - 4w^3 - 20w^2 + 56w + 128 = 0$

All coefficients are divisible by 2:

$3w^4 - 2w^3 - 10w^2 + 28w + 64 = 0$

b $w = 3x - 2$, so $x = \frac{w+2}{3}$

Substitute $x = \frac{w+2}{3}$ into

$6x^4 - 2x^3 - 5x^2 + 7x + 8 = 0$

$6\left(\frac{w+2}{3}\right)^4 - 2\left(\frac{w+2}{3}\right)^3 - 5\left(\frac{w+2}{3}\right)^2$

$+7\left(\frac{w+2}{3}\right) + 8 = 0$

Multiply by 81 to eliminate fractions:

$6(w+2)^4 - 6(w+2)^3 - 45(w+2)^2$

$+189(w+2) + 648 = 0$

All coefficients are divisible by 3:

$2(w+2)^4 - 2(w+2)^3 - 15(w+2)^2$

$+63(w+2) + 216 = 0$

$2(w^4 + 8w^3 + 24w^2 + 32w + 16)$

$-2(w^3 + 6w^2 + 12w + 8)$

$-15(w^2 + 4w + 4) + 63(w+2) + 216 = 0$

$2w^4 + 16w^3 + 48w^2 + 64w + 32$

$-2w^3 - 12w^2 - 24w - 16$

$-15w^2 - 60w - 60 + 63w + 126 + 216 = 0$

$2w^4 + 14w^3 + 21w^2 + 43w + 298 = 0$

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Challenge

1 Multiply by w^3 :

$1-3w+3w^2-w^3+4w\left(1-2w+w^2\right)$

$-5w^2\left(1-w\right)-7w^3=0$

$1-3w+3w^2-w^3+4w-8w^2+4w^3$

$-5w^2+5w^3-7w^3=0$

$w^3-10w^2+w+1=0$

2 $x^3+2x^2-3x-5=0$

So $a=1,\ b=2,\ c=-3$ and $d=-5$.

$\alpha+\beta+\gamma=-\dfrac{b}{a}=-2$

Hence $\alpha+\beta+\gamma=-2$

So, $\alpha+\beta=-2-\gamma$, $\beta+\gamma=-2-\alpha$ and

$\gamma+\alpha=-2-\beta$

$w=-2-x$, so $x=-2-w$

Substitute $x=-2-w$ into

$x^3+2x^2-3x-5=0$

$\left(-2-w\right)^3+2\left(-2-w\right)^2-3\left(-2-w\right)-5=0$

$-8-12w-6w^2-w^3+2\left(4+4w+w^2\right)$

$+6+3w-5=0$

$-8-12w-6w^2-w^3+8+8w+2w^2$

$+6+3w-5=0$

$-w^3-4w^2-w+1=0$

Divide by -1 :

$w^3+4w^2+w-1=0$

3 $x^4+2x^2-5x+2=0$

$w=x^2+1$, so $x^2=w-1$.

Therefore $x=\sqrt{w-1}$.

Substitute $x=\sqrt{w-1}$ into

$x^4+2x^2-5x+2=0$.

$\left(\sqrt{w-1}\right)^4+2\left(\sqrt{w-1}\right)^2-5\left(\sqrt{w-1}\right)+2=0$

$\left(w-1\right)^2+2\left(w-1\right)-5\sqrt{w-1}+2=0$

$w^2-2w+1+2w-2-5\sqrt{w-1}+2=0$

$w^2-5\sqrt{w-1}+1=0$

Multiply by $w^2+5\sqrt{w-1}+1$.

$w^4+2w^2-25\left(w-1\right)+1=0$

$w^4+2w^2-25w+26=0$

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Volumes of revolution 5A

1 a $V = \pi\int_0^2 \left(10x^2\right)^2 \, dx$

$= \pi\int_0^2 100x^4 \, dx$

$= \pi\left[20x^5\right]_0^2$

$= \pi\left(640 - 0\right)$

$= 640\pi$

b $V = \pi\int_3^5 \left(5 - x\right)^2 \, dx$

$= \pi\int_3^5 \left(25 - 10x + x^2\right) dx$

$= \pi\left[25x - 5x^2 + \frac{x^3}{3}\right]_3^5$

$= \pi\left(\left(125 - 125 + \frac{125}{3}\right) - \left(75 - 45 + 9\right)\right)$

$= \pi\left(\frac{125}{3} - 39\right)$

$= \frac{8}{3}\pi$

c $V = \pi\int_2^{10} \left(\sqrt{x}\right)^2 \, dx$

$= \pi\int_2^{10} x \, dx$

$= \pi\left[\frac{x^2}{2}\right]_2^{10}$

$= \pi\left(\frac{100}{2} - \frac{4}{2}\right)$

$= 48\pi$

d $V = \pi\int_1^2 \left(1 + \frac{1}{x^2}\right)^2 \, dx$

$V = \pi\int_1^2 \left(1 + \frac{2}{x^2} + \frac{1}{x^4}\right) dx$

$= \pi\left[x - \frac{2}{x} - \frac{1}{3x^3}\right]_1^2$

$= \pi\left(\left(2 - 1 - \frac{1}{24}\right) - \left(1 - 2 - \frac{1}{3}\right)\right)$

$= \pi\left(\frac{23}{24} + \frac{4}{3}\right)$

$= \frac{55}{24}\pi$

2 $5 + 4x - x^2 = 0$

$\left(5 - x\right)\left(1 + x\right) = 0$

$x > 0 \Rightarrow x = 5$

$V = \pi\int_0^5 \left(5 + 4x - x^2\right)^2 \, dx$

$= \pi\int_0^5 \left(25 + 40x + 6x^2 - 8x^3 + x^4\right) dx$

$= \pi\left[25x + 20x^2 + 2x^3 - 2x^4 + \frac{x^5}{5}\right]_0^5$

$= \pi\left(125 + 500 + 250 - 1250 + 625\right)$

$= 250\pi$

3 $V = \pi\int_1^8 \left(3 - \sqrt[3]{x}\right)^2 \, dx$

$V = \pi\int_1^8 \left(9 - 6x^{\frac{1}{3}} + x^{\frac{2}{3}}\right) dx$

$= \pi\left[9x - \frac{9}{2}x^{\frac{4}{3}} + \frac{3}{5}x^{\frac{5}{3}}\right]_1^8$

$= \pi\left(\left(72 - 72 + \frac{96}{5}\right) - \left(9 - \frac{9}{2} + \frac{3}{5}\right)\right)$

$= \pi\left(\frac{96}{5} - \frac{51}{10}\right)$

$= \frac{141}{10}\pi$

4 $\sqrt{x+2} = 0 \Rightarrow x = -2$

$V = \pi\int_{-2}^2 \left(\sqrt{x+2}\right)^2 \, dx$

$= \pi\int_{-2}^2 \left(x + 2\right) dx$

$= \pi\left[\frac{x^2}{2} + 2x\right]_{-2}^2$

$= \pi\left(\left(2 + 4\right) - \left(2 - 4\right)\right)$

$= 8\pi$

5 a $9x^{\frac{3}{2}} - 3x^{\frac{5}{2}} = 0$

$3x^{\frac{3}{2}}\left(3 - x\right) = 0$

$x = 0$ or $x = 3$

Coordinates of A are (3, 0)

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5 **b** $V = \pi\int_0^3 \left(9x^{\frac{3}{2}} - 3x^{\frac{5}{2}}\right)^2 \,\mathrm{d}x$

$= \pi\int_0^3 \left(81x^3 - 54x^4 + 9x^5\right)\mathrm{d}x$

$= \pi\left[\frac{81}{4}x^4 - \frac{54}{5}x^5 + \frac{3}{2}x^6\right]_0^3$

$= \pi\left(\frac{6561}{4} - \frac{13122}{5} + \frac{2187}{2}\right)$

$= \frac{2187}{20}\pi$

6 $\frac{\sqrt{3x^4-3}}{x^3} = 0 \Rightarrow x = \pm 1$

From the graph $x > 0$ so C cuts the x-axis at $x = 1$.

$V = \pi\int_1^6 \left(\frac{\sqrt{3x^4-3}}{x^3}\right)^2 \,\mathrm{d}x$

$= \pi\int_1^6 \left(\frac{3x^4-3}{x^6}\right)\mathrm{d}x$

$= \pi\int_1^6 \left(\frac{3}{x^2} - \frac{3}{x^6}\right)\mathrm{d}x$

$= \pi\left[-\frac{3}{x} + \frac{3}{5x^5}\right]_1^6$

$= \pi\left(\left(-\frac{3}{6} + \frac{3}{38\,880}\right) - \left(-3 + \frac{3}{5}\right)\right)$

$= 5.97$ (3 s.f.)

7 $5y^2 - x^3 = 2x - 3$

$y^2 = \frac{1}{5}\left(x^3 + 2x - 3\right)$

$V = \frac{\pi}{5}\int_1^4 \left(x^3 + 2x - 3\right)\mathrm{d}x$

$= \frac{\pi}{5}\left[\frac{x^4}{4} + x^2 - 3x\right]_1^4$

$= \frac{\pi}{5}\left(\left(64 + 16 - 12\right) - \left(\frac{1}{4} + 1 - 3\right)\right)$

$= \frac{\pi}{5}\left(68 + \frac{7}{4}\right)$

$= \frac{279}{20}\pi$

8 $V = \pi\int_a^2 \left(x\sqrt{4-x^2}\right)^2 \,\mathrm{d}x$

$= \pi\int_a^2 x^2\left(4 - x^2\right)\mathrm{d}x$

$= \pi\int_a^2 \left(4x^2 - x^4\right)\mathrm{d}x$

$= \pi\left[\frac{4x^3}{3} - \frac{x^5}{5}\right]_a^2$

$= \pi\left(\left(\frac{32}{3} - \frac{32}{5}\right) - \left(\frac{4a^3}{3} - \frac{a^5}{5}\right)\right)$

$= \pi\left(\frac{64}{15} - \frac{20a^3 - 3a^5}{15}\right)$

$= \frac{\pi}{15}\left(64 - 20a^3 + 3a^5\right)$

But $V = \frac{657}{160}\pi$ so $64 - 20a^3 + 3a^5 = \frac{15 \times 657}{160}$

$3a^5 - 20a^3 + \frac{77}{32} = 0$

For $0 < a < 2$,

try $a = 1 \Rightarrow -\frac{467}{32} \neq 0$,

try $a = \frac{1}{2} \Rightarrow 0$

Hence, $\left(a - \frac{1}{2}\right)$ is a factor of $3a^5 - 20a^3 + \frac{77}{32}$.

Hence, a solution of $3a^5 - 20a^3 + \frac{77}{32} = 0$ is $a = \frac{1}{2}$.

9 Equation of line is $y = r$

$V = \pi\int_0^h r^2 \,dx$

$= \pi\left[r^2 x\right]_0^h$

$= \pi r^2 h$

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Challenge

$x^2 - 7x + 10 = 0$

$(x-2)(x-5) = 0 \Rightarrow x = 2$ or $x = 5$

So the x coordinates of the points where the curve touches the x-axis are 2 and 5.

Splitting the region into 3 sections,

R_1, R_2 and R_3

R_1 for $1 \leqslant x \leqslant 2$

R_2 for $2 < x \leqslant 5$

R_3 for $5 < x \leqslant 6$

Volume V_1 is generated by R_1, etc.

$$V_1 = \pi\int_1^2 \left(x^2 - 7x + 10\right)^2 \, dx$$
$$= \pi\int_1^2 \left(x^4 - 14x^3 + 69x^2 - 140x + 100\right) dx$$
$$= \pi\left[\frac{x^5}{5} - \frac{7x^4}{2} + 23x^3 - 70x^2 + 100x\right]_1^2$$
$$= \pi\left(\frac{272}{5} - \frac{497}{10}\right) = \frac{47}{10}\pi$$

$$V_2 = \pi\int_2^5 \left(x^2 - 7x + 10\right)^2 \, dx$$
$$= \pi\int_2^5 \left(x^4 - 14x^3 + 69x^2 - 140x + 100\right) dx$$
$$= \pi\left[\frac{x^5}{5} - \frac{7x^4}{2} + 23x^3 - 70x^2 + 100x\right]_2^5$$
$$= \pi\left(\frac{125}{2} - \frac{272}{5}\right) = \frac{81}{10}\pi$$

$V_3 = V_1$ using the symmetry of the curve.

So total volume generated $= \pi\left(\frac{47}{10} + \frac{81}{10} + \frac{47}{10}\right)$

$$= \frac{175}{10}\pi$$
$$= \frac{35}{2}\pi$$

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Volumes of revolution 5B

1 a $V = \pi\int_2^5 \left(\frac{1}{2}y+1\right)^2 \,\mathrm{d}y$

$= \pi\int_2^5 \left(\frac{1}{4}y^2 + y + 1\right) \mathrm{d}y$

$= \pi\left[\frac{y^3}{12} + \frac{y^2}{2} + y\right]_2^5$

$= \pi\left(\left(\frac{125}{12} + \frac{25}{2} + 5\right) - \left(\frac{8}{12} + \frac{4}{2} + 2\right)\right)$

$= \pi\left(\frac{335}{12} - \frac{14}{3}\right)$

$= \frac{279}{12}\pi$

$= \frac{93}{4}\pi$

b $y = 2\sqrt{x} \Rightarrow x = \frac{y^2}{4}$

$V = \pi\int_0^1 \left(\frac{y^2}{4}\right)^2 \,\mathrm{d}y$

$= \pi\int_0^1 \frac{y^4}{16} \,\mathrm{d}y$

$= \pi\left[\frac{y^5}{80}\right]_0^1$

$= \frac{1}{80}\pi$

c $y = \frac{1}{x} \Rightarrow x = \frac{1}{y}$

$V = \pi\int_1^3 \left(\frac{1}{y}\right)^2 \,\mathrm{d}y$

$= \pi\int_1^3 \frac{1}{y^2} \,\mathrm{d}y$

$= \pi\left[-\frac{1}{y}\right]_1^3$

$= \pi\left(-\frac{1}{3} + 1\right)$

$= \frac{2}{3}\pi$

1 d $y = 2x^2 - 4 \Rightarrow x^2 = \frac{1}{2}(y+4)$

$x = 11 \Rightarrow y = 238$

$x = 5 \Rightarrow y = 46$

$V = \pi\int_{46}^{238} \frac{1}{2}(y+4) \,\mathrm{d}y$

$= \frac{\pi}{2}\left[\frac{y^2}{2} + 4y\right]_{46}^{238}$

$= \frac{\pi}{2}\left((28\,322 + 952) - (1058 + 184)\right)$

$= \frac{\pi}{2}(29\,274 - 1242)$

$= 14\,016\pi$

2 $V = \pi\int_1^4 \left(\frac{1}{2}y^2 + 1\right)^2 \,\mathrm{d}y$

$= \pi\int_1^4 \left(\frac{y^4}{4} + y^2 + 1\right) \mathrm{d}y$

$= \pi\left[\frac{y^5}{20} + \frac{y^3}{3} + y\right]_1^4$

$= \pi\left(\left(\frac{1024}{20} + \frac{64}{3} + 4\right) - \left(\frac{1}{20} + \frac{1}{3} + 1\right)\right)$

$= \pi\left(\frac{3072 + 1280 + 240 - 3 - 20 - 60}{60}\right)$

$= \frac{1503}{20}\pi$

3 a $A = \int_4^9 \left(\sqrt{y} + \frac{1}{y^2}\right) \mathrm{d}y$

$= \left[\frac{2}{3}y^{\frac{3}{2}} - \frac{1}{y}\right]_4^9$

$= \left(18 - \frac{1}{9}\right) - \left(\frac{16}{3} - \frac{1}{4}\right)$

$= \frac{648 - 4 - 192 + 9}{36}$

$= \frac{461}{36}$

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3 b $V = \pi\int_4^9 \left(\sqrt{y} + \frac{1}{y^2}\right)^2 \,\mathrm{d}y$

$= \pi\int_4^9 \left(y + 2y^{-\frac{3}{2}} + y^{-4}\right) \mathrm{d}y$

$= \pi\left[\frac{y^2}{2} - 4y^{-\frac{1}{2}} - \frac{y^{-3}}{3}\right]_4^9$

$= \pi(39.166 - 5.995)$

$= 104.21$ (2 d.p.)

4 a $A = \int_1^4 \left(y^2 - 6y + 10\right) \mathrm{d}y$

$= \left[\frac{y^3}{3} - 3y^2 + 10y\right]_1^4$

$= \left(\left(\frac{64}{3} - 48 + 40\right) - \left(\frac{1}{3} - 3 + 10\right)\right)$

$= 6$

b $V = \pi\int_1^4 \left(y^2 - 6y + 10\right)^2 \,\mathrm{d}y$

$= \pi\int_1^4 \left(y^4 - 12y^3 + 56y^2 - 120y + 100\right) \mathrm{d}y$

$= \pi\left[\frac{y^5}{5} - 3y^4 + \frac{56y^3}{3} - 60y^2 + 100y\right]_1^4$

$= \pi\left(\left(\frac{1024}{5} + \frac{3584}{3} - 1328\right) - \left(\frac{1}{5} + \frac{56}{3} + 37\right)\right)$

$= \frac{78}{5}\pi$

5 $y = 2x^2 + 5$

When $x = 0, y = 5$

So C cuts the y-axis at (0, 5)

Rearranging $x^2 = \frac{1}{2}(y - 5)$

$V = \pi\int_5^{10} \frac{1}{2}(y - 5) \,\mathrm{d}y$

$= \frac{\pi}{2}\left[\frac{y^2}{2} - 5y\right]_5^{10}$

$= \frac{\pi}{2}\left((50 - 50) - \left(\frac{25}{2} - 25\right)\right)$

$= \frac{25}{4}\pi$

6 a $y = x^2 - 2x + 1$

$y = (x - 1)^2$

$\sqrt{y} = x - 1 \quad y > 0$

$x = \sqrt{y} + 1$

$x^2 = \left(\sqrt{y} + 1\right)^2$

$x^2 = y + 2\sqrt{y} + 1$

b $V = \pi\int_1^9 \left(y + 2\sqrt{y} + 1\right) \mathrm{d}y$

$= \pi\left[\frac{y^2}{2} + \frac{4}{3}y^{\frac{3}{2}} + y\right]_1^9$

$= \pi\left(\left(\frac{81}{2} + 36 + 9\right) - \left(\frac{1}{2} + \frac{4}{3} + 1\right)\right)$

$= \frac{248}{3}\pi$

7 $y^3 + x^2 - 2y = 4$

So $x^2 = 4 + 2y - y^3$

$V = \pi\int_0^2 \left(4 + 2y - y^3\right) \,\mathrm{d}y$

$= \pi\left[4y + y^2 - \frac{y^4}{4}\right]_0^2$

$= \pi\left((8 + 4 - 4) - 0\right)$

$= 8\pi$

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8 $y^2 = \dfrac{1}{2x+1}$

So $2x+1 = \dfrac{1}{y^2}$

$$x = \frac{1}{2}\left(\frac{1}{y^2} - 1\right)$$

$$x^2 = \frac{1}{4}\left(\frac{1}{y^4} - \frac{2}{y^2} + 1\right)$$

$$V = \frac{\pi}{4}\int_1^4 \left(\frac{1}{y^4} - \frac{2}{y^2} + 1\right) \mathrm{d}y$$

$$= \frac{\pi}{4}\left[-\frac{1}{3y^3} + \frac{2}{y} + y\right]_1^4$$

$$= \frac{\pi}{4}\left(\left(-\frac{1}{192} + \frac{1}{2} + 4\right) - \left(-\frac{1}{3} + 2 + 1\right)\right)$$

$$= \frac{\pi}{4}\left(\frac{-1 + 96 + 768 + 64 - 384 - 192}{192}\right)$$

$$= \frac{117}{256}\pi$$

$= 1.44$ (2 d.p.)

9 Gradient of hypotenuse $= \dfrac{h}{r}$

So equation of hypotenuse is $y = \dfrac{h}{r}x$

So $x^2 = \dfrac{r^2}{h^2}y^2$

$$V = \pi\int_0^h \frac{r^2}{h^2}y^2 \, \mathrm{d}y$$

$$= \frac{\pi r^2}{h^2}\left[\frac{y^3}{3}\right]_0^h$$

$$= \frac{\pi r^2}{h^2}\left(\frac{h^3}{3} - 0\right)$$

$$= \frac{1}{3}\pi r^2 h$$

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Volumes of revolution 5C

1 a $3x+2y=27$

When $y=0,\ x=9$

$y=\frac{1}{2}(27-3x)$

$y^2=\frac{1}{4}(729-162x+9x^2)$

$V=\frac{\pi}{4}\int_0^9(729-162x+9x^2)\,\mathrm{d}x$

$=\frac{\pi}{4}\left[729x-81x^2+3x^3\right]_0^9$

$=\frac{2187}{4}\pi$

b $3x+2y=27$

When $x=0,\ y=13.5$

$x=\frac{1}{3}(27-2y)$

$x^2=\frac{1}{9}(729-108y+4y^2)$

$V=\frac{\pi}{9}\int_0^{13.5}(729-108y+4y^2)\,\mathrm{d}y$

$=\frac{\pi}{9}\left[729y-54y^2+\frac{4y^3}{3}\right]_0^{13.5}$

$=\frac{729}{2}\pi$

c Rotation about the x-axis: $r=13.5,\ h=9$

$V=\frac{1}{3}\pi r^2h=\frac{1}{3}\pi\times13.5^2\times9=\frac{2187}{4}\pi$

Rotation about the y-axis: $r=9,\ h=13.5$

$V=\frac{1}{3}\pi r^2h=\frac{1}{3}\pi\times9^2\times13.5=\frac{729}{2}\pi$

2 a Substituting $x=2$ into the equation of the line gives $y=8$.

Substituting $x=2$ into the equation of the curve gives $y=\frac{1}{2}\times4\times(2+2)=8$.

So (2, 8) are the coordinates of the point of intersection A.

2 b Volume V_1 is generated by rotating the curve about the x-axis.

$V_1=\pi\int_0^2\left(\frac{1}{2}x^2(x+2)\right)^2\,\mathrm{d}x$

$=\frac{\pi}{4}\int_0^2x^4\left(x^2+4x+4\right)\mathrm{d}x$

$=\frac{\pi}{4}\int_0^2\left(x^6+4x^5+4x^4\right)\mathrm{d}x$

$=\frac{\pi}{4}\left[\frac{x^7}{7}+\frac{2x^6}{3}+\frac{4x^5}{5}\right]_0^2$

$=\frac{\pi}{4}\left(\frac{128}{7}+\frac{128}{3}+\frac{128}{5}\right)$

$=\pi\left(\frac{32}{7}+\frac{32}{3}+\frac{32}{5}\right)$

$=\frac{480+1120+672}{105}\pi$

$=\frac{2272}{105}\pi$

Volume V_2 is generated by rotating the line about the x-axis, so is a cone.

When $y=0$, $x=4$ so $r=8,\ h=2$

$V_2=\frac{1}{3}\pi\times64\times2=\frac{128}{3}\pi$

Total volume $=\frac{2272}{105}\pi+\frac{128}{3}\pi$

$=\frac{2272+4480}{105}\pi$

$=\frac{6752}{105}\pi$

3 a Substituting $x=2$ into the equation of the line gives $y=4$.

Substituting $x=2$ into the equation of the curve gives $y=-\frac{1}{2}\times4\times(2-4)=4$.

So (2, 4) are the coordinates of the point of intersection A.

$y=0\Rightarrow x=4$

so the coordinates of B are (4, 0).

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3 b Volume generated by area under curve is given by $\pi\int_2^4\left(-\frac{1}{2}x^2(x-4)\right)^2\,\mathrm{d}x$

$=\frac{\pi}{4}\int_2^4 x^4\left(x^2-8x+16\right)\,\mathrm{d}x$

$=\frac{\pi}{4}\int_2^4\left(x^6-8x^5+16x^4\right)\,\mathrm{d}x$

$=\frac{\pi}{4}\left[\frac{x^7}{7}-\frac{4x^6}{3}+\frac{16x^5}{5}\right]_2^4$

$=\frac{\pi}{4}\left(\frac{16\,384}{105}-\frac{3712}{105}\right)$

$=\frac{3168}{105}\pi$

Volume generated by line is a cone with $r=4$ and $h=2$.

Volume of cone $=\frac{1}{3}\pi\times16\times2=\frac{32}{3}\pi$

Volume generated by R $=\frac{3168}{105}\pi-\frac{32}{3}\pi$

$=\frac{2048}{105}\pi$

4 a Substituting $x=-2$ into the linear equation $2x-y=-6$ gives $y=2$.
Substituting $x=-2$ into the equation of the curve $y=\frac{1}{2}x^2$ gives $y=2$.
So (−2, 2) are the coordinates of the point of intersection *A*.

Substituting $x=2$ into the linear equation $2x+y=6$ gives $y=2$.
Substituting $x=2$ into the equation of the curve $y=\frac{1}{2}x^2$ gives $y=2$.
So (2, 2) are the coordinates of the point of intersection *B*.

4 b Volume generated by the curve

$=\pi\int_0^2 x^2\,\mathrm{d}y$

$=\pi\int_0^2 2y\,\mathrm{d}y$

$=\pi\left[y^2\right]_0^2$

$=4\pi$

Volume generated by the line $2x+y=6$ is a cone with $r=2$ and $h=4$

Volume of cone $=\frac{1}{3}\pi\times4\times4=\frac{16}{3}\pi$

Total volume $=4\pi+\frac{16}{3}\pi=\frac{28}{3}\pi$

5 Volume generated by *C* about the *y*-axis

$=\pi\int_0^2\left(4-y^2\right)\,\mathrm{d}y$

$=\pi\left[4y-\frac{y^3}{3}\right]_0^2$

$=\pi\left(8-\frac{8}{3}\right)=\frac{16}{3}\pi$

Volume of cylinder with $r=3$ and $h=3$
$=\pi\times9\times3=27\pi$

Volume generated by *R*

$=27\pi-\frac{16}{3}\pi$

$=\frac{65}{3}\pi$

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6 $y = -\frac{1}{5}x^2 + 5$ intersects the y-axis at (0, 5)

Rearranging… $x^2 = 25 - 5y$

Volume generated by the curve about the y-axis

$$= \pi\int_0^5 x^2 \, dy$$
$$= \pi\int_0^5 (25 - 5y) \, dy$$
$$= \pi\left[25y - \frac{5y^2}{2}\right]_0^5$$
$$= \pi\left(125 - \frac{125}{2}\right)$$
$$= \frac{125}{2}\pi$$

Volume to be subtracted is a cone with $r = 4$ and $h = 4$.

Volume of cone $= \frac{1}{3}\pi \times 16 \times 4 = \frac{64}{3}\pi$

Volume generated by $R = \frac{125}{2}\pi - \frac{64}{3}\pi$

$$= \frac{247}{6}\pi$$

7 $6y^2 - x^3 + 4x = 0$

$y = 0 \Rightarrow 4x - x^3 = 0$

$x(4 - x^2) = 0$

$x = 2$ since $x > 0$

So C cuts the x-axis at (2, 0).

Volume generated by the curve about the x-axis

$$= \pi\int_2^4 y^2 \, dx$$
$$= \frac{\pi}{6}\int_2^4 (x^3 - 4x) \, dx$$
$$= \frac{\pi}{6}\left[\frac{x^4}{4} - 2x^2\right]_2^4$$
$$= \frac{\pi}{6}((64 - 32) - (4 - 8))$$
$$= \frac{36\pi}{6}$$
$$= 6\pi$$

Volume generated by lines is a cone with $r = 4$ and $h = 3$

Volume of cone $= \frac{1}{3}\pi \times 16 \times 3 = 16\pi$

Required volume $= 16\pi - 6\pi = 10\pi$

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8 Volume generated by rotating the curve $y = 4 - x^2$ about the x-axis

$$= \pi\int_0^1 \left(4 - x^2\right)^2 \, dx$$

$$= \pi\int_0^1 \left(16 - 8x^2 + x^4\right) dx$$

$$= \pi\left[16x - \frac{8x^3}{3} + \frac{x^5}{5}\right]_0^1$$

$$= \pi\left(16 - \frac{8}{3} + \frac{1}{5}\right)$$

$$= \frac{203}{15}\pi$$

Volume generated by rotating the curve $y = \sqrt[3]{x}$ about the x-axis

$$= \pi\int_0^1 x^{\frac{2}{3}} \, dx$$

$$= \pi\left[\frac{3x^{\frac{5}{3}}}{5}\right]_0^1$$

$$= \frac{3}{5}\pi$$

Required volume $= \frac{203}{15}\pi - \frac{3}{5}\pi = \frac{194}{15}\pi$

9 a $x^2 + y^2 = 11$

$x^2 + \left(x^2 + 1\right)^2 = 11$

$x^2 + x^4 + 2x^2 + 1 = 11$

$x^4 + 3x^2 - 10 = 0$

$\left(x^2 + 5\right)\left(x^2 - 2\right) = 0$

$x = \pm\sqrt{2}$

So the x coordinates of the points of intersection are $-\sqrt{2}$ and $\sqrt{2}$.

9 b Volume generated by $x^2 + y^2 = 11$ between $x = \pm\sqrt{2}$ is $\pi\int_{-\sqrt{2}}^{\sqrt{2}} \left(11 - x^2\right) dx$

$$= \pi\left[11x - \frac{x^3}{3}\right]_{-\sqrt{2}}^{\sqrt{2}}$$

$$= \frac{\pi}{3}\left(\left(33\sqrt{2} - 2\sqrt{2}\right) - \left(-33\sqrt{2} + 2\sqrt{2}\right)\right)$$

$$= \frac{62\sqrt{2}}{3}\pi$$

Volume generated by $y = x^2 + 1$ between $x = \pm\sqrt{2}$ is $\pi\int_{-\sqrt{2}}^{\sqrt{2}} \left(x^2 + 1\right)^2 \, dx$

$$= \pi\int_{-\sqrt{2}}^{\sqrt{2}} \left(x^4 + 2x^2 + 1\right) dx$$

$$= \pi\left[\frac{x^5}{5} + \frac{2x^3}{3} + x\right]_{-\sqrt{2}}^{\sqrt{2}}$$

$$= \pi\left(\frac{4\sqrt{2}}{5} + \frac{4\sqrt{2}}{3} + \sqrt{2}\right) - \left(-\frac{4\sqrt{2}}{5} - \frac{4\sqrt{2}}{3} - \sqrt{2}\right)$$

$$= \pi\left(\frac{8\sqrt{2}}{5} + \frac{8\sqrt{2}}{3} + 2\sqrt{2}\right)$$

$$= \frac{94\sqrt{2}}{15}\pi$$

Required volume

$$= \frac{62\sqrt{2}}{3}\pi - \frac{94\sqrt{2}}{15}\pi$$

$$= \frac{72\sqrt{2}}{5}\pi$$

$= 63.98$ (2 d.p.)

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Challenge

Required volume V will be the volume V_1 of the cone with $r = 5$ and $h = \frac{40}{3}$

minus the volume V_2 of the cone with $r = 2$ and $h = \frac{40}{3} - 8 = \frac{16}{3}$

minus the volume V_3 generated by the curve $y = \frac{64}{x^3}$ between $y = 1$ and $y = 8$.

$$V_1 = \frac{1}{3}\pi \times 25 \times \frac{40}{3} = \frac{1000}{9}\pi$$

$$V_2 = \frac{1}{3}\pi \times 4 \times \frac{16}{3} = \frac{64}{9}\pi$$

$$V_3 = \pi\int_1^8 x^2 \, dy$$

$$= \pi\int_1^8 \left(\sqrt[3]{\frac{64}{y}}\right)^2 dy$$

$$= 16\pi\int_1^8 y^{-\frac{2}{3}} \, dy$$

$$= 48\pi\left[y^{\frac{1}{3}}\right]_1^8$$

$$= 48\pi$$

$$V = V_1 - V_2 - V_3$$

$$= \frac{1000}{9}\pi - \frac{64}{9}\pi - 48\pi$$

$$= 56\pi$$

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Volumes of revolution 5D

1 a k represents the height in metres of the tent at its centre, say $5 \leqslant k \leqslant 15$.

b $V = \pi\int_0^k x^2\ dy$

$= 100\pi\int_0^k \left(k^2 - y^2\right) \mathrm{d}y$

$= 100\pi\left[k^2 y - \frac{y^3}{3}\right]_0^k$

$= 100\pi\left(\frac{2}{3}k^3\right)$

$= \frac{200}{3}k^3\pi \text{ m}^3$

c The actual tent would probably not follow the exact same shape as the model. For example it would probably not be exactly circular looking down on it.

2 $y^2 = 4(16 - x)$

$x = 16 - \frac{y^2}{4}$

$V = \pi\int_{-8}^{8}\left(16 - \frac{y^2}{4}\right)^2 \mathrm{d}y$

$= \pi\int_{-8}^{8}\left(256 - 8y^2 + \frac{y^4}{16}\right) \mathrm{d}y$

$= \pi\left[256y - \frac{8y^3}{3} + \frac{y^5}{80}\right]_{-8}^{8}$

$= 2\pi\left(2048 - \frac{4096}{3} + \frac{32\,768}{80}\right)$

$= 4096\pi\left(1 - \frac{2}{3} + \frac{1}{5}\right)$

$= 4096\pi\left(\frac{8}{15}\right)$

$= \frac{32\,768}{15}\pi \text{ cm}^3$

3 a $\frac{x^2}{9} + \frac{y^2}{4} = 1 \Rightarrow y^2 = 4\left(1 - \frac{x^2}{9}\right)$

$V = 4\pi\int_{-3}^{3}\left(1 - \frac{x^2}{9}\right) \mathrm{d}x$

$= 4\pi\left[x - \frac{x^3}{27}\right]_{-3}^{3}$

$= 4\pi\left((3-1) - (-3+1)\right)$

$= 16\pi \text{ cm}^3$

b $\frac{x^2}{9} + \frac{y^2}{4} = 1 \Rightarrow x^2 = 9\left(1 - \frac{y^2}{4}\right)$

$V = 9\pi\int_{-2}^{2}\left(1 - \frac{y^2}{4}\right) \mathrm{d}y$

$= 9\pi\left[y - \frac{y^3}{12}\right]_{-2}^{2}$

$= 9\pi\left(\left(2 - \frac{2}{3}\right) - \left(-2 + \frac{2}{3}\right)\right)$

$= 24\pi \text{ cm}^3$

c The solid formed by rotating the ellipse about the x-axis will be more like the shape of an egg. The solid formed by rotating about the y-axis will have a flatter disc shape.

4 Volume of sand which flows in 5 minutes is 40 cm^3.

Let b be the required height of the sand.

$40 = \pi\int_0^b \left(\sqrt[3]{y}\right)^2 \mathrm{d}y$

$40 = \pi\left[\frac{3}{5}y^{\frac{5}{3}}\right]_0^b$

$40 = \frac{3}{5}\pi b^{\frac{5}{3}}$

$b^{\frac{5}{3}} = \frac{200}{3\pi}$

$b = \left(\frac{200}{3\pi}\right)^{\frac{3}{5}} = 6.25 \text{ cm}$

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5 a $y = 0.02x^3$

$y = 18 \Rightarrow x^3 = 900$

Diameter of bowl $= 2 \times \sqrt[3]{900} = 19.3$ cm

b $V = \pi \int_0^{18} x^2 \, dy$

$= \pi \int_0^{18} \left(\sqrt[3]{50y}\right)^2 \, dy$

$= \sqrt[3]{2500}\pi \left[\frac{3}{5} y^{\frac{5}{3}}\right]_0^{18}$

$= \sqrt[3]{2500}\pi \left(\frac{3}{5} 18^{\frac{5}{3}}\right)$

$= 3162.8 \text{ cm}^3$

c Area of $R = \int_0^{12} \sqrt[3]{50y} \, dy$

$= \sqrt[3]{50} \times \left[\frac{3}{4} y^{\frac{4}{3}}\right]_0^{12}$

$= 75.9 \text{ cm}^2$

d Let V_p be the volume of liquid mixed by the paddle.

$V_p = \pi \int_0^{12} \left(\sqrt[3]{50y}\right)^2 \, dy$

$= \sqrt[3]{2500}\pi \left[\frac{3}{5} y^{\frac{5}{3}}\right]_0^{12}$

$= \sqrt[3]{2500}\pi \left(\frac{3}{5} 12^{\frac{5}{3}}\right)$

$= 1609.1 \text{ cm}^3$

Proportion of volume in bowl that can be mixed $= \frac{1609.1}{3162.8} = 0.509$

Proportion could be found directly:

$\left(\frac{12}{18}\right)^{\frac{5}{3}} = \left(\frac{2}{3}\right)^{\frac{5}{3}} = 0.509$

6 a $V = \pi \int_0^{20} x^2 \, dy$

$= \pi \int_0^{20} \left(5 - \sqrt{y}\right)^2 \, dy$

$= \pi \int_0^{20} \left(25 - 10\sqrt{y} + y\right) \, dy$

$= \pi \left[25y - \frac{20}{3} y^{\frac{3}{2}} + \frac{1}{2} y^2\right]_0^{20}$

$= \pi \left(500 - \frac{20}{3} \times 20^{\frac{3}{2}} + 200\right)$

$= 325.8 \text{ cm}^3$

b Volume of water to height 10 cm

$= \pi \left[25y - \frac{20}{3} y^{\frac{3}{2}} + \frac{1}{2} y^2\right]_0^{10}$

$= \pi \left(250 - \frac{20}{3} \times 10^{\frac{3}{2}} + 50\right)$

$= 280.2 \text{ cm}^3$

So adding another 50 cm^3 will take the volume over the 325.8 cm^3 capacity of the vase and it will therefore overflow.

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7 $y = 2x + 18$ intersects the x-axis at $(-9, 0)$ and the y-axis at $(0, 18)$ so the uppermost part of the top is a cone with $r = 9$ and $h = 18$.

$$\text{Volume of this cone} = \frac{1}{3}\pi \times 81 \times 18$$
$$= 486\pi \text{ cm}^3$$

The curve intersects the y-axis at $(0, -6)$ so the volume of the lower part of the top is

$$\pi\int_{-6}^{0} x^2 \, dy = \pi\int_{-6}^{0}\left(\frac{y^2}{4} - 9\right)^2 \, dy$$
$$= \pi\int_{-6}^{0}\left(\frac{y^4}{16} - \frac{9y^2}{2} + 81\right) dy$$
$$= \pi\left[\frac{1}{80}y^5 - \frac{3}{2}y^3 + 81y\right]_{-6}^{0}$$
$$= \pi\left((0) - \left(-\frac{486}{5} + 324 - 486\right)\right)$$
$$= \pi\left(\frac{486 - 1620 + 2430}{5}\right)$$
$$= \frac{1296}{5}\pi$$
$$= 259.2\pi \text{ cm}^3$$

$$\text{Total volume of top} = 486\pi + 259.2\pi$$
$$= 745.2\pi \text{ cm}^3$$

Challenge

Volume of frustum generated by the line $3y + 4x = 24$

$$= \pi\int_0^4 x^2 \, dy$$
$$= \pi\int_0^4 \left(6 - \frac{3y}{4}\right)^2 \, dy$$
$$= \pi\int_0^4 \left(36 - 9y + \frac{9y^2}{16}\right) dy$$
$$= \pi\left[36y - \frac{9}{2}y^2 + \frac{3}{16}y^3\right]_0^4$$
$$= \pi(144 - 72 + 12)$$
$$= 84\pi \text{ cm}^3$$

Volume removed from top of frustum by rotating the curve $y = \frac{1}{9}x^2 + 3$ about the y-axis

$$= \pi\int_3^4 x^2 \, dy$$
$$= \pi\int_3^4 (9y - 27) \, dy$$
$$= \pi\left[\frac{9}{2}y^2 - 27y\right]_3^4$$
$$= \pi\left((72 - 108) - \left(\frac{81}{2} - 81\right)\right)$$
$$= \pi\left(-36 - \frac{81}{2} + 81\right)$$
$$= \frac{9}{2}\pi \text{ cm}^3$$

Volume of place-holder

$$= 84\pi - \frac{9}{2}\pi$$
$$= \frac{159}{2}\pi \text{ cm}^3$$

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Volumes of revolution ME 5

1 $V = \pi\int_0^3 y^2 \, \mathrm{d}x$

$= \pi\int_0^3 x^4\left(9 - x^2\right) \mathrm{d}x$

$= \pi\int_0^3 \left(9x^4 - x^6\right) \mathrm{d}x$

$= \pi\left[\frac{9x^5}{5} - \frac{x^7}{7}\right]_0^3$

$= \pi\left(\frac{2187}{5} - \frac{2187}{7}\right)$

$= \frac{4374}{35}\pi$

2 a $2y^2 - 6\sqrt{x} + 3 = 0$

$y = 0 \Rightarrow 3 = 6\sqrt{x} \Rightarrow x = \frac{1}{4}$

Curve cuts the x-axis when $x = \frac{1}{4}$.

b $V = \pi\int_{0.25}^4 y^2 \, \mathrm{d}x$

$= \pi\int_{0.25}^4 \left(3\sqrt{x} - \frac{3}{2}\right) \mathrm{d}x$

$= 3\pi\int_{0.25}^4 \left(\sqrt{x} - \frac{1}{2}\right) \mathrm{d}x$

$= 3\pi\left[\frac{2}{3}x^{\frac{3}{2}} - \frac{1}{2}x\right]_{0.25}^4$

$= 3\pi\left(\left(\frac{16}{3} - 2\right) - \left(\frac{1}{12} - \frac{1}{8}\right)\right)$

$= 3\pi\left(\frac{10}{3} + \frac{1}{24}\right)$

$= 3\pi \times \frac{81}{24}$

$= \frac{81}{8}\pi$

3 a $\mathrm{f}(x) = x^2 + 4x + 4$

$y = \mathrm{f}(x) \Rightarrow y = x^2 + 4x + 4$

$y = (x+2)^2$

$\sqrt{y} = x + 2$

$x = \sqrt{y} - 2$

$x^2 = \left(\sqrt{y} - 2\right)^2$

$x^2 = 4 - 4\sqrt{y} + y$

b $V = \pi\int_4^9 x^2 \, \mathrm{d}y$

$= \pi\int_4^9 \left(4 - 4\sqrt{y} + y\right) \mathrm{d}y$

$= \pi\left[4y - \frac{8}{3}y^{\frac{3}{2}} + \frac{1}{2}y^2\right]_4^9$

$= \pi\left(\left(36 - 72 + \frac{81}{2}\right) - \left(16 - \frac{64}{3} + 8\right)\right)$

$= \pi\left(\frac{9}{2} - \frac{8}{3}\right)$

$= \frac{11}{6}\pi$

4 a $V = \pi\int_0^1 y^2 \, \mathrm{d}x$

$= \pi\int_0^1 \left(x^2 + 3\right)^2 \, \mathrm{d}x$

$= \pi\int_0^1 \left(x^4 + 6x^2 + 9\right) \mathrm{d}x$

$= \pi\left[\frac{1}{5}x^5 + 2x^3 + 9x\right]_0^1$

$= \frac{56}{5}\pi$

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4 b $x=1 \Rightarrow y=4$ and $x=0 \Rightarrow y=3$

Volume generated by curve when rotated about the *y*-axis

$$= \pi\int_3^4 x^2 \, \mathrm{d}y$$
$$= \pi\int_3^4 (y-3) \, \mathrm{d}y$$
$$= \pi\left[\frac{1}{2}y^2 - 3y\right]_3^4$$
$$= \pi\left((8-12)-\left(\frac{9}{2}-9\right)\right)$$
$$= \frac{1}{2}\pi$$

Volume of cylinder generated when the line $x=1$ is rotated about the *y*-axis
$= \pi \times 1 \times 4 = 4\pi$

Volume obtained when region is rotated about the *y*-axis = $4\pi - \frac{1}{2}\pi = \frac{7}{2}\pi$

5 a Substituting $x=2$ and $y=4.5$ into the equation of the line gives
$3\times 2 + 4 \times 4.5 = 24$
Substituting $x=2$ into the equation of the gives $y = \frac{1}{4}\times 2 \times (2+1)^2 = 4.5$
So (2, 4.5) are the coordinates of the point of intersection *A*.

5 b Volume V_1 is generated by the curve between *O* and *A*.

$$V_1 = \pi\int_0^2 y^2 \, \mathrm{d}x$$
$$= \frac{\pi}{16}\int_0^2 x^2(x+1)^4 \, \mathrm{d}x$$
$$= \frac{\pi}{16}\int_0^2 x^2\left(x^4+4x^3+6x^2+4x+1\right) \, \mathrm{d}x$$
$$= \frac{\pi}{16}\int_0^2 \left(x^6+4x^5+6x^4+4x^3+x^2\right) \, \mathrm{d}x$$
$$= \frac{\pi}{16}\left[\frac{1}{7}x^7+\frac{2}{3}x^6+\frac{6}{5}x^5+x^4+\frac{1}{3}x^3\right]_0^2$$
$$= \frac{\pi}{16}\left(\frac{128}{7}+\frac{128}{3}+\frac{192}{5}+16+\frac{8}{3}\right)$$
$$= \pi\left(\frac{8}{7}+\frac{8}{3}+\frac{12}{5}+1+\frac{1}{6}\right)$$
$$= \pi\left(\frac{240+560+504+210+35}{210}\right)$$
$$= \frac{1549}{210}\pi$$

Volume V_2 generated by the line $3x+4y=24$ is a cone with $r=4.5$ and $h=6$

$$V_2 = \frac{1}{3}\pi \times \left(\frac{9}{2}\right)^2 \times 6$$
$$= \frac{81}{2}\pi$$

Total volume $= \frac{1549}{210}\pi + \frac{81}{2}\pi = \frac{5027}{105}\pi$

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6 a y coordinate of points A and B
$= 0.1 \times 2^2 + 4 = 4.4$

Volume of cylinder with $r = 2$ and $h = 4.4$
$= \pi \times 2^2 \times 4.4 = 17.6\pi \text{ cm}^3$

Volume generated by curve
$= \pi \int_4^{4.4} x^2 \, dy$
$= 10\pi \int_4^{4.4} (y - 4) \, dy$
$= 10\pi \left[\frac{1}{2} y^2 - 4y \right]_4^{4.4}$
$= 10\pi \left((9.68 - 17.6) - (8 - 16) \right)$
$= 10\pi \left(-7.92 - (-8) \right)$
$= 0.8\pi$

Volume of bronze required
$= 17.6\pi - 0.8\pi$
$= 16.8\pi \text{ cm}^3$

b For example, the shape of the holder is unlikely to exactly follow the curve, or it doesn't allow for any wastage.

7 Volume of the cap of the mushroom
$= \pi \int_0^4 x^2 \, dy$
$= 4\pi \int_0^4 \left(8\sqrt{y} - 4y \right) \, dy$
$= 4\pi \left[\frac{16}{3} y^{\frac{3}{2}} - 2y^2 \right]_0^4$
$= 4\pi \left(\frac{128}{3} - 32 \right)$
$= \frac{128}{3} \pi$

The stem of mushroom is a cylinder with $r = 1$ and $h = 4$

Volume of stem $= \pi \times 1 \times 4 = 4\pi$

Total volume of mushroom
$= \frac{128}{3} \pi + 4\pi = \frac{140}{3} \pi \text{ cm}^3$

8 The y coordinate of the points of intersection of the two curves will be given by solving simultaneously

$3y^2 + x^2 - 11y = 0$ (1)
$y = 2x^2$ (2)

$3y^2 + \frac{y}{2} - 11y = 0$
$6y^2 - 21y = 0$
$2y^2 - 7y = 0$
$y(2y - 7) = 0$
$y = 0$ or $y = \frac{7}{2}$

Volume V_1 is generated by the curve
$3y^2 + x^2 - 11y = 0$ between $y = 0$ and $y = \frac{7}{2}$

$V_1 = \pi \int_0^{\frac{7}{2}} \left(11y - 3y^2 \right) \, dy$
$= \pi \left[\frac{11}{2} y^2 - y^3 \right]_0^{\frac{7}{2}}$
$= \pi \left(\frac{539}{8} - \frac{343}{8} \right)$
$= \frac{49}{2} \pi$

Volume V_2 is generated by the curve $y = 2x^2$ between $y = 0$ and $y = \frac{7}{2}$

$V_2 = \pi \int_0^{\frac{7}{2}} \frac{y}{2} \, dy$
$= \pi \left[\frac{y^2}{4} \right]_0^{\frac{7}{2}}$
$= \frac{49}{16} \pi$

Volume generated by shaded region
$= V_1 - V_2$
$= \frac{49}{2} \pi - \frac{49}{16} \pi$
$= \frac{343}{16} \pi$

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Challenge

a Using Pythagoras' theorem:

$r^2 = x^2 + y^2$

So $y^2 = r^2 - x^2$

Area of disc $= \pi y^2 = \pi\left(r^2 - x^2\right)$

b Volume of sphere will be obtained by integrating this disc area between the limits $x = -r$ to $x = r$

$$= \pi\int_{-r}^{r}\left(r^2 - x^2\right)\,\mathrm{d}x$$

$$= \pi\left[r^2x - \frac{1}{3}x^3\right]_{-r}^{r}$$

$$= \pi\left(\frac{2}{3}r^3 - \left(-\frac{2}{3}r^3\right)\right)$$

$$= \frac{4}{3}\pi r^3$$

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Review exercise 1

1 a $z_1 - z_2$
$= 4 - 5i - pi$
$= 4 - (5 + p)i$

b $z_1 z_2$
$= (4 - 5i)pi$
$= 4pi - 5pi^2$
$= 4pi + 5p$
$= 5p + 4pi$

c $\frac{z_1}{z_2}$
$= \frac{4 - 5i}{pi}$
$= \frac{i(4 - 5i)}{-p}$
$= \frac{4i + 5}{-p}$
$= -\frac{5}{p} - \frac{4}{p}i$

2 a $z^3 - kz^2 + 3z$
$= z(z^2 - kz + 3)$
So if there are 2 imaginary roots, the discriminant of $z^2 - kz + 3 < 0$
$\Rightarrow (-k)^2 - 12 < 0$
$k^2 < 12$
$-2\sqrt{3} < k < 2\sqrt{3}$

b $z^3 - 2z^2 + 3z = 0$
$\Rightarrow z(z^2 - 2z + 3) = 0$
$\Rightarrow z = 0, z = \frac{2 \pm \sqrt{-8}}{2}$
$\Rightarrow z = 0, z = 1 \pm i\sqrt{2}$

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3

$$z = \frac{5 \pm \sqrt{25-52}}{2}$$
$$= \frac{5 \pm \sqrt{-27}}{2}$$
$$= \frac{5}{2} \pm \frac{3\sqrt{3}}{2}i$$
So $z_1, z_2 = \frac{5}{2} + \frac{3\sqrt{3}}{2}i, \frac{5}{2} - \frac{3\sqrt{3}}{2}i$

4 $(2-i)x-(1+3i)y-7=0$
$\Rightarrow (2x-y-7)+(-x-3y)\mathrm{i}=0$
$\Rightarrow 2x-y=7, x+3y=0$
$\Rightarrow x=3, y=-1$

5 a $\frac{2+3\mathrm{i}}{5+\mathrm{i}} \times \frac{5-\mathrm{i}}{5-\mathrm{i}} = \frac{10-2\mathrm{i}+15\mathrm{i}+3}{26}$
$= \frac{13+13\mathrm{i}}{26} = \frac{1}{2} + \frac{1}{2}\mathrm{i}$
$= \frac{1}{2}(1+\mathrm{i})$
$\lambda = \frac{1}{2}$

$(5+\mathrm{i})(5-\mathrm{i}) = 5^2 + 1^2 = 26$ You should practise doing such calculations mentally.

You use the result from part **a** to simplify the working in part **b**.

b $\left(\frac{2+3\mathrm{i}}{5+\mathrm{i}}\right)^4 = \left[\frac{1}{2}(1+\mathrm{i})\right]^4$
$= \frac{1}{16}(1+4\mathrm{i}+6\mathrm{i}^2+4\mathrm{i}^3+\mathrm{i}^4)$
$= \frac{1}{16}(1+4\mathrm{i}-6-4\mathrm{i}+1)$
$= \frac{1}{16} \times -4 = -\frac{1}{4}$, a real number

$(1+\mathrm{i})^4$ is expanded using the binomial expansion
$(a+b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$
$\mathrm{i}^3 = \mathrm{i}^2 \times \mathrm{i} = -1 \times \mathrm{i} = -\mathrm{i}$
$\mathrm{i}^4 = \mathrm{i}^2 \times \mathrm{i}^2 = -1 \times -1 = 1$

6 $-1+i$ is a root $\Rightarrow -1-i$ is also a root
$\Rightarrow (z+1-i)(z+1+i)$ is a factor
$\Rightarrow z^2+2z+2$ is a factor
$\Rightarrow z^3+5z^2+8z+6 = (z^2+2z+2)(z+3)$
$\Rightarrow z = -3, -1 \pm i$

7 a $f(2-3i)=0$
$\Rightarrow (2-3i)^3 - 6(2-3i)^2 + k(2-3i) - 26 = 0$
$\Rightarrow 8-36i-54+27i-24+72i+54+2k-3ki-26=0$
Equating real coefficients $-42+2k=0 \Rightarrow k=21$

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7 b $2+3i$ must also be a factor

$\Rightarrow (z-2+3i)(z-2-3i) = z^2-4z+13$ is a factor

$\Rightarrow z^3-6z^2+21z-26 = (z^2-4z+13)(z-2)$

$\Rightarrow z = 2, 2+3i$ are the other two factors

8 a $b-3=-1 \Rightarrow b=2$

$-4c=-16 \Rightarrow c=4$

$\Rightarrow z^4-z^3-6z^2-20z-16 = (z^2-3z-4)(z^2+2z+4)$

b $z^4-z^3-6z^2-20z-16 = (z-4)(z+1)(z^2+2z+4)$

$\Rightarrow z = 4, -1, \dfrac{-2\pm\sqrt{12}}{2}$

$\Rightarrow z = 4, -1, -1\pm\sqrt{3}i$

9 $(z-1-2i)(z-1+2i)$ must be a factor

$\Rightarrow z^2-2z+5$ is a factor

$\Rightarrow z^4-8z^3+27z^2-50z+50$

$= (z^2-2z+5)(z^2+kz+10)$

Equating coefficients of z^3

$-2+k=-8 \Rightarrow k=-6$

$\Rightarrow (z^2-2z+5)(z^2-6z+10)=0$

$\Rightarrow z = 1\pm 2i, \dfrac{6\pm\sqrt{-4}}{2}$

$\Rightarrow z = 1\pm 2i, 3\pm i$

10 a Comparing constant coefficients

$\alpha \times \dfrac{4}{\alpha} \times (\alpha + \dfrac{4}{\alpha} + 1) = 12$

$\Rightarrow 4(\alpha + \dfrac{4}{\alpha} + 1) = 12$

$\Rightarrow \alpha^2 + 4 + \alpha = 3\alpha$

$\Rightarrow \alpha^2 - 2\alpha + 4 = 0$

$\Rightarrow \alpha = 1\pm\sqrt{3}i$

So the roots are $1\pm\sqrt{3}i, 3$

b $f(z) = (z-3)(z-1-\sqrt{3}i)(z-1-\sqrt{3}i)$

$= (z-3)(z^2-2z+4)$

$= z^3-5z^2+10z-12$

$\Rightarrow p=-5, q=10$

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11 a $\dfrac{3z-1}{2-\mathrm{i}} = \dfrac{4}{1+2\mathrm{i}}$

$3z-1 = \dfrac{8-4\mathrm{i}}{1+2\mathrm{i}} \times \dfrac{1-2\mathrm{i}}{1-2\mathrm{i}}$

$= \dfrac{8-16\mathrm{i}-4\mathrm{i}-8}{5} = \dfrac{-20\mathrm{i}}{5} = -4\mathrm{i}$

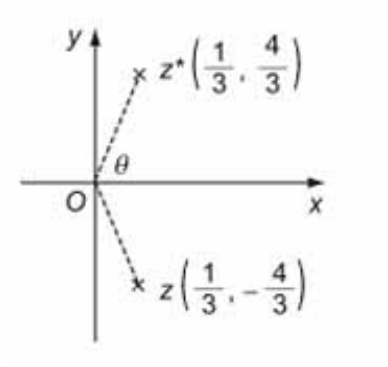
$3z = 1-4\mathrm{i}$

$z = \dfrac{1}{3} - \dfrac{4}{3}\mathrm{i}$

You multiply both sides of the equation by $2-\mathrm{i}$.

Then multiply the numerator and denominator by the conjugate complex of the denominator.

b



You place the points in the Argand diagram which represent conjugate complex numbers symmetrically about the real x-axis.

Label the points so it is clear which is the original number $(\mathbf{z})$ and which is the conjugate $(\mathbf{z^*})$.

c $|z|^2 = \left(\dfrac{1}{3}\right)^2 + \left(-\dfrac{4}{3}\right)^3 = \dfrac{1}{9} + \dfrac{16}{9} = \dfrac{17}{9}$

$|z| = \dfrac{\sqrt{17}}{3}$

$\tan\theta = \dfrac{\frac{4}{3}}{\frac{1}{3}} = 4 \Rightarrow \theta \approx 76^\circ$

z is in the fourth quadrant.

$\arg z = -76^\circ$, to the nearest degree.

$z = \dfrac{\sqrt{17}}{3}\cos(-76^\circ) + \mathrm{i}\dfrac{\sqrt{17}}{3}\sin(-76^\circ)$

$z^* = \dfrac{\sqrt{17}}{3}\cos 76^\circ + \mathrm{i}\dfrac{\sqrt{17}}{3}\sin 76^\circ$

The diagram you have drawn in part **b** shows that z is in the fourth quadrant. There is no need to draw it again.

It is always true $|z^*| = |z|$ and $\arg z^* = -\arg z$, so you just write down the final answer without further working.

12 z lies on a circle radius 1, centre $4i$

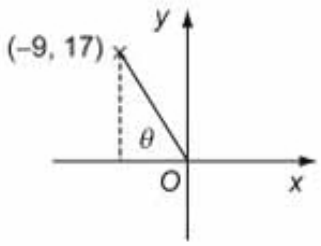
So the maximum and minimum arguments lie on the tangents to the circle from O

$\Rightarrow$ minimum = $\arctan(\sqrt{15}) = 1.318$

maximum = $\pi - \arctan(\sqrt{15}) = 1.823$

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13 a



b $\tan\theta = \frac{17}{9} \Rightarrow \theta = 1.084...$

z is in the second quadrant.

$\arg z = \pi - 1.084\ldots = 2.057\ldots$

$= 2.06$, in radians to 2 d.p.

You have to give your answer to 2 decimal places. To do this accurately you must work to at least 3 decimal places. This avoids rounding errors and errors due to premature approximation.

c $w = \frac{25+35\text{i}}{z} = \frac{25+35\text{i}}{-9+17\text{i}} = \frac{25+35\text{i}}{-9+17\text{i}} \times \frac{-9-17\text{i}}{-9-17\text{i}}$

$= \frac{-225-425\text{i}-315\text{i}+595}{(-9)^2+17^2}$

$= \frac{370-740\text{i}}{370} = 1-2\text{i}$

In this question, the arithmetic gets complicated. Use a calculator to help you with this. However, when you use a calculator, remember to show sufficient working to make your method clear.

14 a $|z_1|^2 = 5^2+1^2 = 26$

$|z_2|^2 = (-2)^2+3^2 = 4+9 = 13$

$26 = 2\times 13$

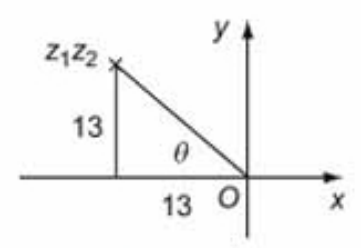
Hence $|z_1|^2 = 2|z_2|^2$, as required.

If $z = a+\text{i}b$, then $|z|^2 = a^2+b^2$

When you are asked to show or prove a result, you should conclude by saying that you have proved or shown the result. You can write the traditional q.e.d. if you like!

b $z_1z_2 = (5+\text{i})(-2+3\text{i})$

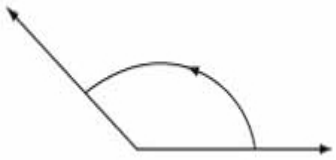
$= -10+15\text{i}-2\text{i}-3 = -13+13\text{i}$



$\tan\theta = \frac{13}{13} = 1 \Rightarrow \theta = \frac{\pi}{4}$

z_1z_2 is in the second quadrant.

$\arg(z_1z_2) = \pi - \frac{\pi}{4} = \frac{3\pi}{4}$



The argument is the angle with the positive x-axis. Anti-clockwise is positive.

As the question has not specified that you should work in radians or degrees, you could work in either and 135° would also be an acceptable answer.

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15 a $z^2 = (2+\mathrm{i})^2 = 4 - 4\mathrm{i} + \mathrm{i}^2$
$= 4 - 4\mathrm{i} - 1$
$= 3 - 4\mathrm{i}$, as required.

You square using the formula $(a-b)^2 = a^2 - 2ab + b^2$

b From part **a**, the square roots of $3 - 4\mathrm{i}$ are $2 - \mathrm{i}$ and $-2 + \mathrm{i}$.

Taking square roots of both sides of the equation $(z+\mathrm{i})^2 = 3 - 4\mathrm{i}$

$z + \mathrm{i} = 2 - \mathrm{i} \Rightarrow z = 2 - 2\mathrm{i}$
$z + \mathrm{i} = -2 + \mathrm{i} \Rightarrow z = -2$

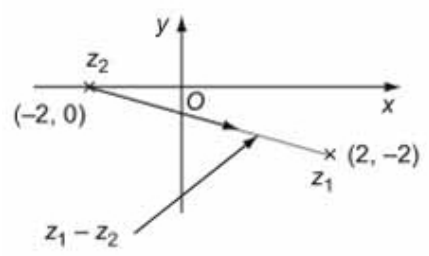
$z_1 = 2 - 2\mathrm{i}$, say, and $z_2 = -2$

The square root of any number k, real or complex, is a root of $z^2 = k$. Hence, part **a** shows that one square root of $3 - 4\mathrm{i}$ is $2 - \mathrm{i}$.

If one square root of $3 - 4\mathrm{i}$ is $2 - \mathrm{i}$, then the other is $-(2 - \mathrm{i})$.

z_1 and z_2 could be the other way round but that would make no difference to $|z_1 - z_2|$ or $z_1 - z_2$, the expressions you are asked about in parts **d** and **e**.

c



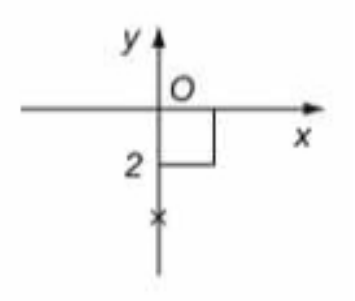
$z_1 - z_2$ can be represented on the diagram you drew in part **c** by the vector joining the point representing z_1 to the point representing z_2. The modulus of $z_1 - z_2$ is then just the length of the line joining these two points and this length can be found using coordinate geometry.

d Using the formula

$d^2 = (x_1 - x_2)^2 + (y_1 - y_2)^2$
$= (2 - (-2))^2 + (-2 - 0)^2$
$= 4^2 + 2^2 = 20$

Hence $|z_1 - z_2| = \sqrt{20} = 2\sqrt{5}$

e $z_1 + z_2 = 2 - 2\mathrm{i} - 2 = -2\mathrm{i}$

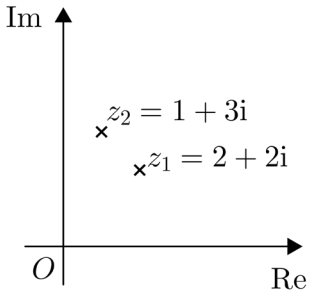


$\arg(z_1 + z_2) = -\frac{\pi}{2}$

The argument of any number on the negative imaginary axis is $-\frac{\pi}{2}$ or $-90°$.

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16 a



b $|z_1|^2 = 2^2 + 2^2 = 8 = 4 \times 2 \Rightarrow |z_1| = 2\sqrt{2}$

$|z_2|^2 = 1^2 + 3^2 = 10 \Rightarrow |z_2| = \sqrt{10}$

P has coordinates (2, 2) and Q (1, 3)

$PQ^2 = (1-2)^2 + (3-2)^2 = (-1)^2 + 1^2 = 2$

$PQ = \sqrt{2}$

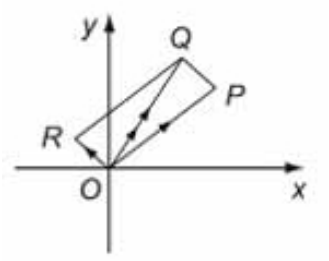
You use the formula $PQ^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2$ from Coordinate Geometry to calculate PQ^2.

c From **b**, $OP = 2\sqrt{2}$ and $OQ = \sqrt{10}$.

$$OP^2 + PQ^2 = (2\sqrt{2})^2 + (\sqrt{2})^2$$
$$= 8 + 2 = 10$$
$$= OQ^2$$

By the converse of Pythagoras' Theorem, ΔOPQ is right-angled.

d



You use the representation of the addition of complex numbers in an Argand diagram. The diagonal OQ of the parallelogram represents the addition of the two adjacent sides, OP and OR, of the parallelogram. (A rectangle is a special case of a parallelogram).

$\overline{OP} + \overline{OR} = \overline{OQ}$

$2 + 2i + z_3 = 1 + 3i$

$z_3 = -1 + i$

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17

$$\frac{\cos 2x + i\sin 2x}{\cos 9x - i\sin 9x}$$

$$= \frac{(\cos 2x + i\sin 2x)(\cos 9x + i\sin 9x)}{(\cos 9x - i\sin 9x)(\cos 9x - i\sin 9x)}$$

$$= \frac{\cos 2x\cos 9x - \sin 2x\sin 9x + i(\sin 2x\cos 9x + \cos 2x\sin 9x)}{\cos^2 9x + \sin^2 9x}$$

Use $\cos^2 A + \sin^2 A \equiv 1$

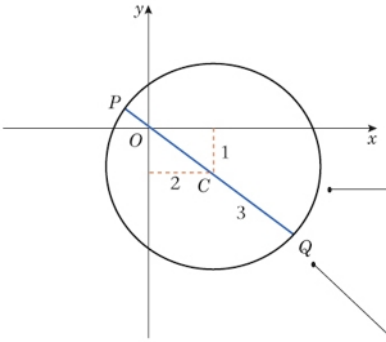
$$= \cos 2x\cos 9x - \sin 2x\sin 9x + i(\sin 2x\cos 9x + \cos 2x\sin 9x)$$

$$= \cos 11x + i\sin 11x$$

$n = 11$

Use $\cos(A+B) \equiv \cos A\cos B - \sin A\sin B$ and $\sin(A+B) \equiv \sin A\cos B + \cos A\sin B$ with A=2x and B=9x

18 a



The locus of $|z-a| = k$, where a is a complex number and k is a real number, is a circle with radius k and centre the point representing a. Rewriting the relation in the question as $|z-(2-\mathrm{i})| = 3$, this locus is a circle of radius 3 with centre $(2,-1)$.

$|z|$ is the distance of the point representing z from the origin. The point on the circle furthest from O is marked by Q on the diagram and the point closest to O by P. The distances of Q and P from O represent the maximum and minimum values of $|z|$ respectively.

b $OC^2 = 1^2 + 2^2 = 5 \Rightarrow OC = \sqrt{5}$

$OQ = OC + CQ = \sqrt{5} + 3$

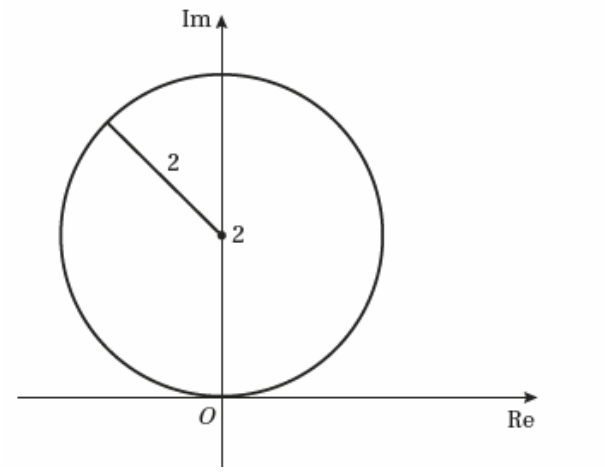
Hence the maximum value of $|z|$ is $3+\sqrt{5}$.

$OP = CP - CO = 3 - \sqrt{5}$

Hence the minimum value of $|z|$ is $3-\sqrt{5}$.

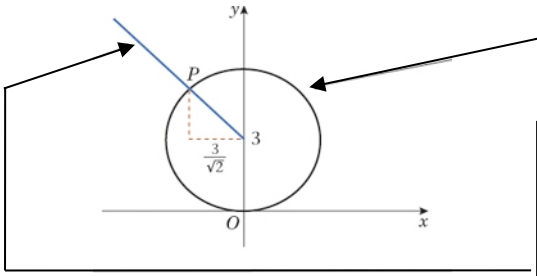
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19 a The locus of z is a circle of radius 2 and centre at $2i$



b $|z| \leqslant 2+2$ because it is the 3rd side of a triangle whose other sides are both 2. The maximum value of 4 occurs when $z = 4i$.

20 a



The locus of P is the circle with centre (0, 3) and radius 3. The coordinates (0, 3) represent the complex number 3i in the Argand diagram.

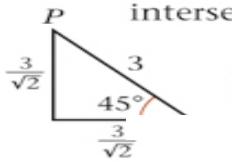
The half-line representing $\arg(z-3\mathrm{i}) = \frac{3\pi}{4}$ has been added to the diagram. This starts at (0, 3) and makes an angle of $\frac{3\pi}{4}$ with the positive x-direction. It is a common error to turn this half into a Full line. The half line has a different equation, 

$\arg(z-3\mathrm{i}) = -\frac{\pi}{4}$.

b From the diagram, z is the intersection of the circle and the half line marked P in the diagram.

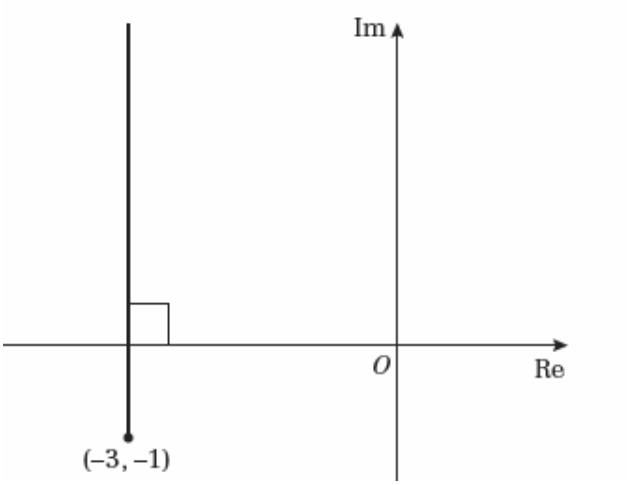
$$z = -\frac{3}{\sqrt{2}} + \mathrm{i}\left(3 + \frac{3}{\sqrt{2}}\right)$$
$$= -\frac{3\sqrt{2}}{2} + \mathrm{i}\left(3 + \frac{3\sqrt{2}}{2}\right)$$

The geometry of the point of intersection is shown here. The coordinates of P can then be just written down.

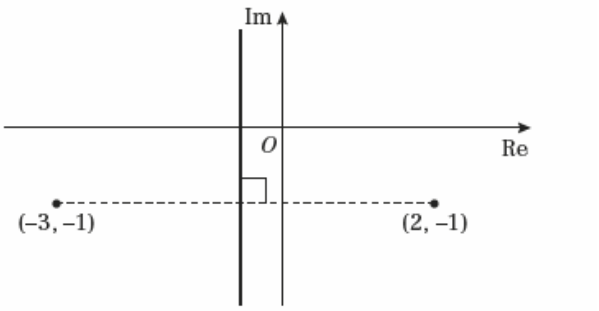


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21 $(z+3+i)=(z-(-3-i)$ so the locus is a vertical half line from the point $-3-i$.



22 a The locus is the perpendicular bisector of the line joining the points $-3-i$ and $2-i$



b The minimum value of $|z|$ is the minimum distance of O from the perpendicular bisector, so where the locus crosses the real axis. So minimum $=\frac{1}{2}$

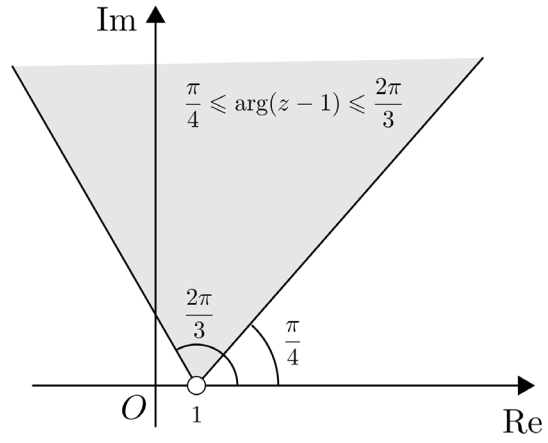
c From the diagram, when $\arg z=-\frac{3\pi}{4}$

the point on the locus is $-\frac{1}{2}-\frac{1}{2}i$ because

$\frac{x}{y}=\tan\left(-\frac{3\pi}{4}\right)=1$ and x must $=-\frac{1}{2}$

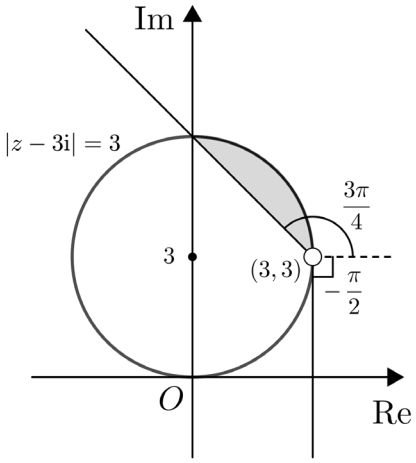
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23 $\frac{\pi}{4} \leqslant \arg(z-1) \leqslant \frac{2\pi}{3}$ is the region between the half lines from z=1 making angles of $\frac{\pi}{4}$ and $\frac{2\pi}{3}$ with the real axis.



24 $-\frac{\pi}{2} < \arg(z-3-3i) \leqslant \frac{3\pi}{4}$ is the region between the half lines from 3+3i making angles of $-\frac{\pi}{2}, \frac{3\pi}{4}$ with the real axis.

$|z-3i| \leqslant 3$ is the inside of a circle centre 3i and radius 3.



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25 $\sum_{r=1}^{n}(2r-1)^2 = \sum_{r=1}^{n}(4r^2-4r+1)$

$= 4\sum_{r=1}^{n}r^2 - 4\sum_{r=1}^{n}r + \sum_{r=1}^{n}1$

$= \frac{4n(n+1)(2n+1)}{6} - \frac{4n(n+1)}{2} + n$

$= \frac{2n(n+1)(2n+1)}{3} - \frac{6n(n+1)}{3} + \frac{3n}{3}$

$= \frac{n}{3}\left[2(n+1)(2n+1) - 6(n+1) + 3\right]$

$= \frac{n}{3}\left[4n^2 + 6n + 2 - 6n - 6 + 3\right]$

$= \frac{1}{3}n\left(4n^2 - 1\right)$, as required.

$\sum_{r=1}^{n}1 = \underbrace{1+1+1+\ldots+1}_{n \text{ times}} = n$

It is a common error to write $\sum_{r=1}^{n}1 = 1$.

After 'cancelling' the fractions, you should put all terms over a common denominator, here 3.

26 $\sum_{r=1}^{n}r(r^2-3) = \sum_{r=1}^{n}r^3 - 3\sum_{r=1}^{n}r$

$= \frac{n^2(n+1)^2}{4} - \frac{3n(n+1)}{2}$

$= \frac{n^2(n+1)^2}{4} - \frac{6n(n+1)}{4}$

$= \frac{n(n+1)}{4}\left[n(n+1) - 6\right]$

$= \frac{n(n+1)}{4}\left[n^2 + n - 6\right]$

$= \frac{1}{4}n(n+1)(n-2)(n+3)$, as required.

After putting both terms over a common denominator, look for the common factors of the terms, here shown in **bold**;

$$\frac{\boldsymbol{n^2(n+1)^2}}{4} - \frac{\boldsymbol{6n(n+1)}}{4}.$$

You take these, together with the common denominator 4, outside a bracket;

$$\frac{\boldsymbol{n(n+1)}}{4}\left[n(n+1) - 6\right].$$

You need to be careful with the squared terms.

27 a $\sum_{r=1}^{n}r(2r-1) = \sum_{r=1}^{n}(2r^2 - r)$

$= 2\sum_{r=1}^{\pi}r^2 - \sum_{r=1}^{\pi}r$

$= \frac{2n(n+1)(2n+1)}{6} - \frac{n(n+1)}{2}$

$= \frac{2n(n+1)(2n+1)}{6} - \frac{3n(n+1)}{6}$

$= \frac{n(n+1)}{6}\left[2(2n+1) - 3\right]$

$= \frac{n(n+1)}{6}\left[4n + 2 - 3\right]$

$= \frac{n(n+1)(4n-1)}{6}$, as required

You put the expressions over a common denominator, here 6, and then look for the common factors of the expressions, here n and $(n+1)$.

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27 b $\sum_{r=1}^{30} r(2r-1) = \sum_{r=1}^{30} r(2r-1) - \sum_{r=1}^{10} r(2r-1)$

Substituting $n = 30$ and $n = 10$ into the result in part (a).

$$\sum_{r=11}^{30} r(2r-1) = \frac{30\times31\times119}{6} - \frac{10\times11\times39}{6}$$
$$= 18\,445 - 715$$
$$= 17\,730$$

$$\sum_{r=11}^{30} f(r) = \sum_{r=1}^{30} f(r) - \sum_{r=1}^{10} f(r).$$

You find the sum from the 11th to the 30th term by subtracting the sum from the first to the 10th term from the sum from the first to the 30th term. It is a common error to subtract one term too many, in this case the 11th term. The sum you are finding starts with the 11th term. You must not subtract it from the series – you have to leave it in the series.

28 a $\sum_{r=1}^{n}(6r^2+4r-5) = 6\sum_{r=1}^{n} r^2 + 4\sum_{r=1}^{n} r - \sum_{r=1}^{n} 5$

$$= \frac{\cancel{6}n(n+1)(2n+1)}{\cancel{6}} + \frac{\cancel{4}n(n+1)}{\cancel{2}} - 5n$$
$$= n(n+1)(2n+1) + 2n(n+1) - 5n$$
$$= n[(n+1)(2n+1) + 2(n+1) - 5]$$
$$= n[2n^2 + 3n + 1 + 2n + 2 - 5]$$
$$= n(2n^2 + 5n - 2), \text{as required}$$

A common error with the last term is to write $-\sum_{r=1}^{n} 5 = -5$.

Correctly:

$$-\sum_{r=1}^{n} 5 = -(5+5+5+\ldots+5)$$
$$= -5(\underbrace{1+1+1\ldots+1}_{n \text{ times}})$$
$$= -5n$$

b $\sum_{r=10}^{25}(6r^2+4r-5) = \sum_{r=1}^{25}(6r^2+4r-5) - \sum_{r=1}^{9}(6r^2+4r-5)$

Substituting $n = 25$ and $n = 9$ into the result in part (a)

$$\sum_{r=10}^{25}(6r^2+4r-5)$$
$$= 25(2\times25^2 + 5\times25 - 2) - 9(2\times9^2 + 5\times9 - 2)$$
$$= 34\,325 - 1845 = 32\,480$$

29 a $\sum_{r=1}^{n} r(r+1) = \sum_{r=1}^{n} r^2 + \sum_{r=1}^{n} r$

$$= \frac{n(n+1)(2n+1)}{6} + \frac{n(n+1)}{2}$$
$$= \frac{n(n+1)(2n+1)}{6} + \frac{3(n+1)}{6}$$
$$= \frac{n(n+1)}{6}[2n+1+3]$$
$$= \frac{n(n+1)\cancel{2}^{1}(n+2)}{\cancel{6}^{3}}$$
$$= \frac{1}{3}n(n+1)(n+2), \text{as required.}$$

After putting the expressions over a common denominator 6, you look for any factors common to both expressions. Here there are two, n and $(n+1)$.

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29 b $\sum_{r=1}^{3n} r(r+1) = \sum_{r=1}^{3n} r(r+1) - \sum_{r=1}^{n-1} r(r+1)$

$= \frac{1}{3}3n(3n+1)(3n+2) - \frac{1}{3}(n-1)n(n+1)$

$= \frac{1}{3}n[3(3n+1)(3n+2) - (n-1)(n+1)]$

$= \frac{1}{3}n[27n^2 + 27n + 6 - (n^2 - 1)]$

$= \frac{1}{3}n(26n^2 + 27n + 7)$

$= \frac{1}{3}n(2n+1)(13n+7)$

$p = 13, q = 7$

To find an expression for $\sum_{r=1}^{n=1} r(r+1)$, you replace the n in the result in part (a) by $n - 1$;

$\frac{1}{3}n(n+1)(n+2)$

becomes $\frac{1}{3}(n-1)\big((n-1)+1\big)\big((n-1)+2\big)$

$= \frac{1}{3}(n-1)n(n+1).$

As you are given that $(2n+1)$ is one factor of $26n^2 + 27n + 7$, the other can just be written down. $(2n+1)(pn+q) = 26n^2 + 27n + 7$, only if $2p = 26$ and $1q = 7$

30 a $\sum_{r=1}^{n} r^2(r-1) = \sum_{r=1}^{n} r^3 - \sum_{r=1}^{n} r^2$

$= \frac{n^2(n+1)^2}{4} - \frac{n(n+1)(2n+1)}{6}$

$= \frac{3n^2(n+1)^2}{12} - \frac{2n(n+1)(2n+1)}{12}$

$= \frac{n(n+1)}{12}[3n(n+1) - 2(2n+1)]$

$= \frac{n(n+1)}{12}[3n^2 + 3n - 4n - 2]$

$= \frac{1}{12}n(n+1)(3n^2 - n - 2)$

$p = 3, q = -1, r = -2$

After putting the expressions over a common denominator 12, you look for any factors common to both expressions. Here there are two, n and $(n+1)$.

b $\sum_{r=50}^{100} r^2(r-1) = \sum_{r=1}^{100} r^2(r-1) - \sum_{r=1}^{49} r^2(r-1)$

$= \frac{1}{12} \times 100 \times 101 \times (3 \times 100^2 - 100 - 2)$

$- \frac{1}{2} \times 49 \times 50 \times (3 \times 49^2 - 49 - 2)$

$= 25\,164150 - 1\,460\,200$

$= 23\,703\,950$

$$\sum_{r=50}^{100} \mathrm{f}(r) = \sum_{r=1}^{100} \mathrm{f}(r) - \sum_{r=1}^{49} \mathrm{f}(r).$$

You find the sum from the 50th to the 100th term by subtracting the sum from the first to the 49th Term from the sum from the first to the 100th term. It is a common error to subtract one term too many, in this case the 50th term. The sum you are finding starts with the 50th term. You must not remove it from the series.

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31 a $\alpha\beta+\beta\gamma+\gamma\alpha=\frac{k}{3}$

$\Rightarrow k=-12$

b In the equation $ax^3+bx^2+cx+d=0$,

$\alpha+\beta+\gamma=-\frac{b}{a}=0$

$\alpha\beta\gamma=-\frac{d}{a}=-\frac{11}{3}$

c $(1-\alpha)(1-\beta)(1-\gamma)$

$=1-(\alpha+\beta+\gamma)+(\alpha\beta+\beta\gamma+\gamma\alpha)-\alpha\beta\gamma$

$=1-0-4+\frac{11}{3}$

$=\frac{2}{3}$

32 a $\alpha\beta\gamma\delta=\frac{d}{a}$ so $\frac{-4}{a}=-1\Rightarrow a=4$

b $\sum\alpha=-\frac{b}{a}=-\frac{7}{4}$

$\sum\alpha\beta=\frac{c}{a}=\frac{5}{4}$

$\sum\alpha\beta\gamma=-\frac{d}{a}=-\frac{3}{4}$

c $\alpha^2+\beta^2+\gamma^2+\delta^2=(\alpha+\beta+\gamma+\delta)^2-2\sum\alpha\beta$

$=\left(-\frac{7}{4}\right)^2-2\times\frac{5}{4}$

$=\frac{9}{16}$

33 Let $w=2x+1$

$\Rightarrow x=\frac{w-1}{2}$

$\left(\frac{w-1}{2}\right)^3+3\left(\frac{w-1}{2}\right)^2+5\times\frac{w-1}{2}-1=0$

$\Rightarrow\frac{w^3-3w^2+3w-1}{8}$

$+\frac{3(w^2-2w+1)}{4}+5\times\frac{w+1}{2}-1=0$

$\Rightarrow w^3-3w^2+3w-1+6w^2-12w+$

$6+20w-20-8=0$

$\Rightarrow w^3+3w^2+11w-23=0$

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34 a Let $w = 3x$

$\Rightarrow x = \frac{w}{3}$

$\left(\frac{w}{3}\right)^4 - \left(\frac{w}{3}\right)^3 - 2\left(\frac{w}{3}\right)^2 + 3 \times \frac{w}{3} + 4 = 0$

$\Rightarrow \frac{w^4}{81} - \frac{w^3}{27} - \frac{2w^2}{9} + w + 4 = 0$

$\Rightarrow w^4 - 3w^3 - 18w^2 + 81w + 324 = 0$

b Let $w = 2x - 1$

$\Rightarrow x = \frac{w+1}{2}$

$\left(\frac{w+1}{2}\right)^4 - \left(\frac{w+1}{2}\right)^3 - 2\left(\frac{w+1}{2}\right)^2 + 3 \times \frac{w+1}{2} + 4 = 0$

$\Rightarrow \frac{w^4 + 4w^3 + 6w^2 + 4w + 1}{16} - \frac{w^3 + 3w^2 + 3w + 1}{8}$

$- \frac{2(w^2 + 2w + 1)}{4} + 3 \times \frac{w+1}{2} + 4 = 0$

$\Rightarrow w^4 + 4w^3 + 6w^2 + 4w + 1 - 2w^3 - 6w^2 - 6w -$

$2 - 8w^2 - 16w - 8 + 24w + 24 + 64 = 0$

$\Rightarrow w^4 + 2w^3 - 8w^2 + 6w + 79 = 0$

35 a Crosses x-axis at $x = a$ so

$a > 0$ and $a\sqrt{1 - a^2} = 0$

So $a = 1$

b $\pi \int_0^1 x^2(1 - x^2)dx$

$= \pi \left[\frac{1}{3}x^3 - \frac{1}{5}x^5\right]_0^1$

$= \frac{2\pi}{15}$

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36 $y=\sqrt{x^2+3} \Rightarrow y^2=x^2+3$

$x^2=y^2-3$

$\text{Volume}=\pi\int_2^k (y^2-3)\mathrm{d}y$

$=\pi\left[\frac{1}{3}x^3-3x\right]_2^k$

$=\pi\left(\frac{1}{3}k^3-3k-\frac{8}{3}+6\right)$

$=\pi\left(\frac{1}{3}k^3-3k+\frac{10}{3}\right)$

So $\frac{1}{3}k^3-3k+\frac{10}{3}=30$

$\Rightarrow k^3-9k+10=90$

$\Rightarrow k^3-9k-80=0$

$\Rightarrow (k-5)(k^2+5k+16)=0$

$\Rightarrow k=5$

37 Curve and line cross when $4-x^2=2x+1$

$\Rightarrow x^2+2x-3=0$

$\Rightarrow (x+3)(x-1)=0$

So $x=1$

So volume $=\pi\int_0^1 (2x+1)^2\mathrm{d}x+\pi\int_1^2 (4-x^2)^2\mathrm{d}x$

$=\pi\int_0^1 (4x^2+4x+1)^2\mathrm{d}x+\pi\int_1^2 (16-8x^2+x^4)^2\mathrm{d}x$

$=\pi\left[\frac{4}{3}x^3+2x^2+x\right]_0^1+\pi\left[16x-\frac{8}{3}x^3+\frac{x^5}{5}\right]_1^2$

$=\pi\left(\frac{4}{3}+2+1\right)+\pi\left(32-\frac{64}{3}+\frac{32}{5}-16+\frac{8}{3}-\frac{1}{5}\right)$

$=24.71\ldots\simeq 24.7$

38 a $x^2+(y-k)^2=100$ is the equation of a circle centre $(0,k)$ and radius 10 so $k=20$

b $V=\pi\int_a^b 100-(y-20)^2\,\mathrm{d}y=\pi\int_a^b 40y-y^2-300\,\mathrm{d}y$

$=\pi\left[20y^2-\frac{y^3}{3}-300y\right]_a^b$

$=\frac{\pi}{3}\left(60\left(b^2-a^2\right)-\left(b^3-a^3\right)-900\left(b-a\right)\right)$

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38 c Volume of stand
=Volume of cylinder - volume of cut out section

$=\pi\times10^2\times20-\frac{\pi}{3}\left(60(20^2-10^2)-(20^3-10^3)-900\times10\right)$

$=4188.79$

So cost $=4188.79\times.025$

$=£104.72$ to nearest penny

39 a $y=12-x^2$ crosses the y-axis when y=12.
So the maximum outer radius =12 mm and maximum outer diameter =24 mm.

b Curves cross when $12-x^2=8-0.2x^2$

$\Rightarrow 0.8x^2=5$

$\Rightarrow x=\pm\sqrt{5}$

So volume $=\pi\int_{-\sqrt{5}}^{\sqrt{5}}(12-x^2)^2-(8-0.2x^2)\mathrm{d}x$

$=\pi\int_{-\sqrt{5}}^{\sqrt{5}}(80-20.8x^2+0.96x^4)\mathrm{d}x$

$=\pi\left[80x-\frac{20.8x^3}{3}+\frac{0.96x^5}{5}\right]_{-\sqrt{5}}^{\sqrt{5}}$

$=704.355$

So mass $=704.355\times19.3\div1000$

$=13.6\,\mathrm{g}$

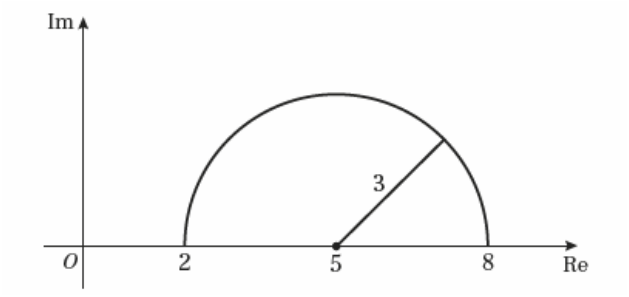
c Any valid reason, e.g. the gold may have voids or impurities, the actual dimensions may differ from those modelled, answer is given to too great a degree of accuracy.

Challenge

1 a $\arg\left(\frac{z-8}{z-2}\right)=\frac{\pi}{2}$

$\Rightarrow \arg(z-8)-\arg(z-2)=\frac{\pi}{2}$

So the angle between the lines joining z to 8 and 2 is $\frac{\pi}{2}$. So it is a semi-circle radius 3 and centre 5.



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Challenge

1 b $|z-5| =$ radius of semi-circle = 3.

2 a Let $u_r = ar + b$

$$\sum_{r=1}^{n} u_r = \frac{a}{2}n(n+1) + bn = n^2 + 5n$$

$$\Rightarrow a = 2 \text{ and } \frac{a}{2} + b = 5$$

$$\Rightarrow u_r = 2r + 4$$

b
$$\sum_{r=n}^{2n} u_r = \sum_{r=1}^{2n} u_r - \sum_{r=1}^{n-1} u_r$$
$$= \left((2n)^2 + 5(2n)\right) - \left((n-1)^2 - 5(n-1)\right)$$
$$= 3n^2 + 7n + 4 = (n+1)(3n+4)$$

3 Substitute $w = x^2 + 1 \Rightarrow x^2 = w - 1 \Rightarrow x = \sqrt{w-1}$

So the equation becomes

$\sqrt{w-1}(w-1) - 5(w-1) + 11\sqrt{w-1} - 15 = 0$

$\Rightarrow \sqrt{w-1}(w-1+11) = 5w + 10$

Now squaring both sides

$(w-1)(w+10)^2 = (5w+10)^2$

$\Rightarrow (w-1)(w^2 + 20w + 100) = 25w^2 + 100w + 100$

$\Rightarrow w^3 + 19w^2 + 80w - 100 = 25w^2 + 100w + 100$

$\Rightarrow w^3 - 6w^2 - 20w - 200 = 0$

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Matrices 6A

1 a $\begin{pmatrix} 1 & 0 \\ -1 & 3 \end{pmatrix}$

The size is 2×2.

b $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$

The size is 2×1.

c $\begin{pmatrix} 1 & 2 & 1 \\ 3 & 0 & -1 \end{pmatrix}$

The size is 2×3.

d $\begin{pmatrix} 1 & 2 & 3 \end{pmatrix}$

The size is 1×3.

e $\begin{pmatrix} 3 & -1 \end{pmatrix}$

The size is 1×2.

f $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

The size is 3×3.

2 $\mathbf{I}_4 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

3 Compare corresponding elements:

$a = 6$

$b = 2$

4 a $\begin{pmatrix} 2 & -1 \\ 1 & 3 \end{pmatrix} + \begin{pmatrix} 6 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 8 & -1 \\ 1 & 4 \end{pmatrix}$

b $\begin{pmatrix} 4 & 1 \\ -1 & -2 \end{pmatrix} - \begin{pmatrix} 2 & -1 \\ 1 & 3 \end{pmatrix} = \begin{pmatrix} 2 & 2 \\ -2 & -5 \end{pmatrix}$

c $\begin{pmatrix} 2 & -1 \\ 1 & 3 \end{pmatrix} + \begin{pmatrix} 4 & 1 \\ -1 & -2 \end{pmatrix} - \begin{pmatrix} 6 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

5 a $A + B$ is $(2 \times 1) + (1 \times 2)$ Not possible

b $A - E = \begin{pmatrix} 1 \\ 2 \end{pmatrix} - \begin{pmatrix} 3 \\ -1 \end{pmatrix} = \begin{pmatrix} -2 \\ 3 \end{pmatrix}$

c $F - D + C = \begin{pmatrix} 2 & 1 & 3 \end{pmatrix} - \begin{pmatrix} 0 & 1 & -1 \end{pmatrix} + \begin{pmatrix} -1 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 4 \end{pmatrix}$

d $B + C$ is $(1 \times 2) + (1 \times 3)$ Not possible

e $F - (D + C) = \begin{pmatrix} 2 & 1 & 3 \end{pmatrix} - \left[\begin{pmatrix} 0 & 1 & -1 \end{pmatrix} + \begin{pmatrix} -1 & 1 & 0 \end{pmatrix} \right] = \begin{pmatrix} 2 & 1 & 3 \end{pmatrix} - \begin{pmatrix} -1 & 2 & -1 \end{pmatrix} = \begin{pmatrix} 3 & -1 & 4 \end{pmatrix}$

f $A - F = (2 \times 1) - (1 \times 3)$ Not possible.

g $C - (F - D) = \begin{pmatrix} -1 & 1 & 0 \end{pmatrix} - \left[\begin{pmatrix} 2 & 1 & 3 \end{pmatrix} - \begin{pmatrix} 0 & 1 & -1 \end{pmatrix} \right] = \begin{pmatrix} -1 & 1 & 0 \end{pmatrix} - \begin{pmatrix} 2 & 0 & 4 \end{pmatrix} = \begin{pmatrix} -3 & 1 & -4 \end{pmatrix}$

6

$a - 1 = 5 \Rightarrow a = 6$

$2 - c = 0 \Rightarrow c = 2$

$-1 - d = 0 \Rightarrow d = -1$

$b - (-2) = 5 \Rightarrow b = 3$

7

$1 + a = c \quad (1)$

$2 + b = 5 \Rightarrow b = 3$

$0 + c = c$

$a + 1 = c$

$b + 2 = c \quad (2)$

$c + 0 = c$

Use $b = 3$ in **(2)** $\Rightarrow c = 5$

Use $c = 5$ in **(1)** $\Rightarrow a = 4$

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8 $5+a=7 \Rightarrow a=2$

$3+b=1 \Rightarrow b=-2$

$0+c=2 \Rightarrow c=2$

$-1+d=0 \Rightarrow d=1$

$2+e=1 \Rightarrow e=-1$

$1+f=4 \Rightarrow f=3$

9 a $\mathbf{A}+\mathbf{B}=\begin{pmatrix}1&-1&3\\2&0&2\\3&4&0\end{pmatrix}+\begin{pmatrix}6&3&-4\\1&1&2\\-2&0&-3\end{pmatrix}$

$=\begin{pmatrix}7&2&-1\\3&1&4\\1&4&-3\end{pmatrix}$

b $\mathbf{B}-\mathbf{C}=\begin{pmatrix}6&3&-4\\1&1&2\\-2&0&-3\end{pmatrix}-\begin{pmatrix}-2&0&3\\2&8&-6\\-1&1&1\end{pmatrix}$

$=\begin{pmatrix}8&3&-7\\-1&-7&8\\-1&-1&-4\end{pmatrix}$

c $\mathbf{C}+\mathbf{A}=\begin{pmatrix}-2&0&3\\2&8&-6\\-1&1&1\end{pmatrix}+\begin{pmatrix}1&-1&3\\2&0&2\\3&4&0\end{pmatrix}$

$=\begin{pmatrix}-1&-1&6\\4&8&-4\\2&5&1\end{pmatrix}$

d i $\mathbf{A}+\mathbf{M}=\begin{pmatrix}1&-1&3\\2&0&2\\3&4&0\end{pmatrix}+\begin{pmatrix}5&-6&b\\4&a&6\\2&0&c\end{pmatrix}$

$=\begin{pmatrix}6&-7&3+b\\6&a&8\\5&4&c\end{pmatrix}$

Equate with given $\mathbf{A}+\mathbf{M}$:

$\begin{pmatrix}6&-7&3+b\\6&a&8\\5&4&c\end{pmatrix}=\begin{pmatrix}6&-7&-2\\6&3&8\\5&4&6\end{pmatrix}$

Compare corresponding elements:

$a=3$

$3+b=-2 \Rightarrow b=-5$

$c=6$

d ii $\mathbf{M}-\mathbf{B}=\begin{pmatrix}5&-6&b\\4&a&6\\2&0&c\end{pmatrix}-\begin{pmatrix}6&3&-4\\1&1&2\\-2&0&-3\end{pmatrix}$

$=\begin{pmatrix}-1&-9&b+4\\3&a-1&4\\4&0&c+3\end{pmatrix}$

Equate with given $\mathbf{M}-\mathbf{B}$:

$\begin{pmatrix}-1&-9&b+4\\3&a-1&4\\4&0&c+3\end{pmatrix}=\begin{pmatrix}-1&-9&-5\\-2&7&-1\\-3&2&2\end{pmatrix}$

Compare corresponding elements:

$a-1=7 \Rightarrow a=8$

$b+4=-5 \Rightarrow b=-9$

$c+3=2 \Rightarrow c=-1$

10 a $3\begin{pmatrix}2&0\\4&-6\end{pmatrix}=\begin{pmatrix}6&0\\12&-18\end{pmatrix}$

b $\frac{1}{2}\begin{pmatrix}2&0\\4&-6\end{pmatrix}=\begin{pmatrix}1&0\\2&-3\end{pmatrix}$

c $2\begin{pmatrix}1\\-1\end{pmatrix}=\begin{pmatrix}2\\-2\end{pmatrix}$

d **A** and **B** are not the same size, so you can't subtract them.

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11 a $3\mathbf{A}+2\mathbf{B}=3\begin{pmatrix}3&-2\\1&0\end{pmatrix}+2\begin{pmatrix}2&1\\-2&3\end{pmatrix}$

$=\begin{pmatrix}9&-6\\3&0\end{pmatrix}+\begin{pmatrix}4&2\\-4&6\end{pmatrix}$

$=\begin{pmatrix}13&-4\\-1&6\end{pmatrix}$

b $2\mathbf{A}-4\mathbf{B}=2\begin{pmatrix}3&-2\\1&0\end{pmatrix}-4\begin{pmatrix}2&1\\-2&3\end{pmatrix}$

$=\begin{pmatrix}6&-4\\2&0\end{pmatrix}-\begin{pmatrix}8&4\\-8&12\end{pmatrix}$

$=\begin{pmatrix}-2&-8\\10&-12\end{pmatrix}$

c $5\mathbf{A}-2\mathbf{B}=5\begin{pmatrix}3&-2\\1&0\end{pmatrix}-2\begin{pmatrix}2&1\\-2&3\end{pmatrix}$

$=\begin{pmatrix}15&-10\\5&0\end{pmatrix}-\begin{pmatrix}4&2\\-4&6\end{pmatrix}$

$=\begin{pmatrix}11&-12\\9&-6\end{pmatrix}$

d $\frac{1}{2}\mathbf{A}+\frac{3}{2}\mathbf{B}=\frac{1}{2}\begin{pmatrix}3&-2\\1&0\end{pmatrix}+\frac{3}{2}\begin{pmatrix}2&1\\-2&3\end{pmatrix}$

$=\begin{pmatrix}\frac{3}{2}&-1\\\frac{1}{2}&0\end{pmatrix}+\begin{pmatrix}3&\frac{3}{2}\\-3&\frac{9}{2}\end{pmatrix}$

$=\begin{pmatrix}\frac{9}{2}&\frac{1}{2}\\-\frac{5}{2}&\frac{9}{2}\end{pmatrix}$

12 a $\mathbf{M}+2\mathbf{N}=\begin{pmatrix}2&4&-1\\1&-3&-1\\0&2&2\end{pmatrix}+2\begin{pmatrix}6&-2&5\\3&-3&1\\1&-1&0\end{pmatrix}$

$=\begin{pmatrix}2&4&-1\\1&-3&-1\\0&2&2\end{pmatrix}+\begin{pmatrix}12&-4&10\\6&-6&2\\2&-2&0\end{pmatrix}$

$=\begin{pmatrix}14&0&9\\7&-9&1\\2&0&2\end{pmatrix}$

12 b $3\mathbf{M}-\mathbf{N}=3\begin{pmatrix}2&4&-1\\1&-3&-1\\0&2&2\end{pmatrix}-\begin{pmatrix}6&-2&5\\3&-3&1\\1&-1&0\end{pmatrix}$

$=\begin{pmatrix}6&12&-3\\3&-9&-3\\0&6&6\end{pmatrix}-\begin{pmatrix}6&-2&5\\3&-3&1\\1&-1&0\end{pmatrix}$

$=\begin{pmatrix}0&14&-8\\0&-6&-4\\-1&7&6\end{pmatrix}$

c $4\mathbf{M}+5\mathbf{N}=4\begin{pmatrix}2&4&-1\\1&-3&-1\\0&2&2\end{pmatrix}+5\begin{pmatrix}6&-2&5\\3&-3&1\\1&-1&0\end{pmatrix}$

$=\begin{pmatrix}8&16&-4\\4&-12&-4\\0&8&8\end{pmatrix}+\begin{pmatrix}30&-10&25\\15&-15&5\\5&-5&0\end{pmatrix}$

$=\begin{pmatrix}38&6&21\\19&-27&1\\5&3&8\end{pmatrix}$

d $\frac{2}{3}\mathbf{M}-\frac{1}{2}\mathbf{N}=\frac{2}{3}\begin{pmatrix}2&4&-1\\1&-3&-1\\0&2&2\end{pmatrix}-\frac{1}{2}\begin{pmatrix}6&-2&5\\3&-3&1\\1&-1&0\end{pmatrix}$

$=\begin{pmatrix}\frac{4}{3}&\frac{8}{3}&-\frac{2}{3}\\\frac{2}{3}&-2&-\frac{2}{3}\\0&\frac{4}{3}&\frac{4}{3}\end{pmatrix}-\begin{pmatrix}3&-1&\frac{5}{2}\\\frac{3}{2}&-\frac{3}{2}&\frac{1}{2}\\\frac{1}{2}&-\frac{1}{2}&0\end{pmatrix}$

$=\begin{pmatrix}-\frac{5}{3}&\frac{11}{3}&-\frac{19}{6}\\-\frac{5}{6}&-\frac{1}{2}&-\frac{7}{6}\\-\frac{1}{2}&\frac{11}{6}&\frac{4}{3}\end{pmatrix}$

13 $1+2k=7$

$\Rightarrow \quad 2k=6$

$\Rightarrow \quad k=3.$

$2-k=x$

$\Rightarrow \quad 2-3=x$

$\therefore \quad x=-1$

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14 $2a-3=3 \Rightarrow 2a=6$

$\Rightarrow a=3$

$0-3c=3 \Rightarrow c=-1$

$2-3d=-4 \Rightarrow -3d=-6$

$\Rightarrow d=2$

$2b+3=-4 \Rightarrow 2b=-7$

$\Rightarrow b=-3.5$

15 $5-2c=9$

$\Rightarrow -4=2c$

$\Rightarrow c=-2.$

$a-4=1$

$\Rightarrow a=5.$

$b-2=3$

$\Rightarrow b=5.$

$0+2=d$

$\Rightarrow d=2$

16 $-3+2k^2=k$

$\Rightarrow 2k^2-k-3=0$

$(2k-3)(k+1)=0$

$\therefore k=\frac{3}{2}$ or -1

AND $k+2k^2=6$

$\Rightarrow 2k^2+k-6=0$

$(2k-3)(k+2)=0$

$\therefore k=\frac{3}{2}$ or -2

So common value is $k=\frac{3}{2}$

17 a $\mathbf{A}-k\mathbf{B}=\mathbf{I}_3$

$$\begin{pmatrix} p & 0 & 0 \\ 0 & q^2 & r \\ 0 & 0 & 5 \end{pmatrix} - k\begin{pmatrix} 2q & 0 & 0 \\ 0 & 4 & 6 \\ 0 & 0 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} p-2kq & 0 & 0 \\ 0 & q^2-4k & r-6k \\ 0 & 0 & 5-2k \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Compare corresponding elements:

$5-2k=1$

$2k=4$

$k=2$

b Using $k=2$,

$r-6k=0$

$r-6(2)=0 \Rightarrow r=12$

$q^2-4k=1$

$q^2-4(2)=1$

$q^2=9 \Rightarrow q=3$

$p-2kq=1$

$p-2(2)(3)=1 \Rightarrow p=13$

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18 $\mathbf{P}-k\mathbf{Q}=\begin{pmatrix}0&2&c\\a&0&0\\0&b&-1\end{pmatrix}-k\begin{pmatrix}0&-1&-1\\3&d&0\\0&2&e\end{pmatrix}=\begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix}$

$=\begin{pmatrix}0&2&c\\a&0&0\\0&b&-1\end{pmatrix}-\begin{pmatrix}0&-k&-k\\3k&dk&0\\0&2k&ek\end{pmatrix}=\begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix}$

$=\begin{pmatrix}0&2+k&c+k\\a-3k&-dk&0\\0&b-2k&-1-ek\end{pmatrix}=\begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix}$

Compare corresponding elements:

$2+k=0\Rightarrow k=-2$

$a-3k=0$

$a-3(-2)=0$

$a+6=0\Rightarrow a=-6$

$b-2k=0$

$b-2(-2)=0$

$b+4=0\Rightarrow b=-4$

$c+k=0\Rightarrow c=2$

$-dk=0\Rightarrow d=0$

$-1-ek=0$

$-1-e(-2)=0$

$-1+2e=0$

$2e=1\Rightarrow e=\frac{1}{2}$

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Matrices 6B

1 a $(1\times 2)\cdot(2\times 2)=1\times 2$

b $(3\times 2)\cdot(2\times 3)=3\times 3$

c $(1\times 3)\cdot(3\times 2)=1\times 2$

d $(2\times 3)\cdot(3\times 2)=2\times 2$

e $(2\times 2)\cdot(2\times 3)=2\times 3$

f $(3\times 2)\cdot(2\times 2)=3\times 2$

2 a $\begin{pmatrix}1 & 2\\ 2 & 4\end{pmatrix}$ is a 2×2 matrix and $\begin{pmatrix}-1\\ 2\end{pmatrix}$ is a 2×1 matix.

They can be multiplied and the product will be a 2×1 matrix.

$$\begin{pmatrix}1 & 2\\ 2 & 4\end{pmatrix}\times\begin{pmatrix}-1\\ 2\end{pmatrix}=\begin{pmatrix}a\\ b\end{pmatrix}$$

$a=1\times(-1)+2\times 2$

$=3$

$b=2\times(-1)+4\times 2$

$=6$

b $\begin{pmatrix}1 & 2\\ 3 & 4\end{pmatrix}\begin{pmatrix}0 & 5\\ -1 & -2\end{pmatrix}=\begin{pmatrix}-2 & 1\\ -4 & 7\end{pmatrix}$

3 a $\begin{pmatrix}-1 & -2\\ 0 & 3\end{pmatrix}\begin{pmatrix}1 & 0 & 1\\ 1 & 1 & 0\end{pmatrix}=\begin{pmatrix}-3 & -2 & -1\\ 3 & 3 & 0\end{pmatrix}$

b $\begin{pmatrix}-1 & -2\\ 0 & 3\end{pmatrix}\begin{pmatrix}-1 & -2\\ 0 & 3\end{pmatrix}=\begin{pmatrix}1 & -4\\ 0 & 9\end{pmatrix}$

4 a **AB** is $(2\times 1)\cdot(2\times 2)$ Not possible

b $\mathbf{AC}=\begin{pmatrix}2\\ 1\end{pmatrix}(-3 \quad -2)=\begin{pmatrix}-6 & -4\\ -3 & -2\end{pmatrix}$

c **BC** is $(2\times 2)\cdot(1\times 2)$ Not possible

d $\mathbf{BA}=\begin{pmatrix}3 & 1\\ -1 & 2\end{pmatrix}\begin{pmatrix}2\\ 1\end{pmatrix}=\begin{pmatrix}7\\ 0\end{pmatrix}$

e $\mathbf{CA}=(-3 \quad -2)\begin{pmatrix}2\\ 1\end{pmatrix}=(-8)$

f $\mathbf{CB}=(-3 \quad -2)\begin{pmatrix}3 & 1\\ -1 & 2\end{pmatrix}=(-7 \quad -7)$

5 $\begin{pmatrix}2 & a\\ 1 & -1\end{pmatrix}\begin{pmatrix}1 & 3 & 0\\ 0 & -1 & 2\end{pmatrix}=\begin{pmatrix}2 & 6-a & 2a\\ 1 & 4 & -2\end{pmatrix}$

6 $\begin{pmatrix}3 & 2\\ -1 & x\end{pmatrix}\begin{pmatrix}x & -2\\ 1 & 3\end{pmatrix}=\begin{pmatrix}3x+2 & 0\\ 0 & 3x+2\end{pmatrix}$

7 a $\mathbf{AB}-\mathbf{C}=\begin{pmatrix}2 & -1\\ 3 & 4\end{pmatrix}\begin{pmatrix}1 & 0\\ -3 & 2\end{pmatrix}-\begin{pmatrix}-3 & 1\\ 1 & 2\end{pmatrix}$

$=\begin{pmatrix}2\times 1+(-1)\times(-3) & 2\times 0+(-1)\times 2\\ 3\times 1+4\times(-3) & 3\times 0+4\times 2\end{pmatrix}-\begin{pmatrix}-3 & 1\\ 1 & 2\end{pmatrix}$

$=\begin{pmatrix}5 & -2\\ -9 & 8\end{pmatrix}-\begin{pmatrix}-3 & 1\\ 1 & 2\end{pmatrix}$

$=\begin{pmatrix}5+3 & -2-1\\ -9-1 & 8-2\end{pmatrix}$

$=\begin{pmatrix}8 & -3\\ -10 & 6\end{pmatrix}$

b $\mathbf{BC}+3\mathbf{A}=\begin{pmatrix}1 & 0\\ -3 & 2\end{pmatrix}\times\begin{pmatrix}-3 & 1\\ 1 & 2\end{pmatrix}+3\begin{pmatrix}2 & -1\\ 3 & 4\end{pmatrix}$

$=\begin{pmatrix}-3 & 1\\ 11 & 1\end{pmatrix}+\begin{pmatrix}6 & -3\\ 9 & 12\end{pmatrix}$

$=\begin{pmatrix}3 & -2\\ 20 & 13\end{pmatrix}$

c $4\mathbf{B}-3\mathbf{CA}=4\begin{pmatrix}1 & 0\\ -3 & 2\end{pmatrix}-3\begin{pmatrix}-3 & 1\\ 1 & 2\end{pmatrix}\begin{pmatrix}2 & -1\\ 3 & 4\end{pmatrix}$

$=\begin{pmatrix}4 & 0\\ -12 & 8\end{pmatrix}-3\begin{pmatrix}-3 & 7\\ 8 & 7\end{pmatrix}$

$=\begin{pmatrix}4 & 0\\ -12 & 8\end{pmatrix}-\begin{pmatrix}-9 & 21\\ 24 & 21\end{pmatrix}$

$=\begin{pmatrix}13 & -21\\ -36 & -13\end{pmatrix}$

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8 a $\mathbf{MN} = \begin{pmatrix} 3 & k \\ k & 1 \end{pmatrix} \times \begin{pmatrix} 1 & k \\ k & -1 \end{pmatrix}$

$= \begin{pmatrix} 3\times 1 + k \times k & 3 \times k + k \times (-1) \\ k \times 1 + 1 \times k & k \times k + 1 \times (-1) \end{pmatrix}$

$= \begin{pmatrix} 3 + k^2 & 2k \\ 2k & k^2 - 1 \end{pmatrix}$

b $\mathbf{NM} = \begin{pmatrix} 1 & k \\ k & -1 \end{pmatrix} \times \begin{pmatrix} 3 & k \\ k & 1 \end{pmatrix}$

$= \begin{pmatrix} 1\times 3 + k \times k & 1 \times k + k \times 1 \\ k \times 3 + (-1) \times k & k \times k + (-1) \times 1 \end{pmatrix}$

$= \begin{pmatrix} 3 + k^2 & 2k \\ 2k & k^2 - 1 \end{pmatrix}$

c $3\mathbf{M} - 2\mathbf{N} = 3\begin{pmatrix} 3 & k \\ k & 1 \end{pmatrix} - 2\begin{pmatrix} 1 & k \\ k & -1 \end{pmatrix}$

$= \begin{pmatrix} 9 & 3k \\ 3k & 3 \end{pmatrix} - \begin{pmatrix} 2 & 2k \\ 2k & -2 \end{pmatrix}$

$= \begin{pmatrix} 7 & k \\ k & 5 \end{pmatrix}$

d $2\mathbf{NM} + 3\mathbf{N} = 2\begin{pmatrix} 3 & k \\ k & 1 \end{pmatrix} \times \begin{pmatrix} 1 & k \\ k & -1 \end{pmatrix} + 3\begin{pmatrix} 1 & k \\ k & -1 \end{pmatrix}$

$= 2\begin{pmatrix} 3 + k^2 & 2k \\ 2k & k^2 - 1 \end{pmatrix} + 3\begin{pmatrix} 1 & k \\ k & -1 \end{pmatrix}$

$= \begin{pmatrix} 6 + 2k^2 & 4k \\ 4k & 2k^2 - 2 \end{pmatrix} + \begin{pmatrix} 3 & 3k \\ 3k & -3 \end{pmatrix}$

$= \begin{pmatrix} 9 + 2k^2 & 7k \\ 7k & 2k^2 - 5 \end{pmatrix}$

9 a $\mathbf{A}^2 = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 4 \\ 0 & 1 \end{pmatrix}$

b $\mathbf{A}^3 = \mathbf{A}\mathbf{A}^2 = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 4 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 6 \\ 0 & 1 \end{pmatrix}$

Note $\mathbf{A}^2 = \begin{pmatrix} 1 & 2\times 2 \\ 0 & 1 \end{pmatrix}$

$\mathbf{A}^3 = \begin{pmatrix} 1 & 2\times 3 \\ 0 & 1 \end{pmatrix}$

c Suggests $\mathbf{A}^k = \begin{pmatrix} 1 & 2\times k \\ 0 & 1 \end{pmatrix}$

10 a $\mathbf{A}^2 = \begin{pmatrix} a & 0 \\ b & 0 \end{pmatrix}\begin{pmatrix} a & 0 \\ b & 0 \end{pmatrix} = \begin{pmatrix} a^2 & 0 \\ ab & 0 \end{pmatrix}$

b $\mathbf{A}^2 = 3\mathbf{A} \Rightarrow \begin{pmatrix} a^2 & 0 \\ ab & 0 \end{pmatrix} = \begin{pmatrix} 3a & 0 \\ 3b & 0 \end{pmatrix}$

$\Rightarrow \quad a^2 = 3a \quad \Rightarrow \quad a = 3 \text{ (or } 0)$

and $\quad ab = 3b \quad \Rightarrow a = 3$

$\therefore \quad a = 3$

11 a $\mathbf{BAC} = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}\begin{pmatrix} -1 & 3 \end{pmatrix}\begin{pmatrix} 4 & -2 \\ 0 & -3 \end{pmatrix}$

$= \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}\begin{pmatrix} -4 & -7 \end{pmatrix}$

$= \begin{pmatrix} -8 & -14 \\ -4 & -7 \\ 0 & 0 \end{pmatrix}$

b $\mathbf{AC}^2 = \begin{pmatrix} -1 & 3 \end{pmatrix}\begin{pmatrix} 4 & -2 \\ 0 & -3 \end{pmatrix}\begin{pmatrix} 4 & -2 \\ 0 & -3 \end{pmatrix}$

$= \begin{pmatrix} -4 & -7 \end{pmatrix}\begin{pmatrix} 4 & -2 \\ 0 & -3 \end{pmatrix}$

$= \begin{pmatrix} -16 & 29 \end{pmatrix}$

12 a $\mathbf{ABA} = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}\begin{pmatrix} 3 & -2 & -3 \end{pmatrix}\begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$

$= \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}\begin{pmatrix} -1 \end{pmatrix}$

$= \begin{pmatrix} -1 \\ 1 \\ -2 \end{pmatrix}$

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12 b $\mathbf{BAB} = \begin{pmatrix}3 & -2 & -3\end{pmatrix}\begin{pmatrix}1\\-1\\2\end{pmatrix}\begin{pmatrix}3 & -2 & -3\end{pmatrix}$

$= (-1)\begin{pmatrix}3 & -2 & -3\end{pmatrix}$

$= \begin{pmatrix}-3 & 2 & 3\end{pmatrix}$

13 a $\mathbf{I}_2$ is the 2×2 identity matrix in which the elements on the leading diagonal are all 1 and the remaining elements are 0.

$\mathbf{I}_2 = \begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}$

b $\mathbf{AI} = \begin{pmatrix}2 & -2\\1 & 3\end{pmatrix}\times\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}$

$= \begin{pmatrix}2 & -2\\1 & 3\end{pmatrix} = \mathbf{A}$

$\mathbf{IA} = \begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}\times\begin{pmatrix}2 & -2\\1 & 3\end{pmatrix}$

$= \begin{pmatrix}2 & -2\\1 & 3\end{pmatrix} = \mathbf{A}$

Hence, $\mathbf{AI} = \mathbf{IA} = \mathbf{A}$

14 $\mathbf{AB} + \mathbf{AC} = \mathbf{A}(\mathbf{B} + \mathbf{C})$

Consider LHS:

$\mathbf{AB} + \mathbf{AC}$

$= \begin{pmatrix}2 & -1\\3 & 2\end{pmatrix}\begin{pmatrix}4 & 2\\-1 & 0\end{pmatrix} + \begin{pmatrix}2 & -1\\3 & 2\end{pmatrix}\begin{pmatrix}1 & 2\\0 & -1\end{pmatrix}$

$= \begin{pmatrix}9 & 4\\10 & 6\end{pmatrix} + \begin{pmatrix}2 & 5\\3 & 4\end{pmatrix}$

$= \begin{pmatrix}11 & 9\\13 & 10\end{pmatrix}$

Consider RHS:

$\mathbf{A}(\mathbf{B} + \mathbf{C})$

$= \begin{pmatrix}2 & -1\\3 & 2\end{pmatrix}\times\left[\begin{pmatrix}4 & 2\\-1 & 0\end{pmatrix} + \begin{pmatrix}1 & 2\\0 & -1\end{pmatrix}\right]$

$= \begin{pmatrix}2 & -1\\3 & 2\end{pmatrix}\times\begin{pmatrix}5 & 4\\-1 & -1\end{pmatrix}$

$= \begin{pmatrix}11 & 9\\13 & 10\end{pmatrix} = \mathbf{AB} + \mathbf{AC}$

15 $\mathbf{A}^2 = 2\mathbf{A} + 5\mathbf{I}$

Consider LHS:

$\mathbf{A}^2 = \begin{pmatrix}1 & 2\\3 & 1\end{pmatrix}\times\begin{pmatrix}1 & 2\\3 & 1\end{pmatrix}$

$= \begin{pmatrix}7 & 4\\6 & 7\end{pmatrix}$

Consider RHS:

$2\mathbf{A} + 5\mathbf{I} = 2\begin{pmatrix}1 & 2\\3 & 1\end{pmatrix} + 5\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}$

$= \begin{pmatrix}2 & 4\\6 & 2\end{pmatrix} + \begin{pmatrix}5 & 0\\0 & 5\end{pmatrix}$

$= \begin{pmatrix}7 & 4\\6 & 7\end{pmatrix} = \mathbf{A}^2$

16 $\mathbf{M}^2 = \begin{pmatrix}1 & 2 & c\\a & -1 & 1\\1 & b & 0\end{pmatrix}\times\begin{pmatrix}1 & 2 & c\\a & -1 & 1\\1 & b & 0\end{pmatrix}$

$= \begin{pmatrix}2a + c + 1 & bc & 2 + c\\1 & 2a + b + 1 & ac - 1\\ab + 1 & 2 - b & b + c\end{pmatrix}$

17 $\mathbf{A}^2 = \begin{pmatrix}1 & -1 & b\\a & 2 & 0\\1 & 0 & 3\end{pmatrix}\times\begin{pmatrix}1 & -1 & b\\a & 2 & 0\\1 & 0 & 3\end{pmatrix}$

$= \begin{pmatrix}1 - a + b & -3 & 4b\\3a & 4 - a & ab\\4 & -1 & 9 + b\end{pmatrix}$

Compare corresponding elements

$\begin{pmatrix}1 - a + b & -3 & 4b\\3a & 4 - a & ab\\4 & -1 & 9 + b\end{pmatrix} = \begin{pmatrix}-4 & -3 & -8\\9 & 1 & -6\\4 & -1 & 7\end{pmatrix}$

$3a = 9 \Rightarrow a = 3$

$ab = -6$

$3b = -6 \Rightarrow b = -2$

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18 $\mathbf{AB} = \mathbf{BA}$

Consider LHS:

$$\mathbf{AB} = \begin{pmatrix} p & 3 \\ 6 & p \end{pmatrix} \times \begin{pmatrix} q & 2 \\ 4 & q \end{pmatrix}$$

$$= \begin{pmatrix} pq+12 & 2p+3q \\ 4p+6q & 12+pq \end{pmatrix}$$

Consider RHS:

$$\mathbf{BA} = \begin{pmatrix} q & 2 \\ 4 & q \end{pmatrix} \times \begin{pmatrix} p & 3 \\ 6 & p \end{pmatrix}$$

$$= \begin{pmatrix} pq+12 & 2p+3q \\ 4p+6q & 12+pq \end{pmatrix} = \mathbf{AB}$$

19 $\mathbf{A}^2 = \begin{pmatrix} 3 & p \\ -4 & q \end{pmatrix} \times \begin{pmatrix} 3 & p \\ -4 & q \end{pmatrix}$

$$= \begin{pmatrix} 9-4p & 3p+pq \\ -12-4q & q^2-4p \end{pmatrix}$$

$$\mathbf{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

so $\begin{pmatrix} 9-4p & 3p+pq \\ -12-4q & q^2-4p \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

Compare corresponding elements

$9-4p=1$

$4p=8 \Rightarrow p=2$

$3p+pq=0$

$6+2q=0$

$2q=-6 \Rightarrow q=-3$

Challenge

a $\mathbf{A} = \begin{pmatrix} (-1^k)mn & (-1^k)n^2 \\ -(-1^k)m^2 & -(-1^k)mn \end{pmatrix}$

where n, m are real numbers and $k = 0$ or 1.

E.g. $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$

b Let $\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$$\mathbf{A}^2 = \begin{pmatrix} a^2+bc & b(a+d) \\ c(a+d) & bc+d^2 \end{pmatrix}$$

For $\mathbf{A}^2 = \mathbf{0}$ equate each element to 0

$a^2+bc=0$

$b(a+d)=0$

$c(a+d)=0$

$bc+d^2=0$

Set $a+d=0$ then $b=c=0$, subsequently giving $a=d=0$ (inadmissible).

So $a=-d$ and $bc=-a^2$ which has many solutions.

E.g. $\begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}$

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Matrices 6C

1 a $\begin{vmatrix} 3 & 4 \\ -1 & 2 \end{vmatrix} = 3\times 2 - 4\times(-1) = 10$

b $\begin{vmatrix} 4 & 2 \\ 1 & 2 \end{vmatrix} = 4\times 2 - 2\times 1 = 6$

c $\begin{vmatrix} -2 & 1 \\ 1 & 0 \end{vmatrix} = (-2)\times 0 - 1\times 3 = -3$

d $\begin{vmatrix} -4 & -4 \\ 1 & 1 \end{vmatrix} = (-4)\times 1 - (-4)\times 1 = 0$

e $\begin{vmatrix} 7 & -4 \\ 0 & 3 \end{vmatrix} = 7\times 3 - (-4)\times 0 = 21$

f $\begin{vmatrix} -1 & -1 \\ -6 & -10 \end{vmatrix} = (-1)\times(-10) - (-1)\times(-6) = 4$

2 a $\det\begin{vmatrix} a & 1+a \\ 3 & 2 \end{vmatrix} = 2a - 3(1+a)$

$= 2a - 3 - 3a$

$= -3 - a$

Matrix is singular for $a = -3$

b Let $\mathbf{A} = \begin{pmatrix} 1+a & 3-a \\ a+2 & 1-a \end{pmatrix}$

$\det \mathbf{A} = (1+a)(1-a) - (3-a)(a+2)$

$= 1 - a^2 - (-a^2 + a + 6)$

$= 1 - a^2 + a^2 - a - 6$

$= -a - 5$

$\det \mathbf{A} = 0 \Rightarrow a = -5$

c Let $\mathbf{B} = \begin{pmatrix} 2+a & 1-a \\ 1-a & a \end{pmatrix}$

$\det \mathbf{B} = 2a + a^2 - (1-a)^2$

$= 2a + a^2 - 1 + 2a - a^2$

$= 4a - 1$

$\det \mathbf{B} = 0 \Rightarrow a = \frac{1}{4}$

3 $\det \mathbf{M} = \begin{vmatrix} -2 & 1-k \\ k-1 & k \end{vmatrix} = (-2)\times k - (1-k)\times(k-1)$

$= k^2 - 4k + 1$

For singular matrix $k^2 - 4k + 1 = 0$

$k = \frac{-(-4) \pm \sqrt{(-4)^2 - 4\times 1\times 1}}{2\times 1}$

$= \frac{4 \pm \sqrt{12}}{2}$

$k = 2 + \sqrt{3},\ k = 2 - \sqrt{3}$

4 $\det \mathbf{P} = \begin{vmatrix} 3k & 4-k \\ k-2 & -k \end{vmatrix}$

$= 3k\times(-k) - (4-k)\times(k-2)$

$= -2k^2 - 6k + 8$

For singular matrix $2k^2 + 6k - 8 = 0$

$(k+4)(k-1) = 0$

$k = -4$ and $k = 1$

5 a $\mathbf{A} = \begin{pmatrix} a & 2a \\ b & 2b \end{pmatrix} \Rightarrow \det \mathbf{A} = 2ab - 2ab = 0$

$\mathbf{B} = \begin{pmatrix} 2b & -2a \\ -b & a \end{pmatrix} \Rightarrow \det \mathbf{B} = 2ab - 2ab = 0$

b $\mathbf{AB} = \begin{pmatrix} a & 2a \\ b & 2b \end{pmatrix}\begin{pmatrix} 2b & -2a \\ -b & a \end{pmatrix}$

$= \begin{pmatrix} 2ab - 2ab & -2a^2 + 2a^2 \\ 2b^2 - 2b^2 & -2ab + 2ab \end{pmatrix}$

$= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

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6 a $\begin{vmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{vmatrix} = 1\begin{vmatrix} 2 & 0 \\ 0 & 3 \end{vmatrix} - 0\begin{vmatrix} 0 & 0 \\ 0 & 3 \end{vmatrix} + 0\begin{vmatrix} 0 & 2 \\ 0 & 0 \end{vmatrix}$

$= 1(6-0) - 0 + 0 = 6$

b $\begin{vmatrix} 0 & 4 & 0 \\ 5 & -2 & 3 \\ 2 & 1 & 4 \end{vmatrix} = 0\begin{vmatrix} -2 & 3 \\ 1 & 4 \end{vmatrix} - 4\begin{vmatrix} 5 & 3 \\ 2 & 4 \end{vmatrix} + 0\begin{vmatrix} 5 & 2 \\ 2 & 1 \end{vmatrix}$

$= 0 - 4(20-6) + 0 = -56$

c $\begin{vmatrix} 1 & 0 & 1 \\ 2 & 4 & 1 \\ 3 & 5 & 2 \end{vmatrix} = 1\begin{vmatrix} 4 & 1 \\ 5 & 2 \end{vmatrix} - 0\begin{vmatrix} 2 & 1 \\ 3 & 2 \end{vmatrix} + 1\begin{vmatrix} 2 & 4 \\ 3 & 5 \end{vmatrix}$

$= 1(8-5) - 0 + 1(10-12)$

$= 3 - 2 = 1$

d $\begin{vmatrix} 2 & -3 & 4 \\ 2 & 2 & 2 \\ 5 & 5 & 5 \end{vmatrix} = 2\begin{vmatrix} 2 & 2 \\ 5 & 5 \end{vmatrix} - (-3)\begin{vmatrix} 2 & 2 \\ 5 & 5 \end{vmatrix} + 4\begin{vmatrix} 2 & 2 \\ 5 & 5 \end{vmatrix}$

$= 2(10-10) + 3(10-10)$

$+ 4(10-10) = 0$

7 a $\begin{vmatrix} 4 & 3 & -1 \\ 2 & -2 & 0 \\ 0 & 4 & -2 \end{vmatrix}$

$= 4\begin{vmatrix} -2 & 0 \\ 4 & -2 \end{vmatrix} - 3\begin{vmatrix} 2 & 0 \\ 0 & -2 \end{vmatrix} + (-1)\begin{vmatrix} 2 & -2 \\ 0 & 4 \end{vmatrix}$

$= 4(4-0) - 3(-4-0) - 1(8-0)$

$= 16 + 12 - 8 = 20$

b $\begin{vmatrix} 3 & -2 & 1 \\ 4 & 1 & -3 \\ 7 & 2 & -4 \end{vmatrix}$

$= 3\begin{vmatrix} 1 & -3 \\ 2 & -4 \end{vmatrix} - (-2)\begin{vmatrix} 4 & -3 \\ 7 & -4 \end{vmatrix} + 1\begin{vmatrix} 4 & 1 \\ 7 & 2 \end{vmatrix}$

$= 3(-4+6) + 2(-16+21) + 1(8-7)$

$= 6 + 10 + 1 = 17$

7 c $\begin{vmatrix} 5 & -2 & -3 \\ 6 & 4 & 2 \\ -2 & -4 & -3 \end{vmatrix}$

$= 5\begin{vmatrix} 4 & 2 \\ -4 & -3 \end{vmatrix} - (-2)\begin{vmatrix} 6 & 2 \\ -2 & -3 \end{vmatrix} + (-3)\begin{vmatrix} 6 & 4 \\ -2 & -4 \end{vmatrix}$

$= 5(-12+8) + 2(-18+4) - 3(-24+8)$

$= 5 \times (-4) + 2 \times (-14) - 3 \times (-16)$

$= -20 - 28 + 48 = 0$

8 $\det(\mathbf{A}) = \begin{vmatrix} 2 & 1 & -4 \\ 2k+1 & 3 & k \\ 1 & 0 & 1 \end{vmatrix}$

$= 2\begin{vmatrix} 3 & k \\ 0 & 1 \end{vmatrix} - 1\begin{vmatrix} 2k+1 & k \\ 1 & 1 \end{vmatrix} + (-4)\begin{vmatrix} 2k+1 & 3 \\ 1 & 0 \end{vmatrix}$

$= 2(3-0) - 1(2k+1-k) - 4(0-3)$

$= 6 - k - 1 + 12 = 17 - k$

As $\mathbf{A}$ is singular,

$\det(\mathbf{A}) = 0$

$17 - k = 0$

$k = 17$

9 $\det(\mathbf{A}) = \begin{vmatrix} 2 & -1 & 3 \\ k & 2 & 4 \\ -2 & 1 & k+3 \end{vmatrix}$

$= 2\begin{vmatrix} 2 & 4 \\ 1 & k+3 \end{vmatrix} - (-1)\begin{vmatrix} k & 4 \\ -2 & k+3 \end{vmatrix} + 3\begin{vmatrix} k & 2 \\ -2 & 1 \end{vmatrix}$

$= 2(2k+6-4) + 1(k^2+3k+8) + 3(k+4)$

$= 4k + 4 + k^2 + 3k + 8 + 3k + 12$

$= k^2 + 10k + 24$

As $\det(\mathbf{A}) = 8$

$k^2 + 10k + 24 = 8$

$k^2 + 10k + 16 = 0$

$(k+8)(k+2) = 0$

$k = -8, -2$

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10 a $\det(\mathbf{A}) = \begin{vmatrix} 2 & 5 & 3 \\ -2 & 0 & 4 \\ 3 & 10 & 8 \end{vmatrix}$

$= 2\begin{vmatrix} 0 & 4 \\ 10 & 8 \end{vmatrix} - 5\begin{vmatrix} -2 & 4 \\ 3 & 8 \end{vmatrix} + 3\begin{vmatrix} -2 & 0 \\ 3 & 10 \end{vmatrix}$

$= 2(0-40) - 5(-16-12) + 3(-20-0)$

$= -80 + 140 - 60 = 0$

Hence **A** is singular.

b $\mathbf{AB} = \begin{pmatrix} 2 & 5 & 3 \\ -2 & 0 & 4 \\ 3 & 10 & 8 \end{pmatrix}\begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 2 \\ 0 & -2 & -1 \end{pmatrix}$

$= \begin{pmatrix} 2+5+0 & 2+10-6 & 0+10-3 \\ -2+0+0 & -2+0-8 & 0+0-4 \\ 3+10+0 & 3+20-16 & 0+20-8 \end{pmatrix}$

$= \begin{pmatrix} 7 & 6 & 7 \\ -2 & -10 & -4 \\ 13 & 7 & 12 \end{pmatrix}$

c $\det(\mathbf{AB}) = \begin{vmatrix} 7 & 6 & 7 \\ -2 & -10 & -4 \\ 13 & 7 & 12 \end{vmatrix}$

$= 7\begin{vmatrix} -10 & -4 \\ 7 & 12 \end{vmatrix} - 6\begin{vmatrix} -2 & -4 \\ 13 & 12 \end{vmatrix} + 7\begin{vmatrix} -2 & -10 \\ 13 & 7 \end{vmatrix}$

$= 7(-120+28) - 6(-24+52) + 7(-14+130)$

$= 7\times(-92) - 6\times 28 + 7\times 116$

$= -644 - 168 + 812 = 0$

Hence **AB** is also singular.

11 $\begin{vmatrix} 0 & a & -b \\ -a & 0 & c \\ b & -c & 0 \end{vmatrix} = 0\begin{vmatrix} 0 & c \\ -c & 0 \end{vmatrix}$

$- a\begin{vmatrix} -a & c \\ b & 0 \end{vmatrix} + (-b)\begin{vmatrix} -a & 0 \\ b & -c \end{vmatrix}$

$= 0 - a(0-cb) - b(ac-0)$

$= abc - abc = 0$

Hence the matrix is singular for all a, b and c.

12 $\begin{vmatrix} 2 & -2 & 4 \\ 3 & x & -2 \\ -1 & 3 & x \end{vmatrix} = 2\begin{vmatrix} x & -2 \\ 3 & x \end{vmatrix}$

$-(-2)\begin{vmatrix} 3 & -2 \\ -1 & x \end{vmatrix} + 4\begin{vmatrix} 3 & x \\ -1 & 3 \end{vmatrix}$

$= 2(x^2+6) + 2(3x-2) + 4(9+x)$

$= 2x^2 + 12 + 6x - 4 + 36 + 4x$

$= 2x^2 + 10x + 44$

$= 2(x^2+5)x + 44$

$= 2\left(x^2 + 5x + \left(\frac{5}{2}\right)^2\right) + 44 - 2\times\left(\frac{5}{2}\right)^2$

$= 2\left(x + \frac{5}{2}\right)^2 + 31\frac{1}{2} \geq 31\frac{1}{2}$, for all real x

Hence the determinant cannot be zero and the matrix is non-singular for all real x.

13 $\begin{vmatrix} x-3 & -2 & 0 \\ 1 & x & -2 \\ -2 & -1 & x+1 \end{vmatrix}$

$= (x-3)\begin{vmatrix} x & -2 \\ -1 & x+1 \end{vmatrix} - (-2)\begin{vmatrix} 1 & -2 \\ -2 & x+1 \end{vmatrix} + 0\begin{vmatrix} 1 & x \\ -2 & -1 \end{vmatrix}$

$= (x-3)(x^2+x-2) + 2(x+1-4) + 0$

$= x^3 + x^2 - 2x - 3x^2 - 3x + 6 + 2x - 6$

$= x^3 - 2x^2 - 3x$

From the matrix to be singular, the determinant must be zero.

$x^3 - 2x^2 - 3x = x(x^2 - 2x - 3)$

$= x(x-3)(x+1)$

$= 0$

$x = -1, 0, 3$

14 a $\det \mathbf{M} = \begin{vmatrix} 1 & -3 \\ 2 & 1 \end{vmatrix} = 1\times 1 - (-3)\times 2 = 7$

14 b $\det \mathbf{N} = \begin{vmatrix} -1 & k \\ 4 & 3 \end{vmatrix} = (-1)\times 1 - k\times 4 = -3 - 4k$

$\det \mathbf{N} = 7$

$-3 - 4k = 7$

$4k = -10 \Rightarrow k = -2.5$

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c $\mathbf{MN} = \begin{pmatrix} 1 & -3 \\ 2 & 1 \end{pmatrix} \times \begin{pmatrix} -1 & -2.5 \\ 4 & 3 \end{pmatrix}$

$= \begin{pmatrix} -13 & -11.5 \\ 2 & -2 \end{pmatrix}$

d $\det \mathbf{MN} = \begin{vmatrix} -13 & -11.5 \\ 2 & -2 \end{vmatrix}$

$= (-13) \times (-2) - (-11.5) \times 2 = 49$

$\det \mathbf{M} = 7$

$\det \mathbf{N} = -3 + (-4) \times (-2.5) = 7$

$\det \mathbf{MN} = 7 \times 7 = 49$

15 a $\det \mathbf{A} = \begin{vmatrix} 2 & 1 & -1 \\ 1 & 0 & 4 \\ -4 & 2 & 1 \end{vmatrix}$

$= 2(0 - 8) - 1(1 + 16) - 1(2 - 0)$

$= -16 - 17 - 2$

$= -35$

b $\det \mathbf{B} = \begin{vmatrix} 3 & 1 & 2 \\ k & 4 & 5 \\ 0 & 2 & 3 \end{vmatrix}$

$= 3(12 - 10) - 1(3k - 0) + 2(2k - 0)$

$= 6 - 3k + 4k$

$= 6 + k$

$\det \mathbf{B} = 2$

$6 + k = 2 \Rightarrow k = -4$

c $\mathbf{AB} = \begin{pmatrix} 2 & 1 & -1 \\ 1 & 0 & 4 \\ -4 & 2 & 1 \end{pmatrix} \times \begin{pmatrix} 3 & 1 & 2 \\ -4 & 4 & 5 \\ 0 & 2 & 3 \end{pmatrix}$

$= \begin{pmatrix} 2 & 4 & 6 \\ 3 & 9 & 14 \\ -20 & 6 & 5 \end{pmatrix}$

15 d $\det \mathbf{AB} = \begin{vmatrix} 2 & 4 & 6 \\ 3 & 9 & 14 \\ -20 & 6 & 5 \end{vmatrix}$

$= 2(45 - 84) - 4(15 + 280) + 6(18 + 180)$

$= -78 - 1180 + 1188$

$= -70$

$= (-35) \times 2 = \det \mathbf{A} \times \det \mathbf{B}$

Challenge

a $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc = 0$ for singular matrix

$ad - bc = 0 \Rightarrow ad = bc$

The possibilities are:

$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}, \begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix},$

$\begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}, \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}, \begin{pmatrix} -1 & -1 \\ -1 & -1 \end{pmatrix}$

b $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},$

$\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$

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Matrices 6D

1 a $\det \begin{vmatrix} 3 & -1 \\ -4 & 2 \end{vmatrix} = 6-(-4)\times(-1)$

$= 6-4$

$= 2 \quad \neq 0$

$\therefore$ the matrix is non-singular

So inverse is $\frac{1}{2}\begin{pmatrix} 2 & 1 \\ 4 & 3 \end{pmatrix}$

or $\begin{pmatrix} 1 & 0.5 \\ 2 & 1.5 \end{pmatrix}$

b $\det \begin{vmatrix} 3 & 3 \\ -1 & -1 \end{vmatrix} = -3-(-1)\times 3$

$= -3+3$

$= 0$

$\therefore$ matrix is singular.

c $\det \begin{vmatrix} 2 & 5 \\ 0 & 0 \end{vmatrix} = 0-0$

$= 0$

$\therefore$ matrix is singular

d $\det \begin{vmatrix} 1 & 2 \\ 3 & 5 \end{vmatrix} = 5-6$

$= -1 \quad \neq 0$

$\therefore$ matrix is non-singular

Inverse is $\frac{1}{-1}\begin{pmatrix} 5 & -2 \\ -3 & 1 \end{pmatrix} = \begin{pmatrix} -5 & 2 \\ 3 & -1 \end{pmatrix}$

e $\det \begin{vmatrix} 6 & 3 \\ 4 & 2 \end{vmatrix} = 12-12$

$= 0$

$\therefore$ matrix is singular

f $\det \begin{vmatrix} 4 & 3 \\ 6 & 2 \end{vmatrix} = 8-18$

$= -10 \quad \neq 0$

$\therefore$ matrix is non-singular

Inverse is $\frac{1}{-10}\begin{pmatrix} 2 & -3 \\ -6 & 4 \end{pmatrix}$

$= \begin{pmatrix} -0.2 & 0.3 \\ 0.6 & -0.4 \end{pmatrix}$

2 a Let $\mathbf{A} = \begin{pmatrix} a & 1+a \\ 1+a & 2+a \end{pmatrix}$

$\det \mathbf{A} = 2a + a^2 - (1+a)^2$

$= 2a + a^2 - 1 - 2a - a^2$

$= -1$

$\mathbf{A}^{-1} = \frac{1}{-1}\begin{pmatrix} 2+a & -(1+a) \\ -(1+a) & a \end{pmatrix}$

$= \begin{pmatrix} -(2+a) & (1+a) \\ (1+a) & -a \end{pmatrix}$

b Let $\mathbf{B} = \begin{pmatrix} 2a & 3b \\ -a & -b \end{pmatrix}$

$\det \mathbf{B} = -2ab - (-a)\times 3b$

$= -2ab + 3ab$

$= ab$

$\mathbf{B}^{-1} = \frac{1}{ab}\begin{pmatrix} -b & -3b \\ a & 2a \end{pmatrix}$

$= \begin{pmatrix} -\frac{1}{a} & -\frac{3}{a} \\ \frac{1}{b} & \frac{2}{b} \end{pmatrix}$ **provided that** $ab \neq 0$

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3 a $\mathbf{ABC} = 1$

$\Rightarrow \mathbf{A}^{-1}\mathbf{ABC} = \mathbf{A}^{-1}\mathbf{I}$

$\Rightarrow \mathbf{BC} = \mathbf{A}^{-1}$

$\Rightarrow \mathbf{BCC}^{-1} = \mathbf{A}^{-1}\mathbf{C}^{-1}$

$\Rightarrow \mathbf{B} = \mathbf{A}^{-1}\mathbf{C}^{-1} = (\mathbf{CA})^{-1}$

$\therefore \mathbf{B}^{-1} = \mathbf{CA}$

b $\mathbf{CA} = \begin{pmatrix} 2 & 1 \\ -3 & -1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & -6 \end{pmatrix}$

$= \begin{pmatrix} -1 & -4 \\ 1 & 3 \end{pmatrix}$

$\therefore (\mathbf{CA})^{-1} = \frac{1}{-3+4}\begin{pmatrix} 3 & 4 \\ -1 & -1 \end{pmatrix}$

$\mathbf{B} = \begin{pmatrix} 3 & 4 \\ -1 & -1 \end{pmatrix}$

4 a $\mathbf{AB} = \mathbf{C}$

$\Rightarrow \mathbf{A}^{-1}\mathbf{AB} = \mathbf{A}^{-1}\mathbf{C}$

$\Rightarrow \mathbf{B} = \mathbf{A}^{-1}\mathbf{C}$

b $\mathbf{A} = \begin{pmatrix} 2 & -1 \\ 4 & 3 \end{pmatrix} \Rightarrow \det \mathbf{A} = 6 - -4 = 10$

$\therefore \mathbf{A}^{-1} = \frac{1}{10}\begin{pmatrix} 3 & 1 \\ -4 & 2 \end{pmatrix}$

$\therefore \mathbf{B} = \mathbf{A}^{-1}\mathbf{C}$

$= \frac{1}{10}\begin{pmatrix} 3 & 1 \\ -4 & 2 \end{pmatrix}\begin{pmatrix} 3 & 6 \\ 1 & 22 \end{pmatrix}$

$= \frac{1}{10}\begin{pmatrix} 10 & 40 \\ -10 & 20 \end{pmatrix}$

$= \begin{pmatrix} 1 & 4 \\ -1 & 2 \end{pmatrix}$

5 a $\mathbf{BAC} = \mathbf{B}$

$\Rightarrow \mathbf{B}^{-1}\mathbf{BAC} = \mathbf{B}^{-1}\mathbf{B}$

$\Rightarrow \mathbf{AC} = \mathbf{I}$

$\Rightarrow \mathbf{A} = \mathbf{C}^{-1}.$

b $\mathbf{C} = \begin{pmatrix} 5 & 3 \\ 3 & 2 \end{pmatrix}$

$\det \mathbf{C} = 10 - 9 = 1$

$\therefore \mathbf{C}^{-1} = \frac{1}{1}\begin{pmatrix} 2 & -3 \\ -3 & 5 \end{pmatrix}$

$\therefore \mathbf{A} = \begin{pmatrix} 2 & -3 \\ -3 & 5 \end{pmatrix}.$

6 $\mathbf{A} = \begin{pmatrix} 2 & -1 \\ -4 & 3 \end{pmatrix} \Rightarrow$

$\det \mathbf{A} = 6 - (-4) \times (-1) = 2$

$\therefore \mathbf{A}^{-1} = \frac{1}{2}\begin{pmatrix} 3 & 1 \\ 4 & 2 \end{pmatrix}$

$\mathbf{AB} = \begin{pmatrix} 4 & 7 & -8 \\ -8 & -13 & 18 \end{pmatrix}$ ($\times$ **on left by** $\mathbf{A}^{-1}$)

$\Rightarrow \mathbf{B} = \frac{1}{2}\begin{pmatrix} 3 & 1 \\ 4 & 2 \end{pmatrix}\begin{pmatrix} 4 & 7 & -8 \\ -8 & -13 & 18 \end{pmatrix}$

$\mathbf{B} = \frac{1}{2}\begin{pmatrix} 4 & 8 & -6 \\ 0 & 2 & 4 \end{pmatrix}$

$\mathbf{B} = \begin{pmatrix} 2 & 4 & -3 \\ 0 & 1 & 2 \end{pmatrix}$

7 $\mathbf{B} = \begin{pmatrix} 5 & -4 \\ 2 & 1 \end{pmatrix} \Rightarrow \det \mathbf{B} = 5 + 8 = 13$

$\mathbf{B}^{-1} = \frac{1}{13}\begin{pmatrix} 1 & 4 \\ -2 & 5 \end{pmatrix}$

$\mathbf{AB} = \begin{pmatrix} 11 & -1 \\ -8 & 9 \\ -2 & -1 \end{pmatrix}$ ($\times$ **on right by** $\mathbf{B}^{-1}$)

$\Rightarrow \mathbf{ABB}^{-1} = \begin{pmatrix} 11 & -1 \\ -8 & 9 \\ -2 & -1 \end{pmatrix}\mathbf{B}^{-1}$

$\therefore \mathbf{A} = \frac{1}{13}\begin{pmatrix} 11 & -1 \\ -8 & 9 \\ -2 & -1 \end{pmatrix}\begin{pmatrix} 1 & 4 \\ -2 & 5 \end{pmatrix}$

$= \frac{1}{13}\begin{pmatrix} 13 & 39 \\ -26 & 13 \\ 0 & -13 \end{pmatrix}$

$= \begin{pmatrix} 1 & 3 \\ -2 & 1 \\ 0 & -1 \end{pmatrix}$

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8 a $\mathbf{A}=\begin{pmatrix}3a & b\\ 4a & 2b\end{pmatrix} \Rightarrow$

$\det \mathbf{A} = 6ab - 4ab = 2ab$

$\therefore \mathbf{A}^{-1} = \frac{1}{2ab}\begin{pmatrix}2b & -b\\ -4a & 3a\end{pmatrix}.$

b $\mathbf{B} = \mathbf{XA}$

$\Rightarrow \mathbf{BA}^{-1} = \mathbf{XAA}^{-1}$

$\therefore \mathbf{X} = \mathbf{BA}^{-1}$

So $\mathbf{X} = \begin{pmatrix}-a & b\\ 3a & 2b\end{pmatrix}\begin{pmatrix}2b & -b\\ -4a & 3a\end{pmatrix} \times \frac{1}{2ab}$

$= \frac{1}{2ab}\begin{pmatrix}-6ab & 4ab\\ -2ab & 3ab\end{pmatrix}$

$\therefore \mathbf{X} = \begin{pmatrix}-3 & 2\\ -1 & \frac{3}{2}\end{pmatrix}$

9 a Given $\mathbf{AB} = \mathbf{BA}$
and $\mathbf{ABA} = \mathbf{B}$

$\Rightarrow \mathbf{A(AB)} = \mathbf{B}$

$\Rightarrow \mathbf{A}^2\mathbf{B} = \mathbf{B}$

$\Rightarrow \mathbf{A}^2\mathbf{BB}^{-1} = \mathbf{BB}^{-1}$

$\Rightarrow \mathbf{A}^2 = \mathbf{I}$

b $\mathbf{AB} = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}a & b\\ c & d\end{pmatrix} = \begin{pmatrix}c & d\\ a & b\end{pmatrix}$

$\mathbf{BA} = \begin{pmatrix}a & b\\ c & d\end{pmatrix}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} = \begin{pmatrix}b & a\\ d & c\end{pmatrix}$

$\mathbf{AB} = \mathbf{BA} \Rightarrow b = c$

$d = a$

i.e. $a = d$ and $b = c$

10 a $\det \mathbf{M} = \begin{vmatrix}2 & 3\\ k & -1\end{vmatrix} = 2\times(-1) - 3\times k$

$= -2 - 3k \neq 0$ for inverse to exist.

Hence $-2 - 3k \Rightarrow k \neq -\frac{2}{3}$

b $\mathbf{M}^{-1} = \frac{1}{-(2+3k)}\begin{pmatrix}-1 & -3\\ -k & 2\end{pmatrix}$

$= \frac{1}{3k+2}\begin{pmatrix}1 & 3\\ k & -2\end{pmatrix}$

11 a $\det \mathbf{A} = \begin{vmatrix}4 & p\\ -2 & -2\end{vmatrix} = 4\times(-2) - p\times(-2)$

$= -8 + 2p \neq 0$ for inverse to exist.

Hence $-8 + 2p \Rightarrow p \neq 4$

$\mathbf{A}^{-1} = \frac{1}{2p-8}\begin{pmatrix}-2 & -p\\ 2 & 4\end{pmatrix}$

$= \frac{1}{p-4}\begin{pmatrix}-1 & \frac{-p}{2}\\ 1 & 2\end{pmatrix}$

b $\mathbf{A} + \mathbf{A}^{-1} = \begin{pmatrix}4 & p\\ -2 & -2\end{pmatrix} + \begin{pmatrix}-\frac{1}{p-4} & \frac{p}{2p-8}\\ \frac{1}{p-4} & \frac{2}{p-4}\end{pmatrix}$

$= \begin{pmatrix}\frac{4p-17}{p-4} & \frac{2p^2-9p}{2p-8}\\ \frac{9-2p}{p-4} & \frac{10-2p}{p-4}\end{pmatrix} = \begin{pmatrix}5 & \frac{9}{2}\\ -3 & -4\end{pmatrix}$

Compare a corresponding element to find p

$\frac{4p-17}{p-4} = 5$

$4p - 17 = 5(p-4)$

$4p - 17 = 5p - 20$

$5p - 4p = -17 + 20$

$p = 3$

12 a $\det \mathbf{M} = \begin{vmatrix}k & -3\\ 4 & k+3\end{vmatrix} = k\times(k+3) - (-3)\times 4$

$= k^2 + 3k + 12$

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12 b Complete the square

$$k^2+3k+12=\left(k^2+3k+\frac{9}{4}\right)+12-\frac{9}{4}$$

$$=\left(k+\frac{3}{2}\right)^2+9.75\geq 9.75 \text{ for all } k$$

Hence, **M** is non-singular

c $10\mathbf{M}^{-1}+\mathbf{M}=\mathbf{I}$

$$\mathbf{M}^{-1}=\frac{1}{k^2+3k+12}\begin{pmatrix}k+3 & 3\\ -4 & k\end{pmatrix}$$

$$10\mathbf{M}^{-1}=\frac{10}{k^2+3k+12}\begin{pmatrix}k+3 & 3\\ -4 & k\end{pmatrix}$$

$$10\mathbf{M}^{-1}+\mathbf{M}=\frac{10}{k^2+3k+12}\begin{pmatrix}k+3 & 3\\ -4 & k\end{pmatrix}+\begin{pmatrix}k & -3\\ 4 & k+3\end{pmatrix}$$

Top right element is $\frac{30}{k^2+3k+12}-3$

Compare with top right element of **I**

$$\frac{30}{k^2+3k+12}-3=0$$

$$\frac{30}{k^2+3k+12}=3$$

$$3\left(k^2+3k+12\right)=30$$

$$3k^2+9k+6=0$$

$$(3k+6)(k+1)=0$$

$k=-2$ inadmissible, does not give **I**

$k=-1$

13 a $\det\mathbf{A}=\begin{vmatrix}a & 2\\ 3 & 2a\end{vmatrix}=a\times 2a-2\times 3$

$$=2a^2-6$$

$$\mathbf{A}^{-1}=\frac{1}{2a^2-6}\begin{pmatrix}2a & -2\\ -3 & a\end{pmatrix}$$

b $2a^2-6$

$2a^2=6$

$a^2=3$

$a=\sqrt{3},\ a=-\sqrt{3}$

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Matrices 6E

1 a Let $\mathbf{A} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix}$

$$\det(\mathbf{A}) = 1\begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} - 0\begin{vmatrix} 0 & 1 \\ 0 & 2 \end{vmatrix} + 0\begin{vmatrix} 0 & 2 \\ 0 & 1 \end{vmatrix}$$
$$= 1(4-1) - 0 + 0 = 3$$

The matrix of minors is given by

$$\mathbf{M} = \begin{pmatrix} \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} & \begin{vmatrix} 0 & 1 \\ 0 & 2 \end{vmatrix} & \begin{vmatrix} 0 & 2 \\ 0 & 1 \end{vmatrix} \\ \begin{vmatrix} 0 & 0 \\ 1 & 2 \end{vmatrix} & \begin{vmatrix} 1 & 0 \\ 0 & 2 \end{vmatrix} & \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} \\ \begin{vmatrix} 0 & 0 \\ 2 & 1 \end{vmatrix} & \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} & \begin{vmatrix} 1 & 0 \\ 0 & 2 \end{vmatrix} \end{pmatrix}$$
$$= \begin{pmatrix} 3 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix}$$

The matrix of cofactors is given by

$$\mathbf{C} = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}$$

$$\mathbf{C}^{\mathrm{T}} = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}$$

$$\mathbf{A}^{-1} = \frac{1}{\det(\mathbf{A})}\mathbf{C}^{\mathrm{T}} = \frac{1}{3}\begin{pmatrix} 3 & 0 & 0 \\ 0 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}$$
$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{2}{3} & -\frac{1}{3} \\ 0 & -\frac{1}{3} & \frac{2}{3} \end{pmatrix}$$

b By inspection

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{3} \end{pmatrix}$$

c Let $\mathbf{A} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{3}{5} & \frac{-4}{5} \\ 0 & \frac{4}{5} & \frac{3}{5} \end{pmatrix}$

$$\det(\mathbf{A}) = 1\begin{vmatrix} \frac{3}{5} & -\frac{4}{5} \\ \frac{4}{5} & \frac{3}{5} \end{vmatrix} - 0\begin{vmatrix} 0 & -\frac{4}{5} \\ 0 & \frac{3}{5} \end{vmatrix} + 0\begin{vmatrix} 0 & -\frac{3}{5} \\ 0 & \frac{4}{5} \end{vmatrix}$$
$$= 1\left(\frac{9}{25} + \frac{6}{25}\right) - 0 + 0 = 1$$

The matrix of minors is given by

$$\mathbf{M} = \begin{pmatrix} \begin{vmatrix} \frac{3}{5} & -\frac{4}{5} \\ \frac{4}{5} & \frac{3}{5} \end{vmatrix} & \begin{vmatrix} 0 & -\frac{4}{5} \\ 0 & \frac{3}{5} \end{vmatrix} & \begin{vmatrix} 0 & -\frac{3}{5} \\ 0 & \frac{4}{5} \end{vmatrix} \\ \begin{vmatrix} 0 & 0 \\ \frac{4}{5} & \frac{3}{5} \end{vmatrix} & \begin{vmatrix} 1 & 0 \\ 0 & \frac{3}{5} \end{vmatrix} & \begin{vmatrix} 1 & 0 \\ 0 & \frac{4}{5} \end{vmatrix} \\ \begin{vmatrix} 0 & 0 \\ \frac{3}{5} & -\frac{4}{5} \end{vmatrix} & \begin{vmatrix} 1 & 0 \\ 0 & -\frac{4}{5} \end{vmatrix} & \begin{vmatrix} 1 & 0 \\ 0 & \frac{3}{5} \end{vmatrix} \end{pmatrix}$$
$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{3}{5} & \frac{4}{5} \\ 0 & -\frac{4}{5} & \frac{3}{5} \end{pmatrix}$$

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1 c cont. The matrix of cofactors is given by

$$\mathbf{C} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{3}{5} & \frac{-4}{5} \\ 0 & \frac{4}{5} & \frac{3}{5} \end{pmatrix}$$

$$\mathbf{C}^{\mathrm{T}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{3}{5} & \frac{4}{5} \\ 0 & -\frac{4}{5} & \frac{3}{5} \end{pmatrix}$$

$$\mathbf{A}^{-1} = \frac{1}{\det(\mathbf{A})}\mathbf{C}^{\mathrm{T}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{3}{5} & \frac{4}{5} \\ 0 & -\frac{4}{5} & \frac{3}{5} \end{pmatrix}$$

2 a Let $\mathbf{A} = \begin{pmatrix} 1 & -3 & 2 \\ 0 & -2 & 1 \\ 3 & 0 & 2 \end{pmatrix}$

$$\begin{aligned} \det(\mathbf{A}) &= 1\begin{vmatrix} -2 & 1 \\ 0 & 2 \end{vmatrix} - (-3)\begin{vmatrix} 0 & 1 \\ 3 & 2 \end{vmatrix} + 2\begin{vmatrix} 0 & -2 \\ 3 & 0 \end{vmatrix} \\ &= 1(-4-0) + 3(0-3) + 2(0+6) \\ &= -4-9+12 = -1 \end{aligned}$$

The matrix of minors is given by

$$\mathbf{M} = \begin{pmatrix} \begin{vmatrix} -2 & 1 \\ 0 & 2 \end{vmatrix} & \begin{vmatrix} 0 & 1 \\ 3 & 2 \end{vmatrix} & \begin{vmatrix} 0 & -2 \\ 3 & 0 \end{vmatrix} \\ \begin{vmatrix} -3 & 2 \\ 0 & 2 \end{vmatrix} & \begin{vmatrix} 1 & 2 \\ 3 & 2 \end{vmatrix} & \begin{vmatrix} 1 & -3 \\ 3 & 0 \end{vmatrix} \\ \begin{vmatrix} -3 & 2 \\ -2 & 1 \end{vmatrix} & \begin{vmatrix} 1 & 2 \\ 0 & 1 \end{vmatrix} & \begin{vmatrix} 1 & -3 \\ 0 & -2 \end{vmatrix} \end{pmatrix}$$

$$= \begin{pmatrix} -4 & -3 & 6 \\ -6 & -4 & 9 \\ 1 & 1 & -2 \end{pmatrix}$$

The matrix of cofactors is given by

$$\mathbf{C} = \begin{pmatrix} -4 & 3 & 6 \\ 6 & -4 & -9 \\ 1 & -1 & -2 \end{pmatrix}$$

$$\mathbf{C}^{\mathrm{T}} = \begin{pmatrix} -4 & 6 & 1 \\ 3 & -4 & -1 \\ 6 & -9 & -2 \end{pmatrix}$$

$$\mathbf{A}^{-1} = \frac{1}{\det(\mathbf{A})}\mathbf{C}^{\mathrm{T}} = \frac{1}{-1}\begin{pmatrix} -4 & 6 & 1 \\ 3 & -4 & -1 \\ 6 & -9 & -2 \end{pmatrix}$$

$$= \begin{pmatrix} 4 & -6 & -1 \\ -3 & 4 & 1 \\ -6 & 9 & 2 \end{pmatrix}$$

2 b Let $\mathbf{A} = \begin{pmatrix} 2 & 3 & 2 \\ 3 & -2 & 1 \\ 2 & 1 & 1 \end{pmatrix}$

$$\begin{aligned} \det(\mathbf{A}) &= 2\begin{vmatrix} -2 & 1 \\ 1 & 1 \end{vmatrix} - 3\begin{vmatrix} 3 & 1 \\ 2 & 1 \end{vmatrix} + 2\begin{vmatrix} 3 & -2 \\ 2 & 1 \end{vmatrix} \\ &= 2(-2-1) - 3(3-2) + 2(3+4) \\ &= -6-3+14 = 5 \end{aligned}$$

The matrix of minors is given by

$$\mathbf{M} = \begin{pmatrix} \begin{vmatrix} -2 & 1 \\ 1 & 1 \end{vmatrix} & \begin{vmatrix} 3 & 1 \\ 2 & 1 \end{vmatrix} & \begin{vmatrix} 3 & -2 \\ 2 & 1 \end{vmatrix} \\ \begin{vmatrix} 3 & 2 \\ 1 & 1 \end{vmatrix} & \begin{vmatrix} 2 & 2 \\ 2 & 1 \end{vmatrix} & \begin{vmatrix} 2 & 3 \\ 2 & 1 \end{vmatrix} \\ \begin{vmatrix} 3 & 2 \\ -2 & 1 \end{vmatrix} & \begin{vmatrix} 2 & 2 \\ 3 & 1 \end{vmatrix} & \begin{vmatrix} 2 & 3 \\ 3 & -2 \end{vmatrix} \end{pmatrix}$$

$$= \begin{pmatrix} -3 & 1 & 7 \\ 1 & -2 & -4 \\ 7 & -4 & -13 \end{pmatrix}$$

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2 b cont.

The matrix of cofactors is given by

$$\mathbf{C} = \begin{pmatrix} -3 & -1 & 7 \\ -1 & -2 & 4 \\ 7 & 4 & -13 \end{pmatrix}$$

$$\mathbf{C}^{\mathrm{T}} = \begin{pmatrix} -3 & -1 & 7 \\ -1 & -2 & 4 \\ 7 & 4 & -13 \end{pmatrix}$$

$$\mathbf{A}^{-1} = \frac{1}{\det(\mathbf{A})}\mathbf{C}^{\mathrm{T}} = \frac{1}{5}\begin{pmatrix} -3 & 1 & 7 \\ -1 & -2 & 4 \\ 7 & 4 & -13 \end{pmatrix}$$

$$= \begin{pmatrix} -\frac{3}{5} & -\frac{1}{5} & \frac{7}{5} \\ -\frac{1}{5} & -\frac{2}{5} & \frac{4}{5} \\ \frac{7}{5} & \frac{4}{5} & \frac{-13}{5} \end{pmatrix}$$

c Let

$$\mathbf{A} = \begin{pmatrix} 3 & 2 & -7 \\ 1 & -3 & 1 \\ 0 & 2 & -2 \end{pmatrix}$$

$$\begin{aligned} \det(\mathbf{A}) &= 3\begin{vmatrix} -3 & 1 \\ 2 & -2 \end{vmatrix} - 2\begin{vmatrix} 1 & 1 \\ 0 & -2 \end{vmatrix} + (-7)\begin{vmatrix} 1 & -3 \\ 0 & 2 \end{vmatrix} \\ &= 3(6-2) - 2(-2-0) - 7(2-0) \\ &= 12 + 4 - 14 = 2 \end{aligned}$$

The matrix of minors is given by

$$\mathbf{M} = \begin{pmatrix} \begin{vmatrix} -3 & 1 \\ 2 & -2 \end{vmatrix} & \begin{vmatrix} 1 & 1 \\ 0 & -2 \end{vmatrix} & \begin{vmatrix} 1 & -3 \\ 0 & 2 \end{vmatrix} \\ \begin{vmatrix} 2 & -7 \\ 2 & -2 \end{vmatrix} & \begin{vmatrix} 3 & -7 \\ 0 & -2 \end{vmatrix} & \begin{vmatrix} 3 & 2 \\ 0 & 2 \end{vmatrix} \\ \begin{vmatrix} 2 & -7 \\ -3 & 1 \end{vmatrix} & \begin{vmatrix} 3 & -7 \\ 1 & 1 \end{vmatrix} & \begin{vmatrix} 3 & 2 \\ 1 & -3 \end{vmatrix} \end{pmatrix}$$

$$= \begin{pmatrix} 4 & -2 & 2 \\ 10 & -6 & 6 \\ -19 & 10 & -11 \end{pmatrix}$$

The matrix of cofactors is given by

$$\mathbf{C} = \begin{pmatrix} 4 & 2 & 2 \\ -10 & -6 & -6 \\ -19 & -10 & -11 \end{pmatrix}$$

$$\mathbf{C}^{\mathrm{T}} = \begin{pmatrix} 4 & -10 & -19 \\ 2 & -6 & -10 \\ 2 & -6 & -11 \end{pmatrix}$$

$$\mathbf{A}^{-1} \frac{1}{\det(\mathbf{A})}\mathbf{C}^{\mathrm{T}} = \frac{1}{2}\begin{pmatrix} 4 & -10 & -19 \\ 2 & -6 & -10 \\ 2 & -6 & -11 \end{pmatrix}$$

$$= \begin{pmatrix} 2 & -5 & -\frac{19}{2} \\ 1 & -3 & -5 \\ 1 & -3 & -\frac{11}{2} \end{pmatrix}$$

3 a $$\det(\mathbf{A}) = 1\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} - 0\begin{vmatrix} 0 & 0 \\ 2 & 1 \end{vmatrix} + 1\begin{vmatrix} 0 & 1 \\ 2 & 0 \end{vmatrix}$$

$$= 1 - 0 - 2 = -1$$

The matrix of minors is given by

$$\mathbf{M} = \begin{pmatrix} \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} & \begin{vmatrix} 0 & 0 \\ 2 & 1 \end{vmatrix} & \begin{vmatrix} 0 & 1 \\ 2 & 2 \end{vmatrix} \\ \begin{vmatrix} 0 & 1 \\ 0 & 1 \end{vmatrix} & \begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix} & \begin{vmatrix} 1 & 0 \\ 2 & 0 \end{vmatrix} \\ \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} & \begin{vmatrix} 1 & 1 \\ 0 & 0 \end{vmatrix} & \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & -2 \\ 0 & -1 & 0 \\ -1 & 0 & 1 \end{pmatrix}$$

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3 a cont. The matrix of cofactors is given by

$$\mathbf{C} = \begin{pmatrix} 1 & 0 & -2 \\ 0 & -1 & 0 \\ -1 & 0 & 1 \end{pmatrix}$$

$$\mathbf{C}^{\mathrm{T}} = \begin{pmatrix} 1 & 0 & -1 \\ 0 & -1 & 0 \\ -2 & 0 & 1 \end{pmatrix}$$

$$\mathbf{A}^{-1} = \frac{1}{\det(\mathbf{A})}\mathbf{C}^{\mathrm{T}} = \frac{1}{-1}\begin{pmatrix} 1 & 0 & -1 \\ 0 & -1 & 0 \\ -2 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} -1 & 0 & 1 \\ 0 & 1 & 0 \\ 2 & 0 & -1 \end{pmatrix}$$

b $\det(\mathbf{B}) = 2\begin{vmatrix} 0 & 1 \\ 2 & 1 \end{vmatrix} - 1\begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} + (-1)\begin{vmatrix} 1 & 0 \\ 1 & 2 \end{vmatrix}$

$= -4 - 0 - 2 = -6$

The matrix of minors is given by

$$\mathbf{M} = \begin{pmatrix} \begin{vmatrix} 0 & 1 \\ 2 & 1 \end{vmatrix} & \begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} & \begin{vmatrix} 1 & 0 \\ 1 & 2 \end{vmatrix} \\ \begin{vmatrix} 1 & -1 \\ 2 & 1 \end{vmatrix} & \begin{vmatrix} 2 & -1 \\ 1 & 1 \end{vmatrix} & \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} \\ \begin{vmatrix} 1 & -1 \\ 0 & 1 \end{vmatrix} & \begin{vmatrix} 2 & -1 \\ 1 & 1 \end{vmatrix} & \begin{vmatrix} 2 & 1 \\ 1 & 0 \end{vmatrix} \end{pmatrix}$$

$$= \begin{pmatrix} -2 & 0 & 2 \\ 3 & 3 & 3 \\ 1 & 3 & -1 \end{pmatrix}$$

The matrix of cofactors is given by

$$\mathbf{C} = \begin{pmatrix} -2 & 0 & 2 \\ -3 & 3 & -3 \\ 1 & -3 & -1 \end{pmatrix}$$

$$\mathbf{C}^{\mathrm{T}} = \begin{pmatrix} -2 & -3 & 1 \\ 0 & 3 & -3 \\ 2 & -3 & -1 \end{pmatrix}$$

$$\mathbf{B}^{-1} = \frac{1}{\det(\mathbf{B})}\mathbf{C}^{\mathrm{T}} = \frac{1}{-6}\begin{pmatrix} -2 & -3 & 1 \\ 0 & 3 & -3 \\ 2 & -3 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{3} & \frac{1}{2} & -\frac{1}{6} \\ 0 & -\frac{1}{2} & \frac{1}{2} \\ -\frac{1}{3} & \frac{1}{2} & \frac{1}{6} \end{pmatrix}$$

c $B^{1}A^{-1} = \begin{pmatrix} \frac{1}{3} & \frac{1}{2} & -\frac{1}{6} \\ 0 & -\frac{1}{2} & \frac{1}{2} \\ -\frac{1}{3} & \frac{1}{2} & \frac{1}{6} \end{pmatrix}\begin{pmatrix} -1 & 0 & 1 \\ 0 & 1 & 0 \\ 2 & 0 & -1 \end{pmatrix}$

$$= \begin{pmatrix} -\frac{1}{3}+0-\frac{1}{3} & 0+\frac{1}{2}+0 & \frac{1}{3}+0+\frac{1}{6} \\ 0+0+1 & 0-\frac{1}{2}+0 & 0+0-\frac{1}{2} \\ \frac{1}{3}+0+\frac{1}{3} & 0+\frac{1}{2}+0 & -\frac{1}{3}+0-\frac{1}{6} \end{pmatrix}$$

$$= \begin{pmatrix} -\frac{2}{3} & \frac{1}{2} & \frac{1}{2} \\ 1 & -\frac{1}{2} & -\frac{1}{2} \\ \frac{2}{3} & \frac{1}{2} & -\frac{1}{2} \end{pmatrix} = (\mathbf{AB})^{-1}, \text{ as required.}$$

4 a $\det(\mathbf{A}) = 2\begin{vmatrix} 1 & 1 \\ 1 & 4 \end{vmatrix} - 0\begin{vmatrix} k & 1 \\ 1 & 4 \end{vmatrix} + 3\begin{vmatrix} k & 1 \\ 1 & 1 \end{vmatrix}$

$= 2(4-1) - 0 + 3(k-1)$

$= 6 + 3k - 3 = 3k + 3 = 3(k+1),$

as required.

b The matrix of minors is given by

$$\mathbf{M} = \begin{pmatrix} \begin{vmatrix} 1 & 1 \\ 1 & 4 \end{vmatrix} & \begin{vmatrix} k & 1 \\ 1 & 4 \end{vmatrix} & \begin{vmatrix} k & 1 \\ 1 & 1 \end{vmatrix} \\ \begin{vmatrix} 0 & 3 \\ 1 & 4 \end{vmatrix} & \begin{vmatrix} 2 & 3 \\ 1 & 4 \end{vmatrix} & \begin{vmatrix} 2 & 0 \\ 1 & 1 \end{vmatrix} \\ \begin{vmatrix} 0 & 3 \\ 1 & 1 \end{vmatrix} & \begin{vmatrix} 2 & 3 \\ k & 1 \end{vmatrix} & \begin{vmatrix} 2 & 0 \\ k & 1 \end{vmatrix} \end{pmatrix}$$

$$= \begin{pmatrix} 3 & 4k-1 & k-1 \\ -3 & 5 & 2 \\ -3 & 2-3k & 2 \end{pmatrix}$$

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4 b cont. The matrix of cofactors is given by

$$\mathbf{C}=\begin{pmatrix}3 & 1-4k & k-1\\ -3 & 5 & -2\\ -3 & 3k-2 & 2\end{pmatrix}$$

$$\mathbf{C}^{\mathrm{T}}=\begin{pmatrix}3 & 3 & -3\\ 1-4k & 5 & 3k-2\\ k-1 & -2 & 2\end{pmatrix}$$

$$\mathbf{A}^{-1}=\frac{1}{\det(\mathbf{A})}\mathbf{C}^{\mathrm{T}}=\frac{1}{3(k+1)}\begin{pmatrix}3 & 3 & -3\\ 1-4k & 5 & 3k-2\\ k-1 & -2 & 2\end{pmatrix}$$

5 $\mathbf{A}=\mathbf{A}^{-1}$

Multiplying throughout by $\mathbf{A}$

$\mathbf{AA}=\mathbf{AA}^{-1}$

$\mathbf{A}^2=\mathbf{I}$

$$\mathbf{A}^2=\begin{pmatrix}5 & a & 4\\ b & -7 & 8\\ 2 & -2 & c\end{pmatrix}\begin{pmatrix}5 & a & 4\\ b & -7 & 8\\ 2 & -2 & c\end{pmatrix}$$

$$=\begin{pmatrix}ab+33 & -2a-8 & 8a+4c+20\\ 16-2b & ab+33 & 4b+8c-56\\ -2b+2c+10 & 2a-2c+14 & c^2-8\end{pmatrix}$$

$$=\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}$$

Equating the second elements in the first row

$-2a-8=0\Rightarrow a=-4$

Equating the first elements in the second row

$16-2b=0\Rightarrow b=8$

Equating the first elements in the third row using $b=8$

$-2b+2c+10=0\Rightarrow -16+2c+10=0$

$2c=6\Rightarrow c=3$

$a=-4, b=8, c=3$

6 a

$$\mathbf{A}^2=\begin{pmatrix}2 & -1 & 1\\ 4 & -3 & 0\\ -3 & 3 & 1\end{pmatrix}\begin{pmatrix}2 & -1 & 1\\ 4 & -3 & 0\\ -3 & 3 & 1\end{pmatrix}$$

$$=\begin{pmatrix}4-4-3 & -2+3+3 & 2+0+1\\ 8-12+0 & -4+9+0 & 4+0+0\\ -6+12-3 & 3-9+3 & -3+0+1\end{pmatrix}$$

$$=\begin{pmatrix}-3 & 4 & 3\\ -4 & 5 & 4\\ 3 & -3 & -2\end{pmatrix}$$

$\mathbf{A}^3=\mathbf{A}^2\mathbf{A}$

$$=\begin{pmatrix}-3 & 4 & 3\\ -4 & 5 & 4\\ 3 & -3 & -2\end{pmatrix}\begin{pmatrix}2 & -1 & 1\\ 4 & -3 & 0\\ -3 & 3 & 1\end{pmatrix}$$

$$=\begin{pmatrix}-6+16-9 & 3-12+9 & -3+0+3\\ -8+20-12 & 4-15+12 & -4+0+4\\ 6-12+6 & -3+9-6 & 3+0-2\end{pmatrix}$$

$$=\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}=\mathbf{I}, \text{as required.}$$

b $\mathbf{A}^3=\mathbf{AA}^2=\mathbf{I}$

Comparing with the definition of an inverse

$\mathbf{AA}^{-1}=\mathbf{I}$

$$\mathbf{A}^{-1}=\mathbf{A}^2=\begin{pmatrix}-3 & 4 & 3\\ -4 & 5 & 4\\ 3 & -3 & -2\end{pmatrix}$$

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7 a $\mathbf{A}^2 = \begin{pmatrix} 1 & 1 & 0 \\ 3 & -3 & 1 \\ 0 & 3 & 2 \end{pmatrix}\begin{pmatrix} 1 & 1 & 0 \\ 3 & -3 & 1 \\ 0 & 3 & 2 \end{pmatrix}$

$= \begin{pmatrix} 1+3+0 & 1-3+0 & 0+1+0 \\ 3-9+0 & 3+9+3 & 0-3+2 \\ 0+9+0 & 0-9+6 & 0+3+4 \end{pmatrix}$

$= \begin{pmatrix} 4 & -2 & 1 \\ -6 & 15 & -1 \\ 9 & -3 & 7 \end{pmatrix}$

$\mathbf{A}^3 = \mathbf{A}^2\mathbf{A} = \begin{pmatrix} 4 & -2 & 1 \\ -6 & 15 & -1 \\ 9 & -3 & 7 \end{pmatrix}\begin{pmatrix} 1 & 1 & 0 \\ 3 & -3 & 1 \\ 0 & 3 & 2 \end{pmatrix}$

$= \begin{pmatrix} 4-6+0 & 4+6+3 & 0-2+2 \\ -6+45+0 & -6-45-3 & 0+15-2 \\ 9-9+0 & 9+9+21 & 0-3+14 \end{pmatrix}$

$= \begin{pmatrix} -2 & 13 & 0 \\ 39 & -54 & 13 \\ 0 & 39 & 11 \end{pmatrix}$

$13\mathbf{A} - 15\mathbf{I} = \begin{pmatrix} 13 & 13 & 0 \\ 39 & -39 & 13 \\ 0 & 39 & 26 \end{pmatrix} - \begin{pmatrix} 15 & 0 & 0 \\ 0 & 15 & 0 \\ 0 & 0 & 15 \end{pmatrix}$

$= \begin{pmatrix} -2 & 13 & 0 \\ 39 & -54 & 13 \\ 0 & 39 & 11 \end{pmatrix} = \mathbf{A}^3$

Hence

$\mathbf{A}^3 = 13\mathbf{A} + 15\mathbf{I}$, as required.

b Multiply the result of part a throughout by $\mathbf{A}^{-1}$

$\mathbf{A}^3\mathbf{A}^{-1} = 13\mathbf{A}\mathbf{A}^{-1} - 15\mathbf{I}\mathbf{A}^{-1}$

$\mathbf{A}^2 = 13\mathbf{I} - 15\mathbf{A}^{-1}$

Rearranging

$15\mathbf{A}^{-1} = 13\mathbf{I} - \mathbf{A}^2$, as required

c Using the result of part **b**

$15\mathbf{A}^{-1} = 13\mathbf{I} - \mathbf{A}^2 = \begin{pmatrix} 13 & 0 & 0 \\ 0 & 13 & 0 \\ 0 & 0 & 13 \end{pmatrix} - \begin{pmatrix} 4 & -2 & 1 \\ -6 & 15 & -1 \\ 9 & -3 & 7 \end{pmatrix}$

$= \begin{pmatrix} 9 & 2 & -1 \\ 6 & -2 & 1 \\ -9 & 3 & 6 \end{pmatrix}$

Hence

$\mathbf{A}^{-1} = \frac{1}{15}\begin{pmatrix} 9 & 2 & -1 \\ 6 & -2 & 1 \\ -9 & 3 & 6 \end{pmatrix}$

8 a $\det(\mathbf{A}) = 2\begin{vmatrix} 3 & -2 \\ 3 & -4 \end{vmatrix} - 0\begin{vmatrix} 4 & -2 \\ 0 & -4 \end{vmatrix} + 1\begin{vmatrix} 4 & 3 \\ 0 & 3 \end{vmatrix}$

$= 2(-12+6) - 0 + 1(12-0)$

$= -12 + 12 = 0$

Hence $\mathbf{A}$ is singular.

b The matrix of minors is given by

$\mathbf{M} = \begin{pmatrix} \begin{vmatrix} 3 & -2 \\ 3 & -4 \end{vmatrix} & \begin{vmatrix} 4 & -2 \\ 0 & -4 \end{vmatrix} & \begin{vmatrix} 4 & 3 \\ 0 & 3 \end{vmatrix} \\ \begin{vmatrix} 0 & 1 \\ 3 & -4 \end{vmatrix} & \begin{vmatrix} 2 & 1 \\ 0 & -4 \end{vmatrix} & \begin{vmatrix} 2 & 0 \\ 0 & 3 \end{vmatrix} \\ \begin{vmatrix} 0 & 1 \\ 3 & -2 \end{vmatrix} & \begin{vmatrix} 2 & 1 \\ 4 & -2 \end{vmatrix} & \begin{vmatrix} 2 & 0 \\ 4 & 3 \end{vmatrix} \end{pmatrix}$

$= \begin{pmatrix} -6 & -16 & 12 \\ -3 & -8 & 6 \\ -3 & -8 & 6 \end{pmatrix}$

The matrix of cofactors is given by

$\mathbf{C} = \begin{pmatrix} -6 & 16 & 12 \\ 3 & -8 & -6 \\ -3 & 8 & 6 \end{pmatrix}$

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8 c $\mathbf{AC}^{\mathrm{T}} = \begin{pmatrix} 2 & 0 & 1 \\ 4 & 3 & -2 \\ 0 & 3 & -4 \end{pmatrix}\begin{pmatrix} -6 & 3 & -3 \\ 16 & -8 & 8 \\ 12 & -6 & 6 \end{pmatrix}$

$= \begin{pmatrix} -12+0+12 & 6+0-6 & -6+0+6 \\ -24+48-24 & 12-24+12 & -12+24-12 \\ 0+48-48 & 0-24+24 & 0+24-24 \end{pmatrix}$

$= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = 0$, as required.

9 a All values of k.

b The matrix of minors is given by

$\begin{pmatrix} 2 & k & 3 \\ -1 & 2 & 1 \\ 1 & -1 & -1 \end{pmatrix}$

$\det \mathbf{M} = 2\begin{vmatrix} 2 & 1 \\ -1 & -1 \end{vmatrix} - k\begin{vmatrix} -1 & 1 \\ 1 & -1 \end{vmatrix} + 3\begin{vmatrix} -1 & 2 \\ 1 & -1 \end{vmatrix}$

$= -2 - 3 = -5$

Matrix of minors

$= \begin{pmatrix} \begin{vmatrix} 2 & 1 \\ -1 & -1 \end{vmatrix} & \begin{vmatrix} -1 & 1 \\ 1 & -1 \end{vmatrix} & \begin{vmatrix} -1 & 2 \\ 1 & -1 \end{vmatrix} \\ \begin{vmatrix} k & 3 \\ -1 & -1 \end{vmatrix} & \begin{vmatrix} 2 & 3 \\ 1 & -1 \end{vmatrix} & \begin{vmatrix} 2 & k \\ 1 & -1 \end{vmatrix} \\ \begin{vmatrix} k & 3 \\ 2 & 1 \end{vmatrix} & \begin{vmatrix} 2 & 3 \\ -1 & 1 \end{vmatrix} & \begin{vmatrix} 2 & k \\ -1 & 2 \end{vmatrix} \end{pmatrix}$

$= \begin{pmatrix} -1 & 0 & -1 \\ 3-k & -5 & -2-k \\ k-6 & 5 & 4+k \end{pmatrix}$

$\mathbf{C} = \begin{pmatrix} -1 & 0 & -1 \\ k-3 & -5 & k+2 \\ k-6 & -5 & k+4 \end{pmatrix}$

$\mathbf{C}^{\mathrm{T}} = \begin{pmatrix} -1 & k-3 & k-6 \\ 0 & -5 & -5 \\ -1 & k+2 & k+4 \end{pmatrix}$

$\text{Inverse} = \frac{1}{5}\begin{pmatrix} 1 & 3-k & 6-k \\ 0 & 5 & 5 \\ 1 & -k-2 & -k-4 \end{pmatrix}$

10 $\mathbf{A} = \begin{pmatrix} p & 2p & 3 \\ 4 & -1 & 1 \\ 1 & -2 & 0 \end{pmatrix}$

$\det \mathbf{A} = p\begin{vmatrix} -1 & 1 \\ -2 & 0 \end{vmatrix} - 2p\begin{vmatrix} 4 & 1 \\ 1 & 0 \end{vmatrix} + 3\begin{vmatrix} 4 & -1 \\ 1 & -2 \end{vmatrix}$

$= -2p + 2p + 21 = 4p - 21$

$\mathbf{M} = \begin{pmatrix} \begin{vmatrix} -1 & 1 \\ -2 & 0 \end{vmatrix} & \begin{vmatrix} 4 & 1 \\ 1 & 0 \end{vmatrix} & \begin{vmatrix} 4 & -1 \\ 1 & -2 \end{vmatrix} \\ \begin{vmatrix} 2p & 3 \\ -2 & 0 \end{vmatrix} & \begin{vmatrix} p & 3 \\ 1 & 0 \end{vmatrix} & \begin{vmatrix} p & 2p \\ 1 & -2 \end{vmatrix} \\ \begin{vmatrix} 2p & 3 \\ -1 & 1 \end{vmatrix} & \begin{vmatrix} p & 3 \\ 4 & 1 \end{vmatrix} & \begin{vmatrix} p & 2p \\ 4 & -1 \end{vmatrix} \end{pmatrix}$

$= \begin{pmatrix} 2 & -1 & -7 \\ 6 & -3 & -4p \\ 2p+3 & p-12 & -p-8p \end{pmatrix}$

$\mathbf{C} = \begin{pmatrix} 2 & 1 & -7 \\ -6 & -3 & 4p \\ 2p+3 & 12-p & -9p \end{pmatrix}$

$\mathbf{C}^{\mathrm{T}} = \begin{pmatrix} 2 & -6 & 2p+3 \\ 1 & -3 & 12-p \\ -7 & 4p & -9p \end{pmatrix}$

$\text{Inverse} = \frac{1}{4p-21}\begin{pmatrix} 2 & -6 & 2p+3 \\ 1 & -3 & 12-p \\ -7 & 4p & -9p \end{pmatrix}$

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Matrices 6F

1 a Write the system of equations using matrices:

$$\begin{pmatrix} 2 & -6 & 4 \\ 3 & 2 & -9 \\ -2 & 4 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 32 \\ -49 \\ -3 \end{pmatrix}$$

Find the inverse of the left-hand matrix:

$$\det \begin{vmatrix} 2 & -6 & 4 \\ 3 & 2 & -9 \\ -2 & 4 & 1 \end{vmatrix} = 50$$

$$\mathbf{M} = \begin{pmatrix} \begin{vmatrix} 2 & -9 \\ 4 & 1 \end{vmatrix} & \begin{vmatrix} 3 & -9 \\ -2 & 1 \end{vmatrix} & \begin{vmatrix} 3 & 2 \\ -2 & 4 \end{vmatrix} \\ \begin{vmatrix} -6 & 4 \\ 4 & 1 \end{vmatrix} & \begin{vmatrix} 2 & 4 \\ -2 & 1 \end{vmatrix} & \begin{vmatrix} 2 & -6 \\ -2 & 4 \end{vmatrix} \\ \begin{vmatrix} -6 & 4 \\ 2 & -9 \end{vmatrix} & \begin{vmatrix} 2 & 4 \\ 3 & -9 \end{vmatrix} & \begin{vmatrix} 2 & -6 \\ 3 & 2 \end{vmatrix} \end{pmatrix}$$

$$= \begin{pmatrix} 38 & -15 & 16 \\ -22 & 10 & -4 \\ 46 & -30 & 22 \end{pmatrix}$$

$$\mathbf{C} = \begin{pmatrix} 38 & 15 & 16 \\ 22 & 10 & 4 \\ 46 & 30 & 22 \end{pmatrix}$$

$$\mathbf{C}^{\mathrm{T}} = \begin{pmatrix} 38 & 22 & 46 \\ 15 & 10 & 30 \\ 16 & 4 & 22 \end{pmatrix}$$

$$\text{Inverse} = \frac{1}{50} \begin{pmatrix} 38 & 22 & 46 \\ 15 & 10 & 30 \\ 16 & 4 & 22 \end{pmatrix}$$

$$\mathbf{A}^{-1}\mathbf{v} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$= \frac{1}{50} \begin{pmatrix} 38 & 22 & 46 \\ 15 & 10 & 30 \\ 16 & 4 & 22 \end{pmatrix} \begin{pmatrix} 32 \\ -49 \\ -3 \end{pmatrix}$$

$$= \frac{1}{50} \begin{pmatrix} 0 \\ -100 \\ 250 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ -2 \\ 5 \end{pmatrix}$$

$x = 0, y = -2, z = 5$

b Write the system of equations using matrices:

$$\begin{pmatrix} -4 & 6 & -2 \\ 3 & 3 & -2 \\ -6 & -7 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -22 \\ 1 \\ 3 \end{pmatrix}$$

Find the inverse of the left-hand matrix:

$$\det \begin{vmatrix} -4 & 6 & -2 \\ 3 & 3 & -2 \\ -6 & -7 & 3 \end{vmatrix} = 44$$

$$\mathbf{M} = \begin{pmatrix} \begin{vmatrix} 3 & -2 \\ -7 & 3 \end{vmatrix} & \begin{vmatrix} 3 & -2 \\ -6 & 3 \end{vmatrix} & \begin{vmatrix} 3 & 3 \\ -6 & -7 \end{vmatrix} \\ \begin{vmatrix} 6 & -2 \\ -7 & 3 \end{vmatrix} & \begin{vmatrix} -4 & -2 \\ -6 & 3 \end{vmatrix} & \begin{vmatrix} -4 & 6 \\ -6 & -7 \end{vmatrix} \\ \begin{vmatrix} 6 & -2 \\ 3 & -2 \end{vmatrix} & \begin{vmatrix} -4 & -2 \\ 3 & -2 \end{vmatrix} & \begin{vmatrix} -4 & 6 \\ 3 & 3 \end{vmatrix} \end{pmatrix}$$

$$= \begin{pmatrix} -5 & -3 & -3 \\ 4 & -24 & 64 \\ -6 & 14 & -30 \end{pmatrix}$$

$$\mathbf{C} = \begin{pmatrix} -5 & 3 & -3 \\ -4 & -24 & -64 \\ -6 & -14 & -30 \end{pmatrix}$$

$$\mathbf{C}^{\mathrm{T}} = \begin{pmatrix} -5 & -4 & -6 \\ 3 & -24 & -14 \\ -3 & -64 & -30 \end{pmatrix}$$

$$\text{Inverse} = \frac{1}{44} \begin{pmatrix} -5 & -4 & -6 \\ 3 & -24 & -14 \\ -3 & -64 & -30 \end{pmatrix}$$

$$\mathbf{A}^{-1}\mathbf{v} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$= \frac{1}{44} \begin{pmatrix} -5 & -4 & -6 \\ 3 & -24 & -14 \\ -3 & -64 & -30 \end{pmatrix} \begin{pmatrix} -22 \\ 1 \\ 3 \end{pmatrix}$$

$$= \frac{1}{44} \begin{pmatrix} 88 \\ -132 \\ -88 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ -3 \\ -2 \end{pmatrix}$$

$x = 2, y = -3, z = -2$

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1 c Write the system of equations using matrices:

$$\begin{pmatrix} 4 & 7 & -2 \\ -10 & -1 & -7 \\ -2 & 1 & -4 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 21 \\ 0 \\ 9 \end{pmatrix}$$

Find the inverse of the left-hand matrix:

$$\det \begin{vmatrix} 4 & 7 & -2 \\ -10 & -1 & -7 \\ -2 & 1 & -4 \end{vmatrix} = -114$$

$$\mathbf{M} = \begin{pmatrix} \begin{vmatrix} -1 & -7 \\ 1 & -4 \end{vmatrix} & \begin{vmatrix} -10 & -7 \\ -2 & -4 \end{vmatrix} & \begin{vmatrix} -10 & -1 \\ -2 & 1 \end{vmatrix} \\ \begin{vmatrix} 7 & -2 \\ 1 & -4 \end{vmatrix} & \begin{vmatrix} 4 & -2 \\ -2 & -4 \end{vmatrix} & \begin{vmatrix} 4 & 7 \\ -2 & 1 \end{vmatrix} \\ \begin{vmatrix} 7 & -2 \\ -1 & -7 \end{vmatrix} & \begin{vmatrix} 4 & -2 \\ -10 & -7 \end{vmatrix} & \begin{vmatrix} 4 & 7 \\ -10 & -1 \end{vmatrix} \end{pmatrix}$$

$$= \begin{pmatrix} 11 & 26 & -12 \\ -26 & -20 & 18 \\ -51 & -48 & 66 \end{pmatrix}$$

$$\mathbf{C} = \begin{pmatrix} 11 & -26 & -12 \\ 26 & -20 & -18 \\ -51 & 48 & 66 \end{pmatrix}$$

$$\mathbf{C}^{\mathrm{T}} = \begin{pmatrix} 11 & 26 & -51 \\ -26 & -20 & 48 \\ -12 & -18 & 66 \end{pmatrix}$$

$$\text{Inverse} = \frac{1}{-114} \begin{pmatrix} 11 & 26 & -51 \\ -26 & -20 & 48 \\ -12 & -18 & 66 \end{pmatrix}$$

$$\mathbf{A}^{-1}\mathbf{v} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$= \frac{1}{-114} \begin{pmatrix} 11 & 26 & -51 \\ -26 & -20 & 48 \\ -12 & -18 & 66 \end{pmatrix} \begin{pmatrix} 21 \\ 0 \\ 9 \end{pmatrix}$$

$$= \frac{1}{-114} \begin{pmatrix} -228 \\ -114 \\ 342 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix}$$

$x = 2, y = 1, z = -3$

d Write the system of equations using matrices:

$$\begin{pmatrix} -3 & -6 & 4 \\ -3 & 6 & -10 \\ 3 & 7 & -3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -23 \\ -1 \\ 27 \end{pmatrix}$$

Find the inverse of the left-hand matrix:

$$\det \begin{vmatrix} -3 & -6 & 4 \\ -3 & 6 & -10 \\ 3 & 7 & -3 \end{vmatrix} = -78$$

$$\mathbf{M} = \begin{pmatrix} \begin{vmatrix} -6 & -10 \\ 7 & -3 \end{vmatrix} & \begin{vmatrix} -3 & -10 \\ 3 & -3 \end{vmatrix} & \begin{vmatrix} -3 & -6 \\ 3 & 7 \end{vmatrix} \\ \begin{vmatrix} -6 & 4 \\ 7 & -3 \end{vmatrix} & \begin{vmatrix} -3 & 4 \\ 3 & -3 \end{vmatrix} & \begin{vmatrix} -3 & -6 \\ 3 & 7 \end{vmatrix} \\ \begin{vmatrix} -6 & 4 \\ -6 & -10 \end{vmatrix} & \begin{vmatrix} -3 & 4 \\ -3 & -10 \end{vmatrix} & \begin{vmatrix} -3 & -6 \\ -3 & -6 \end{vmatrix} \end{pmatrix}$$

$$= \begin{pmatrix} 88 & 39 & -3 \\ -10 & -3 & -3 \\ 84 & 42 & 0 \end{pmatrix}$$

$$\mathbf{C} = \begin{pmatrix} 88 & -39 & -3 \\ 10 & -3 & 3 \\ 84 & -42 & 0 \end{pmatrix}$$

$$\mathbf{C}^{\mathrm{T}} = \begin{pmatrix} 88 & 10 & 84 \\ -39 & -3 & -42 \\ -3 & 3 & 0 \end{pmatrix}$$

$$\text{Inverse} = \frac{1}{-42} \begin{pmatrix} 88 & 10 & 84 \\ -39 & -3 & -42 \\ -3 & 3 & 0 \end{pmatrix}$$

$$\mathbf{A}^{-1}\mathbf{v} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$= \frac{1}{-42} \begin{pmatrix} 88 & 10 & 84 \\ -39 & -3 & -42 \\ -3 & 3 & 0 \end{pmatrix} \begin{pmatrix} -23 \\ -1 \\ 27 \end{pmatrix}$$

$$= \frac{1}{-42} \begin{pmatrix} 234 \\ -234 \\ 66 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -\frac{234}{42} \\ \frac{234}{42} \\ -\frac{66}{42} \end{pmatrix} = \begin{pmatrix} -\frac{39}{7} \\ \frac{39}{7} \\ -\frac{11}{7} \end{pmatrix}$$

$x = -\frac{39}{7}, y = \frac{39}{7}, z = -\frac{11}{7}$

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2 Write the system of equations using matrices:

$$\mathbf{A}\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}3\\30\\-3\end{pmatrix},\ \text{where } \mathbf{A}=\begin{pmatrix}1&-3&-4\\6&5&-7\\1&4&6\end{pmatrix}$$

$$\text{so } \begin{pmatrix}1&-3&-4\\6&5&-7\\1&4&6\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}3\\30\\-3\end{pmatrix}$$

Find the inverse of the left-hand matrix:

$$\det\begin{vmatrix}1&-3&-4\\6&5&-7\\1&4&6\end{vmatrix}=111$$

A is invertible, there is a unique solution to the set of equations

$$\mathbf{M}=\begin{pmatrix}\begin{vmatrix}5&-7\\4&6\end{vmatrix}&\begin{vmatrix}6&-7\\1&6\end{vmatrix}&\begin{vmatrix}6&5\\1&4\end{vmatrix}\\\begin{vmatrix}-3&-4\\4&6\end{vmatrix}&\begin{vmatrix}1&-4\\1&6\end{vmatrix}&\begin{vmatrix}1&-3\\1&4\end{vmatrix}\\\begin{vmatrix}-3&-4\\5&-7\end{vmatrix}&\begin{vmatrix}1&-4\\6&-7\end{vmatrix}&\begin{vmatrix}1&-3\\6&5\end{vmatrix}\end{pmatrix}$$

$$=\begin{pmatrix}58&43&19\\-2&10&7\\41&17&23\end{pmatrix}$$

$$\mathbf{C}=\begin{pmatrix}58&-43&19\\2&10&-7\\41&-17&23\end{pmatrix}$$

$$\mathbf{C}^{\mathrm{T}}=\begin{pmatrix}58&2&41\\-43&10&-17\\19&-7&23\end{pmatrix}$$

$$\text{Inverse}=\frac{1}{111}\begin{pmatrix}58&2&41\\-43&10&-17\\19&-7&23\end{pmatrix}$$

$\mathbf{A}^{-1}\mathbf{v}=\begin{pmatrix}x\\y\\z\end{pmatrix}$ is the coordinates of point of intersection

$$=\frac{1}{111}\begin{pmatrix}58&2&41\\-43&10&-17\\19&-7&23\end{pmatrix}\begin{pmatrix}3\\30\\-3\end{pmatrix}$$

$$=\frac{1}{111}\begin{pmatrix}111\\222\\-222\end{pmatrix}$$

$$\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}1\\2\\-2\end{pmatrix}$$

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3 Write the system of equations using matrices:

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \\ 1.01 & 1.025 & 0.985 \end{pmatrix} \begin{pmatrix} \mathbf{A} \\ \mathbf{B} \\ \mathbf{C} \end{pmatrix} = \begin{pmatrix} 3000 \\ 190 \\ 3041 \end{pmatrix}$$

Find the inverse of the left-hand matrix:

$$\det \begin{vmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \\ 1.01 & 1.025 & 0.985 \end{vmatrix} = 0.065$$

$$\mathbf{M} = \begin{pmatrix} \begin{vmatrix} -1 & 0 \\ 1.025 & 0.985 \end{vmatrix} & \begin{vmatrix} 1 & 0 \\ 1.01 & 0.985 \end{vmatrix} & \begin{vmatrix} 1 & -1 \\ 1.01 & 1.025 \end{vmatrix} \\ \begin{vmatrix} 1 & 1 \\ 1.025 & 0.985 \end{vmatrix} & \begin{vmatrix} 1 & 1 \\ 1.01 & 0.985 \end{vmatrix} & \begin{vmatrix} 1 & 1 \\ 1.01 & 1.025 \end{vmatrix} \\ \begin{vmatrix} 1 & 1 \\ -1 & 0 \end{vmatrix} & \begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} & \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} \end{pmatrix}$$

$$= \begin{pmatrix} -0.985 & 0.985 & 2.035 \\ -0.04 & -0.025 & 0.015 \\ 1 & -1 & -2 \end{pmatrix}$$

$$\mathbf{C} = \begin{pmatrix} -0.985 & -0.985 & 2.035 \\ 0.04 & -0.025 & -0.015 \\ 1 & 1 & -2 \end{pmatrix}$$

$$\mathbf{C}^{\mathrm{T}} = \begin{pmatrix} -0.985 & 0.04 & 1 \\ -0.985 & -0.025 & 1 \\ 2.035 & -0.015 & -2 \end{pmatrix}$$

$$\text{Inverse} = \frac{1}{0.065} \begin{pmatrix} -0.985 & 0.04 & 1 \\ -0.985 & -0.025 & 1 \\ 2.035 & -0.015 & -2 \end{pmatrix}$$

$$\mathbf{A}^{-1}\mathbf{v} = \begin{pmatrix} \mathbf{A} \\ \mathbf{B} \\ \mathbf{C} \end{pmatrix}$$

$$= \frac{1}{0.065} \begin{pmatrix} -0.985 & 0.04 & 1 \\ -0.985 & -0.025 & 1 \\ 2.035 & -0.015 & -2 \end{pmatrix} \begin{pmatrix} 3000 \\ 190 \\ 3041 \end{pmatrix}$$

$$= \frac{1}{0.065} \begin{pmatrix} 93.6 \\ 81.25 \\ 20.15 \end{pmatrix}$$

$$\begin{pmatrix} \mathbf{A} \\ \mathbf{B} \\ \mathbf{C} \end{pmatrix} = \begin{pmatrix} 1440 \\ 1250 \\ 310 \end{pmatrix}$$

$\mathrm{A} = £1440,\ \mathrm{B} = £1250,\ \mathrm{C} = £310$

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4 Write the system of equations using matrices:

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \\ 0.99 & 0.98 & 1.04 \end{pmatrix} \begin{pmatrix} \text{Br} \\ \text{G} \\ \text{Bl} \end{pmatrix} = \begin{pmatrix} 2000 \\ 250 \\ 2040 \end{pmatrix}$$

Find the inverse of the left-hand matrix:

$$\det \begin{vmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \\ 0.99 & 0.98 & 1.04 \end{vmatrix} = -0.11$$

$$\mathbf{M} = \begin{pmatrix} \begin{vmatrix} -1 & 0 \\ 0.98 & 1.04 \end{vmatrix} & \begin{vmatrix} 1 & 0 \\ 0.99 & 1.04 \end{vmatrix} & \begin{vmatrix} 1 & -1 \\ 0.99 & 0.98 \end{vmatrix} \\ \begin{vmatrix} 1 & 1 \\ 0.98 & 1.04 \end{vmatrix} & \begin{vmatrix} 1 & 1 \\ 0.99 & 1.04 \end{vmatrix} & \begin{vmatrix} 1 & 1 \\ 0.99 & 0.98 \end{vmatrix} \\ \begin{vmatrix} 1 & 1 \\ -1 & 0 \end{vmatrix} & \begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} & \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} \end{pmatrix}$$

$$= \begin{pmatrix} -1.04 & 1.04 & 1.97 \\ 0.06 & 0.05 & -0.01 \\ 1 & -1 & -2 \end{pmatrix}$$

$$\mathbf{C} = \begin{pmatrix} -1.04 & -1.04 & 1.97 \\ -0.06 & 0.05 & 0.01 \\ 1 & 1 & -2 \end{pmatrix}$$

$$\mathbf{C}^{\mathrm{T}} = \begin{pmatrix} -1.04 & -0.06 & 1 \\ -1.04 & 0.05 & 1 \\ 1.97 & 0.01 & -2 \end{pmatrix}$$

$$\text{Inverse} = -\frac{1}{0.11}\begin{pmatrix} -1.04 & -0.06 & 1 \\ -1.04 & 0.05 & 1 \\ 1.97 & 0.01 & -2 \end{pmatrix}$$

$$\mathbf{A}^{-1}\mathbf{v} = \begin{pmatrix} \text{Br} \\ \text{G} \\ \text{Bl} \end{pmatrix}$$

$$= -\frac{1}{0.11}\begin{pmatrix} -1.04 & -0.06 & 1 \\ -1.04 & 0.05 & 1 \\ 1.97 & 0.01 & -2 \end{pmatrix}\begin{pmatrix} 2000 \\ 250 \\ 2040 \end{pmatrix}$$

$$= -\frac{1}{0.11}\begin{pmatrix} -55 \\ -27.5 \\ -137.5 \end{pmatrix}$$

$$\begin{pmatrix} \text{Br} \\ \text{G} \\ \text{Bl} \end{pmatrix} = \begin{pmatrix} 500 \\ 250 \\ 1250 \end{pmatrix}$$

Brown = 500, Grey = 250, Black = 1250

5 a Write the system of equations using matrices :

$$\begin{pmatrix} 1 & a & 2 \\ 1 & -1 & -1 \\ 1 & 4 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} a \\ a \\ 0 \end{pmatrix}$$

$$\det \begin{vmatrix} 1 & a & 2 \\ 1 & -1 & -1 \\ 1 & 4 & 4 \end{vmatrix} = -5(a-2)$$

$-5(a-2) = -5a - 10$

$5a = 10 \Rightarrow a = 2$

b The system of equations is inconsistent and has no solution. The three planes meet in three distinct parallel lines to form a prism.

6 a

$$\begin{vmatrix} 1 & 4 & q \\ 2 & 3 & -3 \\ q & q & -2 \end{vmatrix} = 1\begin{vmatrix} 3 & -3 \\ q & -2 \end{vmatrix} - 4\begin{vmatrix} 2 & -3 \\ q & -2 \end{vmatrix} + q\begin{vmatrix} 2 & 3 \\ q & q \end{vmatrix}$$

$= (-6 + 3q) - 4(-4 + 3q) + q(2q - 3q)$

$= -6 + 3q + 16 - 12q - q^2$

$= -q^2 - 9q + 10$

$\det \mathbf{M} = 0 \Rightarrow -q^2 - 9q + 10$

$= 0 \Rightarrow q^2 + 9q - 10 = 0$

b i Consistent, infinity of solutions, planes meet in a sheaf

ii Consistent, unique solution, planes meet in a point

iii Inconsistent, no solutions, planes meet in a prism

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Matrices Mixed exercise 6

1 $\mathbf{A} = \begin{pmatrix} 1 & -3 \\ 2 & 1 \end{pmatrix} \Rightarrow \det \mathbf{A} = 1 + 6 = 7$

$\therefore \quad \mathbf{A}^{-1} = \frac{1}{7}\begin{pmatrix} 1 & 3 \\ -2 & 1 \end{pmatrix}$

$\mathbf{A}^{-1}(\mathbf{AB}) = \frac{1}{7}\begin{pmatrix} 1 & 3 \\ -2 & 1 \end{pmatrix}\begin{pmatrix} 4 & 1 & 9 \\ 1 & 9 & 4 \end{pmatrix}$

$\therefore \quad \mathbf{B} = \frac{1}{7}\begin{pmatrix} 7 & 28 & 21 \\ -7 & 7 & -14 \end{pmatrix}$

$\therefore \quad \mathbf{B} = \begin{pmatrix} 1 & 4 & 3 \\ -1 & 1 & -2 \end{pmatrix}$

2 a $\mathbf{A} = \begin{pmatrix} a & b \\ 2a & 3b \end{pmatrix}$

$\Rightarrow \det \mathbf{A} = 3ab - 2ab = ab$

$\therefore \quad \mathbf{A}^{-1} = \frac{1}{ab}\begin{pmatrix} 3b & -b \\ -2a & a \end{pmatrix}$

$= \begin{pmatrix} \frac{3}{a} & -\frac{1}{a} \\ \frac{-2}{b} & \frac{1}{b} \end{pmatrix}$

b $\mathbf{XA} = \mathbf{Y} \quad \Rightarrow \quad \mathbf{X} = \mathbf{YA}^{-1}$

$\therefore \quad \mathbf{X} = \begin{pmatrix} a & 2b \\ 2a & b \end{pmatrix}\begin{pmatrix} \frac{3}{a} & -\frac{1}{a} \\ -\frac{2}{b} & \frac{1}{b} \end{pmatrix}$

$= \begin{pmatrix} -1 & 1 \\ 4 & -1 \end{pmatrix}$

3 a $\mathbf{XB} = \mathbf{BA}$

$\therefore \quad \mathbf{(XB)B}^{-1} = \mathbf{BAB}^{-1}$

i.e. $\mathbf{X} = \mathbf{BAB}^{-1} \qquad (\because \mathbf{BB}^{-1} = \mathbf{I})$

3 b $\mathbf{B} = \begin{pmatrix} 2 & 1 \\ -1 & -1 \end{pmatrix} \Rightarrow \det \mathbf{B} = -2 - (-1) = -1$

$\therefore \quad \mathbf{B}^{-1} = \frac{1}{-1}\begin{pmatrix} -1 & -1 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -1 & -2 \end{pmatrix}$

$\therefore \quad \mathbf{X} = \begin{pmatrix} 2 & 1 \\ -1 & -1 \end{pmatrix}\begin{pmatrix} 5 & 3 \\ 0 & -2 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ -1 & -2 \end{pmatrix}$

$= \begin{pmatrix} 2 & 1 \\ -1 & -1 \end{pmatrix}\begin{pmatrix} 2 & -1 \\ 2 & 4 \end{pmatrix}$

$\mathbf{X} = \begin{pmatrix} 6 & 2 \\ -4 & -3 \end{pmatrix}$

4 $\mathbf{A}^2 = \begin{pmatrix} a & 2 & -1 \\ -1 & 1 & -1 \\ b & 2 & 1 \end{pmatrix} \times \begin{pmatrix} a & 2 & -1 \\ -1 & 1 & -1 \\ b & 2 & 1 \end{pmatrix}$

$= \begin{pmatrix} a^2 - 2 - b & 2a & -a - 2 - 1 \\ -a - 1 - b & -3 & -1 \\ ab - 2 + b & 2b + 4 & -b - 1 \end{pmatrix}$

Compare corresponding elements with given $\mathbf{A}^2$:

$\begin{pmatrix} a^2 - 2 - b & 2a & -a - 2 - 1 \\ -a - 1 - b & -3 & -1 \\ ab - 2 + b & 2b + 4 & -b - 1 \end{pmatrix} = \begin{pmatrix} -4 & 2 & -4 \\ -5 & -3 & -1 \\ 4 & 10 & -4 \end{pmatrix}$

$2a = 2 \Rightarrow a = 1$

$-b - 1 = -4 \Rightarrow b = 3$

5 $\begin{vmatrix} 1 & 0 & 2 \\ t & 3 & 1 \\ -2 & -1 & 1 \end{vmatrix} = 1\begin{vmatrix} 3 & 1 \\ -1 & 1 \end{vmatrix} - 0\begin{vmatrix} t & 1 \\ -2 & 1 \end{vmatrix} + 2\begin{vmatrix} t & 3 \\ -2 & -1 \end{vmatrix}$

$= 1(3 + 1) + 2(-t + 6) = 16 - 2t$

As $\mathbf{A}$ is singular

$\det(\mathbf{A}) = 16 - 2t = 0 \Rightarrow t = 8$

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6 $\det(\mathbf{M}) = \begin{vmatrix} 1 & 0 & 0 \\ x & 2 & 0 \\ 3 & 1 & 1 \end{vmatrix} = 1\begin{vmatrix} 2 & 0 \\ 1 & 1 \end{vmatrix} - 0\begin{vmatrix} x & 0 \\ 3 & 1 \end{vmatrix} + 0\begin{vmatrix} x & 2 \\ 3 & 1 \end{vmatrix}$

$= 2 - 0 + 0 = 2$

The matrix of minors is

$$\begin{pmatrix} \begin{vmatrix} 2 & 0 \\ 1 & 1 \end{vmatrix} & \begin{vmatrix} x & 0 \\ 3 & 1 \end{vmatrix} & \begin{vmatrix} x & 2 \\ 3 & 1 \end{vmatrix} \\ \begin{vmatrix} 0 & 0 \\ 1 & 1 \end{vmatrix} & \begin{vmatrix} 1 & 0 \\ 3 & 1 \end{vmatrix} & \begin{vmatrix} 1 & 0 \\ 3 & 1 \end{vmatrix} \\ \begin{vmatrix} 0 & 0 \\ 2 & 0 \end{vmatrix} & \begin{vmatrix} 1 & 0 \\ x & 0 \end{vmatrix} & \begin{vmatrix} 1 & 0 \\ x & 2 \end{vmatrix} \end{pmatrix} = \begin{pmatrix} 2 & x & x-6 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{pmatrix}$$

The matrix of cofactors is given by

$$\mathbf{C} = \begin{pmatrix} 2 & -x & x-6 \\ 0 & 1 & -1 \\ 0 & 0 & 2 \end{pmatrix}$$

$$\mathbf{C}^{\mathrm{T}} = \begin{pmatrix} 2 & 0 & 0 \\ -x & 1 & 0 \\ x-6 & -1 & 2 \end{pmatrix}$$

$$\mathbf{M}^{-1} = \frac{1}{\det(\mathbf{M})}\mathbf{C}^{\mathrm{T}} = \frac{1}{2}\begin{pmatrix} 2 & 0 & 0 \\ -x & 1 & 0 \\ x-6 & -1 & 2 \end{pmatrix}$$

7 a $\det \mathbf{A} = \begin{vmatrix} k & -2 \\ -4 & k \end{vmatrix} = k \times (k) - (-2) \times (-4)$

$= k^2 - 8 \neq 0$ for inverse to exist.

Hence $k^2 - 8 \Rightarrow k \neq \pm 2\sqrt{2}$

b $\mathbf{A}^{-1} = \frac{1}{k^2 - 8}\begin{pmatrix} k & 2 \\ 4 & k \end{pmatrix}$

8 a $\det \mathbf{B} = \begin{vmatrix} k & 6 \\ -1 & k-2 \end{vmatrix} = k \times (k-2) - 6 \times (-1)$

$= k^2 - 2k + 6$

b Complete the square on the determinant

$\det \mathbf{B} = k^2 - 2k + 6 = (k^2 - 2k + 1) + 6 - 1$

$= (k-1)^2 + 5 \geq 5$ for all k,

so $\det \mathbf{B} \neq 0$ and $\mathbf{B}$ is non-singular.

8 c $21\mathbf{B}^{-1} + \mathbf{B} = -8\mathbf{I}$

$\mathbf{B}^{-1} = \frac{1}{k^2 - 2k + 6}\begin{pmatrix} k-2 & -6 \\ 1 & k \end{pmatrix}$

$21\mathbf{B}^{-1} = \frac{21}{k^2 - 2k + 6}\begin{pmatrix} k-2 & -6 \\ 1 & k \end{pmatrix}$

$21\mathbf{B}^{-1} + \mathbf{B} = \frac{21}{k^2 - 2k + 6}\begin{pmatrix} k-2 & -6 \\ 1 & k \end{pmatrix} + \begin{pmatrix} k & 6 \\ -1 & k-2 \end{pmatrix}$

Top right element is $\frac{21}{k^2 - 2k + 6} \times (-6) + 6$

$= 6 - \frac{126}{k^2 - 2k + 6}$

Compare with top right element of $\mathbf{I} = \begin{pmatrix} -8 & 0 \\ 0 & -8 \end{pmatrix}$

$6 - \frac{126}{k^2 - 2k + 6} = 0$

$6(k^2 - 2k + 6) - 126 = 0$

$6k^2 - 12k + 36 - 126 = 0$

$(6k - 30)(k + 3) = 0$

$k = 5$ inadmissible, does not give $\mathbf{I}$

$k = -3$

9 a $\mathbf{M}$ is singular when $\det \mathbf{M} = 0$

$\det \mathbf{M} \begin{vmatrix} 2 & -m \\ m & -1 \end{vmatrix} = 2 \times (-1) - (-m) \times m$

$= m^2 - 2$

$m^2 - 2 = 0 \Rightarrow m^2 = 2$

$m = \sqrt{2}, m = -\sqrt{2}$

b $\mathbf{M}^{-1} = \frac{1}{m^2 - 2}\begin{pmatrix} -1 & m \\ -m & 2 \end{pmatrix}$

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10 a $\det \mathbf{A} = \begin{vmatrix} 3 & 4 & 5 \\ 1 & a & -1 \\ -2 & 1 & 1 \end{vmatrix}$

$= 3\begin{vmatrix} a & -1 \\ 1 & 1 \end{vmatrix} - 4\begin{vmatrix} 1 & -1 \\ -2 & 1 \end{vmatrix} + 5\begin{vmatrix} 1 & a \\ -2 & 1 \end{vmatrix}$

$= 3(a+1) - 4(1-2) + 5(1+2a)$

$= 13a + 12 \neq 0$ for $\mathbf{A}$ to have an inverse.

Hence, $a \neq -\frac{12}{13}$

b $\mathbf{M} = \begin{pmatrix} \begin{vmatrix} a & -1 \\ 1 & 1 \end{vmatrix} & \begin{vmatrix} 1 & -1 \\ -2 & 1 \end{vmatrix} & \begin{vmatrix} 1 & a \\ -2 & 1 \end{vmatrix} \\ \begin{vmatrix} 4 & 5 \\ 1 & 1 \end{vmatrix} & \begin{vmatrix} 3 & 5 \\ -2 & 1 \end{vmatrix} & \begin{vmatrix} 3 & 4 \\ -2 & 1 \end{vmatrix} \\ \begin{vmatrix} 4 & 5 \\ a & -1 \end{vmatrix} & \begin{vmatrix} 3 & 5 \\ 1 & -1 \end{vmatrix} & \begin{vmatrix} 3 & 4 \\ 1 & a \end{vmatrix} \end{pmatrix}$

$= \begin{pmatrix} a+1 & -1 & 1+2a \\ -1 & 13 & 11 \\ -4-5a & -8 & 3a-4 \end{pmatrix}$

$\mathbf{C} = \begin{pmatrix} a+1 & 1 & 1+2a \\ 1 & 13 & -11 \\ -4-5a & 8 & 3a-4 \end{pmatrix}$

$\mathbf{C}^{\mathrm{T}} = \begin{pmatrix} a+1 & 1 & -4-5a \\ 1 & 13 & 8 \\ 1+2a & -11 & 3a-4 \end{pmatrix}$

$\mathbf{A}^{-1} = \frac{1}{13a+12}\begin{pmatrix} a+1 & 1 & -4-5a \\ 1 & 13 & 8 \\ 1+2a & -11 & 3a-4 \end{pmatrix}$

11 Write the system of equations using matrices:

$\mathbf{A}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 6 \\ -2 \\ 0 \end{pmatrix}$, where $\mathbf{A} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & -4 & 2 \\ 2 & 1 & -3 \end{pmatrix}$

so $\begin{pmatrix} 1 & 1 & 1 \\ 1 & -4 & 2 \\ 2 & 1 & -3 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 6 \\ -2 \\ 0 \end{pmatrix}$

Find the inverse of the left-hand matrix:

$\det \begin{vmatrix} 1 & 1 & 1 \\ 1 & -4 & 2 \\ 2 & 1 & -3 \end{vmatrix} = 26$

So $\mathbf{A}$ is non-singular and has an inverse.

There is a unique solution to the set of equations.

$\mathbf{M} = \begin{pmatrix} \begin{vmatrix} -4 & 2 \\ 1 & -3 \end{vmatrix} & \begin{vmatrix} 1 & 2 \\ 2 & -3 \end{vmatrix} & \begin{vmatrix} 1 & -4 \\ 2 & 1 \end{vmatrix} \\ \begin{vmatrix} 1 & 1 \\ 1 & -3 \end{vmatrix} & \begin{vmatrix} 1 & 1 \\ 2 & -3 \end{vmatrix} & \begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix} \\ \begin{vmatrix} 1 & 1 \\ -4 & 2 \end{vmatrix} & \begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} & \begin{vmatrix} 1 & 1 \\ 1 & -4 \end{vmatrix} \end{pmatrix}$

$= \begin{pmatrix} 10 & -7 & 9 \\ -4 & -5 & -1 \\ 6 & 1 & -5 \end{pmatrix}$

$\mathbf{C} = \begin{pmatrix} 10 & 7 & 9 \\ 4 & -5 & 1 \\ 6 & -1 & -5 \end{pmatrix}$

$\mathbf{C}^{\mathrm{T}} = \begin{pmatrix} 10 & 4 & 6 \\ 7 & -5 & -1 \\ 9 & 1 & -5 \end{pmatrix}$

$\mathbf{A}^{-1} = \frac{1}{26}\begin{pmatrix} 10 & 4 & 6 \\ 7 & -5 & -1 \\ 9 & 1 & -5 \end{pmatrix}$

$\mathbf{A}^{-1}\mathbf{v} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$

$= \frac{1}{26}\begin{pmatrix} 10 & 4 & 6 \\ 7 & -5 & -1 \\ 9 & 1 & -5 \end{pmatrix}\begin{pmatrix} 6 \\ -2 \\ 0 \end{pmatrix}$

$= \frac{1}{26}\begin{pmatrix} 52 \\ 52 \\ 52 \end{pmatrix}$

$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix}$

The planes intersect at a single point, (2, 2, 2)

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12 Write the system of equations using matrices:

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & -1 \\ 1.06 & 1.04 & 1.03 \end{pmatrix}\begin{pmatrix} \mathbf{H} \\ \mathbf{D} \\ \mathbf{W} \end{pmatrix}=\begin{pmatrix} 2500 \\ 300 \\ 2610 \end{pmatrix}$$

Find the inverse of the left-hand matrix:

$$\det\begin{vmatrix} 1 & 1 & 1 \\ 1 & 0 & -1 \\ 1.06 & 1.04 & 1.03 \end{vmatrix}=-0.01$$

$$\mathbf{M}=\begin{pmatrix} \begin{vmatrix} 0 & -1 \\ 1.04 & 1.03 \end{vmatrix} & \begin{vmatrix} 1 & -1 \\ 1.06 & 1.03 \end{vmatrix} & \begin{vmatrix} 1 & 0 \\ 1.06 & 1.04 \end{vmatrix} \\ \begin{vmatrix} 1 & 1 \\ 1.04 & 1.03 \end{vmatrix} & \begin{vmatrix} 1 & 1 \\ 1.06 & 1.03 \end{vmatrix} & \begin{vmatrix} 1 & 1 \\ 1.06 & 1.04 \end{vmatrix} \\ \begin{vmatrix} 1 & 1 \\ 0 & -1 \end{vmatrix} & \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} & \begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} \end{pmatrix}$$

$$=\begin{pmatrix} 1.04 & 2.09 & 1.04 \\ -0.01 & -0.03 & -0.02 \\ -1 & -2 & -1 \end{pmatrix}$$

$$\mathbf{C}=\begin{pmatrix} 1.04 & -2.09 & 1.04 \\ 0.01 & -0.03 & 0.02 \\ -1 & 2 & -1 \end{pmatrix}$$

$$\mathbf{C}^{\mathrm{T}}=\begin{pmatrix} 1.04 & 0.01 & -1 \\ -2.09 & -0.03 & 2 \\ 1.04 & 0.02 & -1 \end{pmatrix}$$

$$\text{Inverse}=-\frac{1}{0.01}\begin{pmatrix} 1.04 & 0.01 & -1 \\ -2.09 & -0.03 & 2 \\ 1.04 & 0.02 & -1 \end{pmatrix}$$

$$\mathbf{A}^{-1}\mathbf{v}=\begin{pmatrix} \mathbf{H} \\ \mathbf{D} \\ \mathbf{W} \end{pmatrix}$$

$$=-\frac{1}{0.01}\begin{pmatrix} 1.04 & 0.01 & -1 \\ -2.09 & -0.03 & 2 \\ 1.04 & 0.02 & -1 \end{pmatrix}\begin{pmatrix} 2500 \\ 300 \\ 2610 \end{pmatrix}$$

$$=-\frac{1}{0.01}\begin{pmatrix} -7 \\ -14 \\ -4 \end{pmatrix}=\begin{pmatrix} 700 \\ 1400 \\ 400 \end{pmatrix}$$

700 Hampshire, 1400 Dorset horn,
400 Wiltshire horn

13 a Write the system of equations using matrices:

$$\begin{pmatrix} 2 & 3 & 1 \\ -1 & 1 & 2 \\ a & 1 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}=\begin{pmatrix} 6 \\ 7 \\ b \end{pmatrix}$$

$$\det\begin{vmatrix} 2 & 3 & 1 \\ -1 & 1 & 2 \\ a & 1 & 4 \end{vmatrix}=5a+15$$

$5a+15=0$

$5a=-15\Rightarrow a=-3$

$$\mathbf{M}=\begin{pmatrix} \begin{vmatrix} 1 & 2 \\ 1 & 4 \end{vmatrix} & \begin{vmatrix} -1 & 2 \\ -3 & 4 \end{vmatrix} & \begin{vmatrix} -1 & 1 \\ -3 & 1 \end{vmatrix} \\ \begin{vmatrix} 3 & 1 \\ 1 & 4 \end{vmatrix} & \begin{vmatrix} 2 & 1 \\ -3 & 4 \end{vmatrix} & \begin{vmatrix} 2 & 3 \\ -3 & 1 \end{vmatrix} \\ \begin{vmatrix} 3 & 1 \\ 1 & 2 \end{vmatrix} & \begin{vmatrix} 2 & 1 \\ -1 & 2 \end{vmatrix} & \begin{vmatrix} 2 & 3 \\ -1 & 1 \end{vmatrix} \end{pmatrix}$$

$$=\begin{pmatrix} 2 & 2 & 2 \\ 11 & 11 & 11 \\ 5 & 5 & 5 \end{pmatrix}$$

$$\mathbf{C}=\begin{pmatrix} 2 & -2 & 2 \\ -11 & 11 & -11 \\ 5 & -5 & 5 \end{pmatrix}$$

$$\mathbf{C}^{\mathrm{T}}=\begin{pmatrix} 2 & -11 & 5 \\ -2 & 11 & -5 \\ 2 & -11 & 5 \end{pmatrix}$$

$$\text{Inverse}=\frac{1}{5a+15}\begin{pmatrix} 2 & -11 & 5 \\ -2 & 11 & -5 \\ 2 & -11 & 5 \end{pmatrix}$$

$$\mathbf{A}^{-1}\mathbf{v}=\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$=\frac{1}{5a+15}\begin{pmatrix} 2 & -11 & 5 \\ -2 & 11 & -5 \\ 2 & -11 & 5 \end{pmatrix}\begin{pmatrix} 6 \\ 7 \\ b \end{pmatrix}$$

Consider the last row to find b:

$$\frac{1}{5a+15}\left(2\times 6+(-11)\times 7+5\times b\right)=0$$

$$=\frac{1}{5a+15}\left(5b-65\right)=0$$

$5b=65\Rightarrow b=13$

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13 b The three planes form a sheaf.

Challenge

Let $\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} h & j \\ k & l \end{pmatrix}$

$\det \mathbf{A} = ad - bc$,

$\det \mathbf{B} = hl - jk$

$\det \mathbf{A} \det \mathbf{B} = (ad - bc)(hl - jk)$

$= adhl - adjk - bchl - bcjk$

$\mathbf{AB} = \begin{pmatrix} ah + bk & aj + bl \\ ch + dk & cj + dl \end{pmatrix}$

$\det(\mathbf{AB}) = (ah + bk)(cj + dl) - (aj + bl)(ch + dk)$

$= adhl - adjk - bchl - bcjk$

$= \det \mathbf{A} \det \mathbf{B}$

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Exercise 7A

1 a **P** is not linear because $(0,0) \to (0,1)$ is not linear

b **Q** is not linear because $x \to x^2$ is not linear

c **R** is not linear because $y \to x + xy$ is not linear

d **S** is linear

e **T** is not linear because $(0,0) \to (3,3)$ is not linear

f **U** is linear.

2 a **S** is represented by $\begin{pmatrix} 2 & -1 \\ 3 & 0 \end{pmatrix}$

b **T** is not linear because $(0,0) \to (1,-1)$

c **U** is not linear because $x \to xy$ is not linear

d **V** is represented by $\begin{pmatrix} 0 & 2 \\ -1 & 0 \end{pmatrix}$

e **W** is represented by $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

3 a **S** is not linear because $x \to x^2$ and $y \to y^2$ are not linear

b **T** is represented by $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

c **U** is represented by $\begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}$

d **V** is represented by $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

e **W** is represented by $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

4 a **P:** $\begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} y+2x \\ -y \end{pmatrix} = \begin{pmatrix} 2x+y \\ 0x-y \end{pmatrix}$ is represented by $\begin{pmatrix} 2 & 1 \\ 0 & -1 \end{pmatrix}$

b **Q:** $\begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} 0-y \\ x+2y \end{pmatrix}$ is represented by $\begin{pmatrix} 0 & -1 \\ 1 & 2 \end{pmatrix}$

5 a $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} -1 & 2 & 5 \\ 1 & 3 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -2 & -5 \\ 1 & 3 & 1 \end{pmatrix}$

$\therefore$ vertices of image of T are at $(1,1);(-2,3);(-5,1)$

b $\begin{pmatrix} 1 & 4 \\ 0 & -2 \end{pmatrix}\begin{pmatrix} -1 & 2 & 5 \\ 1 & 3 & 1 \end{pmatrix} = \begin{pmatrix} 3 & 14 & 9 \\ -2 & -6 & -2 \end{pmatrix}$

$\therefore$ vertices of image of T are at $(3,-2);(14,-6);(9,-2)$

c $\begin{pmatrix} 0 & -2 \\ 2 & 0 \end{pmatrix}\begin{pmatrix} -1 & 2 & 5 \\ 1 & 3 & 1 \end{pmatrix} = \begin{pmatrix} -2 & -6 & -2 \\ -2 & 4 & 10 \end{pmatrix}$

$\therefore$ vertices of image of T are $(-2,-2);(-6,4);(-2,10)$

6 a $\begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}\begin{pmatrix} -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 \end{pmatrix} = \begin{pmatrix} -2 & 0 & 2 & 0 \\ 0 & 3 & 0 & -3 \end{pmatrix}$

$\therefore$ vertices of the image of S are $(-2,0):(0,3);(2,0);(0,-3)$

b $\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 \end{pmatrix} = \begin{pmatrix} -1 & -1 & 1 & 1 \\ -1 & 1 & 1 & -1 \end{pmatrix}$

$\therefore$ vertices of the image of S are $(-1,-1);(-1,1);(1,1);(1,-1)$

c $\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 \end{pmatrix} = \begin{pmatrix} -1 & 1 & 1 & -1 \\ -1 & -1 & 1 & 1 \end{pmatrix}$

$\therefore$ vertices of the image of S are $(-1,-1);(1,-1);(1,1);(-1,1)$

7 a

$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} -2 \\ -1 \end{pmatrix}$

$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 4 \\ 1 \end{pmatrix} = \begin{pmatrix} -4 \\ -1 \end{pmatrix}$

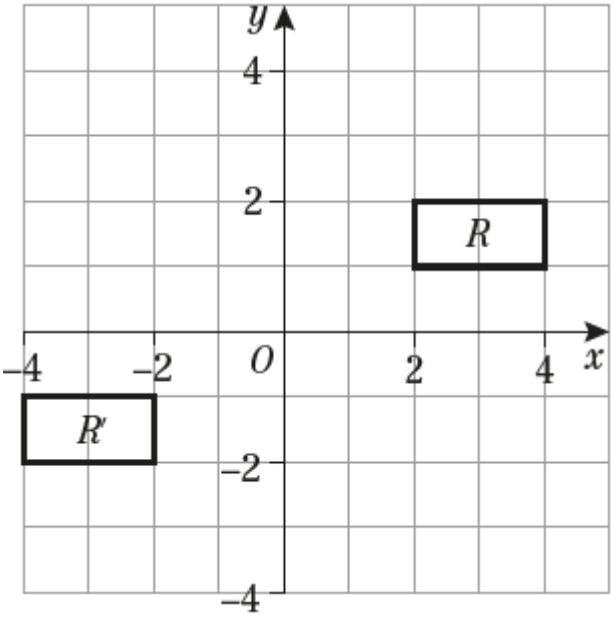
$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 4 \\ 2 \end{pmatrix} = \begin{pmatrix} -4 \\ -2 \end{pmatrix}$

$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 2 \\ 2 \end{pmatrix} = \begin{pmatrix} -2 \\ -2 \end{pmatrix}$

So vertices are $(-2,-1), (-4,-1), (-4,-2), (-2,-2)$

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7 b



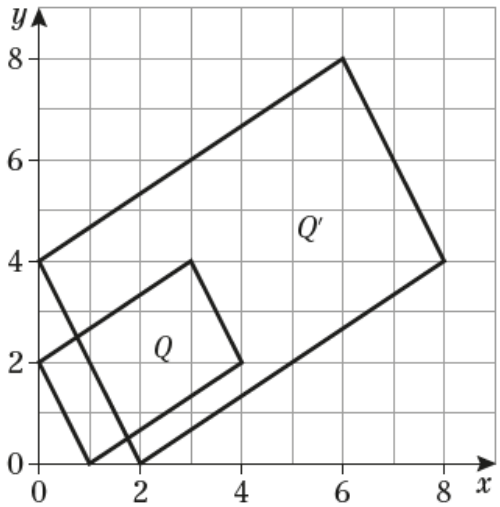
c Rotation through 180° about (0, 0)

8 a

$$\begin{pmatrix}2&0\\0&2\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}=\begin{pmatrix}2\\0\end{pmatrix}$$
$$\begin{pmatrix}2&0\\0&2\end{pmatrix}\begin{pmatrix}4\\2\end{pmatrix}=\begin{pmatrix}8\\4\end{pmatrix}$$
$$\begin{pmatrix}2&0\\0&2\end{pmatrix}\begin{pmatrix}3\\4\end{pmatrix}=\begin{pmatrix}6\\8\end{pmatrix}$$
$$\begin{pmatrix}2&0\\0&2\end{pmatrix}\begin{pmatrix}0\\2\end{pmatrix}=\begin{pmatrix}0\\4\end{pmatrix}$$

so vertices are (2,0), (8,4), (6,8), (0,4)

b



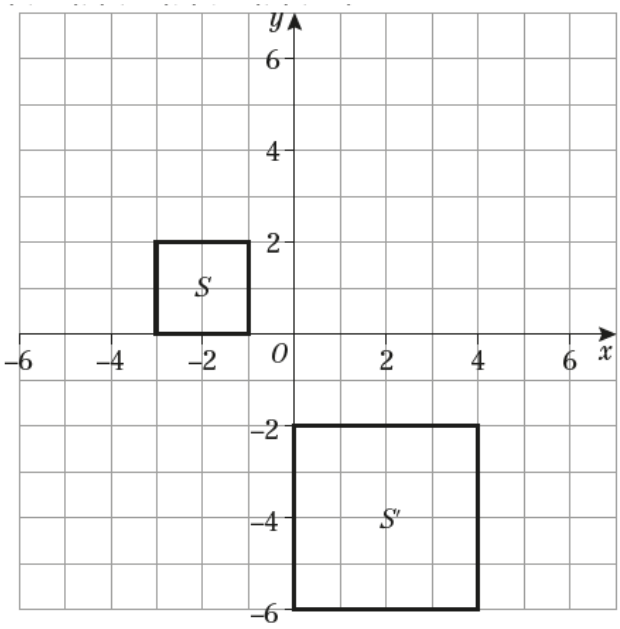
c Enlargement, centre (0, 0), scale factor 2

9 a

$$\begin{pmatrix}0&2\\2&0\end{pmatrix}\begin{pmatrix}-1\\0\end{pmatrix}=\begin{pmatrix}0\\-2\end{pmatrix}$$
$$\begin{pmatrix}0&2\\2&0\end{pmatrix}\begin{pmatrix}-3\\0\end{pmatrix}=\begin{pmatrix}0\\-6\end{pmatrix}$$
$$\begin{pmatrix}0&2\\2&0\end{pmatrix}\begin{pmatrix}-3\\2\end{pmatrix}=\begin{pmatrix}4\\-6\end{pmatrix}$$
$$\begin{pmatrix}0&2\\2&0\end{pmatrix}\begin{pmatrix}-1\\2\end{pmatrix}=\begin{pmatrix}4\\-2\end{pmatrix}$$

so vertices are $(0,-2)$, $(0,6)$, $(4,-6)$, $(4,-2)$

b



c Reflection in $y = x$ and enlargement, centre (0, 0), scale factor 2

10 a

$$\begin{pmatrix}1&0\\0&1\end{pmatrix}\begin{pmatrix}4\\1\end{pmatrix}=\begin{pmatrix}4\\1\end{pmatrix}$$
$$\begin{pmatrix}1&0\\0&1\end{pmatrix}\begin{pmatrix}4\\3\end{pmatrix}=\begin{pmatrix}4\\3\end{pmatrix}$$
$$\begin{pmatrix}1&0\\0&1\end{pmatrix}\begin{pmatrix}1\\3\end{pmatrix}=\begin{pmatrix}1\\3\end{pmatrix}$$

so vertices are (4,1), (4,3), (1,3)

b The transformation represented by the identity matrix leaves T unchanged.

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Challenge

a $T = \begin{pmatrix} 2 & -3 \\ 1 & 1 \end{pmatrix}$ so

$$T\begin{pmatrix} kx \\ ky \end{pmatrix} = \begin{pmatrix} 2 & -3 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} kx \\ ky \end{pmatrix} = \begin{pmatrix} 2kx - 3ky \\ kx + ky \end{pmatrix}$$

$$= k\begin{pmatrix} 2x - 3y \\ x + y \end{pmatrix} = kT\begin{pmatrix} x \\ y \end{pmatrix}$$

b $T\left(\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}\right) = T\begin{pmatrix} x_1 + x_2 \\ y_1 + y_2 \end{pmatrix}$

$$= \begin{pmatrix} 2 & -3 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} x_1 + x_2 \\ y_1 + y_2 \end{pmatrix} = \begin{pmatrix} 2(x_1 + x_2) - 3(y_1 + y_2) \\ x_1 + x_2 + y_1 + y_2 \end{pmatrix}$$

$$= \begin{pmatrix} 2x_1 - 3y_1 \\ x_1 + y_1 \end{pmatrix} + \begin{pmatrix} 2x_2 - 3y_2 \\ x_2 + y_2 \end{pmatrix} = T\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + T\begin{pmatrix} x_2 \\ y_2 \end{pmatrix}$$

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Exercise 7B

1 a $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

b $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 \\ -3 \end{pmatrix}$ so $A' = (1, -3)$

$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 3 \\ 3 \end{pmatrix} = \begin{pmatrix} 3 \\ -3 \end{pmatrix}$ so $B' = (3, -3)$

$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 3 \\ 2 \end{pmatrix} = \begin{pmatrix} 3 \\ -2 \end{pmatrix}$ so $C' = (3, -2)$

2 a $\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$

b $\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \end{pmatrix}$ so $P' = (-1, -1)$

$\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 3 \end{pmatrix} = \begin{pmatrix} -3 \\ -1 \end{pmatrix}$ so $Q' = (-3, -1)$

$\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} -3 \\ -2 \end{pmatrix}$ so $R' = (-3, -2)$

$\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ -2 \end{pmatrix}$ so $S' = (-1, -2)$

3 a $\begin{pmatrix} \cos 90^0 & -\sin 90^0 \\ \sin 90^0 & \cos 90^0 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

b $\begin{pmatrix} \cos 270^0 & -\sin 270^0 \\ \sin 270^0 & \cos 270^0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

c $\begin{pmatrix} \cos 45^0 & -\sin 45^0 \\ \sin 45^0 & \cos 45^0 \end{pmatrix}$

$= \begin{pmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{pmatrix}$

$= \frac{1}{2}\begin{pmatrix} \sqrt{2} & -\sqrt{2} \\ \sqrt{2} & \sqrt{2} \end{pmatrix}$

d $\begin{pmatrix} \cos 210^0 & -\sin 210^0 \\ \sin 210^0 & \cos 210^0 \end{pmatrix}$

$= \begin{pmatrix} -\frac{\sqrt{3}}{2} & \frac{1}{2} \\ -\frac{1}{2} & -\frac{\sqrt{3}}{2} \end{pmatrix}$

$= \frac{1}{2}\begin{pmatrix} -\sqrt{3} & 1 \\ -1 & -\sqrt{3} \end{pmatrix}$

e $\begin{pmatrix} \cos 225^0 & -\sin 225^0 \\ \sin 225^0 & \cos 225^0 \end{pmatrix} = \begin{pmatrix} -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \end{pmatrix}$

$= \frac{1}{2}\begin{pmatrix} -\sqrt{2} & \sqrt{2} \\ -\sqrt{2} & -\sqrt{2} \end{pmatrix}$

4 a $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 4 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 4 \end{pmatrix}$

$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 4 \\ 2 \end{pmatrix} = \begin{pmatrix} -2 \\ 4 \end{pmatrix}$

So $A' = (-1,1)$, $B' = (-1,4)$, $C' = (-2,4)$

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4 b

$$\begin{pmatrix} -\frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{\sqrt{3}}{2} \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -\frac{\sqrt{3}+1}{2} \\ \frac{1-\sqrt{3}}{2} \end{pmatrix}$$

$$\begin{pmatrix} -\frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{\sqrt{3}}{2} \end{pmatrix}\begin{pmatrix} 4 \\ 1 \end{pmatrix} = \begin{pmatrix} -\frac{4\sqrt{3}+1}{2} \\ \frac{4-\sqrt{3}}{2} \end{pmatrix}$$

$$\begin{pmatrix} -\frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{\sqrt{3}}{2} \end{pmatrix}\begin{pmatrix} 4 \\ 2 \end{pmatrix} = \begin{pmatrix} -\frac{4\sqrt{3}+2}{2} \\ \frac{4-2\sqrt{3}}{2} \end{pmatrix}$$

So $A' = \left(-\frac{\sqrt{3}}{2} - \frac{1}{2}, \frac{1}{2} - \frac{\sqrt{3}}{2}\right)$,

$B' = \left(-2\sqrt{3} - \frac{1}{2}, 2 - \frac{\sqrt{3}}{2}\right)$,

$C' = \left(-2\sqrt{3} - 1, 2 - \sqrt{3}\right)$

5 a

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 2 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ -2 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 3 \\ -2 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 4 \\ 3 \end{pmatrix} = \begin{pmatrix} 3 \\ -4 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 4 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ -4 \end{pmatrix}$$

So $P' = (2, -2)$,

$Q' = (3, -2)$, $R' = (3, -4)$,

$S' = (2, -4)$

5 b

$$\begin{pmatrix} -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \end{pmatrix}\begin{pmatrix} 2 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ -2\sqrt{2} \end{pmatrix}$$

$$\begin{pmatrix} -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \end{pmatrix}\begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} \frac{\sqrt{2}}{2} \\ -\frac{5\sqrt{2}}{2} \end{pmatrix}$$

$$\begin{pmatrix} -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \end{pmatrix}\begin{pmatrix} 4 \\ 3 \end{pmatrix} = \begin{pmatrix} -\frac{\sqrt{2}}{2} \\ -\frac{7\sqrt{2}}{2} \end{pmatrix}$$

$$\begin{pmatrix} -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \end{pmatrix}\begin{pmatrix} 4 \\ 2 \end{pmatrix} = \begin{pmatrix} -\sqrt{2} \\ -3\sqrt{2} \end{pmatrix}$$

So $P' = (0, -2\sqrt{2})$,

$Q' = \left(\frac{\sqrt{2}}{2}, -\frac{5\sqrt{2}}{2}\right)$, $R' = \left(-\frac{\sqrt{2}}{2}, -\frac{7\sqrt{2}}{2}\right)$,

$S' = \left(-\sqrt{2}, -3\sqrt{2}\right)$

6 a **A** represents a reflection in the x-axis.
B represents a rotation through 270° anticlockwise about (0, 0).

b (3, −2)

c $b = a - 3b \Rightarrow a - 4b = 0$

$-a = 2a - 2b \Rightarrow 3a - 2b = 0$

$\Rightarrow a = 0, b = 0$

7 a Rotation through 225° anticlockwise about (0, 0)

b $-\frac{1}{\sqrt{2}}p + \frac{1}{\sqrt{2}}q = -\sqrt{2}$

$\Rightarrow -p + q = -2 \Rightarrow p - q = 2$

$-\frac{1}{\sqrt{2}}p = \frac{1}{\sqrt{2}}q = -2\sqrt{2}$

$\Rightarrow -p - q = -4 \Rightarrow p + q = 4$

$\Rightarrow p = 3, q = 1$

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7 c $\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$

d $\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}\begin{pmatrix} -\sqrt{2} \\ -2\sqrt{2} \end{pmatrix} = \begin{pmatrix} -3 \\ -1 \end{pmatrix}$

So $C' = (-3, -1)$

Rotation through 45° clockwise about (0,0)

8 a Reflection in the line $y = x$

b Lines of the form $y = -x + k$ for any value of k

c $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

9 a $a = -0.5$

b $\cos\theta = -0.5 \Rightarrow \theta = 120^\circ$ or 240°

So possible matrices are

$\begin{pmatrix} \cos 120^\circ & -\sin 120^\circ \\ \sin 120^\circ & \cos 120^\circ \end{pmatrix} = \begin{pmatrix} -0.5 & -0.866 \\ 0.866 & -0.5 \end{pmatrix}$

$\begin{pmatrix} \cos 240^\circ & -\sin 240^\circ \\ \sin 240^\circ & \cos 240^\circ \end{pmatrix} = \begin{pmatrix} -0.5 & 0.866 \\ -0.866 & -0.5 \end{pmatrix}$

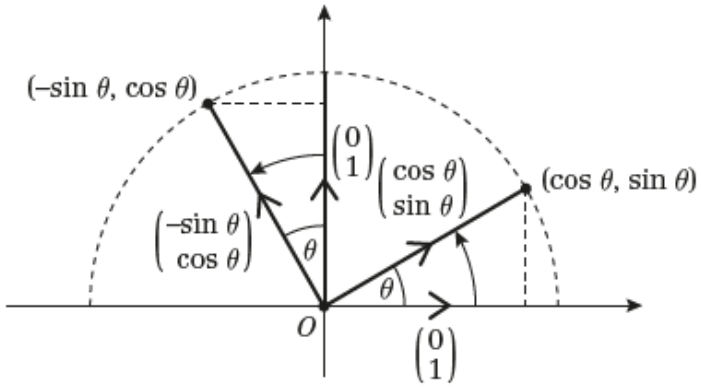
10 a $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

b $a - 5b = -b \Rightarrow a = 4b$

$3b = a \Rightarrow a = 3b$

So $a = 0, b = 0$

Challenge



The diagram shows that rotating $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ by θ takes it to $\begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}$ and rotating $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ by θ takes it to $\begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix}$. We know that the image of $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ is the first column of the matrix representing the transformation and that the image of $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ is the second column of the matrix representing the transformation. So the matrix representing the transformation is $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$

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Exercise 7C

1 a $\begin{pmatrix} 4 & 0 \\ 0 & 1 \end{pmatrix}$

b $\begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}$

c $\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$

d $\begin{pmatrix} 5 & 0 \\ 0 & 0.5 \end{pmatrix}$

2 a 4
b 3
c 4
d 2.5

3 a (0, 0)
b (0,0), (3,0), (3,4), (0,4)
(0,0), (3,0), (3,4), (0,4)
So area $= 3 \times 4 = 12$

4 a $\begin{pmatrix} -2 & 0 \\ 0 & 1 \end{pmatrix}$

b $\begin{pmatrix} -3 & 0 \\ 0 & 4 \end{pmatrix}$

c $\begin{pmatrix} -\frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}$

5 $\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}\begin{pmatrix} x \\ kx \end{pmatrix} = \begin{pmatrix} ax \\ kax \end{pmatrix}$; $kax = k(ax)$, so this point lies on $y = kx$ and the line is invariant.

6 a Stretch parallel to the x-axis, scale factor 2 and stretch parallel to the y-axis, scale factor -3

b $\det \mathbf{M} = -6$ so the area has been multiplied by 6. So $k = 4$.

7 a $\begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}\begin{pmatrix} 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 3 \\ 9 \end{pmatrix}$

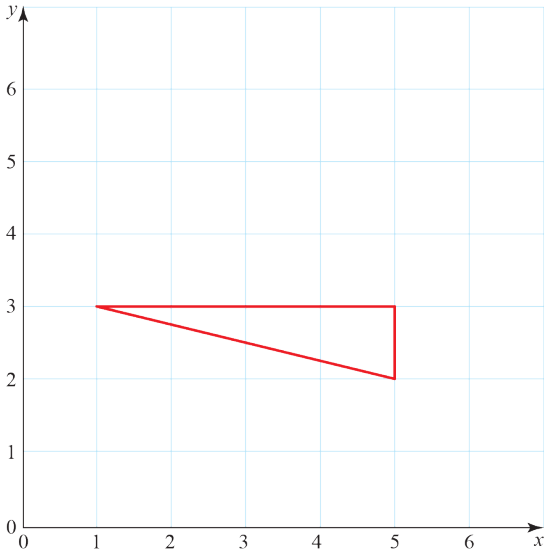
$\begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}\begin{pmatrix} 5 \\ 3 \end{pmatrix} = \begin{pmatrix} 15 \\ 9 \end{pmatrix}$

$\begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}\begin{pmatrix} 5 \\ 2 \end{pmatrix} = \begin{pmatrix} 15 \\ 6 \end{pmatrix}$

So vertices are (3,9), (15,9), (15,6)

7 b Area of original triangle $= \frac{1}{2} \times 4 \times 1 = 2$

$\det \mathbf{M} = 9$ so new area = 18



8 a $\begin{pmatrix} 2 & 0 \\ 0 & -3 \end{pmatrix}\begin{pmatrix} 2 \\ 0 \end{pmatrix} = \begin{pmatrix} 4 \\ 0 \end{pmatrix}$

$\begin{pmatrix} 2 & 0 \\ 0 & -3 \end{pmatrix}\begin{pmatrix} 4 \\ 0 \end{pmatrix} = \begin{pmatrix} 8 \\ 0 \end{pmatrix}$

$\begin{pmatrix} 2 & 0 \\ 0 & -3 \end{pmatrix}\begin{pmatrix} 4 \\ 5 \end{pmatrix} = \begin{pmatrix} 8 \\ -15 \end{pmatrix}$

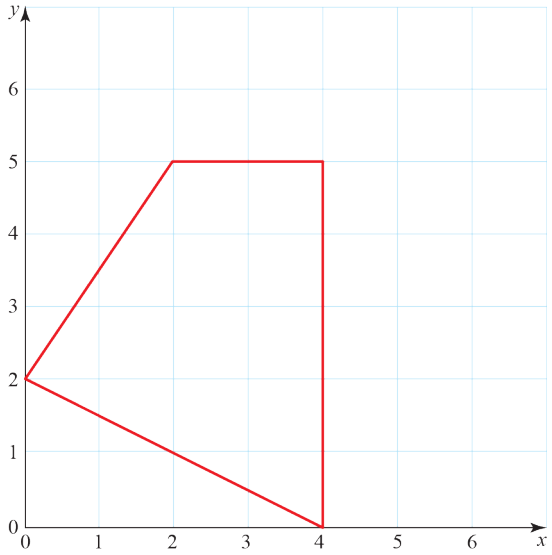
$\begin{pmatrix} 2 & 0 \\ 0 & -3 \end{pmatrix}\begin{pmatrix} 2 \\ 5 \end{pmatrix} = \begin{pmatrix} 4 \\ -15 \end{pmatrix}$

So vertices are (4,0), (8,0), (8,-15), (4,-15)

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8 b Area of original rectangle $= 2\times 5 = 10$

$\det \mathbf{M} = -6$ so new area $= 60$



9 a Enlargement, centre (0, 0), scale factor $2\sqrt{5}$

b Area of original triangle

$$= \left|\frac{1}{2}\times(4-a)\times 2\right| = |4-a|$$

$\det \mathbf{M} = 20 \Rightarrow |4-a| = 3$

$\Rightarrow a = 7$ or 1

10 a $\mathbf{M}^2 = \begin{pmatrix} p & 1 \\ p & q \end{pmatrix}\begin{pmatrix} p & 1 \\ p & q \end{pmatrix}$

$$= \begin{pmatrix} p^2+p & p+q \\ p^2+pq & p+q^2 \end{pmatrix}$$

b $\mathbf{M}^2 = \begin{pmatrix} 6 & 0 \\ 0 & 6 \end{pmatrix} \Rightarrow p^2+p = 6$

$\Rightarrow (p+3)(p-2) = 0$

$\Rightarrow p = -3$ or 2

But also $p+q=0$ so p must be -3 because $q > 0$

So $p = -3, q = 3$

11 a $\mathbf{M} = \begin{pmatrix} 5 & -1 \\ 2 & -2 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 2 & 5 \end{pmatrix} = \begin{pmatrix} 8 & 0 \\ 0 & -8 \end{pmatrix}$

b Stretch parallel to the x-axis, scale factor 8;
and stretch parallel to the y-axis, scale factor -8. Or enlargement scale factor 8 and
centre (0, 0) and reflection in the x-axis.

c $\det \mathbf{M} = -6$

so original area $= 320 \div 64 = 5$

$\Rightarrow |2(k-1)| = 5$

$\Rightarrow k = -\frac{3}{2}$ or $\frac{7}{2}$

12 $\det \mathbf{M} = -1\times -1 - (-\sqrt{2}\times\sqrt{2})$

$= 1 - -2 = 3$

So area of $P' = 12\times 3 = 36$

13 $\det \mathbf{M} = 8 + k$

Area of $T = \frac{1}{2}(4-k)(k-1)$

$\Rightarrow \frac{1}{2}(4-k)(k-1)(k+8) = 10$

$\Rightarrow (4-k)(k-1)(k+8) = 20$

$\Rightarrow (k^2-5k+4)(k+8) = -20$

$\Rightarrow k^3+3k^2-36k+52 = 0$

$\Rightarrow (k-2)(k^2+5k-26) = 0$

$\Rightarrow k = 2$ or $k = \frac{-5\pm\sqrt{129}}{2}$

$\Rightarrow k = 2$

for integer values of k.

14 a $\begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$

b $\begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$\begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}\begin{pmatrix} 7 \\ 7 \end{pmatrix} = \begin{pmatrix} 0 \\ 7\sqrt{2} \end{pmatrix}$

$\begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}\begin{pmatrix} 3 \\ -2 \end{pmatrix} = \begin{pmatrix} \frac{5}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$

So vertices of T' are

$$(0,0), (0,7\sqrt{2}), \left(\frac{5}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$$

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14 c $\det \mathbf{M} = \frac{1}{\sqrt{2}} \times \frac{1}{\sqrt{2}} - \left(-\frac{1}{\sqrt{2}} \times \frac{1}{\sqrt{2}} \right)$

$= \frac{1}{2} - \left(-\frac{1}{2} \right) = 1$

d Area of $T' = \frac{1}{2} \times 7\sqrt{2} \times \frac{5}{\sqrt{2}} = 17.5$

So area of $T = 17.5$

Challenge

a 0

b For any x, y, $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 7 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 7x \\ 0 \end{pmatrix}$

The y-coordinate is 0, so all points (x, y) map onto the x-axis.

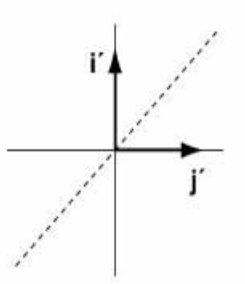
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Exercise 7D

1 a $\mathbf{AB} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \to \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$\begin{pmatrix} 0 \\ 1 \end{pmatrix} \to \begin{pmatrix} 1 \\ 0 \end{pmatrix}$



Reflection in $y = x$

b $\mathbf{BA} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ Reflection in $y = x$

c $\mathbf{AC} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix}$ Enlargement scale factor -2 centre (0, 0)

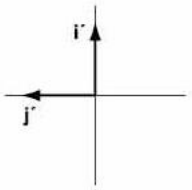
d $\mathbf{A}^2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ Identity (No transformation)

[This can be thought of as a rotation of 180° +180°=360°]

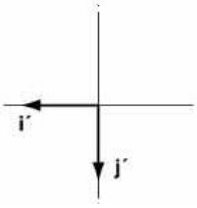
e $\mathbf{C}^2 = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix}$

Enlargement scale factor 4 centre (0, 0**)**

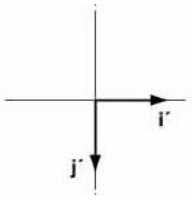
2 a Rotation of 90° anticlockwise $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$



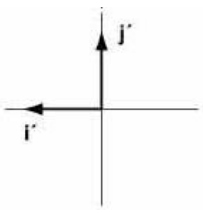
Rotation of 180° about (0, 0) $B = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$



Reflection in x-axis $C = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$



Reflection in y-axis $D = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$



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2 b i $BC = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ $(= D)$

Reflection in y-axis

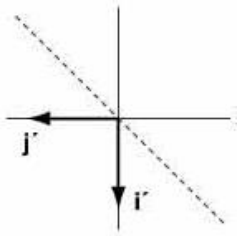
ii $CB = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ $(= D)$

Reflection in y-axis

iii $CD = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ $(= B)$

Rotation of 180° about (0, 0)

iv $AD = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$



Reflection in line $y = -x$.

v $BB = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

Rotation of 360° about (0, 0) or Identity

vi $DAC = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

$= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

$= \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ $(= A)$

Rotation of 90° anticlockwise about (0, 0)

vii $CBD = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$

$= \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$

$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

Identify –no transformation

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3 a $\mathbf{RS} = \begin{pmatrix} 3 & 0 \\ 0 & -2 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 3 \\ 2 & 0 \end{pmatrix}$

Reflection in $y = x$ with a stretch by scale factor 3 parallel to the x-axis and by scale factor 2 parallel to the y-axis.

b $\mathbf{RT} = \begin{pmatrix} 3 & 0 \\ 0 & -2 \end{pmatrix}\begin{pmatrix} 5 & 0 \\ 0 & 5 \end{pmatrix} = \begin{pmatrix} 15 & 0 \\ 0 & -10 \end{pmatrix}$

Stretch by scale factor 15 parallel to the x-axis and by scale factor -10 parallel to the y-axis

c $\mathbf{TS} = \begin{pmatrix} 5 & 0 \\ 0 & 5 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 5 \\ -5 & 0 \end{pmatrix}$

Enlargement by scale factor 5 about (0,0) and rotation through 270° anti-clockwise.

d $\mathbf{TR} = \begin{pmatrix} 5 & 0 \\ 0 & 5 \end{pmatrix}\begin{pmatrix} 3 & 0 \\ 0 & -2 \end{pmatrix} = \begin{pmatrix} 15 & 0 \\ 0 & -10 \end{pmatrix}$

Stretch by scale factor 15 parallel to the x-axis and by scale factor -10 parallel to the y-axis.

e $\mathbf{ST} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 5 & 0 \\ 0 & 5 \end{pmatrix} = \begin{pmatrix} 0 & 5 \\ -5 & 0 \end{pmatrix}$

Enlargement by scale factor 5 about (0,0) and rotation through 270° anti-clockwise.

f $\mathbf{RST} = \begin{pmatrix} 3 & 0 \\ 0 & -2 \end{pmatrix}\begin{pmatrix} 0 & 5 \\ -5 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 15 \\ 10 & 0 \end{pmatrix}$

Reflection in $y = x$ with a stretch by scale factor 15 parallel to the x-axis and by scale factor 10 parallel to the y-axis.

4 a $\mathbf{A}: \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}, \mathbf{B}: \begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix}, \mathbf{C}: \begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix}$

b i $\mathbf{AB} = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}\begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix} = \begin{pmatrix} -4 & 0 \\ 0 & -6 \end{pmatrix}$

ii $\mathbf{AC} = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}\begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix} = \begin{pmatrix} 8 & 0 \\ 0 & 12 \end{pmatrix}$

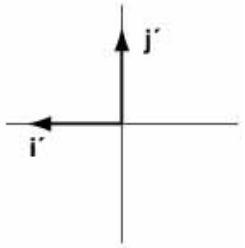
iii $\mathbf{CB} = \begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix}\begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix} = \begin{pmatrix} -8 & 0 \\ 0 & -8 \end{pmatrix}$

iv $\mathbf{C}^2 = \begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix}\begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix} = \begin{pmatrix} 16 & 0 \\ 0 & 16 \end{pmatrix}$

v $\mathbf{ABC} = \begin{pmatrix} -4 & 0 \\ 0 & -6 \end{pmatrix}\begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix} = \begin{pmatrix} -16 & 0 \\ 0 & -24 \end{pmatrix}$

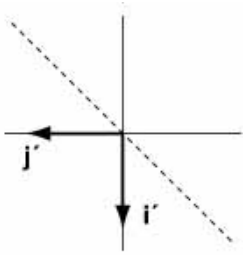
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5 Reflection in y-axis



$\therefore$ Matrix is $\mathbf{Y} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$

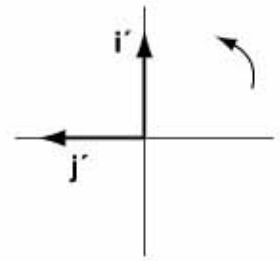
Reflection in $y = -x$



$\therefore$ Matrix $\mathbf{R} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$

$$\mathbf{RY} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \to \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$
$$\begin{pmatrix} 0 \\ 1 \end{pmatrix} \to \begin{pmatrix} -1 \\ 0 \end{pmatrix}$$



i.e Rotation of 90° anticlockwise about (0, 0).

6 $T = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and $U = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

$$UT = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$TU = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \neq UT$$

7 a $$PQ = \begin{pmatrix} -4 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix} = \begin{pmatrix} -4k & 0 \\ 0 & 2k \end{pmatrix}$$

b Stretch by scale factor $-4k$ parallel to the x-axis and by scale factor $2k$ parallel to the y-axis.

c $$\mathbf{QP} = \begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix}\begin{pmatrix} -4 & 0 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} -4k & 0 \\ 0 & 2k \end{pmatrix} = \mathbf{PQ}$$ (from part a)

8 a $$\mathbf{C}^2 = \begin{pmatrix} 3 & 0 \\ 0 & 4 \end{pmatrix}\begin{pmatrix} 3 & 0 \\ 0 & 4 \end{pmatrix} = \begin{pmatrix} 9 & 0 \\ 0 & 16 \end{pmatrix}$$

b Stretch by scale factor 9 parallel to the x-axis and by scale factor 16 parallel to the y-axis.

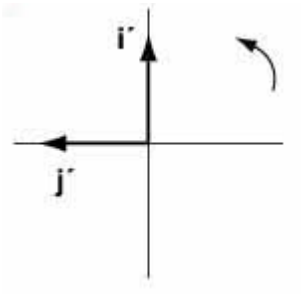
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8 c $\mathbf{B}^2 = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} = \begin{pmatrix} a^2 & 0 \\ 0 & b^2 \end{pmatrix}$

Stretch by scale factor a^2 parallel to the x-axis and by scale factor b^2 parallel to the y-axis.

9 a $\mathbf{R}^2 = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}\begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

b



i.e. $\mathbf{R}^2$ represents rotation of 90° anticlockwise about (0, 0)

c $\mathbf{R}$ represents a rotation of 45° anticlockwise about (0, 0)

d $\mathbf{R}^8$ will represent rotation of $8 \times 45^\circ = 360^\circ$

This is equivalent to no transformation

$\therefore \quad \mathbf{R}^8 = \mathbf{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

10 a $\det \mathbf{M} = \frac{9}{2} + \frac{9}{2} = 9$

$\Rightarrow k = -3$ since $k < 0$

b $\mathbf{M} = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}\begin{pmatrix} -3 & 0 \\ 0 & -3 \end{pmatrix}$

$\Rightarrow \theta = 45^\circ$

11 $\mathbf{AB} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 5 & 0 \\ 0 & 5 \end{pmatrix} = \begin{pmatrix} 0 & 5 \\ -5 & 0 \end{pmatrix}$

$\det \mathbf{M} = 25$

$\Rightarrow$ Area of $T = 75 \div 25 = 3$

12 a $\begin{pmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}$

b $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

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12 c $TU = \begin{pmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}$

13 a $\mathbf{A}^2 = \begin{pmatrix} k & \sqrt{3} \\ \sqrt{3} & -k \end{pmatrix}\begin{pmatrix} k & \sqrt{3} \\ \sqrt{3} & -k \end{pmatrix} = \begin{pmatrix} k^2+3 & 0 \\ 0 & k^2+3 \end{pmatrix}$

b Enlargement centre (0,0) with scale factor k^2+3

14 $\mathbf{P}^2 = \begin{pmatrix} a & b \\ b & -a \end{pmatrix}\begin{pmatrix} a & b \\ b & -a \end{pmatrix} = \begin{pmatrix} a^2+b^2 & ab-ba \\ ab-ba & b^2+a^2 \end{pmatrix}$

$= \begin{pmatrix} a^2+b^2 & 0 \\ 0 & a^2+b^2 \end{pmatrix}$

Enlargement centre (0,0) with scale factor $\mathbf{a}^2+\mathbf{b}^2$

Challenge

a $\mathbf{P}^2 = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$

$= \begin{pmatrix} \cos^2\theta-\sin^2\theta & -2\sin\theta\cos\theta \\ 2\sin\theta\cos\theta & \cos^2\theta-\sin^2\theta \end{pmatrix} = \begin{pmatrix} \cos 2\theta & -\sin 2\theta \\ \sin 2\theta & \cos 2\theta \end{pmatrix}$

b Two successive anticlockwise rotations about the origin by an angle θ are equivalent to a single anticlockwise rotation by an angle 2θ.

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Exercise 7E

1 a $\begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

b $\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

c $\begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$

d $\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

e $\begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$

f $\begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{\sqrt{3}}{2} \\ 0 & -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}$

2 a Reflection in the plane $z = 0$

b Rotation anticlockwise 90° about y-axis

c Rotation anticlockwise 135° about z-axis

3 a Rotation anticlockwise 90° about x-axis

b $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} 3 \\ -1 \\ 4 \end{pmatrix} = \begin{pmatrix} 3 \\ -4 \\ -1 \end{pmatrix}$

So coordinates of A′ are (3, -4, -1)

c $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} a \\ -a \\ 2a-1 \end{pmatrix} = \begin{pmatrix} a \\ a-5 \\ -a \end{pmatrix}$

$\Rightarrow \begin{pmatrix} a \\ 1-2a \\ -a \end{pmatrix} = \begin{pmatrix} a \\ a-5 \\ -a \end{pmatrix}$

$\Rightarrow 1-2a = a-5 \Rightarrow 3a = 6 \Rightarrow a = 2$

4 a $\begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} & 0 \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}$

b $\begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} & 0 \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 3 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{-3+\sqrt{3}}{2} \\ \frac{3\sqrt{3}+1}{2} \\ 0 \end{pmatrix}$

So coordinates of Q' are

$\left(\frac{-3+\sqrt{3}}{2}, \frac{3\sqrt{3}+1}{2}, 0\right)$

c $\begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} & 0 \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} k \\ 0 \\ k \end{pmatrix} = \begin{pmatrix} \frac{-k}{2} \\ \frac{\sqrt{3}k}{2} \\ k \end{pmatrix}$

So coordinates of R' are

$\left(\frac{-k}{2}, \frac{\sqrt{3}k}{2}, k\right)$

5 a $\mathbf{A} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

b $\begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} -a \\ b \\ c \end{pmatrix}$

So coordinates of P′ are (-a, b, c)

c $\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} -a \\ b \\ c \end{pmatrix} = \begin{pmatrix} -a \\ -b \\ c \end{pmatrix}$

So coordinates of P'' are $(-a, -b, c)$

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6 a Since $\mathbf{M} = \begin{pmatrix} \cos 210^\circ & 0 & \sin 210^\circ \\ 0 & 1 & 0 \\ -\sin 210^\circ & 0 & \cos 210^\circ \end{pmatrix}$ and $\begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix}$ represents an anti-clockwise rotation about the y-axis, $\mathbf{M}$ represents an anti-clockwise roation of 210° about the y-axis.

b $\begin{pmatrix} -\frac{\sqrt{3}}{2} & 0 & -\frac{1}{2} \\ 0 & 1 & 0 \\ \frac{1}{2} & 0 & -\frac{\sqrt{3}}{2} \end{pmatrix} \begin{pmatrix} k \\ -k \\ 0 \end{pmatrix} = \begin{pmatrix} -\frac{k\sqrt{3}}{2} \\ -k \\ \frac{k}{2} \end{pmatrix}$

So coordinates of M' are $\left(-\frac{k\sqrt{3}}{2}, -k, \frac{k}{2}\right)$.

7 a $\begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{pmatrix}$

b $\begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} \sqrt{2} \\ 0 \\ 0 \end{pmatrix}$

$\begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} \sqrt{2} \\ 1 \\ 0 \end{pmatrix}$

$\begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} 3 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \\ 3\sqrt{2} \end{pmatrix}$

$\begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

So coordinates of T' are

$\left(\sqrt{2}, 0, 0\right), \left(\sqrt{2}, 1, 0\right)$

$\left(0, 2, 3\sqrt{2}\right), (0, 0, 0)$

c Volume of a tetrahedron:

$= \frac{1}{3} \times$ base area $\times$ height

To find the volume of T':

Area of base on plane z

$= 0 = \frac{1}{2} \times \sqrt{2} \times 1 = \frac{1}{2}\sqrt{2}$

So volume of T'

$= \frac{1}{3} \times \frac{1}{2}\sqrt{2} \times 3\sqrt{2} = 1$

So volume of $T = 1$ as the determinant of the transformation matrix $= 1$.

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Challenge

a $\begin{pmatrix}1&0&0\\0&-1&0\\0&0&1\end{pmatrix}\begin{pmatrix}-1&0&0\\0&1&0\\0&0&1\end{pmatrix}=\begin{pmatrix}-1&0&0\\0&-1&0\\0&0&1\end{pmatrix}$

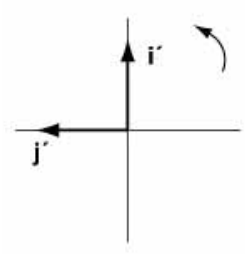
b

$$\begin{pmatrix}-1&0&0\\0&1&0\\0&0&1\end{pmatrix}\begin{pmatrix}1&0&0\\0&\frac{1}{\sqrt{2}}&-\frac{1}{\sqrt{2}}\\0&\frac{1}{\sqrt{2}}&\frac{1}{\sqrt{2}}\end{pmatrix}=\begin{pmatrix}-1&0&0\\0&\frac{1}{\sqrt{2}}&-\frac{1}{\sqrt{2}}\\0&\frac{1}{\sqrt{2}}&\frac{1}{\sqrt{2}}\end{pmatrix}$$

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Exercise 7F

1 a $(1,0) \to (0,1)$
$(0,1) \to (-1,0)$



R represents a rotation of 90° anticlockwise about (0, 0)

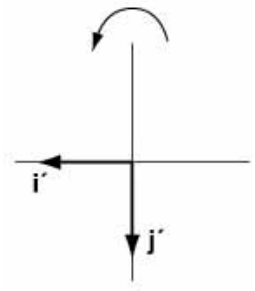
b det $\mathbf{R} = 0 - (-1) = 1$

$\therefore \mathbf{R}^{-1} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

c $\mathbf{R}^{-1}$ represents a rotation of –90° anticlockwise about (0, 0)

(or … 90° clockwise … or … 270° anticlockwise…)

2 a i $(1,0) \to (-1,0)$
$(0,1) \to (0,-1)$



S represents a rotation of 180° about (0, 0)

ii $\mathbf{S}^2$ will be rotation of 180 + 180 = 360° about (0, 0) $\therefore \mathbf{S}^2 = \mathbf{I}$

or $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \mathbf{I}$

iii $\mathbf{S}^{-1} = \mathbf{S} =$ rotation of 180° about (0, 0)

b i $(1,0) \to (0,-1)$
$(0,1) \to (-1,0)$

T represents a reflection in the line $y = -x$

ii $\mathbf{T}^2 = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \mathbf{I}$

iii $\mathbf{T}^{-1} = \mathbf{T} =$ reflection in the line $y = -x$

c det $\mathbf{S} = 1 - 0 = 1$

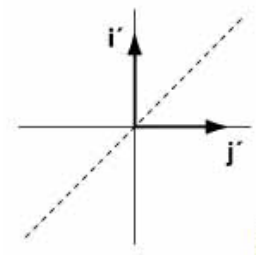
det $\mathbf{T} = 0 - 1 = -1$

For both **S** and **T**, area is unaltered

T represents a reflection and $\therefore$ has a negative determinant. Orientation is reversed

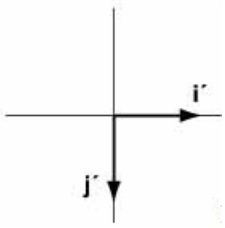
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3 a $\mathbf{A} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$



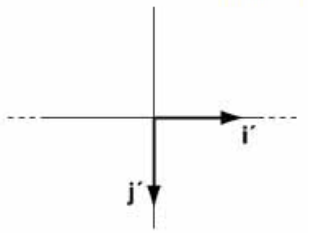
Reflection in $y = x$

$\mathbf{B} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$



Rotation of 270° (about (0, 0))

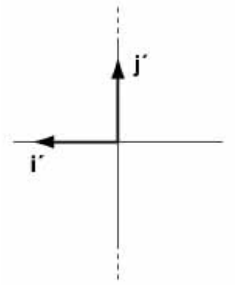
$\mathbf{C} = \mathbf{BA} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$



C represents a reflection in the line $y = 0$ (or the x-axis)

b $\mathbf{C}^{-1} = \mathbf{C} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ is a reflection in the line $y = 0$

c $\mathbf{D} = \mathbf{AB} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$



D represents a reflection in the line $x = 0$ (or the y-axis)

d $\mathbf{D}^{-1} = \mathbf{D} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ is a reflection in the line $x = 0$

4 a $\mathbf{A}^{-1} = \frac{1}{4}\begin{pmatrix} -2 & -2 \\ 3 & 1 \end{pmatrix}$

$$\frac{1}{4}\begin{pmatrix} -2 & -2 \\ 3 & 1 \end{pmatrix}\begin{pmatrix} 5 \\ 8 \end{pmatrix} = \frac{1}{4}\begin{pmatrix} -26 \\ 23 \end{pmatrix} = \begin{pmatrix} -\frac{13}{2} \\ \frac{23}{4} \end{pmatrix}$$

So coordinates of P are $\left(-\frac{13}{2}, \frac{23}{4}\right)$

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4 b

$$\mathbf{UA} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$$

$$\Rightarrow \mathbf{U} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}\mathbf{A}^{-1}$$

$$\Rightarrow \mathbf{U} = \frac{1}{4}\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} -2 & -2 \\ 3 & 1 \end{pmatrix}$$

$$= \frac{1}{4}\begin{pmatrix} -3 & -1 \\ 2 & 2 \end{pmatrix} = \begin{pmatrix} -\frac{3}{4} & -\frac{1}{4} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

5 a Enlargement, scale factor 4, centre (0, 0)

b $\begin{pmatrix} \frac{1}{4} & 0 \\ 0 & \frac{1}{4} \end{pmatrix}$

c $\begin{pmatrix} \frac{1}{4} & 0 \\ 0 & \frac{1}{4} \end{pmatrix}\begin{pmatrix} 4 \\ 6 \end{pmatrix} = \begin{pmatrix} 1 \\ \frac{3}{2} \end{pmatrix}$

$\begin{pmatrix} \frac{1}{4} & 0 \\ 0 & \frac{1}{4} \end{pmatrix}\begin{pmatrix} 9 \\ 7 \end{pmatrix} = \begin{pmatrix} \frac{9}{4} \\ \frac{7}{4} \end{pmatrix}$

$\begin{pmatrix} \frac{1}{4} & 0 \\ 0 & \frac{1}{4} \end{pmatrix}\begin{pmatrix} 3 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{3}{4} \\ \frac{1}{4} \end{pmatrix}$

So coordinates of T are $\left(1, \frac{3}{2}\right), \left(\frac{9}{4}, \frac{7}{4}\right), \left(\frac{3}{4}, \frac{1}{4}\right)$

6 a $\det \mathbf{M} = ab$

$$\Rightarrow \mathbf{M}^{-1} = \frac{1}{ab}\begin{pmatrix} b & 0 \\ 0 & a \end{pmatrix} = \begin{pmatrix} \frac{1}{a} & 0 \\ 0 & \frac{1}{b} \end{pmatrix}$$

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6 b $\begin{pmatrix} \frac{1}{a} & 0 \\ 0 & \frac{1}{b} \end{pmatrix}\begin{pmatrix} -6 \\ 8 \end{pmatrix} = \begin{pmatrix} -\frac{6}{a} \\ \frac{8}{b} \end{pmatrix}$

So coordinates of D are $\left(-\frac{6}{a}, \frac{8}{b}\right)$

7 a Rotation of 330° anticlockwise about (0, 0)

b $\mathbf{R}^{-1} = \begin{pmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix}$

$\begin{pmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix}\begin{pmatrix} \frac{1-2\sqrt{3}}{2} \\ \frac{2+\sqrt{3}}{2} \end{pmatrix} = \begin{pmatrix} \frac{\sqrt{3}-6-2-\sqrt{3}}{4} \\ \frac{1-2\sqrt{3}+2\sqrt{3}+3}{4} \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$

So $p = -2, q = 1$

8 $\mathbf{AB} = \begin{pmatrix} 2 & 4 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} -4 & -2 \\ -3 & -1 \end{pmatrix}$

$\Rightarrow (\mathbf{AB})^{-1} = \frac{1}{-2}\begin{pmatrix} -1 & 2 \\ 3 & -4 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & -1 \\ -\frac{3}{2} & 2 \end{pmatrix}$

9 $\mathbf{A}^{-1} = \frac{1}{-2}\begin{pmatrix} 1 & 2 \\ 4 & 6 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} -1 & -2 \\ -4 & -6 \end{pmatrix}$

$\frac{1}{2}\begin{pmatrix} -1 & -2 \\ -4 & -6 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} -\frac{a}{2} - b \\ -2a - 3b \end{pmatrix}$

So coordinates of P are $\left(-\frac{a}{2} - b, -2a - 3b\right)$

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10 $\mathbf{M}^{-1} = \begin{pmatrix} 2 & \frac{3}{2} & -1 \\ -1 & -1 & 1 \\ -1 & -\frac{3}{2} & 1 \end{pmatrix}$

$$\begin{pmatrix} 2 & \frac{3}{2} & -1 \\ -1 & -1 & 1 \\ -1 & -\frac{3}{2} & 1 \end{pmatrix}\begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 10 \\ -6 \\ -7 \end{pmatrix}$$

So $a = 10,\ b = -6,\ c = -7$

11 a $\mathbf{PQ} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

Reflection in $y = x$

b $\mathbf{M}^{-1} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

Reflection in $y = x$

c $\mathbf{QP} = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

Reflection in $y = -x$

d $\mathbf{N} = \mathbf{QP} \Rightarrow \mathbf{Q}^{-1}\mathbf{N} = \mathbf{P} \Rightarrow \mathbf{P}^{-1}\mathbf{Q}^{-1}\mathbf{N} = 1 \Rightarrow \mathbf{P}^{-1}\mathbf{Q}^{-1} = \mathbf{N}^{-1}$

$\mathbf{N}^{-1} = \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$; reflection in $y = -x$

12 $\mathbf{BA} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$

$$\Rightarrow \mathbf{B} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}\mathbf{A}^{-1}$$

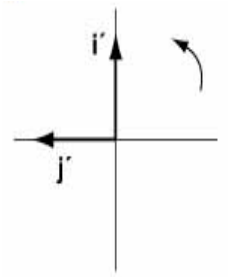
$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}\begin{pmatrix} -0.1 & 0.3 & 0.7 \\ 0.3 & 0.1 & -0.1 \\ -0.2 & 0.6 & 0.4 \end{pmatrix}$$

$$= \begin{pmatrix} -0.1 & 0.3 & 0.7 \\ 0.3 & 0.1 & -0.1 \\ 0.2 & -0.6 & -0.4 \end{pmatrix}$$

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Mixed exercise 7

1 a



$$\mathbf{Y} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

b $\mathbf{AB} = \mathbf{Y} \quad \Rightarrow \quad \mathbf{A} = \mathbf{YB}^{-1}.$

$$\mathbf{B} = \begin{pmatrix} 3 & 2 \\ 2 & 1 \end{pmatrix} \Rightarrow \det \mathbf{B} = 3 - 4 = -1$$

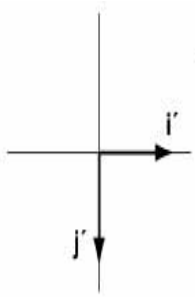
$$\therefore \mathbf{B}^{-1} = \frac{1}{-1}\begin{pmatrix} 1 & -2 \\ -2 & 3 \end{pmatrix} = \begin{pmatrix} -1 & 2 \\ 2 & -3 \end{pmatrix}$$

$$\therefore \mathbf{A} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} -1 & 2 \\ 2 & -3 \end{pmatrix}$$

$$= \begin{pmatrix} -2 & 3 \\ -1 & 2 \end{pmatrix}$$

c $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ (Identity matrix); four successive anticlockwise rotations of 90° about (0, 0).

2 Reflection in x-axis



$$\Rightarrow \mathbf{R} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Enlargement S. F. 2 centre (0, 0)

$$\Rightarrow \mathbf{E} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$$

a $\mathbf{C} = \mathbf{ER} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

$$= \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix}$$

Reflection in the x-axis and enlargement scale factor 2, centre (0, 0).

b $\mathbf{C}^{-1} = \frac{1}{-4}\begin{pmatrix} -2 & 0 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}$

Reflection in the x-axis and enlargement scale factor $\frac{1}{2}$. Centre (0, 0)

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3 a $\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}M = P \Rightarrow M = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}^{-1} P$

$\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}^{-1} = \frac{1}{-1}\begin{pmatrix}0 & -1\\ -1 & 0\end{pmatrix} = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$

$M = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}0 & k\\ k & 0\end{pmatrix} = \begin{pmatrix}k & 0\\ 0 & k\end{pmatrix}$

b $\begin{pmatrix}k & 0\\ 0 & k\end{pmatrix}\begin{pmatrix}-3\\ -2\end{pmatrix} = \begin{pmatrix}9\\ 6\end{pmatrix} \Rightarrow -3k = 9$

$\Rightarrow k = -3$

c $\begin{pmatrix}-3 & 0\\ 0 & -3\end{pmatrix}\begin{pmatrix}x\\ mx\end{pmatrix} = \begin{pmatrix}-3x\\ -3mx\end{pmatrix}$

The point $(-3x, -3mx)$ lies on the line $y = mx$ for $m = 1$ and $m = -1$, so the line is invariant under T for these two values of m.

4 a $\det \mathbf{M} = \left(2\sqrt{3}\right)^2 + 4 = 16$

$\Rightarrow$ scale factor $= 4$

b $\mathbf{M} = \begin{pmatrix}4 & 0\\ 0 & 4\end{pmatrix}R$

$\Rightarrow R = \begin{pmatrix}\frac{1}{4} & 0\\ 0 & \frac{1}{4}\end{pmatrix}\begin{pmatrix}2\sqrt{3} & -2\\ 2 & 2\sqrt{3}\end{pmatrix} = \begin{pmatrix}\frac{\sqrt{3}}{2} & -\frac{1}{2}\\ \frac{1}{2} & \frac{\sqrt{3}}{2}\end{pmatrix}$

$= \begin{pmatrix}\cos 30^\circ & -\sin 30^\circ\\ \sin 30^\circ & \cos 30^\circ\end{pmatrix}$

So angle is 30° anti-clockwise.

c $\mathbf{M}^{-1} = \frac{1}{16}\begin{pmatrix}2\sqrt{3} & 2\\ -2 & 2\sqrt{3}\end{pmatrix}$

$\mathbf{M}\begin{pmatrix}x\\ y\end{pmatrix} = \begin{pmatrix}a\\ b\end{pmatrix} \Rightarrow \begin{pmatrix}x\\ y\end{pmatrix} = \mathbf{M}^{-1}\begin{pmatrix}a\\ b\end{pmatrix}$

$= \frac{1}{16}\begin{pmatrix}2\sqrt{3} & 2\\ -2 & 2\sqrt{3}\end{pmatrix}\begin{pmatrix}a\\ b\end{pmatrix} = \begin{pmatrix}\frac{\sqrt{3}a}{8} + \frac{b}{8}\\ -\frac{a}{8} + \frac{\sqrt{3}b}{8}\end{pmatrix}$

So coordinates of P are $\left(\frac{\sqrt{3}a}{8} + \frac{b}{8}, -\frac{a}{8} + \frac{\sqrt{3}b}{8}\right)$

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5 a **A** represents a reflection in the line $y = x$;
B represents a rotation through 270° anticlockwise about (0, 0)

b $\begin{pmatrix}0 & 1\\1 & 0\end{pmatrix}\begin{pmatrix}0 & 1\\-1 & 0\end{pmatrix}=\begin{pmatrix}-1 & 0\\0 & 1\end{pmatrix}$

$\begin{pmatrix}-1 & 0\\0 & 1\end{pmatrix}\begin{pmatrix}p\\q\end{pmatrix}=\begin{pmatrix}-p\\q\end{pmatrix}$

So coordinates of image are $(-p, q)$

6 a Area of $\mathbf{T} = \frac{1}{2}|k-6|\times 5 = \frac{5}{2}|k-6|$

$\det \mathbf{M} = 8-3 = 5$

$\Rightarrow 5\times\frac{5}{2}|k-6| = 110$

$\Rightarrow |k-6| = 8.8$

$\Rightarrow k = 14.8 \text{ or } -2.8$

b $\begin{pmatrix}-4 & 3\\1 & -2\end{pmatrix}\begin{pmatrix}x\\-\frac{1}{3}x\end{pmatrix}=\begin{pmatrix}-5x\\\frac{5}{3}x\end{pmatrix}$; $(-5x)+3\left(\frac{5}{3}\right)=0$

so the point satisfies the equation of the original line.

7 a $P=\begin{pmatrix}-1 & 0\\0 & 1\end{pmatrix}\begin{pmatrix}4 & 0\\0 & 3\end{pmatrix}=\begin{pmatrix}-4 & 0\\0 & 3\end{pmatrix}$

b $\det \mathbf{P} = -12$

So area of $T = 60\div 12 = 5$

8 a $\mathbf{A}=\begin{pmatrix}\cos 150^\circ & -\sin 150^\circ & 0\\\sin 150^\circ & \cos 150^\circ & 0\\0 & 0 & 1\end{pmatrix}$

So **A** represents a 150° anti-clockwise rotation about the z-axis.

b

$$\begin{pmatrix}-\frac{\sqrt{3}}{2} & -\frac{1}{2} & 0\\\frac{1}{2} & -\frac{\sqrt{3}}{2} & 0\\0 & 0 & 1\end{pmatrix}\begin{pmatrix}a\\b\\-a\end{pmatrix}=\begin{pmatrix}-\frac{a\sqrt{3}}{2}-\frac{b}{2}\\\frac{a}{2}-\frac{b\sqrt{3}}{2}\\-a\end{pmatrix}$$

So coordinates of image are $\left(-\frac{a\sqrt{3}}{2}-\frac{b}{2}, \frac{a}{2}-\frac{b\sqrt{3}}{2}, -a\right)$

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9 a $\mathbf{P}^{-1} = \frac{1}{a^2}\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} = \begin{pmatrix} \frac{1}{a} & 0 \\ 0 & \frac{1}{a} \end{pmatrix}$

b $\mathbf{P}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 4 \\ 7 \end{pmatrix}$

$\begin{pmatrix} x \\ y \end{pmatrix} = \mathbf{P}^{-1}\begin{pmatrix} 4 \\ 7 \end{pmatrix} = \begin{pmatrix} \frac{1}{a} & 0 \\ 0 & \frac{1}{a} \end{pmatrix}\begin{pmatrix} 4 \\ 7 \end{pmatrix} = \begin{pmatrix} \frac{4}{a} \\ \frac{7}{a} \end{pmatrix}$

So coordinates of **A** are $\left(\frac{4}{a}, \frac{7}{a}\right)$

10 The matrix representing a 90° rotation about the origin is $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\mathbf{P} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} -1 & 2 \\ -5 & 8 \end{pmatrix} = \begin{pmatrix} 5 & -8 \\ -1 & 2 \end{pmatrix}$

$\begin{pmatrix} 5 & -8 \\ -1 & 2 \end{pmatrix}^{-1} = \frac{1}{2}\begin{pmatrix} 2 & 8 \\ 1 & 5 \end{pmatrix} = \begin{pmatrix} 1 & 4 \\ \frac{1}{2} & \frac{5}{2} \end{pmatrix}$

11 a $\mathbf{P} = \mathbf{AB} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 4 & -1 \\ 3 & -2 \end{pmatrix} = \begin{pmatrix} -4 & 1 \\ -3 & 2 \end{pmatrix}$

b $\det \mathbf{P} = -8 - -3 = -5$

So area of $T = 35 \div 5 = 7$

c $\mathbf{Q} = \mathbf{P}^{-1} = \frac{1}{-5}\begin{pmatrix} 2 & -1 \\ 3 & -4 \end{pmatrix} = -\begin{pmatrix} -\frac{2}{5} & \frac{1}{5} \\ -\frac{3}{5} & \frac{4}{5} \end{pmatrix}$

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12 $\mathbf{M}^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ 0 & \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \end{pmatrix}$

$\mathbf{M}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \Rightarrow \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \mathbf{M}^{-1}\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$

$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & -\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ 0 & \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \end{pmatrix}\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ -\sqrt{2} \\ 0 \end{pmatrix}$

$\Rightarrow a = 0, b = -\sqrt{2}, c = 0$

13 a $\mathbf{A}^{-1} = \begin{pmatrix} -\frac{4}{3} & \frac{1}{6} & \frac{2}{3} \\ -\frac{7}{3} & \frac{1}{6} & \frac{2}{3} \\ \frac{2}{3} & \frac{1}{6} & -\frac{1}{3} \end{pmatrix}$

$\mathbf{A}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ 7 \\ 4 \end{pmatrix} \Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \mathbf{A}^{-1}\begin{pmatrix} 3 \\ 7 \\ 4 \end{pmatrix}$

$= \begin{pmatrix} -\frac{4}{3} & \frac{1}{6} & \frac{2}{3} \\ -\frac{7}{3} & \frac{1}{6} & \frac{2}{3} \\ \frac{2}{3} & \frac{1}{6} & -\frac{1}{3} \end{pmatrix}\begin{pmatrix} 3 \\ 7 \\ 4 \end{pmatrix} = \begin{pmatrix} -\frac{1}{6} \\ -\frac{19}{6} \\ \frac{11}{6} \end{pmatrix}$

So coordinates of **P** are $\left(-\frac{1}{6}, -\frac{19}{6}, \frac{11}{6}\right)$

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13 b $\mathbf{BA} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$

$$\Rightarrow \mathbf{B} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} -\frac{4}{3} & \frac{1}{6} & \frac{2}{3} \\ -\frac{7}{3} & \frac{1}{6} & \frac{2}{3} \\ \frac{2}{3} & \frac{1}{6} & -\frac{1}{3} \end{pmatrix}$$

$$= \begin{pmatrix} -\frac{4}{3} & \frac{1}{6} & \frac{2}{3} \\ -\frac{2}{3} & -\frac{1}{6} & \frac{1}{3} \\ -\frac{7}{3} & \frac{1}{6} & \frac{2}{3} \end{pmatrix} = \frac{1}{6} \begin{pmatrix} -8 & 1 & 4 \\ -4 & -1 & 2 \\ -14 & 1 & 4 \end{pmatrix}$$

Challenge

1 $\mathbf{M} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$$

2 a Let the point be $P = (a, b)$: $\begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0+b \\ 0+b \end{pmatrix} = \begin{pmatrix} b \\ b \end{pmatrix}$

So P' is (b, b); its x- and y-coordinates are equal, so it is on $y = x$

b $\begin{pmatrix} 0 & 1 \\ 0 & m \end{pmatrix}$

c If c does not equal 0, then the line $ax + by = c$ does not go through the origin. Hence the origin cannot be mapped to itself, and the transformation is not linear.

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Proof by induction 8A

1 Basis: When $n = 1$: LHS = 1; RHS = $\frac{1}{2}(1)(1+1) = 1$

Assumption:

$$\sum_{r=1}^{k} r = \frac{1}{2}k(k+1)$$

Induction:

$$\sum_{r=1}^{k+1} r = \sum_{r=1}^{k} r + (k+1) = \frac{1}{2}k(k+1) + (k+1)$$

$$= \frac{1}{2}k(k+1) + (k+1) \ = \frac{1}{2}(k+1)(k+2)$$

So if the statement holds for $n = k$, it holds for $n = k + 1$.

Conclusion: The statement holds for all $n \in \mathbb{Z}^+$.

2 Basis: When $n = 1$: LHS = 1; RHS = $\frac{1}{4}(1)^2(1+1)^2 = 1$

Assumption:

$$\sum_{r=1}^{k} r^3 = \frac{1}{4}k^2(k+1)^2$$

Induction:

$$\sum_{r=1}^{k+1} r^3 = \sum_{r=1}^{k} r^3 + \ (k+1)^3 = \frac{1}{4}k^2(k+1)^2 + (k+1)^3$$

$$= \frac{1}{4}(k+1)^2\left(k^2 + 4(k+1)\right) = \frac{1}{4}(k+1)^2(k+2)^2$$

So if the statement holds for $n = k$, it holds for $n = k + 1$.

Conclusion: The statement holds for all $n \in \mathbb{Z}^+$.

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3 a Basis: $n = 1$: LHS = 0; RHS = $\frac{1}{3}(1)(1+1)(1-1) = 0$

Assumption:

$$\sum_{r=1}^{k} r(r-1) = \frac{1}{3}k(k+1)(k-1)$$

Induction:

$$\sum_{r=1}^{k+1} r(r-1) = \sum_{r=1}^{k} r(r-1) + (k+1)k$$

$$= \frac{1}{3}k(k+1)(k-1) + k(k+1)$$

$$= \frac{1}{3}k(k+1)(k-1+3) = \frac{1}{3}k(k+1)(k+2)$$

So if the statement holds for $n = k$, it holds for $n = k + 1$.

Conclusion: The statement holds for all $n \in \mathbb{Z}^+$.

b Hence the required expression is

$$\sum_{r=1}^{2n+1} r(r-1) = \frac{1}{3}(2n+1)\big((2n+1)+1\big)\big((2n+1)-1\big)$$

$$= \frac{4}{3}n(2n+1)(n+1)$$

4 a Basis: $n = 1$: LHS = 2; RHS = $(1)^2(1+1) = 2$

Assumption:

$$\sum_{r=1}^{k} r(3r-1) = k^2(k+1)$$

Induction:

$$\sum_{r=1}^{k+1} r(3r-1) = \sum_{r=1}^{k} r(3r-1) + (k+1)(3k+2)$$

$$= k^2(k+1) + (k+1)(3k+2)$$

$$= (k+1)(k^2+3k+2) = (k+1)^2(k+2)$$

So if the statement holds for $n = k$, it holds for $n = k + 1$.

Conclusion: The statement holds for all $n \in \mathbb{Z}^+$.

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4 b We need to solve the equation

$$4n^2(n+1) = \frac{n^2(n+1)^2}{4}$$

Rearranging and cancelling the common factor $n^2(n+1)$ gives

$(n+1) = 16$

$n = 15$

5 a Basis: $n = 1$: LHS $= \frac{1}{2}$; RHS $= 1 - \frac{1}{2} = \frac{1}{2}$

Assumption:

$$\sum_{r=1}^{k}\left(\frac{1}{2}\right)^r = 1 - \frac{1}{2^k}$$

Induction:

$$\sum_{r=1}^{k+1}\left(\frac{1}{2}\right)^r = \sum_{r=1}^{k}\left(\frac{1}{2}\right) + \frac{1}{2^{k+1}} = \frac{1}{2^k} + \frac{1}{2^{k+1}}$$

$$= 1 - \frac{2}{2^{k+1}} + \frac{1}{2^{k+1}} = 1 - \frac{1}{2^{k+1}}$$

So if the statement holds for $n = k$, it holds for $n = k + 1$.

Conclusion: The statement holds for all $n \in \mathbb{Z}^+$.

b Basis: $n = 1$: LHS $= 1 \times 1!$; RHS $= (1 + 1)! - 1 = 1$

Assumption:

$$\sum_{r=1}^{k+1} r(r!) = (n+1) - 1$$

Induction:

$$\sum_{r=1}^{k+1} r(r!) = \sum_{r=1}^{k} r(r!) + (k+1)(k+1)!$$

$$= (k + 1)! - 1 + (k + 1)(k + 1)!$$

$$= (k + 1)!\,(k + 2) - 1 = ((k + 1) + 1)! - 1)$$

So if the statement holds for $n = k$, it holds for $n = k + 1$.

Conclusion: The statement holds for all $n \in \mathbb{Z}^+$.

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5 c Basis: $n = 1$: LHS $= \frac{4}{1\times 3} = \frac{4}{3}$; RHS $= \frac{1\times 8}{2\times 3} = \frac{4}{3}$

Assumption:

$$\sum_{r=1}^{k} \frac{4}{r(r+2)} = \frac{k(3k+5)}{(k+1)(k+2)}$$

Induction:

$$\begin{aligned}\sum_{r=1}^{k+1} \frac{4}{r(r+2)} &= \sum_{r=1}^{k} \frac{4}{r(r+2)} + \frac{4}{(k+1)(k+3)}\\ &= \frac{k(3k+5)}{(k+1)(k+2)} + \frac{4}{(k+1)(k+3)}\\ &= \frac{k(3k+5)(k+3)}{(k+1)(k+2)(k+3)} + \frac{4(k+2)}{(k+1)(k+2)(k+3)}\\ &= \frac{k(3k+5)(k+3)+4(k+2)}{(k+1)(k+2)(k+3)} + \frac{(k+1)(3k+8)}{(k+2)(k+3)}\\ &= \frac{(k+1)(3(k+1)+5)}{((k+1+1)((k+1)+2))}\end{aligned}$$

So if the statement holds for $n = k$, it holds for $n = k + 1$.

Conclusion: The statement holds for all $n \in \mathbb{Z}^+$.

6 a The student has just stated and not shown that the statement is true for $n = k + 1$.

b e.g. $n = 2$: LHS $= (1 + 2)^2 = 9$; RHS $= 1^2 + 2^2 \neq 9$, so that LHS $\neq$ RHS.

7 a The student has not completed the basis step.

b e.g. $n = 1$: LHS $= 1$; RHS $= \frac{1}{2}(1^2 + 1 + 1) = \frac{3}{2} \neq 1$

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Challenge

Basis: $n = 1$: LHS $= (-1)^1 \times 1^2$; RHS $= \frac{1}{2}(-1)^1(1)(1+1)$

Assumption:

$$\sum_{r=1}^{k}(-1)^r r^2 = \frac{1}{2}(-1)^k k(k+1)$$

Induction:

$$\begin{aligned}\sum_{r=1}^{k+1}(-1)^r r^2 &= \sum_{r=1}^{k}(-1)^r r^2 + (-1)^{k+1}(k+1)^2 \\ &= \frac{1}{2}(-1)k(k+1) + (-1)^{k+1}(k+1)^2 \\ &= \frac{1}{2}(-1)^{k+1}(k+1)(-k+2(k+1)) \\ &= \frac{1}{2}(-1)^{k+1}(k+1)(k+2)\end{aligned}$$

So if the statement holds for $n = k$, it holds for $n = k + 1$.

Conclusion: The statement holds for all $n \in \mathbb{Z}^+$.

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Proof by induction 8B

1 a Let $f(n) = 8^n - 1$, where $n \in \mathbb{Z}^+$.

$\therefore f(1) = 8^1 - 1 = 7$, which is divisible by 7.

$\therefore f(n)$ is divisible by 7 when $n = 1$.

Assume that for $n = k$,

$f(k) = 8^k - 1$ is divisible by 7 for $k \in \mathbb{Z}^+$.

$$\therefore f(k+1) = 8^{k+1} - 1$$
$$= 8^k.8^1 - 1$$
$$= 8(8^k) - 1$$
$$\therefore f(k+1) - f(k) = [8(8^k) - 1] - [8^k - 1]$$
$$= 8(8^k) - 1 - 8^k + 1$$
$$= 7(8^k)$$
$$\therefore f(k+1) = f(k) + 7(8^k)$$

As both f(k) and 7(8^k) are divisible by 7 then the sum of these two terms must also be divisible by 7. Therefore f(n) is divisible by 7 when $n = 1$.

If f(n) is divisible by 7 when $n = k$, then it has been shown that f(n) is also divisible by 7 when $n = k + 1$. As f(n) is divisible by 7 when $n = 1$, f(n) is also divisible by 7 for all $n \geqslant 1$ and $n \in \mathbb{Z}^+$ by mathematical induction.

1 b Let $f(n) = 3^{2n} - 1$, where $n \in \mathbb{Z}^+$.

$\therefore f(1) = 3^{2(1)} - 1 = 9 - 1 = 8$, which is divisible by 8.

$\therefore$ f(n) is divisible by 8 when $n = 1$.

Assume that for $n = k$,

$f(k) = 3^{2k} - 1$ is divisible by 8 for $k \in \mathbb{Z}^+$.

$$\therefore f(k+1) = 3^{2(k+1)} - 1$$
$$= 3^{2k+2} - 1$$
$$= 3^{2k}.3^2 - 1$$
$$= 9(3^{2k}) - 1$$
$$\therefore f(k+1) - f(k) = [9(3^{2k}) - 1] - [3^{2k} - 1]$$
$$= 9(3^{2k}) - 1 - 3^{2k} + 1$$
$$= 8(3^{2k})$$
$$\therefore f(k+1) = f(k) + 8(3^{2k})$$

As both f(k) and 8(3^{2k}) are divisible by 8 then the sum of these two terms must also be divisible by 8. Therefore f(n) is divisible by 8 when $n = k + 1$.

If f(n) is divisible by 8 when $n = k$, then it has been shown that f(n) is also divisible by 8 when $n = k + 1$. As f(n) is divisible by 8 when $n = 1$, f(n) is also divisible by 8 for all $n \geqslant 1$ and $n \in \mathbb{Z}^+$ by mathematical induction.

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1 c Let $f(n) = 5^n + 9^n + 2$, where $n \in \mathbb{Z}^+$.

$\therefore\ f(1) = 5^1 + 9^1 + 2 = 5 + 9 + 2 = 16$, which is divisible by 4.

$\therefore f(n)$ is divisible by 4 when $n = 1$.

Assume that for $n = k$,

$f(k) = 5^k + 9^k + 2$ is divisible by 4 for $k \in \mathbb{Z}^+$.

$$\therefore f(k+1) = 5^{k+1} + 9^{k+1} + 2$$
$$= 5^k.5^1 + 9^k.9^1 + 2$$
$$= 5(5^k) + 9(9^k) + 2$$

$$\therefore f(k+1) - f(k) = [5(5^k) + 9(9^k) + 2] - [5^k + 9^k + 2]$$
$$= 5(5^k) + 9(9^k) + 2 - 5^k - 9^k - 2$$
$$= 4(5^k) + 8(9^k)$$
$$= 4[5^k + 2(9)^k]$$

$$\therefore f(k+1) = f(k) + 4[5^k + 2(9)^k]$$

As both $f(k)$ and $4[5^k + 2(9)^k]$ are divisible by 4 then the sum of these two terms must also be divisible by 4. Therefore $f(n)$ is divisible by 4 when $n = k + 1$.

If $f(n)$ is indivisible by 4 when $n = k$, then it has been shown that $f(n)$ is also divisible by 4 when $n = k + 1$. As $f(n)$ is divisible by 4 when $n = 1$, $f(n)$ is also divisible by 4 for all $n \geqslant 1$ and $n \in \mathbb{Z}^+$ by mathematical induction.

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1 d Let $f(n) = 2^{4n} - 1$, where $n \in \mathbb{Z}^+$.

$\therefore\ f(1) = 2^{4(1)} - 1 = 16 - 1 = 15,$ which is divisible by 15.

$\therefore f(n) =$ is divisible by 15 when $n = 1$.

Assume that for $n = k$,

$f(k) = 2^{4k} - 1$ is divisible by 15 for $k \in \mathbb{Z}^+$.

$$\therefore f(k+1) = 2^{4(k+1)} - 1$$
$$= 2^{4k+4} - 1$$
$$= 2^{4k}.2^4 - 1$$
$$= 16(2^{4k}) - 1$$

$$\therefore f(k+1) - f(k) = [16(2^{4k}) - 1] - [2^{4k} - 1]$$
$$= 16(2^{4k}) - 1 - 2^{4k} + 1$$
$$= 15(2^{4k})$$

$$\therefore f(k+1) = f(k) + 15(2^{4k})$$

As both $f(k)$ and $15(2^{4k})$ are divisible by 15 then the sum of these two terms must also be divisible by 15. Therefore $f(n)$ is divisible by 15 when $n = k + 1$.

If $f(n)$ is divisible by 15 when $n = k$, then it has been shown that $f(n)$ is also divisible by 15 when $n = k + 1$. As $f(n)$ is divisible by 15 when $n = 1$, $f(n)$ is also divisible by 15 for all $n \geqslant 1$ and $n \in \mathbb{Z}^+$ by mathematical induction.

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1 e Let $f(n) = 3^{2n-1} + 1$, where $n \in \mathbb{Z}^+$.

$\therefore f(1) = 3^{2(1)-1} + 1 = 3 + 1 = 4$, which is divisible by 4.

$\therefore f(n)$ is divisible by 4 when $n = 1$.

Assume that for $n = k$,

$f(k) = 3^{2k-1} + 1$ is divisible by 4 for $k \in \mathbb{Z}^+$.

$$\begin{aligned}\therefore f(k+1) &= 3^{2(k+1)-1} + 1 \\ &= 3^{2k+2-1} + 1 \\ &= 3^{2k-1}.3^2 + 1 \\ &= 9(3^{2k-1}) + 1\end{aligned}$$

$$\begin{aligned}\therefore f(k+1) - f(k) &= \left[9(3^{2k-1}) + 1\right] - \left[3^{2k-1} + 1\right] \\ &= 9(3^{2k-1}) + 1 - 3^{2k-1} + 1 \\ &= 8(3^{2k-1})\end{aligned}$$

$\therefore f(k+1) = f(k) + 8(3^{2k-1})$

As both f(k) and $8(3^{2k-1})$ are divisible by 4 then the sum of these two terms must also by divisible by 4. Therefore f(n) is divisible by 4 when $n = k + 1$.

If f(n) is divisible by 4 when $n = k$, then it has been shown that f(n) is also divisible by 4 when $n = k + 1$. As f(n) is divisible by 4 when $n = 1$, f(n) is also divisible by 8 for $n \geqslant 1$ and $n \in \mathbb{Z}^+$ by mathematical induction.

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1 f Let $f(n) = n^3 + 6n^2 + 8n$, where $n \geqslant 1$ and $n \in \mathbb{Z}^+$.

$\therefore f(1) = 1 + 6 + 8 = 15$, which is divisible by 3.

$\therefore f(n)$ is divisible by 3 when $n = 1$.

Assume that for $n = k$,

$f(k) = k^3 + 6k^2 + 8k$ is divisible by 3 for $k \in \mathbb{Z}^+$.

$$\begin{aligned}\therefore f(k+1) &= (k+1)^3 + 6(k+1)^2 + 8(k+1)\\ &= k^3 + 3k^2 + 3k + 1 + 6(k^2 + 2k + 1) + 8(k+1)\\ &= k^3 + 3k^2 + 3k + 1 + 6k^2 + 12k + 6 + 8k + 8\\ &= k^3 + 9k^2 + 23k + 15\end{aligned}$$

$$\begin{aligned}\therefore f(k+1) - f(k) &= [k^3 + 9k^2 + 23k + 15] - [k^3 + 6k^2 + 8k]\\ &= 3k^2 + 15k + 15\\ &= 3(k^2 + 5k + 5)\end{aligned}$$

$\therefore f(k+1) = f(k) + 3(k^2 + 5k + 5)$

As both f(k) and $3\left(k^2 + 5k + 5\right)$ are divisible by 3 then the sum of these two terms must also be divisible by 3.

Therefore f(n) is divisible by 3 when $n = k + 1$.

If f(n) is divisible by 3 when $n = k$, then it has been shown that f(n) is also divisible by 3 when $n = k + 1$. As f(n) is divisible by 3 when $n = 1, f(n)$ is also divisible by 3 for all $n \geqslant 1$ and $n \in \mathbb{Z}^+$ by mathematical induction.

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1 g Let $f(n) = n^3 + 5n$, where $n \geqslant 1$ and $n \in \mathbb{Z}^+$.

$\therefore f(1) = 1 + 5 = 6$, which is divisible by 6.

$\therefore f(n)$ is divisible by 6 when $n = 1$.

Assume that for $n = k$,

$f(k) = k^3 + 5k$ is divisible by 6 for $k \in \mathbb{Z}^+$.

$$\begin{aligned}\therefore f(k+1) &= (k+1)^3 + 5(k+1)\\ &= k^3 + 3k^2 + 3k + 1 + 5(k+1)\\ &= k^3 + 3k^2 + 3k + 1 + 5k + 5\\ &= k^3 + 3k^2 + 8k + 6\end{aligned}$$

$$\begin{aligned}\therefore f(k+1) - f(k) &= [k^3 + 3k^2 + 8k + 6] - [k^3 + 5k]\\ &= 3k^2 + 3k + 6\\ &= 3k(k+1) + 6\\ &= 3(2m) + 6\\ &= 6m + 6\\ &= 6(m+1)\end{aligned}$$

Let $k(k+1) = 2m, m \in \mathbb{Z}^+$, as the product of two consecutive integers must be even.

$\therefore f(k+1) = f(k) + 6(m+1)$.

As both $f(k)$ and $6(m+1)$ are divisible by 6 then the sum of these two terms must also be divisible by 6. Therefore f(n) is divisible by 6 when $n = k + 1$.

If $f(n)$ is divisible by 6 when $n = k$, then it has been shown that $f(n)$ is also divisible by 6 when $n = k + 1$. As $f(n)$ is divisible by 6 when $n = 1, f(n)$ is also divisible by 6 for all $n \geqslant 1$ and $n \in \mathbb{Z}^+$ by mathematical induction.

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1 h Let $f(n) = 2^n.3^{2n} - 1$, where $n \in \mathbb{Z}^+$.

$\therefore f(1) = 2^1.3^{2(1)} - 1 = 2(9) - 1 = 18 - 1 = 17$, which is divisible by 17.

$\therefore f(n)$ is divisible by 17 when $n = 1$.

Assume that for $n = k$,

$f(k) = 2^k.3^{2k} - 1$ is divisible by 17 for $k \in \mathbb{Z}^+$.

$$\begin{aligned}\therefore f(k+1) &= 2^{k+1}.3^{2(k+1)} - 1\\ &= 2^k(2)^1(3)^{2k}(3)^2 - 1\\ &= 2^k(2)^1(3)^{2k}(9) - 1\\ &= 18(2^k.3^{2k}) - 1\end{aligned}$$

$$\begin{aligned}\therefore f(k+1) - f(k) &= \left[18(2^k.3^{2k}) - 1\right] - \left[2^k.3^{2k} - 1\right]\\ &= 18(2^k.3^{2k}) - 1 - 2^k.3^{2k} + 1\\ &= 17(2^k.3^{2k})\end{aligned}$$

$\therefore f(k+1) = f(k) + 17(2^k.3^{2k})$

As both $f(k)$ and $17(2^k.3^{2k})$ are divisible by 17 then the sum of these two terms must also be divisible by 17.

Therefore $f(n)$ is divisible by 17 when $n = k + 1$.

If $f(n)$ is divisible by 17 when $n = k$, then it has been shown that $f(n)$ is also divisible by 17 when $n = k + 1$. As f(n) is divisible by 17 when $n = 1$, f(n) is also divisible by 17 for all $n \geqslant 1$ and $n \in \mathbb{Z}^+$ by mathematical induction.

2 a $f(k + 1) = 13^{k+1} - 6^{k+1} = 13 \times 13^k - 6 \times 6^k$
$= 6(13^k - 6^k) + 7 \times 13^k = 6f(k) + 7(13^k)$

b Basis: $n = 1$: $f(1) = 13 - 6 = 7$ is divisible by 7.
Assumption: $f(k)$ is divisible by 7.
Induction: $f(k + 1) = 6f(k) + 7(13^k)$ by part **a**.
So if the statement holds for $n = k$, it holds for $n = k + 1$.
Conclusion: The statement holds for all $n \in \mathbb{Z}^+$.

3 a $g(k + 1) = 5^{2(k+1)} - 6(k + 1) + 8 = 25 \times 52k - 6k + 2$
$= 25(5^{2k} - 6k + 8) + 144k - 198$
$= 25g(k) + 9(16k - 22)$

b Basis: $n = 1$: $g(1) = 5^2 - 6 + 8 = 27$ is divisible by 9.
Assumption: $g(k)$ is divisible by 9.
Induction: $g(k + 1) = 25\, g(k) + 9(16k - 22)$ by part **a**.
So if the statement holds for $n = k$, it holds for $n = k + 1$.
Conclusion: The statement holds for all $n \in \mathbb{Z}^+$.

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4 $f(n) = 8^n - 3^n$, where $n \in \mathbb{Z}^+$.

$\therefore f(1) = 8^1 - 3^1 = 5$, which is divisible by 5.

$\therefore f(n)$ is divisible by 5 when $n = 1$.

Assume that for $n = k$,

$f(k) = 8^k - 3^k$ is divisible by 5 for $k \in \mathbb{Z}^+$.

$$\begin{aligned}\therefore f(k+1) &= 8^{k+1} - 3^{k+1} \\ &= 8^k.8^1 - 3^k.3^1 \\ &= 8(8^k) - 3(3^k)\end{aligned}$$

$$\begin{aligned}\therefore f(k+1) - 3f(k) &= \left[8(8^k) - 3(3^k)\right] - 3\left[8^k - 3^k\right] \\ &= 8(8^k) - 3(3^k) - 3(8^k) + 3(3^k) \\ &= 5(8^k)\end{aligned}$$

From (a), $f(k+1) = f(k) + 5(8^k)$

As both f(k) and $5(8^k)$ are divisible by 5 then the sum of these two terms must also be divisible by 5. Therefore f(n) is divisible by 5 when $n = k + 1$.

If f(n) is divisible by 5 when $n = k$, then it has been shown that $f(n)$ is also be divisible by 5 when $n = k + 1$. As f(n) is divisible by 5 when $n = 1$, f(n) is also divisible by 5 for all $n \geqslant 1$ and $n \in \mathbb{Z}^+$ by mathematical induction.

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5 $f(n) = 3^{2n+2} + 8n - 9$, where $n \in \mathbb{Z}^+$.

$\therefore f(1) = 3^{2(1)+2} + 8(1) - 9$

$= 3^4 + 8 - 9 = 81 - 1 = 80$, which is divisible by 8.

$\therefore f(n)$ is divisible by 8 when $n = 1$.

Assume that for $n = k$,

$f(k) = 3^{2k+2} + 8k - 9$ is divisible by 8 for $k \in \mathbb{Z}^+$.

$$f(k+1) = 3^{2(k+1)+2} + 8(k+1) - 9$$
$$= 3^{2k+2+2} + 8(k+1) - 9$$
$$= 3^{2k+2}.(3^2) + 8^k + 8 - 9$$
$$= 9(3^{2k+2}) + 8k - 1$$

$$\therefore f(k+1) - f(k) = \left[9(3^{2k+2}) + 8k - 1\right] - \left[3^{2k+2} + 8k - 9\right]$$
$$= 9(3^{2k+2}) + 8k - 1 - 3^{2k+2} - 8k + 9$$
$$= 8(3^{2k+2}) + 8$$
$$= 8\left[3^{2k+2} + 1\right]$$

$$\therefore f(k+1) = f(k) + 8\left[3^{2k+2} + 1\right]$$

As both f(k) and $8\left[3^{2k+2} + 1\right]$ are divisible by 8 then the sum of these two terms must also be divisible by 8. Therefore f(n) is divisible by 8 when $n = k + 1$.

If f(n) is divisible by 8 when $n = k$, then it has been shown that f(n) is also divisible by 8 when $n = k + 1$. As f(n) is divisible by 8 when $n = 1$ f(n) is also divisible by 8 for all $n \geqslant 1$ and $n \in \mathbb{Z}^+$ by mathematical induction.

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6 $f(n) = 2^{6n} + 3^{2n-2}$, where $n \in \mathbb{Z}^+$.

$\therefore f(1) = 2^{6(1)} + 3^{2(1)-2} = 2^6 + 3^0 = 64 + 1 = 65$, which is divisible by 5.

$\therefore$ $f(n)$ is divisible by 5 when $n = 1$.

Assume that for $n = k$.

$f(k) = 2^{6k} + 3^{2k-2}$ is divisible by 5 for $k \in \mathbb{Z}^+$.

$$\begin{aligned}\therefore f(k+1) &= 2^{6(k+1)} + 3^{2(k+1)-2} \\ &= 2^{6k+6} + 3^{2k+2-2} \\ &= 2^6(2^{6k}) + 3^2(3^{2k-2}) \\ &= 64(2^{6k}) + 9(3^{2k-2})\end{aligned}$$

$$\begin{aligned}\therefore f(k+1) - f(k) &= \left[64(2^{6k}) + 9(3^{2k-2})\right] - \left[2^{6k} + 3^{2k-2}\right] \\ &= 64(2^{6k}) + 9(3^{2k-2}) - 2^{6k} - 3^{2k-2} \\ &= 63(2^{6k}) + 8(3^{2k-2}) \\ &= 63(2^{6k}) + 63(3^{2k-2}) - 55(3^{2k-2}) \\ &= 63\left[2^{6k} + 3^{2k-2}\right] - 55(3^{2k-2})\end{aligned}$$

$$\begin{aligned}\therefore f(k+1) &= f(k) + 63\left[2^{6k} + 3^{2k-2}\right] - 55(3^{2k-2}) \\ &= f(k) + 63f(k) - 55(3^{2k-2}) \\ &= 64f(k) - 55(3^{2k-2})\end{aligned}$$

$\therefore f(k+1) = 64f(k) - 55(3^{2k-2})$

As both $64\,f(k)$ and $-55(3^{2k-2})$ are divisible by 5 then the sum of these two terms must also be divisible by 5. Therefore $f(n)$ is divisible by 5 when $n = k + 1$.

If $f(n)$ is divisible by 5 when $n = k$, then it has been shown that $f(n)$ is also divisible by 5 when $n = k + 1$. As $f(n)$ is divisible by 5 when $n = 1$, $f(n)$ is also divisible by 5 for all $n \geqslant 1$ and $n \in \mathbb{Z}^+$ by mathematical induction.

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Proof by induction 8C

1 $n=1;\ \text{LHS}=\begin{pmatrix}1 & 2\\ 0 & 1\end{pmatrix}^1=\begin{pmatrix}1 & 2\\ 0 & 1\end{pmatrix}$

$$\text{RHS}=\begin{pmatrix}1 & 2(1)\\ 0 & 1\end{pmatrix}=\begin{pmatrix}1 & 2\\ 0 & 1\end{pmatrix}$$

As LHS = RHS, the matrix equation is true for $n=1$.

Assume that the matrix equation is true for $n=k$.

i.e. $\begin{pmatrix}1 & 2\\ 0 & 1\end{pmatrix}^k=\begin{pmatrix}1 & 2k\\ 0 & 1\end{pmatrix}$

With $n=k+1$ the matrix equation becomes

$$\begin{pmatrix}1 & 2\\ 0 & 1\end{pmatrix}^{k+1}=\begin{pmatrix}1 & 2\\ 0 & 1\end{pmatrix}^k\begin{pmatrix}1 & 2\\ 0 & 1\end{pmatrix}$$

$$=\begin{pmatrix}1 & 2k\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & 2\\ 0 & 1\end{pmatrix}$$

$$=\begin{pmatrix}1+0 & 2+2k\\ 0+0 & 0+1\end{pmatrix}.$$

$$=\begin{pmatrix}1 & 2(k+1)\\ 0 & 1\end{pmatrix}$$

Therefore the matrix equation is true when $n=k+1$.

If the matrix equation is true for $n=k$, then it is shown to be true for $n=k+1$. As the matrix equation is true for $n=1$, it is now also true for all $n\geqslant 1$ and $n\in\mathbb{Z}^+$, by mathematical induction.

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2 $n=1;\ \text{LHS}=\begin{pmatrix}3 & -4\\ 1 & -1\end{pmatrix}^1=\begin{pmatrix}3 & -4\\ 1 & -1\end{pmatrix}$

$$\text{RHS}=\begin{pmatrix}2(1)+1 & -4(1)\\ 1 & -2(1)+1\end{pmatrix}=\begin{pmatrix}3 & -4\\ 1 & -1\end{pmatrix}$$

As $\text{LHS}=\text{RHS},$ the matrix equation is true for $n=1.$

Assume that the matrix equation is true for $n=k.$

i.e. $\begin{pmatrix}3 & -4\\ 1 & -1\end{pmatrix}^k=\begin{pmatrix}2k+1 & -4k\\ k & -2k+1\end{pmatrix}.$

With $n=k+1$ the matrix equation becomes

$$\begin{pmatrix}3 & -4\\ 1 & -1\end{pmatrix}^{k+1}=\begin{pmatrix}3 & -4\\ 1 & -1\end{pmatrix}^k\begin{pmatrix}3 & -4\\ 1 & -1\end{pmatrix}$$

$$=\begin{pmatrix}2k+1 & -4k\\ k & -2k+1\end{pmatrix}\begin{pmatrix}3 & -4\\ 1 & -1\end{pmatrix}$$

$$=\begin{pmatrix}6k+3-4k & -8k-4+4k\\ 3k-2k+1 & -4k+2k-1\end{pmatrix}$$

$$=\begin{pmatrix}2k+3 & -4k-4\\ k+1 & -2k-1\end{pmatrix}$$

$$=\begin{pmatrix}2(k+1)+1 & -4(k+1)\\ (k+1) & -2(k+1)+1\end{pmatrix}$$

Therefore the matrix equation is true when $n=k+1.$

If the matrix equation is true for $n=k,$ then it is shown to be true for $n=k+1.$ As the matrix equation is true for $n=1,$ it is now also true for all $n\geqslant 1$ and $n\in\mathbb{Z}^+$ by mathematical induction.

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3 $n = 1$; LHS $= \begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix}^1 = \begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix}$

RHS $= \begin{pmatrix} 2^1 & 0 \\ 2^1 - 1 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix}$

As LHS=RHS, the matrix equation is true for $n = 1$.

Assume that the matrix equation is true for $n = k$.

ie. $\begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix}^k = \begin{pmatrix} 2^k & 0 \\ 2^k - 1 & 1 \end{pmatrix}$

With $n = k + 1$ the matrix equation becomes

$$\begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix}^{k+1} = \begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix}^k \begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 2^k & 0 \\ 2^k - 1 & 1 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 2(2^k) + 0 & 0 + 0 \\ 2(2^k) - 2 + 1 & 0 + 1 \end{pmatrix}$$

$$= \begin{pmatrix} 2^{1(2^k)} & 0 \\ 2^{1(2^k)} - 1 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 2^{k+1} & 0 \\ 2^{k+1} - 1 & 1 \end{pmatrix}$$

Therefore the matrix equation is true when $n = k + 1$.

If the matrix equation is true for $n = k$, then it is shown to be true for $n = k + 1$. As the matrix equation is true for $n = 1$, it is now also true for all $n \geqslant 1$ and $n \in \mathbb{Z}^+$ by mathematical induction.

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4 a $n=1$; LHS $= \begin{pmatrix} 5 & -8 \\ 2 & -3 \end{pmatrix}^1 = \begin{pmatrix} 5 & -8 \\ 2 & -3 \end{pmatrix}$

$$\text{RHS} = \begin{pmatrix} 4(1)+1 & -8(1) \\ 2(1) & 1-4(1) \end{pmatrix} = \begin{pmatrix} 5 & -8 \\ 2 & -3 \end{pmatrix}$$

As LHS = RHS, the matrix equation is true for $n = 1$.

Assume that the matrix equation is true for $n = k$.

ie. $\begin{pmatrix} 5 & -8 \\ 2 & -3 \end{pmatrix}^k = \begin{pmatrix} 4k+1 & -8k \\ 2k & 1-4k \end{pmatrix}$.

With $n = k + 1$ the matrix equation becomes

$$\begin{pmatrix} 5 & -8 \\ 2 & -3 \end{pmatrix}^{k+1} = \begin{pmatrix} 5 & -8 \\ 2 & -3 \end{pmatrix}^k \begin{pmatrix} 5 & -8 \\ 2 & -3 \end{pmatrix}$$

$$= \begin{pmatrix} 4k+1 & -8k \\ 2k & 1-4k \end{pmatrix} \begin{pmatrix} 5 & -8 \\ 2 & -3 \end{pmatrix}$$

$$= \begin{pmatrix} 20k+5-16k & -32-8+24k \\ 10k+2-8k & -16k-3+12k \end{pmatrix}$$

$$= \begin{pmatrix} 4k+5 & -8k-8 \\ 2k+2 & -4k-3 \end{pmatrix}$$

$$= \begin{pmatrix} 4(k+1)+1 & -8(k+1) \\ 2(k+1) & 1-4(k+1) \end{pmatrix}$$

Therefore the matrix equation is true when $n = k + 1$.

If the matrix equation is true for $n = k$, then it is shown to be true for $n = k + 1$. As the matrix equation is true for $n = 1$, it is now also true for all $n \geqslant 1$ and $n \in \mathbb{Z}^+$ by mathematical induction.

b n = 6

5 a Basis: $n = 1$: LHS = RHS $= \begin{pmatrix} 2 & 5 \\ 0 & 1 \end{pmatrix}$

Assumption: $\mathbf{M}^k = \begin{pmatrix} 2k & 5(2k-1) \\ 0 & 1 \end{pmatrix}$

Induction: With $n = k + 1$ the matrix equation becomes

$$\mathbf{M}^{k+1} = \mathbf{M}^k \begin{pmatrix} 2 & 5 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 2k & 5(2k-1) \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 5 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 2k+1 & 5\times 2k + 5(2k-1) \\ 0 & 1 \end{pmatrix}$$

So if the statement holds for $n = k$, it holds for $n = k + 1$.

Conclusion: The statement holds for all $n \in \mathbb{Z}^+$.

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5 b $\begin{pmatrix} 2^{-n} & 5\left(2^{-n}-1\right) \\ 0 & 1 \end{pmatrix}$

Challenge

Basis: $n = 1$: LHS = RHS = $\begin{pmatrix} 3 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 4 \end{pmatrix}$

Assumption: $\begin{pmatrix} 3 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 4 \end{pmatrix}^k = \begin{pmatrix} 3k & \frac{3k-1}{2} & 0 \\ 0 & 1 & 0 \\ 0 & \frac{1-4k}{3} & 4k \end{pmatrix}$

Induction:

$$\begin{pmatrix} 3 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 4 \end{pmatrix}^{k+1} = \begin{pmatrix} 3 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 4 \end{pmatrix}^k \begin{pmatrix} 3 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 4 \end{pmatrix}$$

$$= \begin{pmatrix} 3k & \frac{3k-1}{2} & 0 \\ 0 & 1 & 0 \\ 0 & \frac{1-4k}{3} & 4k \end{pmatrix} \begin{pmatrix} 3 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 4 \end{pmatrix}$$

$$= \begin{pmatrix} 3k+1 & 3k+\frac{3k-1}{2} & 0 \\ 0 & 1 & 0 \\ 0 & \frac{1-4k}{3}-4k & 4k\times 4 \end{pmatrix} = \begin{pmatrix} 3k+1 & \frac{3k+1-1}{2} & 0 \\ 0 & 1 & 0 \\ 0 & \frac{1-4k+1}{3} & 4k+1 \end{pmatrix}$$

So if the statement holds for $n = k$, it holds for $n = k + 1$.

If the matrix equation is true for $n = k$, then it is shown to be true for $n = k + 1$. As the matrix equation is true for $n = 1$, it is now also true for all $n \geqslant 1$ and $n \in \mathbb{Z}^+$ by mathematical induction.

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Proof by induction ME 8

1 Let $f(n) = 9^n - 1$, where $n \in \mathbb{Z}^+$.

$\therefore f(1) = 9^1 - 1 = 8$, which is divisible by 8.

$\therefore f(n)$ is divisible by 8 when $n = 1$.

Assume that for $n = k$,

$f(k) = 9^k - 1$ is divisible by 8 for $k \in \mathbb{Z}^+$.

$$\therefore f(k+1) = 9^{k+1} - 1$$
$$= 9^k.9^1 - 1$$
$$= 9(9^k) - 1$$

$$\therefore f(k+1) - f(k) = [9(9^k) - 1] - [9^k - 1]$$
$$= 9(9^k) - 1 - 9^k + 1$$
$$= 8(9^k)$$

$\therefore f(k+1) = f(k) + 8(9^k)$

As both $f(k)$ and $8(9^k)$ are divisible by 8 then the sum of these two terms must also be divisible by 8. Therefore $f(n)$ is divisible by 8 when $n = k + 1$.

If $f(n)$ is divisible by 8 when $n = k$, then it has been shown that $f(n)$ is also divisible by 8 when $n = k + 1$. As $f(n)$ is divisible by 8 when $n = 1$, $f(n)$ is also divisible by 8 for all $n \geqslant 1$ and $n \in \mathbb{Z}^+$ by mathematical induction.

2 a $\mathbf{B}^2 = \mathbf{BB} = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix} = \begin{pmatrix} 1+0 & 0+0 \\ 0+0 & 0+9 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 9 \end{pmatrix}$

$\mathbf{B}^3 = \mathbf{B}^2\mathbf{B} = \begin{pmatrix} 1 & 0 \\ 0 & 9 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix} = \begin{pmatrix} 1+0 & 0+0 \\ 0+0 & 0+27 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 27 \end{pmatrix}$

b As $\mathbf{B}^2 = \begin{pmatrix} 1 & 0 \\ 0 & 3^2 \end{pmatrix}$ and $\mathbf{B}^3 = \begin{pmatrix} 1 & 0 \\ 0 & 3^3 \end{pmatrix}$, we suggest that $\mathbf{B}^n = \begin{pmatrix} 1 & 0 \\ 0 & 3^n \end{pmatrix}$.

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2 c $n = 1$; LHS $= \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}^1 = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}$

RHS $= \begin{pmatrix} 1 & 0 \\ 0 & 3^1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}$

As LHS=RHS, the matrix equation is true for $n = 1$.

Assume that the matrix equation is true for $n = k$.

i.e. $\begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}^k = \begin{pmatrix} 1 & 0 \\ 0 & 3^k \end{pmatrix}$

With $n = k + 1$ the matrix equation becomes

$$\begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}^{k+1} = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}^k \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 3^k \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}$$

$$= \begin{pmatrix} 1+0 & 0+0 \\ 0+0 & 0+3(3^k) \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 3^{k+1} \end{pmatrix}$$

Therefore the matrix is true when $n = k+1$.

If the matrix equation is true for $n = k$, then it is shown to be true for $n = k + 1$. As the matrix equation is true for $n = 1$, it is now also true for all $n \geqslant 1$ and $n \in \mathbb{Z}^+$ by mathematical induction.

3 Basis: $n = 1$: LHS $= 3 \times 1 + 4 = 7$; RHS $= \frac{1}{2} \times 1(3 \times 1 + 11) = 7$

Assumption:

$$\sum_{r=1}^{k} (3r + 4) = \frac{1}{2} k (3k + 11)$$

Induction:

$$\sum_{r=1}^{k+1} (3r + 4) = \sum_{r=1}^{k} (3r + 4) + 3(k + 1) + 4$$

$$= \frac{1}{2} k (3k + 11) + 3(k + 1) + 4 = \frac{1}{2} (3k^2 + 17k + 14)$$

$$= \frac{1}{2} (k + 1)(3(k + 1) + 11)$$

So if the statement holds for $n = k$, it holds for $n = k + 1$.

Conclusion: The statement holds for all $n \in \mathbb{Z}^+$.

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4 a $n = 1; \text{LHS} = \begin{pmatrix} 9 & 16 \\ -4 & -7 \end{pmatrix}^1 = \begin{pmatrix} 9 & 16 \\ -4 & -7 \end{pmatrix}$

$$= \begin{pmatrix} 8(1)+1 & 16(1) \\ -4(1) & 1-8(1) \end{pmatrix} = \begin{pmatrix} 9 & 16 \\ -4 & -7 \end{pmatrix}$$

As LHS = RHS, the matrix equation is true for $n = 1$.

Assume that the matrix equation is true for $n = k$.

i.e. $\begin{pmatrix} 9 & 16 \\ -4 & -7 \end{pmatrix}^k = \begin{pmatrix} 8k+1 & 16k \\ -4k & 1-8k \end{pmatrix}$.

With $n = k + 1$ the matrix equation becomes

$$\begin{pmatrix} 9 & 16 \\ -4 & -7 \end{pmatrix}^{k+1} = \begin{pmatrix} 9 & 16 \\ -4 & -7 \end{pmatrix}^k \begin{pmatrix} 9 & 16 \\ -4 & -7 \end{pmatrix}$$

$$= \begin{pmatrix} 8k+1 & 16k \\ -4k & 1-8k \end{pmatrix} \begin{pmatrix} 9 & 16 \\ -4 & -7 \end{pmatrix}$$

$$= \begin{pmatrix} 72k+9-64k & 128k+16-112k \\ -36k-4+32k & -64k-7+56k \end{pmatrix}$$

$$= \begin{pmatrix} 8k+9 & 16k+16 \\ -4k-4 & -8k-7 \end{pmatrix}$$

$$= \begin{pmatrix} 8(k+1)+1 & 16(k+1) \\ -4(k+1) & 1-8(k+1) \end{pmatrix}$$

Therefore the matrix equation is true when $n = k + 1$.

If the matrix equation is true for $n = k$, then it is shown to be true for $n = k + 1$. As the matrix equation is true for $n = 1$, it is now also true for all $n \geqslant 1$ and $n \in \mathbb{Z}^+$ by mathematical induction.

b $\det(\mathbf{A}^n) = (8n+1)(1-8n) - -64n^2$

$$= 8n - 64n^2 + 1 - 8n + 64n^2$$

$$= 1$$

$$\mathbf{B} = (\mathbf{A}^n)^{-1} = \frac{1}{1}\begin{pmatrix} 1-8n & -16n \\ 4n & 8n+1 \end{pmatrix}$$

$$\text{So } \mathbf{B} = \begin{pmatrix} 1-8n & -16n \\ 4n & 8n+1 \end{pmatrix}$$

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5 a $f(n+1) = 5^{2(n+1)-1} + 1$

$= 5^{2n+2-1} + 1$

$= 5^{2n-1}.5^2 + 1$

$= 24(5^{2n-1}) + 1$

$\therefore f(n+1) - f(n) = [25(5^{2n-1}) + 1] - [5^{2n-1} + 1]$

$= 25(5^{2n-1}) + 1 - (5^{2n-1}) - 1$

$= 24(5^{2n-1})$

Therefore, $\mu = 24$.

b $f(n) = 5^{2n-1} + 1$, where $n \in \mathbb{Z}^+$.

$\therefore f(1) = 5^{2(1)-1} + 1 = 5^1 + 1 = 6$, which is divisible by 6.

$\therefore$ f(n) is divisible by 6 when $n = 1$.

Assume that for $n = k$,

$f(k) = 5^{2k-1} + 1$ is divisible by 6 for $k \in \mathbb{Z}^+$.

Using (a) $f(k+1) - f(k) = 24(5^{2k-1})$

$\therefore f(k+1) = f(k) + 24(5^{2k-1})$

As both f(k) and $24(5^{2k-1})$ are divisible by 6 then the sum of these two terms must also be divisible by 6. Therefore f(n) is divisible by 6 when $n = k + 1$.

If f(n) is divisible by 6 when $n = k$, then it has been shown that f(n) is also divisible by 6 when $n = k + 1$. As f(n) is divisible by 6 when $n = 1$n, f(n) is also divisible by 6 for all $n \geqslant 1$ and $n \in \mathbb{Z}^+$ by mathematical induction.

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6 Let $f(n) = 7^n + 4^n + 1$, where $n \in \mathbb{Z}^+$.

$\therefore f(1) = 7^1 + 4^1 + 1 = 7 + 4 + 1 = 12$, which is divisible by 6.

$\therefore$ f(n) is divisible by 6 when $n = 1$.

Assume that for $n = k$,

$f(k) = 7^k + 4^k + 1$ is divisible by 6 for $k \in \mathbb{Z}^+$.

$$\begin{aligned}\therefore f(k+1) &= 7^{k+1} + 4^{k+1} + 1 \\ &= 7^k.7^1 + 4^k.4^1 + 1 \\ &= 7(7^k) + 4(4^k) + 1\end{aligned}$$

$$\begin{aligned}\therefore f(k+1) - f(k) &= [7(7^k) + 4(4^k) + 1] - [7^k + 4^k + 1] \\ &= 7(7^k) + 4(4^k) + 1 - 7^k - 4^k - 1 \\ &= 6(7^k) + 3(4^k)\end{aligned}$$

Where both $6(7^k)$ and $3(4^k)$ are divisible by 6 since 4 is even.

Therefore f(n) is divisible by 6 when $n = k + 1$.

If f(n) is divisible by 6 when $n = k$, then it has been shown that f(n) is also divisible by 6 when $n = k + 1$. As f(n) is divisible by 6 when $n = 1$, f(n) is also divisible by 6 for all $n \geqslant 1$ and $n \in \mathbb{Z}^+$ by mathematical induction.

7 Basis: $n = 1$: LHS = $1 \times 5 = 5$; RHS = $\frac{1}{6} \times 1 \times 2 \times (2 + 13) = 5$

Assumption:

$$\sum_{r=1}^{k} r(r+4) = \frac{1}{6}k(k+1)(2k+13)$$

Induction:

$$\begin{aligned}\sum_{r=1}^{k+1} r(r+4) &= \sum_{r=1}^{k} r(r+4) + (k+1)(k+5) \\ &= \frac{1}{6}k(k+1)(2k+13) + (k+1)(k+5) \\ &= \frac{1}{6}(2k^3 + 21k^2 + 49k + 30) = \frac{1}{6}(k+1)(k+2)(2(k+1)+13)\end{aligned}$$

So if the statement holds for $n = k$, it holds for $n = k + 1$.

Conclusion: The statement holds for all $n \in \mathbb{Z}^+$.

8 a Basis: $n = 1$: LHS = $1 + 4 = 5$; RHS = $\frac{1}{3} \times 1 \times 3 \times 5 = 5$

Assumption:

$$\sum_{r=1}^{2k} r^2 = \frac{1}{3}k(2k+1)(4k+1)$$

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Induction:

$$\sum_{r=1}^{2(k+1)} r^2 = \sum_{r=1}^{2k} r^2 + (2k+1)^2 + (2k+2)^2$$
$$= \frac{1}{3}k(2k+1)(4k+1) + (2k+1)^2 + (2k+2)^2$$
$$= \frac{1}{3}(8k^3 + 30k^2 + 37k + 15)$$
$$= \frac{1}{3}(k+1)(2(k+1)+1)(4(k+1)+1)$$

So if the statement holds for $n = k$, it holds for $n = k + 1$.

Conclusion: The statement holds for all $n \in \mathbb{Z}^+$.

b Using **a** and the formula for $\sum_{r=1}^{n} r^2$

$$\frac{1}{6} \times 2n(2n+1)(4n+1) = \frac{1}{6}kn(n+1)(2n+1)$$
$$2n(2n + 1) + (4n + 1) = kn(n + 1)(2n + 1)$$
$$16n^3 + 12n^3 + 12n^2 + 2n = k(2n^3 + 3n^2)$$
$$\Rightarrow k = \frac{2n(8n^2 + 6n + 1)}{n(2n^2 + 3n + 1)} = \frac{2(2n+1)(4n+1)}{(2n+1)(n+1)} = \frac{8n+2}{n+1}$$
$$\Rightarrow kn + k = 8n + 2 \Rightarrow n(k-8) = 2 - k \Rightarrow n = \frac{2-k}{k-8}$$

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9 a Basis: $n = 1$: LHS = RHS = $\begin{pmatrix} 2c & 1 \\ 0 & c \end{pmatrix}$

Assumption:

$$\mathbf{M}^k = \mathbf{c}^k \begin{pmatrix} 2^k & \dfrac{2^k - 1}{c} \\ 0 & 1 \end{pmatrix}$$

Induction:

$$\mathbf{M}^{k+1} = \mathbf{M}^k \begin{pmatrix} 2c & 1 \\ 0 & c \end{pmatrix}$$

$$= c^k \begin{pmatrix} 2^k & \dfrac{2^k - 1}{c} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 2c & 1 \\ 0 & c \end{pmatrix} = c^{k+1} \begin{pmatrix} 2^k & \dfrac{2^k - 1}{c} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & \dfrac{1}{c} \\ 0 & 1 \end{pmatrix}$$

$$= c^{k+1} \begin{pmatrix} 2^{k+1} & \dfrac{2^k + 2^k - 1}{c} \\ 0 & 1 \end{pmatrix} = c^{k+1} \begin{pmatrix} 2^{k+1} & \dfrac{2^{k+1} - 1}{c} \\ 0 & 1 \end{pmatrix}$$

So if the statement holds for $n = k$, it holds for $n = k + 1$.

Conclusion: The statement holds for all $n \in \mathbb{Z}^+$.

b Consider $n = 1$: $\det \mathbf{M} = 50 \Rightarrow 2c^2 = 50$
So $c = 5$, since c is +ve.

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Challenge

a Basis: $n = 1$: LHS = RHS = $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$

Assumption:

$\mathbf{M}^k = \begin{pmatrix} \cos k\theta & -\sin k\theta \\ \sin k\theta & \cos k\theta \end{pmatrix}$

Induction:

$\mathbf{M}^{k+1} = \mathbf{M}^k \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} = \begin{pmatrix} \cos k\theta & -\sin k\theta \\ \sin k\theta & \cos k\theta \end{pmatrix} \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$

$= \begin{pmatrix} \cos k\theta\cos\theta & -\sin k\theta\sin\theta & -\cos\theta\sin\theta & -\sin\theta\cos\theta \\ \sin k\theta\cos\theta & +\cos k\theta\sin\theta & -\sin k\theta\sin\theta & +\cos k\theta\cos\theta \end{pmatrix}$

$= \begin{pmatrix} \cos\big((k+1)\theta\big) & -\sin\big((k+1)\theta\big) \\ \sin\big((k+1)\theta\big) & \cos\big((k+1)\theta\big) \end{pmatrix}$

So if the statement holds for $n = k$, it holds for $n = k + 1$.

Conclusion: The statement holds for all $n \in \mathbb{Z}^+$.

b The matrix $\mathbf{M}$ represents a rotation through angle θ, and so $\mathbf{M}^n$ represents a rotation through angle $n\theta$.

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Exercise 9A

1 a $\mathbf{r} = \begin{pmatrix} 6 \\ 5 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -3 \\ -1 \end{pmatrix}$

b $\mathbf{r} = \begin{pmatrix} 2 \\ 5 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

c $\mathbf{r} = \begin{pmatrix} -7 \\ 6 \\ 2 \end{pmatrix} + \lambda = \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix}$

d $\mathbf{r} = \begin{pmatrix} 2 \\ 0 \\ 4 \end{pmatrix} + \lambda \begin{pmatrix} -3 \\ 2 \\ 1 \end{pmatrix}$

e $\mathbf{r} = \begin{pmatrix} 6 \\ -11 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 0 \\ 5 \\ -2 \end{pmatrix}$

2 a i $\overrightarrow{PQ} = (5-3)\mathbf{i} + (3-(-4))\mathbf{j} + (-1-2)\mathbf{k}$
$= 2\mathbf{i} + 7\mathbf{j} - 3\mathbf{k}$

ii Since the line passes through P, an equation of the line is
$\mathbf{r} = (3\mathbf{i} - 4\mathbf{j} + 2\mathbf{k}) + \lambda(2\mathbf{i} + 7\mathbf{j} - 3\mathbf{k})$

b i $\overrightarrow{PQ} = (4-2)\mathbf{i} + (-2-1)\mathbf{j} + (1-(-3))\mathbf{k}$
$= 2\mathbf{i} - 3\mathbf{j} + 4\mathbf{k}$

ii $\mathbf{r} = (2\mathbf{i} + \mathbf{j} - 3\mathbf{k}) + \lambda(2\mathbf{i} - 3\mathbf{j} + 4\mathbf{k})$

c i $\overrightarrow{PQ} = (-2-1)\mathbf{i} + (-3-(-2))\mathbf{j} + (2-4)\mathbf{k}$
$= -3\mathbf{i} - \mathbf{j} - 2\mathbf{k}$

ii $\mathbf{r} = (\mathbf{i} - 2\mathbf{j} + 4\mathbf{k}) + \lambda(-3\mathbf{i} - \mathbf{j} - 2\mathbf{k})$

d i $\overrightarrow{PQ} = \begin{pmatrix} -2 \\ 3 \\ 1 \end{pmatrix} - \begin{pmatrix} 3 \\ -1 \\ 4 \end{pmatrix} = \begin{pmatrix} -5 \\ 4 \\ -3 \end{pmatrix}$

ii $\mathbf{r} = \begin{pmatrix} 3 \\ -1 \\ 4 \end{pmatrix} + \lambda \begin{pmatrix} -5 \\ 4 \\ -3 \end{pmatrix}$

e i $\overrightarrow{PQ} = \begin{pmatrix} -2 \\ 2 \\ 4 \end{pmatrix} - \begin{pmatrix} 4 \\ -2 \\ 3 \end{pmatrix} = \begin{pmatrix} -6 \\ 4 \\ 1 \end{pmatrix}$

2 e ii $\mathbf{r} = \begin{pmatrix} 4 \\ -2 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} -6 \\ 4 \\ 1 \end{pmatrix}$

3 Vector $\mathbf{k} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ is in the direction of the z-axis.

The point $(4, -3, 8)$ has position vector $\begin{pmatrix} 4 \\ -3 \\ 8 \end{pmatrix}$

The equation of the line is

$\mathbf{r} = \begin{pmatrix} 4 \\ -3 \\ 8 \end{pmatrix} + \lambda \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

4 a i $\mathbf{a} = \begin{pmatrix} 2 \\ 1 \\ 9 \end{pmatrix}, \mathbf{b} = \begin{pmatrix} 4 \\ -1 \\ 8 \end{pmatrix}$

$\mathbf{b} - \mathbf{a} = \begin{pmatrix} 4 \\ -1 \\ 8 \end{pmatrix} - \begin{pmatrix} 2 \\ 1 \\ 9 \end{pmatrix} = \begin{pmatrix} 2 \\ -2 \\ -1 \end{pmatrix}$

Equation is

$\mathbf{r} = \begin{pmatrix} 2 \\ 1 \\ 9 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -2 \\ -1 \end{pmatrix}$

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4 a ii $\mathbf{a}=\begin{pmatrix}-3\\5\\0\end{pmatrix}, \mathbf{b}=\begin{pmatrix}7\\2\\2\end{pmatrix}$

$\mathbf{b}-\mathbf{a}=\begin{pmatrix}7\\2\\2\end{pmatrix}-\begin{pmatrix}-3\\5\\0\end{pmatrix}=\begin{pmatrix}10\\-3\\2\end{pmatrix}$

Equation is

$\mathbf{r}=\begin{pmatrix}-3\\5\\0\end{pmatrix}+\lambda\begin{pmatrix}10\\-3\\2\end{pmatrix}$

iii $\mathbf{a}=\begin{pmatrix}1\\11\\-4\end{pmatrix}, \mathbf{b}=\begin{pmatrix}5\\9\\2\end{pmatrix}$

$\mathbf{b}-\mathbf{a}=\begin{pmatrix}5\\9\\2\end{pmatrix}-\begin{pmatrix}1\\11\\-4\end{pmatrix}=\begin{pmatrix}4\\-2\\6\end{pmatrix}$

Equation is

$\mathbf{r}=\begin{pmatrix}1\\11\\-4\end{pmatrix}+\lambda\begin{pmatrix}4\\-2\\6\end{pmatrix}$

iv $\mathbf{a}=\begin{pmatrix}-2\\-3\\-7\end{pmatrix}, \mathbf{b}=\begin{pmatrix}12\\4\\-3\end{pmatrix}$

$\mathbf{b}-\mathbf{a}=\begin{pmatrix}12\\4\\-3\end{pmatrix}-\begin{pmatrix}-2\\-3\\-7\end{pmatrix}=\begin{pmatrix}14\\7\\4\end{pmatrix}$

Equation is

$\mathbf{r}=\begin{pmatrix}-2\\-3\\-7\end{pmatrix}+\lambda\begin{pmatrix}14\\7\\4\end{pmatrix}$

4 b i $\dfrac{x-2}{2}=\dfrac{y-1}{-2}=\dfrac{z-9}{-1}$

ii $\dfrac{x+3}{10}=\dfrac{y-5}{-3}=\dfrac{z}{2}$

4 b iii $\dfrac{x-1}{4}=\dfrac{y-11}{-2}=\dfrac{z+4}{6}$

iv $\dfrac{x+2}{14}=\dfrac{y+3}{7}=\dfrac{z+7}{4}$

5 a $\begin{pmatrix}2\\-3\\1\end{pmatrix}+\lambda\begin{pmatrix}1\\-4\\-9\end{pmatrix}=\begin{pmatrix}1\\p\\q\end{pmatrix}$

i component: $2+\lambda=1\Rightarrow\lambda=-1$

j component: $-3+(-1)(-4)=p\Rightarrow p=1$

k component: $1+(-1)(-9)=q\Rightarrow q=10$

b $\begin{pmatrix}-4\\6\\-1\end{pmatrix}+\lambda\begin{pmatrix}2\\-5\\-8\end{pmatrix}=\begin{pmatrix}1\\p\\q\end{pmatrix}$

i component: $-4+2\lambda=1\Rightarrow\lambda=\dfrac{5}{2}$

j component: $6+\left(\dfrac{5}{2}\right)(-5)=p\Rightarrow p=-\dfrac{13}{2}$

$=-6\dfrac{1}{2}$

k component: $-1+\left(\dfrac{5}{2}\right)(-8)=q\Rightarrow q=-21$

c $\begin{pmatrix}16\\-9\\-10\end{pmatrix}+\lambda\begin{pmatrix}3\\2\\1\end{pmatrix}=\begin{pmatrix}1\\p\\q\end{pmatrix}$

i component: $16+3\lambda=1\Rightarrow\lambda=-5$

j component: $-9+(-5)(2)=p\Rightarrow p=-19$

k component: $-10+(-5)(1)=q\Rightarrow q=-15$

6 Direction of l_1: $\begin{pmatrix}-1\\2\\4\end{pmatrix}$, direction of l_2: $\begin{pmatrix}2\\-4\\-8\end{pmatrix}$

$=-2\begin{pmatrix}-1\\2\\4\end{pmatrix}$, so parallel

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7 $l_1 : \mathbf{r} = \begin{pmatrix} 3+2\lambda \\ 2-3\lambda \\ -1+4\lambda \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -3 \\ 4 \end{pmatrix}$

So the direction vector of l_1 is $\begin{pmatrix} 2 \\ -3 \\ 4 \end{pmatrix}$

l_2 has direction vector

$$\overrightarrow{BA} = \overrightarrow{OA} - \overrightarrow{OB} = \begin{pmatrix} 5 \\ 4 \\ -1 \end{pmatrix} - \begin{pmatrix} 3 \\ 7 \\ -5 \end{pmatrix} = \begin{pmatrix} 2 \\ -3 \\ 4 \end{pmatrix}$$

Both direction vectors are the same, so line l_1 is parallel to l_2

8 $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix} - \begin{pmatrix} -3 \\ -4 \\ 5 \end{pmatrix} = \begin{pmatrix} 6 \\ 3 \\ -3 \end{pmatrix}$

$\overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA} = \begin{pmatrix} 9 \\ 2 \\ -1 \end{pmatrix} - \begin{pmatrix} -3 \\ -4 \\ 5 \end{pmatrix} = \begin{pmatrix} 12 \\ 6 \\ -6 \end{pmatrix}$

So $\overrightarrow{AC} = 2\overrightarrow{AB}$

So AC is parallel to AB and there is a common point A in both.

Therefore A, B and C are collinear.

9 $\begin{pmatrix} 3 \\ -1 \\ 8 \end{pmatrix} - \begin{pmatrix} 1 \\ 7 \\ -2 \end{pmatrix} = \begin{pmatrix} 2 \\ -8 \\ 10 \end{pmatrix} = 2\begin{pmatrix} 1 \\ -4 \\ 5 \end{pmatrix}$

$\begin{pmatrix} 10 \\ 4 \\ 0 \end{pmatrix} - \begin{pmatrix} 1 \\ 7 \\ -2 \end{pmatrix} = \begin{pmatrix} 9 \\ -3 \\ 2 \end{pmatrix}$

So not collinear

10 Since P,Q and R are collinear, the vectors $\overrightarrow{PQ}$ and $\overrightarrow{QR}$ are parallel.

$$\overrightarrow{PQ} = \begin{pmatrix} a \\ 5 \\ 1 \end{pmatrix} - \begin{pmatrix} 2 \\ 0 \\ 4 \end{pmatrix} = \begin{pmatrix} a-2 \\ 5 \\ -3 \end{pmatrix}$$

$$\overrightarrow{QR} = \begin{pmatrix} 3 \\ 10 \\ b \end{pmatrix} - \begin{pmatrix} a \\ 5 \\ 1 \end{pmatrix} = \begin{pmatrix} 3-a \\ 5 \\ b-1 \end{pmatrix}$$

$\therefore\ a-2 = 3-a \Rightarrow a = \frac{5}{2}$

$\therefore\ -3 = b-1 \Rightarrow b = -2$

So not collinear

11 Let $\mathbf{a}$ denote the position vector of A

$\therefore\ \mathbf{a} = (8\mathbf{i} - 5\mathbf{j} + 4\mathbf{k}) + (-2)(3\mathbf{i} + \mathbf{j} - 6\mathbf{k})$

$= 2\mathbf{i} - 7\mathbf{j} + 16\mathbf{k}$

A line parallel to l_2 has the same direction vector: $2\mathbf{i} - 4\mathbf{j} + \mathbf{k}$

So the equation of a line through A and parallel to l_2 is

$\mathbf{r} = (2\mathbf{i} - 7\mathbf{j} + 16\mathbf{k}) + \lambda(2\mathbf{i} - 4\mathbf{j} + \mathbf{k})$

12 a $(10\mathbf{i} + 8\mathbf{j} - 12\mathbf{k}) + \lambda(\mathbf{i} - \mathbf{j} + b\mathbf{k}) = 4\mathbf{i} + a\mathbf{j}$

$\mathbf{i}$ component: $10 + \lambda = 4 \Rightarrow \lambda = -6$

$\mathbf{j}$ component: $8 + (-6)(-1) = a \Rightarrow a = 14$

$\mathbf{k}$ component: $-12 + (-6)(b) = 0 \Rightarrow b = -2$

b Substituting $\lambda = -1$:

$\mathbf{r} = (10\mathbf{i} + 8\mathbf{j} - 12\mathbf{k}) + (-1)(\mathbf{i} - \mathbf{j} - 2\mathbf{k})$

$= 9\mathbf{i} + 9\mathbf{j} - 10\mathbf{k}$

Therefore, X has coordinates $(9, 9, -10)$

13 Find the coordinates of A and B:

$\lambda = 5:\ \mathbf{a} = \begin{pmatrix} 3 \\ -5 \\ 9 \end{pmatrix} + 5\begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix} = \begin{pmatrix} 8 \\ 5 \\ -1 \end{pmatrix}$

$\lambda = 2:\ \mathbf{b} = \begin{pmatrix} 3 \\ -5 \\ 9 \end{pmatrix} + 2\begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix} = \begin{pmatrix} 5 \\ -1 \\ 5 \end{pmatrix}$

$\therefore\ \overrightarrow{AB} = \mathbf{b} - \mathbf{a} = \begin{pmatrix} 5 \\ -1 \\ 5 \end{pmatrix} - \begin{pmatrix} 8 \\ 5 \\ -1 \end{pmatrix} = \begin{pmatrix} -3 \\ -6 \\ 6 \end{pmatrix}$

$\therefore\ |\overrightarrow{AB}| = \sqrt{(-3)^2 + (-6)^2 + (6)^2} = 9$

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14 Find the coordinates of C and A:

$\lambda = 4$: $\mathbf{c} = \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix} + 4\begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 9 \\ 2 \\ -1 \end{pmatrix}$

$\lambda = 3$: $\mathbf{a} = \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix} + 3\begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 7 \\ 1 \\ 0 \end{pmatrix}$

$\therefore \overrightarrow{AC} = \mathbf{c} - \mathbf{a} = \begin{pmatrix} 9 \\ 2 \\ -1 \end{pmatrix} - \begin{pmatrix} 7 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}$

$\therefore \mathbf{b} = \mathbf{c} + \overrightarrow{AC} = \begin{pmatrix} 9 \\ 2 \\ -1 \end{pmatrix} + \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 11 \\ 3 \\ -2 \end{pmatrix}$

So B has position vector $\begin{pmatrix} 11 \\ 3 \\ -2 \end{pmatrix}$

15 A vector equation for l is

$\mathbf{r} = \begin{pmatrix} 5 \\ -1 \\ 6 \end{pmatrix} + \lambda\begin{pmatrix} 1 \\ 3 \\ -2 \end{pmatrix}$

Let $\mathbf{c} = \begin{pmatrix} 4 \\ -1 \\ 2 \end{pmatrix}$

$\mathbf{c} - \mathbf{r} = \begin{pmatrix} 4 \\ -1 \\ 2 \end{pmatrix} - \begin{pmatrix} 5 \\ -1 \\ 6 \end{pmatrix} - \lambda\begin{pmatrix} 1 \\ 3 \\ -2 \end{pmatrix} = \begin{pmatrix} -1-\lambda \\ -3\lambda \\ -4+2\lambda \end{pmatrix}$

Since it is given that C intersects l at two distinct points, the equation $|\mathbf{c} - \mathbf{r}| = 3\sqrt{5}$ has two solutions.

$\therefore \sqrt{(-1-\lambda)^2 + (-3\lambda)^2 + (-4+2\lambda)^2} = 3\sqrt{5}$

$\Rightarrow \sqrt{14\lambda^2 - 14\lambda + 17} = 3\sqrt{5}$

$\Rightarrow 14\lambda^2 - 14\lambda + 17 = 45$

$\Rightarrow \lambda^2 - \lambda - 2 = 0$

$\Rightarrow \lambda = -1$ or $\lambda = 2$

Therefore, substituting these values of λ into the vector equation, the coordinates are $(4, -4, 8)$ and $(7, 5, 2)$

16 a $\lambda = 2$: $\mathbf{r} = \begin{pmatrix} -4 \\ 6 \\ 5 \end{pmatrix} + 2\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} -2 \\ 4 \\ 7 \end{pmatrix}$

$\lambda = 5$: $\mathbf{r} = \begin{pmatrix} -4 \\ 6 \\ 5 \end{pmatrix} + 5\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 10 \end{pmatrix}$

$\therefore A\begin{pmatrix} -2 \\ 4 \\ 7 \end{pmatrix}$ and $B\begin{pmatrix} 1 \\ 1 \\ 10 \end{pmatrix}$

b Since l_2 is parallel to l_1, they have the same direction vector $\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$. Therefore a vector equation of l_2 is $\mathbf{r} = \begin{pmatrix} 0 \\ 2 \\ 3 \end{pmatrix} + \lambda\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$

c $|\overrightarrow{AB}| = \sqrt{(1-(-2))^2 + (1-4)^2 + (10-7)^2}$

$= \sqrt{27}$

$\therefore |\overrightarrow{AC}| = |\overrightarrow{AD}| = \sqrt{27}$

$\begin{pmatrix} 0 \\ 2 \\ 3 \end{pmatrix} + \lambda\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} - \begin{pmatrix} -2 \\ 4 \\ 7 \end{pmatrix} = \begin{pmatrix} 2+\lambda \\ -2-\lambda \\ -4+\lambda \end{pmatrix}$

$\therefore \sqrt{(2+\lambda)^2 + (-2-\lambda)^2 + (-4+\lambda)^2} = \sqrt{27}$

$\Rightarrow \sqrt{3\lambda^2 + 24} = \sqrt{27}$

$\Rightarrow 3\lambda^2 + 24 = 27$

$\Rightarrow \lambda^2 = 1 \Rightarrow \lambda = \pm 1$

$\therefore C\begin{pmatrix} 1 \\ 1 \\ 4 \end{pmatrix}$ and $D\begin{pmatrix} -1 \\ 3 \\ 2 \end{pmatrix}$

$\Rightarrow \overrightarrow{CD} = \begin{pmatrix} -2 \\ 2 \\ -2 \end{pmatrix}$

$\begin{pmatrix} 0 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 4 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} -2 \\ 2 \\ -2 \end{pmatrix} = \overrightarrow{OC} + \frac{1}{2}\overrightarrow{CD}$

i.e. P is the midpoint of CD

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17 a A vector equation for the tightrope is

$$\mathbf{r} = \begin{pmatrix} 2 \\ 3 \\ 8 \end{pmatrix} + \lambda \begin{pmatrix} 20 \\ 15 \\ 0 \end{pmatrix}$$

Let $B(14,1,0)$ and $C(6,17,0)$

$$\therefore \overrightarrow{BA} = \begin{pmatrix} -12+20\lambda \\ 2+15\lambda \\ 8 \end{pmatrix}, \ \overrightarrow{CA} = \begin{pmatrix} -4+20\lambda \\ -14+15\lambda \\ 8 \end{pmatrix}$$

$$|\overrightarrow{BA}| = \sqrt{(20\lambda-12)^2+(15\lambda+2)^2+8^2} = 12$$

$$\Rightarrow 625\lambda^2 - 420\lambda + 68 = 0$$

$$\Rightarrow \lambda = \frac{2}{5} \quad \text{or} \quad \lambda = \frac{34}{125}$$

$$|\overrightarrow{CA}| = \sqrt{(20\lambda-4)^2+(15\lambda-14)^2+8^2} = 12$$

$$\Rightarrow 625\lambda^2 - 580\lambda + 132 = 0$$

$$\Rightarrow \lambda = \frac{2}{5} \quad \text{or} \quad \lambda = \frac{66}{125}$$

$$\therefore \lambda = \frac{2}{5}$$

Therefore the position vector of A is

$$\mathbf{a} = \begin{pmatrix} 2 \\ 3 \\ 8 \end{pmatrix} + \frac{2}{5}\begin{pmatrix} 20 \\ 15 \\ 0 \end{pmatrix} = \begin{pmatrix} 10 \\ 9 \\ 8 \end{pmatrix}$$

$$\therefore A(10,9,8)$$

b Tightrope will bow in middle due to acrobat's weight.

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Exercise 9B

1 a Let

$\mathbf{a} = \mathbf{i} + 2\mathbf{j}, \mathbf{b} = (3\mathbf{i} + \mathbf{j} - \mathbf{k}) - (\mathbf{i} + 2\mathbf{j})$

$= 2\mathbf{i} - \mathbf{j} - \mathbf{k}$

and

$\mathbf{c} = (4\mathbf{i} + 3\mathbf{j} + 3\mathbf{k}) - (\mathbf{i} + 2\mathbf{j})$

$= 3\mathbf{i} + \mathbf{j} + 2\mathbf{k}$

$\therefore \mathbf{r} = \mathbf{i} + 2\mathbf{j} + \lambda(2\mathbf{i} - \mathbf{j} - \mathbf{k}) + \mu(3\mathbf{i} + \mathbf{j} + 2\mathbf{k})$

b Let

$\mathbf{a} = 3\mathbf{i} + 4\mathbf{j} + \mathbf{k}, \mathbf{b}$

$= -\mathbf{i} - 2\mathbf{j} - (3\mathbf{i} + 4\mathbf{j} + \mathbf{k})$

$= -4\mathbf{i} - 6\mathbf{j} - \mathbf{k}$

and

$\mathbf{c} = 2\mathbf{i} + \mathbf{j} + 4\mathbf{k} - (3\mathbf{i} + 4\mathbf{j} + \mathbf{k})$

$= -\mathbf{i} - 3\mathbf{j} + 3\mathbf{k}$

$\therefore \mathbf{r} = 3\mathbf{i} + 4\mathbf{j} + \mathbf{k} + \lambda(-4\mathbf{i} - 6\mathbf{j} - \mathbf{k}) + \mu(-\mathbf{i} - 3\mathbf{j} + 3\mathbf{k})$

or $\mathbf{r} = 3\mathbf{i} + 4\mathbf{j} + \mathbf{k} + \lambda'(4\mathbf{i} + 6\mathbf{j} + \mathbf{k}) + \mu(-\mathbf{i} - 3\mathbf{j} + 3\mathbf{k})$

c Let

$\mathbf{a} = 2\mathbf{i} - \mathbf{j} - \mathbf{k}, \mathbf{b}$

$= 3\mathbf{i} + \mathbf{j} + 2\mathbf{k} - (2\mathbf{i} - \mathbf{j} - \mathbf{k})$

$= \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$

and $\mathbf{c} = 4\mathbf{i} + \mathbf{k} - (2\mathbf{i} - \mathbf{j} - \mathbf{k}) = 2\mathbf{i} + \mathbf{j} + 2\mathbf{k}$

$\therefore \mathbf{r} = 2\mathbf{i} - \mathbf{j} - \mathbf{k} + \lambda(\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}) + \mu(2\mathbf{i} + \mathbf{j} + 2\mathbf{k})$

d Let

$\mathbf{a} = -\mathbf{i} + \mathbf{j} + 3\mathbf{k}, \mathbf{b}$

$= -\mathbf{i} + 2\mathbf{j} + 5\mathbf{k} - (-\mathbf{i} + \mathbf{j} + 3\mathbf{k})$

$= \mathbf{j} + 2\mathbf{k}$

and

$\mathbf{c} = 4\mathbf{j} + 4\mathbf{k} - (-\mathbf{i} + \mathbf{j} + 3\mathbf{k})$

$= \mathbf{i} + 3\mathbf{j} + \mathbf{k}$

$\therefore \mathbf{r} = -\mathbf{i} + \mathbf{j} + 3\mathbf{k} + \lambda(\mathbf{j} + 2\mathbf{k}) + \mu(\mathbf{i} + 3\mathbf{j} + \mathbf{k})$

2 The plane has normal vector $\begin{pmatrix} -1 \\ 3 \\ 2 \end{pmatrix}$ so a cartesian equation of the plane takes takes the form

$-x + 3y + 2z = d$

The plane contains the point $(4, -2, 6)$ so

$-(4) + 3(-2) + 2(6) = d \Rightarrow d = 2$

$\therefore \quad -x + 3y + 2z = 2$

3 a $\begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ 2 \\ -2 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix} = \begin{pmatrix} 7 \\ -1 \\ k \end{pmatrix}$

i component: $2 + 3\lambda + \mu$

$= 7 \Rightarrow 3\lambda + \mu = 5$

j component: $-1 + 2\lambda - \mu$

$= -1 \Rightarrow 2\lambda - \mu = 0$

Solving these simultaneous equations:

$\lambda = 1, \ \mu = 2$

k component: $3 + 1(-2) + 2(3)$

$= k \Rightarrow k = 7$

b $\begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ 2 \\ -2 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 \\ k \\ 11 \end{pmatrix}$

i component: $2 + 3\lambda + \mu = 1$

$\Rightarrow 3\lambda + \mu = -1$

k component: $3 - 2\lambda + 3\mu = 11$

$\Rightarrow -2\lambda + 3\mu = 8$

Solving these simultaneous equations:

$\lambda = -1, \ \mu = 2$

j component: $-1 - 2 + 2(-1) = k \Rightarrow k = -5$

c $\begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ 2 \\ -2 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix} = \begin{pmatrix} k \\ -4 \\ 10 \end{pmatrix}$

j component: $-1 + 2\lambda - \mu = -4$

$\Rightarrow 2\lambda - \mu = -3$

k component: $3 - 2\lambda + 3\mu = 10$

$\Rightarrow -2\lambda + 3\mu = 7$

Solving these simultaneous equations:

$\lambda = -\frac{1}{2}, \ \mu = 2$

i component: $2 - \frac{1}{2}(3) + 2(1) = k \Rightarrow k = \frac{5}{2}$

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3 d $\begin{pmatrix}2\\-1\\3\end{pmatrix}+\lambda\begin{pmatrix}3\\2\\-2\end{pmatrix}+\mu\begin{pmatrix}1\\-1\\3\end{pmatrix}=\begin{pmatrix}10\\k\\-k\end{pmatrix}$

j component: $-1+2\lambda-\mu=k$

$\Rightarrow 2\lambda-\mu=k+1$

k component: $3-2\lambda+3\mu=-k$

$\Rightarrow -2\lambda+3\mu=-k-3$

Adding these two equations together:

$2\mu=-2\Rightarrow\mu=-1$

i component: $2+3\lambda+(-1)=10$

$\Rightarrow\lambda=3$

j component: $-1+3(2)+(-1)(-1)=k$

$\Rightarrow k=6$

4 a i $2(1)-3(2)+5(1)=1$

ii $2(2)-3(-4)+5(-3)=1$

b $\mathbf{n}=2\mathbf{i}-3\mathbf{j}+5\mathbf{k}$

5 a l is normal to the plane and passes through $(2,3,-2)$ so an equation for l is

$$\mathbf{r}=\begin{pmatrix}2\\3\\-2\end{pmatrix}+\lambda\begin{pmatrix}5\\-3\\-4\end{pmatrix}$$

b $\mathbf{r}=\begin{pmatrix}2\\3\\-2\end{pmatrix}+\lambda\begin{pmatrix}5\\-3\\-4\end{pmatrix}$

$$\Rightarrow\frac{x-2}{5}=\frac{y-3}{-3}=\frac{z+2}{-4}$$

6 $x=0,\ 0\le y,z\le 3$

$x=3,\ 0\le y,z\le 3$

$y=0,\ 0\le x,z\le 3$

$y=3,\ 0\le x,z\le 3$

$z=0,\ 0\le x,y\le 3$

$z=3,\ 0\le x,y\le 3$

7 Find the plane containing $A(2,2,3)$ $B(1,5,3)$ and $C(4,3,-1)$, then show that $D(3,6,-1)$ lies in this plane.

$$\overrightarrow{AB}=\overrightarrow{OB}-\overrightarrow{OA}=\begin{pmatrix}-1\\3\\0\end{pmatrix}$$

$$\overrightarrow{AC}=\overrightarrow{OC}-\overrightarrow{OA}=\begin{pmatrix}2\\1\\-4\end{pmatrix}$$

So a vector equation of the plane is

$$\mathbf{r}=\begin{pmatrix}2\\2\\3\end{pmatrix}+\lambda\begin{pmatrix}-1\\3\\0\end{pmatrix}+\mu\begin{pmatrix}2\\1\\-4\end{pmatrix}$$

If D also lies in this plane then there exist λ,μ such that

$$\begin{pmatrix}2\\2\\3\end{pmatrix}+\lambda\begin{pmatrix}-1\\3\\0\end{pmatrix}+\mu\begin{pmatrix}2\\1\\-4\end{pmatrix}=\begin{pmatrix}3\\6\\-1\end{pmatrix}$$

k component: $3-4\mu=-1\Rightarrow\mu=1$

j component: $2+3\lambda+1(1)=6\Rightarrow\lambda=1$

Check **i** component: $2+1(-1)+1(2)=3$

Therefore the points are coplanar.

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8 Find the plane containing $A(2,3,4)$ $B(2,-1,3)$ and $C(5,3,-2)$, then show that $D(-1,-9,8)$ lies in this plane.

$$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \begin{pmatrix} 0 \\ -4 \\ -1 \end{pmatrix}$$

$$\overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA} = \begin{pmatrix} 3 \\ 0 \\ -6 \end{pmatrix}$$

So a vector equation of the plane is

$$\mathbf{r} = \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix} + \lambda \begin{pmatrix} 0 \\ -4 \\ -1 \end{pmatrix} + \mu \begin{pmatrix} 3 \\ 0 \\ -6 \end{pmatrix}$$

If D also lies in this plane then there exist λ, μ such that

$$\begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix} + \lambda \begin{pmatrix} 0 \\ -4 \\ -1 \end{pmatrix} + \mu \begin{pmatrix} 3 \\ 0 \\ -6 \end{pmatrix} = \begin{pmatrix} -1 \\ -9 \\ 8 \end{pmatrix}$$

i component: $2 + 3\mu = -1 \Rightarrow \mu = -1$

j component: $3 - 4\lambda = -9 \Rightarrow \lambda = 3$

Check **k** component: $4 + 3(-1) - (-6) = 7 \neq 8$

So the equations are not consistent.

Therefore the points are not coplanar.

9 a $\mathbf{a} = \begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix} + 1 \begin{pmatrix} -2 \\ 3 \\ 5 \end{pmatrix} + 2 \begin{pmatrix} 4 \\ 2 \\ -3 \end{pmatrix} = \begin{pmatrix} 9 \\ 5 \\ 0 \end{pmatrix}$

b $\begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} -2 \\ 3 \\ 5 \end{pmatrix} + \mu \begin{pmatrix} 4 \\ 2 \\ -3 \end{pmatrix} = \begin{pmatrix} 1 \\ -7 \\ -1 \end{pmatrix}$

i component: $3 - 2\lambda + 4\mu = 1 \Rightarrow \lambda - 2\mu = 1$

j component: $-2 + 3\lambda + 2\mu = -7 \Rightarrow 3\lambda + 2\mu = -5$

k component: $1 + 5\lambda - 3\mu = -1 \Rightarrow 5\lambda - 3\mu = -2$

$\lambda = \mu = -1$ satisfies all of the equations, so $B(1,-7,-1)$ lies on the plane.

c $A(9,5,0)$ and $B(1,-7,-1)$

So an equation for the line through A and B is $\mathbf{r} = \begin{pmatrix} 9 \\ 5 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} -8 \\ -12 \\ -1 \end{pmatrix}$

d $|\overrightarrow{OA}| = |\overrightarrow{OC}|$

$|\overrightarrow{OA}| = \sqrt{9^2 + 5^2} = \sqrt{106}$

$\therefore |\overrightarrow{OC}| = \sqrt{(9-8\lambda)^2 + (5-12\lambda)^2 + \lambda^2} = \sqrt{106}$

$\Rightarrow 209\lambda^2 - 264\lambda + 106 = 106$

$\Rightarrow \lambda(19\lambda - 24) = 0$

$\lambda \neq 0$ (otherwise that would give A) so $\lambda = \frac{24}{19}$

Therefore C has position vector

$$\begin{pmatrix} 9 \\ 5 \\ 0 \end{pmatrix} + \frac{24}{19} \begin{pmatrix} -8 \\ -12 \\ -1 \end{pmatrix} = \begin{pmatrix} -\frac{21}{19} \\ -\frac{193}{19} \\ -\frac{24}{19} \end{pmatrix}$$

Challenge

A: $2\mathbf{i} + 6\mathbf{j} + \mathbf{k}$ lies on plane; $\lambda = -2, \mu = 1$

$\overrightarrow{AB} = 5\mathbf{i} - 7\mathbf{j} + 6\mathbf{k}$

B: $7\mathbf{i} - \mathbf{j} + 7\mathbf{k}$ lies on plane; $\lambda = 1, \mu = 2$

So line lies entirely within plane.

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Exercise 9C

1 $\mathbf{a}.\mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos\theta = 3\times 3\times \cos 60^\circ$

$= 3\times 3\times \frac{1}{2} = \frac{9}{2}$

2 a $\mathbf{a}.\mathbf{b} = \begin{pmatrix}5\\2\\3\end{pmatrix}.\begin{pmatrix}2\\-1\\-2\end{pmatrix} = 10-2-6 = 2$

b $\mathbf{a}.\mathbf{b} = \begin{pmatrix}10\\-7\\4\end{pmatrix}.\begin{pmatrix}3\\-5\\-12\end{pmatrix} = 30+35-48 = 17$

c $\mathbf{a}.\mathbf{b} = \begin{pmatrix}1\\1\\-1\end{pmatrix}.\begin{pmatrix}-1\\-1\\4\end{pmatrix} = -1-1-4 = -6$

d $\mathbf{a}.\mathbf{b} = \begin{pmatrix}2\\0\\-1\end{pmatrix}.\begin{pmatrix}6\\-5\\-8\end{pmatrix} = 12+0+8 = 20$

e $\mathbf{a}.\mathbf{b} = \begin{pmatrix}0\\3\\9\end{pmatrix}.\begin{pmatrix}1\\-12\\-4\end{pmatrix} = 0+36-36 = 0$

3 a $\mathbf{a}.\mathbf{b} = \begin{pmatrix}3\\7\end{pmatrix}.\begin{pmatrix}5\\1\end{pmatrix} = 15+7 = 22$

$|\mathbf{a}| = \sqrt{3^2+7^2} = \sqrt{58}$

$|\mathbf{b}| = \sqrt{5^2+1^2} = \sqrt{26}$

$\sqrt{58}\sqrt{26}\cos\theta = 22$

$\cos\theta = \frac{22}{\sqrt{58}\sqrt{26}}$

$\theta = 55.5^\circ$ (1 d.p.)

b $\mathbf{a}.\mathbf{b} = \begin{pmatrix}2\\-5\end{pmatrix}.\begin{pmatrix}6\\3\end{pmatrix} = 12-15 = -3$

$|a| = \sqrt{2^2+(-5)^2} = \sqrt{29}$

$|b| = \sqrt{6^2+3^2} = \sqrt{45}$

$\sqrt{29}\sqrt{45}\cos\theta = -3$

$\cos\theta = -\frac{-3}{\sqrt{29}\sqrt{45}}$

$\theta = 94.8^\circ$ (1 d.p.)

3 c $\mathbf{a}.\mathbf{b} = \begin{pmatrix}1\\-7\\8\end{pmatrix}.\begin{pmatrix}12\\2\\1\end{pmatrix} = 12-14+8+6$

$|\mathbf{a}| = \sqrt{1^2+(-7)^2+8^2} = \sqrt{114}$

$|\mathbf{b}| = \sqrt{12^2+2^2+1^2} = \sqrt{149}$

$\sqrt{114}\sqrt{149}\cos\theta = 6$

$\cos\theta = \frac{6}{\sqrt{114}\sqrt{149}}$

$\theta = 87.4^\circ$ (1 d.p.)

d $\mathbf{a}.\mathbf{b} = \begin{pmatrix}-1\\-1\\5\end{pmatrix}.\begin{pmatrix}11\\-3\\4\end{pmatrix} = -11+3+20 = 12$

$|\mathbf{a}| = \sqrt{(-1)^2+(-1)^2+5^2} = \sqrt{27}$

$|\mathbf{b}| = \sqrt{11^2+(-3)^2+4^2} = \sqrt{146}$

$\sqrt{27}\sqrt{146}\cos\theta = 12$

$\cos\theta = \frac{12}{\sqrt{27}\sqrt{146}}$

$\theta = 79.0^\circ$ 1 (d.p.)

e $\mathbf{a}.\mathbf{b} = \begin{pmatrix}6\\-7\\12\end{pmatrix}.\begin{pmatrix}-2\\1\\1\end{pmatrix} = -12-7+12 = -7$

$|\mathbf{a}| = \sqrt{6^2+(-7)^2+12^2} = \sqrt{229}$

$|\mathbf{b}| = \sqrt{(-2)^2+1^2+1^2} = \sqrt{6}$

$\sqrt{229}\sqrt{6}\cos\theta = -7$

$\cos\theta = \frac{-7}{\sqrt{229}\sqrt{6}}$

$\theta = 100.9^\circ$ (1 d.p.)

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3 f $\mathbf{a}.\mathbf{b} = \begin{pmatrix}4\\0\\5\end{pmatrix}.\begin{pmatrix}6\\-2\\0\end{pmatrix} = 24+0+0+24$

$|\mathbf{a}| = \sqrt{4^2+5^2} = \sqrt{41}$

$|\mathbf{b}| = \sqrt{6^2+(-2)^2} = \sqrt{40}$

$\sqrt{41}\sqrt{40}\cos\theta = 24$

$\cos\theta = \dfrac{24}{\sqrt{41}\sqrt{40}}$

$\theta = 53.7°$ (1 d.p.)

g $\mathbf{a}.\mathbf{b} = \begin{pmatrix}-5\\2\\-3\end{pmatrix}.\begin{pmatrix}2\\-2\\-11\end{pmatrix} = -10-4+33 = 19$

$|\mathbf{a}| = \sqrt{(-5)^2+2^2+(-3)^2} = \sqrt{38}$

$|\mathbf{b}| = \sqrt{2^2-(-2)^2+11^2} = \sqrt{129}$

$\sqrt{38}\sqrt{129}\cos\theta = 19$

$\cos\theta = \dfrac{19}{\sqrt{38}\sqrt{129}}$

$\theta = 74.3°$ (1 d.p.)

h $\mathbf{a}.\mathbf{b} = \begin{pmatrix}1\\1\\1\end{pmatrix}.\begin{pmatrix}1\\-1\\1\end{pmatrix} = 1-1+1 = 1$

$|\mathbf{a}| = \sqrt{1^2+1^2+1^2} = \sqrt{3}$

$|\mathbf{b}| = \sqrt{1^2+(-1)^2+1^2} = \sqrt{3}$

$\sqrt{3}\sqrt{3}\cos\theta = 1$

$\cos\theta = \dfrac{1}{\sqrt{3}\sqrt{3}} = \dfrac{1}{3}$

$\theta = 70.5°$ (1 d.p.)

4 a $\begin{pmatrix}3\\5\end{pmatrix}.\begin{pmatrix}\lambda\\6\end{pmatrix} = 3\lambda+30 = 0$

$\Rightarrow 3\lambda = -30$

$\Rightarrow \lambda = -10$

b $\begin{pmatrix}2\\6\\-1\end{pmatrix}\begin{pmatrix}\lambda\\-4\\-14\end{pmatrix} = 2\lambda-24+14 = 0$

$\Rightarrow 2\lambda = 10$

$\Rightarrow \lambda = 5$

4 c $\begin{pmatrix}3\\\lambda\\-8\end{pmatrix}.\begin{pmatrix}7\\-5\\1\end{pmatrix} = 21-5\lambda-8 = 0$

$\Rightarrow 5\lambda = 13$

$\Rightarrow \lambda = 2\dfrac{3}{5}$

d $\begin{pmatrix}9\\-3\\5\end{pmatrix}.\begin{pmatrix}\lambda\\\lambda\\3\end{pmatrix} = 9\lambda-3\lambda+15 = 0$

$\Rightarrow 6\lambda = -15$

$\Rightarrow \lambda = -2\dfrac{1}{2}$

e $\begin{pmatrix}\lambda\\3\\-2\end{pmatrix}.\begin{pmatrix}\lambda\\\lambda\\5\end{pmatrix} = \lambda^2+3\lambda-10 = 0$

$\Rightarrow (\lambda+5)(\lambda-2) = 0$

$\Rightarrow \lambda = -5$ or $\lambda = 2$

5 a Using $\mathbf{a} = 9\mathbf{i}-5\mathbf{j}+3\mathbf{k}$ and $\mathbf{b} = \mathbf{i}$,

$\mathbf{a}.\mathbf{b} = \begin{pmatrix}9\\-5\\3\end{pmatrix}.\begin{pmatrix}1\\0\\0\end{pmatrix} = 9$

$|\mathbf{a}| = \sqrt{9^2+(-5)^2+3^2} = \sqrt{115}$

$|\mathbf{b}| = 1$

$\sqrt{115}\cos\theta = 9$

$\cos\theta = \dfrac{9}{\sqrt{115}}$

$\theta = 32.9°$

b Using $\mathbf{a} = 9\mathbf{i}-5\mathbf{j}+3\mathbf{k}$ and $\mathbf{b} = \mathbf{j}$,

$\mathbf{a}.\mathbf{b} = \begin{pmatrix}9\\-5\\3\end{pmatrix}.\begin{pmatrix}0\\1\\0\end{pmatrix} = -5$

$|\mathrm{a}| = \sqrt{115}, |\mathrm{b}| = 1$

$\sqrt{115}\cos\theta = -5$

$\cos\theta = \dfrac{-5}{\sqrt{115}}$

$\theta = 117.8°$

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6 a Using $\mathbf{a} = \mathbf{i} + 11\mathbf{j} - 4\mathbf{k}$ and $\mathbf{b} = \mathbf{j}$,

$$\mathbf{a}.\mathbf{b} = \begin{pmatrix} 1 \\ 11 \\ -4 \end{pmatrix} . \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = 11$$

$$|\mathbf{a}| = \sqrt{1^2 + 11^2 + (-4)^2} = \sqrt{138}$$

$$|\mathbf{b}| = 1$$

$$\sqrt{138}\cos\theta = 11$$

$$\cos\theta = \frac{11}{\sqrt{138}}$$

$$\theta = 20.5^\circ$$

b Using $\mathbf{a} = \mathbf{i} + 11\mathbf{j} - 4\mathbf{k}$ and $\mathbf{b} = \mathbf{k}$,

$$\mathbf{a}.\mathbf{b} = \begin{pmatrix} 1 \\ 11 \\ -4 \end{pmatrix} . \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = -4$$

$$|\mathbf{a}| = \sqrt{138}, \ |\mathrm{b}| = 1$$

$$\sqrt{138}\cos\theta = -4$$

$$\cos\theta = \frac{-4}{\sqrt{138}}$$

$$\theta = 109.9^\circ$$

7 Using $\mathbf{a} = \mathbf{i} + \mathbf{j} + \mathbf{k}$ and $\mathbf{b} = 2\mathbf{i} + \mathbf{j} + \mathbf{k}$,

$$\mathbf{a}.\mathbf{b} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} . \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} = 2 + 1 + 1 = 4$$

$$|\mathbf{a}| = \sqrt{1^2 + 1^2 + 1^2} = \sqrt{3}$$

$$|\mathbf{b}| = \sqrt{2^2 + 1^2 + 1^2} = \sqrt{6}$$

$$\sqrt{3}\sqrt{6}\cos\theta = 4$$

$$\cos\theta = \frac{4}{\sqrt{3}\sqrt{6}} = \frac{4}{\sqrt{3}\sqrt{3}\sqrt{2}} = \frac{4}{3\sqrt{2}}$$

$$= \frac{4}{3\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}} = \frac{4\sqrt{2}}{6} = \frac{2\sqrt{2}}{3}$$

8 Using $\mathbf{a} = \mathbf{i} + 3\mathbf{j}$ and $\mathbf{b} = \mathbf{j} + \lambda\mathbf{k}$,

$$\mathbf{a}.\mathbf{b} = \begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix} . \begin{pmatrix} 0 \\ 1 \\ \lambda \end{pmatrix} = 0 + 3 + 0 = 3$$

$$|\mathbf{a}| = \sqrt{1^2 + 3^2} = \sqrt{10}$$

$$|\mathbf{b}| = \sqrt{1^2 + \lambda^2} = \sqrt{1 + \lambda^2}$$

$$\sqrt{10}\sqrt{1 + \lambda^2}\cos 60^\circ = 3$$

$$\sqrt{1 + \lambda^2} = \frac{3}{\sqrt{10}\cos 60^\circ} = \frac{6}{\sqrt{10}}$$

Squaring both sides:

$$1 + \lambda^2 = \frac{36}{10}$$

$$\lambda^2 = \frac{26}{10} = \frac{13}{5}$$

$$\lambda = \pm\sqrt{\frac{13}{5}}$$

9 a Let the required vector be $x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$.
Then

$$\begin{pmatrix} 1 \\ 1 \\ -3 \end{pmatrix} . \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0 \text{ and } \begin{pmatrix} 5 \\ -2 \\ -1 \end{pmatrix} . \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0$$

$$x + y - 3z = 0$$

$$5x - 2y - z = 0$$

Let $z = 1$:

$$x + y = 3 \quad (\times 2)$$

$$5x - 2y = 1$$

$$2x + 2y = 6$$

$$5x - 2y = 1$$

Adding, $7x = 7 \Rightarrow x = 1$

$1 + y = 3$, so $y = 2$

So $x = 1, y = 2$ and $z = 1$

A possible vector is $\mathbf{i} + 2\mathbf{j} + \mathbf{k}$

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9 b Let the required vector be $x\mathbf{i}+y\mathbf{j}+z\mathbf{k}$

Then $\begin{pmatrix}2\\3\\-4\end{pmatrix}.\begin{pmatrix}x\\y\\z\end{pmatrix}=0$ and $\begin{pmatrix}1\\-6\\3\end{pmatrix}.\begin{pmatrix}x\\y\\z\end{pmatrix}=0$

$2x+3y-4z=0$

$x-6y+3z=0$

Let $z=1$:

$2x+3y=4$

$x-6y=-3 \quad (\times 2)$

$2x+3y=4$

$2x-12y=-6$

Subtracting, $15y=10 \Rightarrow y=\frac{2}{3}$

$2x+2=4, x$ so $x=1$

So $x=1, y=\frac{2}{3}$ and $z=1$

A possible vector is $\mathbf{i}+\frac{2}{3}\mathbf{j}+\mathbf{k}$

Another possible vector is

$3\left(\mathbf{i}+\frac{2}{3}\mathbf{j}+\mathbf{k}\right)=3\mathbf{i}+2\mathbf{j}+3\mathbf{k}$

c Let the required vector be $x\mathbf{i}+y\mathbf{j}+z\mathbf{k}$

Then

$\begin{pmatrix}4\\-4\\-1\end{pmatrix}.\begin{pmatrix}x\\y\\z\end{pmatrix}=0$ and $\begin{pmatrix}-2\\-9\\6\end{pmatrix}.\begin{pmatrix}x\\y\\z\end{pmatrix}=0$

$4x-4y-z=0$

$-2x-9y+6z=0$

Let $z=1$:

$4x-4y=1$

$-2x-9y=-6 \quad (\times 2)$

$4x-4y=1$

$-4x-18y=-12$

Adding, $-22y=-11 \Rightarrow y=\frac{1}{2}$

$4x-2=1$, so $x=\frac{3}{4}$

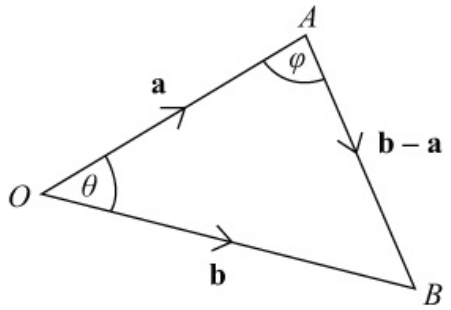
So $x=\frac{3}{4}, y=\frac{1}{2}$ and $z=1$

A possible vector is $\frac{3}{4}\mathbf{i}+\frac{1}{2}\mathbf{j}+\mathbf{k}$

Another possible vector is

$4\left(\frac{3}{4}\mathbf{i}+\frac{1}{2}\mathbf{j}+\mathbf{k}\right)=3\mathbf{i}+2\mathbf{j}+4\mathbf{k}$

10



Using **a** and **b** to find θ:

$\mathbf{a}.\mathbf{b}=\begin{pmatrix}2\\5\\1\end{pmatrix}.\begin{pmatrix}6\\1\\-2\end{pmatrix}=12+5-2=15$

$|\mathbf{a}|=\sqrt{2^2+5^2+1^2}=\sqrt{30}$

$|\mathbf{b}|=\sqrt{6^2+1^2+(-2)^2}=\sqrt{41}$

$\sqrt{30}\sqrt{41}\cos\theta=15$

$\cos\theta=\frac{15}{\sqrt{30}\sqrt{41}}$

$\theta=64.7°$

Using **AO** and **AB** to find ϕ:

$\mathbf{AO}=-\mathbf{a}=\begin{pmatrix}-2\\-5\\-1\end{pmatrix}$

$\mathbf{AB}=\mathbf{b}-\mathbf{a}=\begin{pmatrix}6\\1\\-2\end{pmatrix}-\begin{pmatrix}2\\5\\1\end{pmatrix}=\begin{pmatrix}4\\-4\\-3\end{pmatrix}$

$(-\mathbf{a}).(\mathbf{b}-\mathbf{a})=\begin{pmatrix}-2\\-5\\-1\end{pmatrix}.\begin{pmatrix}4\\-4\\-3\end{pmatrix}$

$=-8+20+3=15$

$|-a|=\sqrt{(-2)^2+(-5)^2+(-1)^2}=\sqrt{30}$

$|b-a|=\sqrt{4^2+(-4)^2+(-3)^2}=\sqrt{41}$

$\cos\phi=\frac{15}{\sqrt{30}\sqrt{41}}$

$\phi=64.7° \quad (1 \text{ d.p.})$

(Since $|\mathbf{b}-\mathbf{a}|=|\mathbf{b}|$, AB = OB, so the triangle is isosceles).

$\angle\mathbf{OBA}=180°-64.7°-64.7°$

$=50.6° \quad (1 \text{ d.p.})$

Angles are

64.7°, 64.7° and 50.6° (all 1 d.p.)

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11 a $|\overrightarrow{AB}| = \sqrt{(2-1)^2+(7-3)^2+(-3-1)^2}$

$= \sqrt{33}$

$|\overrightarrow{BC}| = \sqrt{(4-2)^2+(-5-7)^2+(2-(-3))^2}$

$= \sqrt{173}$

b $\overrightarrow{AB}\cdot\overrightarrow{BC} = \begin{pmatrix}1\\4\\-4\end{pmatrix}\cdot\begin{pmatrix}2\\-12\\5\end{pmatrix} = 1(2)+4(-12)-4(5)$

$= -66$

$\therefore\ -66 = \sqrt{33}\sqrt{173}\cos\theta$

$\Rightarrow \cos\theta = -\dfrac{66}{\sqrt{33}\sqrt{173}}$

$\therefore \theta = 180^o - \cos^{-1}\left(-\dfrac{66}{\sqrt{33}\sqrt{173}}\right)$

$= 29.1^o$ (1d.p.) $= \overrightarrow{BA}\cdot\overrightarrow{BC}$

12 a $\overrightarrow{OA} = \begin{pmatrix}7\\4\\4\end{pmatrix},\ \overrightarrow{OB} = \begin{pmatrix}2\\2\\1\end{pmatrix}$

$\overrightarrow{OA}\cdot\overrightarrow{OB} = \begin{pmatrix}7\\4\\4\end{pmatrix}\cdot\begin{pmatrix}2\\2\\1\end{pmatrix} = 7(2)+4(2)+4(1)$

$= 26$

$= \sqrt{7^2+4^2+4^2}\sqrt{2^2+2^2+1^2}\cos\theta$

$\Rightarrow \cos\theta = \dfrac{26}{27}$

b $\cos\theta = \dfrac{26}{27}$

$\Rightarrow \cos^2\theta = 1-\sin^2\theta = \left(\dfrac{26}{27}\right)^2 = \dfrac{676}{729}$

$\Rightarrow \sin^2\theta = \dfrac{53}{729} \Rightarrow \sin\theta = \dfrac{\sqrt{53}}{27}$

since $0^o < \theta < 180^o$.

$\therefore$ The area of the triangle ΔAOB is

$\dfrac{1}{2}|\overrightarrow{OA}||\overrightarrow{OB}|\sin\theta = \dfrac{1}{2}(9)(3)\sin\theta = \dfrac{\sqrt{53}}{2}$

13 Let $\overrightarrow{OA} = \mathbf{a}$ and $\overrightarrow{OP} = \mathbf{p}$

Then $\overrightarrow{OB} = -\mathbf{a}$

$\therefore \overrightarrow{AP}\cdot\overrightarrow{PB} = \left(\overrightarrow{AO}+\overrightarrow{OP}\right)\cdot\left(\overrightarrow{PO}+\overrightarrow{OB}\right)$

$= (-\mathbf{a}+\mathbf{p})\cdot(-\mathbf{p}-\mathbf{a})$

$= \mathbf{a}\cdot\mathbf{p}+\mathbf{a}\cdot\mathbf{a}-\mathbf{p}\cdot\mathbf{p}-\mathbf{p}\cdot\mathbf{a}$

$= 0$

since $\mathbf{a}\cdot\mathbf{p} = \mathbf{p}\cdot\mathbf{a}$ and $\mathbf{a}\cdot\mathbf{a} = \mathbf{p}\cdot\mathbf{p}$ because both A and P lie on the circle

14 a $\overrightarrow{CA} = -\overrightarrow{OC}+\overrightarrow{OA} = \begin{pmatrix}-6+5\\1-1\\-4+0\end{pmatrix} = \begin{pmatrix}-1\\0\\-4\end{pmatrix}$

$\overrightarrow{CB} = -\overrightarrow{OC}+\overrightarrow{OB} = \begin{pmatrix}-6+2\\1+4\\-4+10\end{pmatrix} = \begin{pmatrix}-4\\5\\6\end{pmatrix}$

b $\overrightarrow{CA}\cdot\overrightarrow{CB} = \begin{pmatrix}-1\\0\\-4\end{pmatrix}\cdot\begin{pmatrix}-4\\5\\6\end{pmatrix} = 4-24 = -20$

$|\overrightarrow{CA}| = \sqrt{(-1)^2+(-4)^2} = \sqrt{17}$

$|\overrightarrow{CB}| = \sqrt{(-4)^2+5^2+6^2} = \sqrt{77}$

$\therefore \cos\theta = -\dfrac{20}{\sqrt{77}\sqrt{17}}$

$\Rightarrow \cos^2\theta = 1-\sin^2\theta = \dfrac{400}{1309}$

$\Rightarrow \sin\theta = \sqrt{\dfrac{909}{1309}}$

$\therefore$ Area is $\dfrac{1}{2}|\overrightarrow{CA}||\overrightarrow{CB}|\sin\theta$

$= \dfrac{1}{2}\sqrt{1309}\sqrt{\dfrac{909}{1309}} = \dfrac{3\sqrt{101}}{2}$

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14 c The following pairings or parallel sides are possible:
1) AD&BC, 2)BD&AC, 3)CD&AB

$$1): \overrightarrow{OD} = \overrightarrow{OA} + \overrightarrow{BC} = \begin{pmatrix} 9 \\ -6 \\ -6 \end{pmatrix}$$

$\Rightarrow D(9,-6,-6)$

$$2): \overrightarrow{OD} = \overrightarrow{OB} + \overrightarrow{CA} = \begin{pmatrix} 1 \\ 4 \\ 6 \end{pmatrix}$$

$\Rightarrow D(1,4,6)$

$$3): \overrightarrow{OD} = \overrightarrow{OC} + \overrightarrow{AB} = \begin{pmatrix} 3 \\ 4 \\ 6 \end{pmatrix}$$

$\Rightarrow D(3,4,6)$

$\therefore (9,2,-6);(1,4,6);(3,4,6)$ all possible coordinates of D

d $3\sqrt{101}$

15 a $\overrightarrow{PQ} = \begin{pmatrix} -3 \\ 6 \\ -2 \end{pmatrix}$, $\overrightarrow{QR} = \begin{pmatrix} 2 \\ -2 \\ -9 \end{pmatrix}$

$$\therefore \overrightarrow{PQ} \cdot \overrightarrow{QR} = \begin{pmatrix} -3 \\ 6 \\ -2 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ -2 \\ -9 \end{pmatrix} = -6 - 12 + 18 = 0$$

$\therefore PQ$ is perpendicular to QR

b Since PQ is perpendicular to QR, the three points lie on a semi-circle with diameter PR. Therefore the centre of the circle containing the three points is the midpoint of PR:

$$\overrightarrow{OP} + \frac{1}{2}\overrightarrow{PR} = \begin{pmatrix} 1 \\ -1 \\ 6 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} -1 \\ 4 \\ -11 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ 1 \\ \frac{1}{2} \end{pmatrix}$$

$\therefore$ Centre $\left(\frac{1}{2}, 1, \frac{1}{2}\right)$

$$\text{Radius} = \frac{1}{2}|\overrightarrow{PR}| = \frac{1}{2}\sqrt{(-1)^2 + 4^2 + (-11)^2}$$

$$= \frac{\sqrt{138}}{2}$$

Challenge

1 $\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos\theta = |\mathbf{b}||\mathbf{a}|\cos\theta = \mathbf{b} \cdot \mathbf{a}$

2 a i $\mathbf{a}.(\mathbf{b}+\mathbf{c}) = |\mathbf{a}||\mathbf{b}+\mathbf{c}|\cos\theta$, but $\cos\theta = \dfrac{PQ}{|\mathbf{b}+\mathbf{c}|}$

so $\mathbf{a}.(\mathbf{b}+\mathbf{c}) = |\mathbf{a}| \times PQ$

ii $\mathbf{a}.\mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos\alpha$, but $\cos\alpha = \dfrac{PR}{|\mathbf{b}|}$

so $\mathbf{a}.\mathbf{b} = |\mathbf{a}| \times PR$

iii $\mathbf{a}.\mathbf{c} = |\mathbf{a}||\mathbf{c}|\cos\beta$, but $\cos\beta = \dfrac{MN}{|\mathbf{c}|} = \dfrac{RQ}{|\mathbf{c}|}$

so $\mathbf{a}.\mathbf{c} = |\mathbf{a}| \times RQ$

b $\mathbf{a}.(\mathbf{b}+\mathbf{c}) = |\mathbf{a}| \times PQ = |\mathbf{a}| \times (PR + RQ)$
$= (|\mathbf{a}| \times PR) + (|\mathbf{a}| \times RQ) = \mathbf{a}.\mathbf{b} + \mathbf{a}.\mathbf{c}$;
so $\mathbf{a}.(\mathbf{b}+\mathbf{c}) = \mathbf{a}.\mathbf{b} + \mathbf{a}.\mathbf{c}$

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Exercise 9D

1 a Direction vectors are $\mathbf{a} = \begin{pmatrix} 3 \\ -5 \\ -1 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 2 \\ 1 \\ -9 \end{pmatrix}$

$$\cos\theta = \frac{\mathbf{a}.\mathbf{b}}{|\mathbf{a}||\mathbf{b}|}$$

$$\mathbf{a}.\mathbf{b} = \begin{pmatrix} 3 \\ -5 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 1 \\ -9 \end{pmatrix} = 6 - 5 + 9 = 10$$

$$|\mathbf{a}| = \sqrt{3^2 + (-5)^2 + (-1)^2} = \sqrt{35}$$

$$|\mathbf{b}| = \sqrt{2^2 + 1^2 + (-9)^2} = \sqrt{86}$$

$$\cos\theta = \frac{10}{\sqrt{35}\sqrt{86}}$$

$\theta = 79.5°$ (1 d.p.)

The acute angle between the lines is 79.5° (1 d.p.)

b Direction vectors are $\mathbf{a} = \begin{pmatrix} -2 \\ -1 \\ 3 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} -4 \\ -2 \\ 1 \end{pmatrix}$

$$\cos\theta = \frac{\mathbf{a}.\mathbf{b}}{|\mathbf{a}||\mathbf{b}|}$$

$$\mathbf{a}.\mathbf{b} = \begin{pmatrix} -2 \\ -1 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} -4 \\ -2 \\ 1 \end{pmatrix} = 8 + 2 + 3 = 13$$

$$|\mathbf{a}| = \sqrt{(-2)^2 + (-1)^2 + 3^2} = \sqrt{14}$$

$$|\mathbf{b}| = \sqrt{(-4)^2 + (-2)^2 + 1^2} = \sqrt{21}$$

$$\cos\theta = \frac{13}{\sqrt{14}\sqrt{21}}$$

$\theta = 40.7°$ (1 d.p.)

The acute angle between the lines is 40.7° (1 d.p.)

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1 c Direction vectors are $\mathbf{a} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 2 \\ -7 \\ 3 \end{pmatrix}$

$$\cos\theta = \frac{\mathbf{a}.\mathbf{b}}{|\mathbf{a}||\mathbf{b}|}$$

$$\mathbf{a}.\mathbf{b} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ -7 \\ 3 \end{pmatrix} = 2 - 7 + 3 = -2$$

$$|\mathbf{a}| = \sqrt{1^2 + 1^2 + 1^2} = \sqrt{3}$$

$$|\mathbf{b}| = \sqrt{2^2 + (-7)^2 + 3^2} = \sqrt{62}$$

$$\cos\theta = \frac{-2}{\sqrt{3}\sqrt{62}}$$

$\theta = 98.4°$ (1 d.p.)

This is the angle between the two vectors.

The acute angle between the lines is 180° – 98.4° = 81.6° (1 d.p.).

d Direction vectors are $\mathbf{a} = \begin{pmatrix} 8 \\ -1 \\ -2 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 1 \\ 3 \\ -7 \end{pmatrix}$

$$\cos\theta = \frac{\mathbf{a}.\mathbf{b}}{|\mathbf{a}||\mathbf{b}|}$$

$$\mathbf{a}.\mathbf{b} = \begin{pmatrix} 8 \\ -1 \\ -2 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 3 \\ -7 \end{pmatrix} = 8 - 3 + 14 = 19$$

$$|\mathbf{a}| = \sqrt{8^2 + (-1)^2 + (-2)^2} = \sqrt{69}$$

$$|\mathbf{b}| = \sqrt{1^2 + 3^2 + (-7)^2} = \sqrt{59}$$

$$\cos\theta = \frac{19}{\sqrt{69}\sqrt{59}}$$

$\theta = 72.7°$ (1 d.p.)

The acute angle between the lines is 72.7° (1 d.p.)

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1 e Direction vectors are $\mathbf{a} = \begin{pmatrix} 11 \\ 5 \\ -3 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} -3 \\ 5 \\ 4 \end{pmatrix}$

$$\cos\theta = \frac{\mathbf{a}.\mathbf{b}}{|\mathbf{a}||\mathbf{b}|}$$

$$\mathbf{a}.\mathbf{b} = \begin{pmatrix} 11 \\ 5 \\ -3 \end{pmatrix} \cdot \begin{pmatrix} -3 \\ 5 \\ 4 \end{pmatrix} = -33 + 25 - 12 = -20$$

$$|\mathbf{a}| = \sqrt{11^2 + 5^2 + (-3)^2} = \sqrt{155}$$

$$|\mathbf{b}| = \sqrt{(-3)^2 + 5^2 + 4^2} = \sqrt{50}$$

$$\cos\theta = \frac{-20}{\sqrt{155}\sqrt{50}}$$

$\theta = 103.1°$ (1 d.p.)

This is the angle between the two vectors.

The acute angle between the lines is 180° – 103.1° = 76.9° (1 d.p.).

2 a $\mathbf{r} \cdot (2\mathbf{i} + \mathbf{j} + \mathbf{k}) = (\mathbf{i} - \mathbf{j} - \mathbf{k}) \cdot (2\mathbf{i} + \mathbf{j} + \mathbf{k})$

$= 2 - 1 - 1$

i.e. $\mathbf{r} \cdot (2\mathbf{i} + \mathbf{j} + \mathbf{k}) = 0$

b $\mathbf{r} \cdot (5\mathbf{i} - \mathbf{j} - 3\mathbf{k}) = (\mathbf{i} + 2\mathbf{j} + \mathbf{k}) \cdot (5\mathbf{i} - \mathbf{j} - 3\mathbf{k})$

$= 5 - 2 - 3$

i.e. $\mathbf{r} \cdot (5\mathbf{i} - \mathbf{j} - 3\mathbf{k}) = 0$

c $\mathbf{r} \cdot (\mathbf{i} + 3\mathbf{j} + 4\mathbf{k}) = (2\mathbf{i} - 3\mathbf{k}) \cdot (\mathbf{i} + 3\mathbf{j} + 4\mathbf{k})$

$= 2 - 12$

i.e. $\mathbf{r} \cdot (\mathbf{i} + 3\mathbf{j} + 4\mathbf{k}) = -10$

d $\mathbf{r} \cdot (4\mathbf{i} + \mathbf{j} - 5\mathbf{k}) = (4\mathbf{i} - 2\mathbf{j} + \mathbf{k}) \cdot (4\mathbf{i} + \mathbf{j} - 5\mathbf{k})$

$= 16 - 2 - 5$

i.e. $\mathbf{r} \cdot (4\mathbf{i} + \mathbf{j} - 5\mathbf{k}) = 9$

3 a $2x + y + z = 0$

b $5x - y - 3z = 0$

c $x + 3y + 4z = -10$

d $4x + y - 5z = 9$

Replace $\mathbf{r}$ by $x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ in each equation.

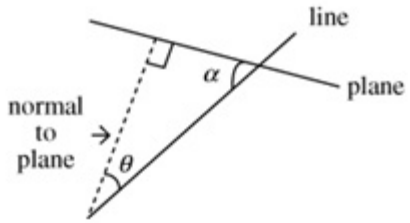
4 Expanding the scalar product gives

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \cdot \begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix} = k$$

$\Rightarrow n_1x + n_2y + n_3z = k$

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5



Find the acute angle between the given line and the normal to the plane, then subtract from 90°.

Let θ be the acute angle between the line and the normal to the plane.

Then $\cos\theta = \left|\dfrac{(4\mathbf{i}+4\mathbf{j}+7\mathbf{k})\cdot(2\mathbf{i}+\mathbf{j}-2\mathbf{k})}{\sqrt{4^2+4^2+7^2}\sqrt{2^2+1^2+(-2)^2}}\right|$

$= \left|\dfrac{8+4-14}{\sqrt{81}\sqrt{9}}\right|$

$= \left|\dfrac{-2}{27}\right| = \dfrac{2}{27}$

Let α be the angle between the line and the plane.

Then $\theta + \alpha = 90°$

So $\sin\alpha = \cos\theta = \dfrac{2}{27}$

$\therefore \alpha = 4.25°$ (3 s.f.)

6 Let θ be the acute angle between the line and the normal to the plane.

Then $\cos\theta = \dfrac{(3\mathbf{i}+4\mathbf{j}-12\mathbf{k})\cdot(4\mathbf{i}-4\mathbf{j}-7\mathbf{k})}{\sqrt{3^2+4^2+(-12)^2}\sqrt{4^2+(-4)^2+(-7)^2}}$

$= \dfrac{12-16+84}{\sqrt{169}\sqrt{81}}$

$= \dfrac{80}{13\times 9}$

$= \dfrac{80}{117}$

Find the acute angle between the given line and the normal to the plane, then subtract from 90°.

Let α be the angle between the line and the plane.

Then $\theta + \alpha = 90°$

So $\sin\alpha = \cos\theta = \dfrac{80}{117}$

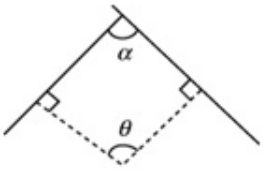
$\therefore \alpha = 43.1°$ (3 s.f.)

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7 Then angle θ between the two normal vectors $\mathbf{i}+2\mathbf{j}-2\mathbf{k}$ and $-4\mathbf{i}+4\mathbf{j}+7\mathbf{k}$ is given by

$$\cos\theta = \frac{(\mathbf{i}+2\mathbf{j}-2\mathbf{k})\cdot(-4\mathbf{i}+4\mathbf{j}+7\mathbf{k})}{|\mathbf{i}+2\mathbf{j}-2\mathbf{k}||-4\mathbf{i}+4\mathbf{j}+7\mathbf{k}|}$$
$$= \frac{-4+8-14}{\sqrt{1^2+2^2+(-2)^2}\sqrt{(-4)^2+4^2+7^2}}$$
$$= \frac{-10}{\sqrt{9}\sqrt{81}}$$
$$= -\frac{10}{27}$$

First find the angle between the two normal vectors.



The acute angle, α, between the two planes is such that
$\alpha+\theta=180°$
So $\cos\alpha = -\cos\theta$
$$= \frac{10}{27}$$
$\therefore \alpha = 68.3°$ (3 s.f.)

8 The angle θ between the two normal vectors $3\mathbf{i}-4\mathbf{j}+12\mathbf{k}$ and $5\mathbf{i}-12\mathbf{k}$ is given by

$$\cos\theta = \frac{(3\mathbf{i}-4\mathbf{j}+12\mathbf{k})\cdot(5\mathbf{i}-12\mathbf{k})}{|3\mathbf{i}-4\mathbf{j}+12\mathbf{k}||5\mathbf{i}-12\mathbf{k}|}$$
$$= \frac{15-144}{\sqrt{3^2+(-4)^2+12^2}\sqrt{5^2+(-12)^2}}$$
$$= \frac{-129}{\sqrt{169}\sqrt{169}}$$
$$= \frac{-129}{169}$$

The acute angle α between the planes is such that $\alpha+\theta=180°$

So $\cos\alpha = -\cos\theta = \frac{129}{169}$

$\therefore \alpha = 40.2°$ (3 s.f.)

9 a Line l_1: $\mathbf{r} = \begin{pmatrix}1\\4\\2\end{pmatrix} + \lambda\begin{pmatrix}8\\5\\1\end{pmatrix}$

When $\lambda = 1$, $\mathbf{r} = \begin{pmatrix}9\\9\\3\end{pmatrix}$

So the point (9, 9, 3) lies on l_1.

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9 b Direction vectors are $\mathbf{a} = \begin{pmatrix} 8 \\ 5 \\ 1 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix}$

$$\mathbf{a} \cdot \mathbf{b} = \begin{pmatrix} 8 \\ 5 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix} = 24 + 5 + 0 = 29$$

$$|\mathbf{a}| = \sqrt{8^2 + 5^2 + 1^2} = \sqrt{90}$$

$$|\mathbf{b}| = \sqrt{3^2 + 1^2 + 0^2} = \sqrt{10}$$

$$\cos\theta = \frac{29}{\sqrt{90}\sqrt{10}} = \frac{29}{\sqrt{900}} = \frac{29}{30}$$

c $\mathbf{PQ} = \sqrt{(9-1)^2 + (9-4)^2 + (3-2)^2}$

$$= \sqrt{8^2 + 5^2 + 1^2} = \sqrt{90}$$

Line l_2: $\mathbf{r} = \begin{pmatrix} 1 \\ 4 \\ 2 \end{pmatrix} + s\begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1+3s \\ 4+s \\ 2 \end{pmatrix}$

Let the coordinates of R be $(1+3s,\ 4+s,\ 2)$

$\mathbf{PR} = \sqrt{(1+3s-1)^2 + (4+s-4)^2 + (2-2)^2}$

$= \sqrt{9s^2 + s^2} = \sqrt{10s^2}$

$PQ^2 = PR^2 : 90 = 10s^2$

$\Rightarrow 5^2 = 9$

$\Rightarrow s = \pm 3$

When $s = 3$, $\mathbf{r} = \begin{pmatrix} 10 \\ 7 \\ 2 \end{pmatrix}$ $R : (10,\ 7,\ 2)$

When $s = -3$, $\mathbf{r} = \begin{pmatrix} -8 \\ 1 \\ 2 \end{pmatrix}$ $R : (-8,\ 1,\ 2)$

10 a $\frac{3-6}{-1} = \frac{3+3}{2} = \frac{7+2}{3} = 3$

and

$\frac{3+5}{2} = \frac{3-15}{-3} = \frac{7-3}{1} = 4$

So A lies on both l_1 and l_2

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10 b $\mathbf{a} = \begin{pmatrix} -1 \\ 2 \\ 3 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix}$

$\therefore \mathbf{a} \cdot \mathbf{b} = -2 - 6 + 3 = -5$

$|\mathbf{a}| = \sqrt{(-1)^2 + 2^2 + 3^2} = \sqrt{14}$

$|\mathbf{b}| = \sqrt{2^2 + (-3)^2 + 1^2} = \sqrt{14}$

$\therefore \cos\theta = \left| \frac{-5}{\sqrt{14}\sqrt{14}} \right| = \frac{5}{14}$

$\Rightarrow \theta = 69.1^o$ (1d.p.)

11 $\begin{pmatrix} 1 \\ 3 \\ 3 \end{pmatrix} + 3\begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 10 \\ 9 \\ 0 \end{pmatrix} \Rightarrow A = (10, 9, 0)$

$\begin{pmatrix} 3 \\ 5 \\ -2 \end{pmatrix} - 2\begin{pmatrix} -4 \\ -3 \\ 1 \end{pmatrix} = \begin{pmatrix} 11 \\ 11 \\ -4 \end{pmatrix} \Rightarrow B = (11, 11, -4)$

$\therefore \overrightarrow{AB} = \mathbf{v}_1 = \begin{pmatrix} 1 \\ 2 \\ -4 \end{pmatrix}$

l_1 has direction vector $\mathbf{v}_2 = \begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix}$

$\mathbf{v}_1 \cdot \mathbf{v}_2 = \begin{pmatrix} 1 \\ 2 \\ -4 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix} = 3 + 4 + 4 = 11$

$|\mathbf{v}_1| = \sqrt{1^2 + 2^2 + (-4)^2} = \sqrt{21}$

$|\mathbf{v}_2| = \sqrt{3^2 + 2^2 + (-1)^2} = \sqrt{14}$

$\therefore \cos\theta = \frac{11}{\sqrt{21}\sqrt{14}} \Rightarrow \theta = 50.1^o$ (1d.p.)

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12 a Find the plane containing $A=(3,5,-1)$; $B=(2,-2,4)$; $C=(4,3,0)$ and show that $D=(1,4,-3)$ does not lie in it.

$\overrightarrow{AB}=\begin{pmatrix}-1\\-7\\5\end{pmatrix}$, $\overrightarrow{AC}=\begin{pmatrix}1\\-2\\1\end{pmatrix}$

So an equation for the plane is

$\mathbf{r}=\begin{pmatrix}3\\5\\-1\end{pmatrix}+\lambda\begin{pmatrix}-1\\-7\\5\end{pmatrix}+\mu\begin{pmatrix}1\\-2\\1\end{pmatrix}$

If D lies in the plane, then there exist λ and μ such that

$\begin{pmatrix}3\\5\\-1\end{pmatrix}+\lambda\begin{pmatrix}-1\\-7\\5\end{pmatrix}+\mu\begin{pmatrix}1\\-2\\1\end{pmatrix}=\begin{pmatrix}1\\4\\-3\end{pmatrix}$

i component: $3-\lambda+\mu=1\Rightarrow-\lambda+\mu=-2$

j component: $5-7\lambda-2\mu=4\Rightarrow7\lambda+2\mu=1$

k component: $-1+5\lambda+\mu=-3\Rightarrow5\lambda+\mu=-2$

i and **j** components $\Rightarrow\ \lambda=\frac{5}{9},\ \mu=-\frac{13}{9}$

Check **k** component: $5\left(\frac{5}{9}\right)-\frac{13}{9}=\frac{4}{3}\neq-2$

So the equations are not consistent.

Therefore the points are not coplanar.

b $\overrightarrow{AD}=\mathbf{b}=\begin{pmatrix}-2\\-1\\-2\end{pmatrix}$

The plane has normal vector $\mathbf{n}=\begin{pmatrix}1\\2\\3\end{pmatrix}$, since this is perpendicular to both $\overrightarrow{AB}$ and $\overrightarrow{AC}$

$\therefore\mathbf{b}\cdot\mathbf{n}=\begin{pmatrix}-2\\-1\\-2\end{pmatrix}\cdot\begin{pmatrix}1\\2\\3\end{pmatrix}=-10$

$|\mathbf{b}|=\sqrt{(-2)^2+(-1)^2+(-2)^2}=3$

$|\mathbf{n}|=\sqrt{1^2+2^2+3^2}=\sqrt{14}$

$\Rightarrow\sin\theta=\left|\frac{-10}{3\sqrt{14}}\right|=\frac{10}{3\sqrt{14}}$

$\Rightarrow\theta=63.0^o$ (1d.p.)

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13 Let E be the midpoint of AD. Then the coordinate of E is $\left(\frac{1}{2},0,\frac{1}{2}\right)$

The angle θ between planes ABD and ADC is the same as the angle between the lines EB and EC

$$\overrightarrow{EB}=\overrightarrow{OB}-\overrightarrow{OE}=\begin{pmatrix}0\\1\\1\end{pmatrix}-\begin{pmatrix}\frac{1}{2}\\0\\\frac{1}{2}\end{pmatrix}=\begin{pmatrix}-\frac{1}{2}\\1\\\frac{1}{2}\end{pmatrix}$$

$$\overrightarrow{EC}=\overrightarrow{OC}-\overrightarrow{OE}=\begin{pmatrix}1\\1\\0\end{pmatrix}-\begin{pmatrix}\frac{1}{2}\\0\\\frac{1}{2}\end{pmatrix}=\begin{pmatrix}\frac{1}{2}\\1\\-\frac{1}{2}\end{pmatrix}$$

$$\overrightarrow{EB}.\overrightarrow{EC}=\begin{pmatrix}-\frac{1}{2}\\1\\\frac{1}{2}\end{pmatrix}.\begin{pmatrix}\frac{1}{2}\\1\\-\frac{1}{2}\end{pmatrix}=-\frac{1}{4}+1-\frac{1}{4}=\frac{1}{2} \qquad (1)$$

and $\overrightarrow{EB}.\overrightarrow{EC}=\left|\begin{pmatrix}-\frac{1}{2}\\1\\\frac{1}{2}\end{pmatrix}\right|\left|\begin{pmatrix}\frac{1}{2}\\1\\-\frac{1}{2}\end{pmatrix}\right|\cos\theta=\sqrt{\left(-\frac{1}{2}\right)^2+1^2+\left(\frac{1}{2}\right)^2}\sqrt{\left(\frac{1}{2}\right)^2+1^2+\left(-\frac{1}{2}\right)^2}\cos\theta$

$$=\sqrt{\frac{3}{2}}\sqrt{\frac{3}{2}}\cos\theta=\frac{3}{2}\cos\theta \qquad (2)$$

So equating equations (1) and (2) gives:

$$\frac{1}{2}=\frac{3}{2}\cos\theta$$

So therefore $\cos\theta=\frac{1}{3}$

Since the tetrahedron is regular, the angle between any two adjacent faces will be the same,

i.e. $\arccos\left(\frac{1}{3}\right)$

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14 Let point F be the point of the top of the pole.

So the coordinate of F is $(0,0,20)$

Let the points of the bases of the ropes be $A=(0,8,2)$, $B=(12,-5,3)$ and $C=(-2,6,5)$ respectively.

Then $\overrightarrow{FA}=\begin{pmatrix}0\\8\\-18\end{pmatrix}$, and $\overrightarrow{FC}=\begin{pmatrix}-2\\6\\-15\end{pmatrix}$

Let θ be the angle between FA and FC

$$\overrightarrow{FA}.\overrightarrow{FC}=\begin{pmatrix}0\\8\\-18\end{pmatrix}.\begin{pmatrix}-2\\6\\-15\end{pmatrix}=(0\times-2)+(8\times6)+(-18\times-15)=318$$

$$\left|\begin{pmatrix}0\\8\\-18\end{pmatrix}\right|=\sqrt{0^2+8^2+(-18)^2}=\sqrt{388} \text{ and } \left|\begin{pmatrix}-2\\6\\-15\end{pmatrix}\right|=\sqrt{(-2)^2+6^2+(-15)^2}=\sqrt{265}$$

So $\cos\theta=\dfrac{318}{\sqrt{388}\sqrt{265}}=0.9917$

and $\theta=\arccos(0.9917)=7.4^\circ$

So the angle between at least one pair of guide ropes is less than 15°. Therefore the flagpole will not be stable.

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Exercise 9E

1 a The line l_1 has equation

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -1 \\ 5 \end{pmatrix}$$

Use column vector form for clarity. Put the two equations equal and compare x, y and z components. Then solve simultaneous equations.

and the line l_1 has equation

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -1 \\ -3 \\ 2 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$$

These lines meet when

$$\begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -1 \\ 5 \end{pmatrix} = \begin{pmatrix} -1 \\ -3 \\ 2 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$$

i.e. $1+\lambda = -1+\mu$ **(1)**

$3-\lambda = -3+\mu$ **(2)**

$5\lambda = 2+2\mu$ **(3)**

Add equations **(1)** and **(2)**

$4 = -4+2\mu$

$\therefore 2\mu = 8$

i.e: $\mu = 4$

Substitute into equation **(1)**

$\therefore 1+\lambda = -1+4$

i.e: $\lambda = 2$

Substitute $\lambda = 2$ into equation for line l_1

$\therefore (x, y, z) = (3, 1, 10)$

Substitute $\mu = 4$ into equation for line l_2

$\therefore (x, y, z) = (3, 1, 10)$

So the two lines do meet at the point (3, 1, 10)

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1 b l_1 has equation $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$ and

l_2 has equation $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4 \\ 3 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix}$

These lines meet when $\begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 4 \\ 3 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix}$

i.e. $3 + \lambda = 4 - \mu$ **(1)**

$2 + \lambda = 3 + \mu$ **(2)**

$1 + 2\lambda = -\mu$ **(3)**

Add equations **(1)** and **(2)**

$\therefore 5 + 2\lambda = 7$

i.e. $\lambda = 1$

Substitute into equation **(1)**

$\therefore 3 + 1 = 4 - \mu$

i.e: $\mu = 0$

Substitute $\lambda = 1$ into equation for line l_1:

$\therefore (x, y, z) = (4, 3, 3)$

Substitute $\mu = 0$ into line l_2:

$\therefore (x, y, z) = (4, 3, 0)$

This is a contradiction and the lines do not meet.

[N.B. $\lambda = 1$ and $\mu = 0$ do not satisfy equation **(3)** above.]

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1 c l_1 has equation $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ 5 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}$ and

l_2 has equation $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 2\frac{1}{2} \\ 2\frac{1}{2} \end{pmatrix} + \mu \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}$

l_1 meets l_2 when $\begin{pmatrix} 1 \\ 3 \\ 5 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 2\frac{1}{2} \\ 2\frac{1}{2} \end{pmatrix} + \mu \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}$

i.e. $1+2\lambda = 1+\mu$ **(1)**

$3+3\lambda = 2\frac{1}{2}+\mu$ **(2)**

$5+\lambda = 2\frac{1}{2}-2\mu$ **(3)**

Substitute equation **(1)** from equation **(2)**

$\therefore 2+\lambda = 1\frac{1}{2}$

i.e. $\lambda = -\frac{1}{2}$

Substitute into equation **(1)**

$\therefore 1-1 = 1+\mu$

i.e. $\mu = -1$

Substitute $\lambda = -\frac{1}{2}$ into equation for line l_1:

$\therefore (x, y, z) = \left(0, 1\frac{1}{2}, 4\frac{1}{2}\right)$

Substitute $\mu = -1$ into equation for line l_2:

$\therefore (x, y, z) = \left(0, 1\frac{1}{2}, 4\frac{1}{2}\right)$

So the two lines do meet at the point $\left(0, 1\frac{1}{2}, 4\frac{1}{2}\right)$

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2 $\begin{pmatrix} -6+\lambda \\ -\lambda \\ 11+\lambda \end{pmatrix} = \begin{pmatrix} 2+2\mu \\ -2+\mu \\ 9-3\mu \end{pmatrix}$

$\therefore \begin{pmatrix} \lambda-2\mu \\ -\lambda-\mu \\ \lambda+3\mu \end{pmatrix} = \begin{pmatrix} 8 \\ -2 \\ -2 \end{pmatrix}$

i and **j** components $\Rightarrow \lambda = 4,\ \mu = -2$

Check **k** component: $4+3(-2) = -2$

So the equations are consistent.

Therefore l_1 and l_2 meet.

Substituting the value of λ into l_1,

$\begin{pmatrix} -6 \\ 0 \\ 11 \end{pmatrix} + 4\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} -2 \\ -4 \\ 15 \end{pmatrix}$

3 $\begin{pmatrix} 3+2\lambda \\ 1+2\lambda \\ -2+3\lambda \end{pmatrix} = \begin{pmatrix} 5+2\mu \\ 4+\mu \\ -\mu \end{pmatrix}$

$\therefore \begin{pmatrix} 2\lambda-2\mu \\ 2\lambda-\mu \\ 3\lambda+\mu \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ 2 \end{pmatrix}$

i and **j** components $\Rightarrow \lambda = 2,\ \mu = 1$

Check **k** component: $3(2)+1 = 7 \neq 2$

So the equations are inconsistent. Therefore l_1 and l_2 do not intersect.

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4 a The line meets the plane when

$$[(1-2\lambda)\mathbf{i}+(1+\lambda)\mathbf{j}+(1-4\lambda)\mathbf{k}]\cdot(3\mathbf{i}-4\mathbf{j}+2\mathbf{k})=16$$

$$\text{i.e. } 3(1-2\lambda)-4(1+\lambda)+2(1-4\lambda)=16$$

$$\therefore\ 3-6\lambda-4-4\lambda+2-8\lambda=16$$

$$\therefore 1-18\lambda=16$$

$$\text{i.e. } -18\lambda=15$$

$$\therefore \lambda=-\frac{15}{18}$$

$$\text{i.e. } \lambda=\frac{-5}{6}$$

Assume that the line meets the plane and perform the scalar product. Solve the resulting equation to find the value of λ. If there is no value for λ, then the line does not meet the plane.

Substitute into the equation of the line

$$\therefore (x,y,z)=\left(1+\frac{10}{6},1-\frac{5}{6},1+\frac{20}{6}\right)$$

$$=\left(2\frac{2}{3},\frac{1}{6},4\frac{1}{3}\right)$$

b The line meets the plane when

$$[\mathbf{i}+(1+2\lambda)\mathbf{j}+(1-2\lambda)\mathbf{k}]\cdot(3\mathbf{i}-\mathbf{j}-6\mathbf{k})=1$$

$$\text{i.e. } 3-(1+2\lambda)-6(1-2\lambda)=1$$

$$\text{i.e. } 3-1-2\lambda-6+12\lambda=1$$

$$\therefore 10\lambda-4=1$$

$$\therefore \lambda=\frac{1}{2}$$

Substitute into the equation of the line

$$\therefore (x,y,z)=(1,\ 1+1,\ 1-1)$$

$$=(1,\ 2,\ 0)$$

5 a If l meets the plane, there exists a λ such that

$$\left(\begin{pmatrix}2\\3\\-2\end{pmatrix}+\lambda\begin{pmatrix}1\\1\\1\end{pmatrix}\right)\cdot\begin{pmatrix}1\\1\\-2\end{pmatrix}=1$$

$$\Rightarrow\begin{pmatrix}2\\3\\-2\end{pmatrix}\cdot\begin{pmatrix}1\\1\\-2\end{pmatrix}+\lambda\begin{pmatrix}1\\1\\1\end{pmatrix}\cdot\begin{pmatrix}1\\1\\-2\end{pmatrix}=1$$

$$\Rightarrow 2+3+4+\lambda(1+1-2)=1$$

$$\Rightarrow 9=1.\text{ This is a contradiction.}$$

Therefore the line does not intersect the plane.

b The line is parallel to the plane and not contained within the plane.

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6 a Since the lines are perpendicular, the scalar product of their direction vectors is 0.

$$\therefore \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} -1 \\ p \\ p \end{pmatrix} = -3 - p + 2p = 0 \Rightarrow p = 3$$

b $$\begin{pmatrix} 5+3\lambda \\ 4-\lambda \\ -1+2\lambda \end{pmatrix} = \begin{pmatrix} -\mu \\ 11+3\mu \\ 3+3\mu \end{pmatrix}$$

$$\therefore \begin{pmatrix} 3\lambda+\mu \\ -\lambda-3\mu \\ 2\lambda-3\mu \end{pmatrix} = \begin{pmatrix} -5 \\ 7 \\ 4 \end{pmatrix}$$

i and **j** components $\Rightarrow \lambda = -1,\ \mu = -2$

Check **k** component: $2(-1) - 3(-2) = 4$

So the equations are consistent

Therefore the lines intersect

Subsituting $\lambda = -1$ into the first line, the point of intersection is

$$\begin{pmatrix} 5 \\ 4 \\ -1 \end{pmatrix} - \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ 5 \\ -3 \end{pmatrix} \text{ so } (2, 5, -3)$$

7 a $$\begin{pmatrix} 5-\lambda \\ 2+\lambda \\ 1+2\lambda \end{pmatrix} = \begin{pmatrix} 4+\mu \\ 1 \\ 1-\mu \end{pmatrix}$$

$$\therefore \begin{pmatrix} -\lambda-\mu \\ \lambda \\ 2\lambda+\mu \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \\ 0 \end{pmatrix}$$

i and **j** components $\Rightarrow \lambda = -1,\ \mu = 2$

Substituting $\lambda = -1$ into l_1,

$$\begin{pmatrix} 5 \\ 2 \\ 1 \end{pmatrix} - \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 6 \\ 1 \\ -1 \end{pmatrix}$$

b Let $\mathbf{a} = \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$

$$\mathbf{a} \cdot \mathbf{b} = \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} = -1 - 2 = -3$$

$$|\mathbf{a}| = \sqrt{(-1)^2 + 1^2 + 2^2} = \sqrt{6}$$

$$|\mathbf{b}| = \sqrt{1^2 + (-1)^2} = \sqrt{2}$$

$$\therefore \cos\theta = \left| \frac{-3}{\sqrt{6}\sqrt{2}} \right| = \frac{3}{\sqrt{12}} = \frac{3}{2\sqrt{3}} = \frac{\sqrt{3}}{2}$$

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8 $\begin{pmatrix} -3\lambda \\ -1+5\lambda \\ 2+4\lambda \end{pmatrix} = \begin{pmatrix} \mu \\ 1-2\mu \\ -5+2\mu \end{pmatrix}$

$-3\lambda = \mu$ (1)
$-1+5\lambda = 1-2\mu$ (2)

(2) $+2\times$ (1) gives $-1+5\lambda-6\lambda = 1-2\mu+2\mu$

$-1-\lambda = 1$
So $\lambda = -2$

Substituting in (1) gives $\mu = 6$

Check for consistency: $2+4\lambda = -6$ and $-5+2\mu = 7$
$2+4\lambda \neq -5+2\mu$, so equations are not consistent.

The direction of l_1 is $\begin{pmatrix} -3 \\ 5 \\ 4 \end{pmatrix}$ and the direction of l_2 is $\begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix}$

The direction vectors are not scalar multiples of each other, so l_1 and l_2 are not parallel.
Hence l_1 and l_2 are skew.

9 a Since l_1 and l_2 are perpendicular, the scalar product of their direction vectors is 0

$\therefore \begin{pmatrix} -1 \\ 3 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} q \\ 2 \\ -1 \end{pmatrix} = -q+6-2 = 0 \Rightarrow q = 4$

b Since l_1 and l_2 intersect, there exist λ and μ such that

$\begin{pmatrix} 8-\lambda \\ 2+3\lambda \\ -12+2\lambda \end{pmatrix} = \begin{pmatrix} -4+4\mu \\ 10+2\mu \\ p-\mu \end{pmatrix}$

$\Rightarrow \begin{pmatrix} -\lambda-4\mu \\ 3\lambda-2\mu \\ 2\lambda+\mu \end{pmatrix} = \begin{pmatrix} -12 \\ 8 \\ p+12 \end{pmatrix}$

i and **j** components $\Rightarrow$ $\lambda = 4$, $\mu = 2$

k component: $2(4)+2 = p+12 \Rightarrow p = -2$

c Substituting $\lambda = 4$ into l_1,

$\begin{pmatrix} 8 \\ 2 \\ -12 \end{pmatrix} + 4\begin{pmatrix} -1 \\ 3 \\ 2 \end{pmatrix} = \begin{pmatrix} 4 \\ 14 \\ -4 \end{pmatrix}$

So $(4, 14, -4)$

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9 d Let the intersection point be P.

$\therefore \overrightarrow{OB} = \overrightarrow{OP} - \overrightarrow{PA}$

$$\Rightarrow \overrightarrow{OB} = \begin{pmatrix} 4 \\ 14 \\ -4 \end{pmatrix} - \begin{pmatrix} 5 \\ -15 \\ -10 \end{pmatrix} = \begin{pmatrix} -1 \\ 29 \\ 6 \end{pmatrix}$$

10 a $$k = \begin{pmatrix} 6 \\ -2 \\ 4 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix} = 12 - 6 - 4 = 2$$

b Expanding the scalar product, $2x + 3y - z = 2$

10 c Find an equation for l: since it passes through $P(6,4,8)$ and is normal to the plane, an equation is

$$\mathbf{r} = \begin{pmatrix} 6 \\ 4 \\ 8 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix}$$

$\therefore$ At N,

$$\left(\left(\begin{pmatrix} 6 \\ 4 \\ 8 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix} \right) \cdot \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix} = 2 \right.$$

$$\Rightarrow \begin{pmatrix} 6 \\ 4 \\ 8 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix} = 2$$

$$\Rightarrow 12 + 12 - 8 + \lambda(4 + 9 + 1) = 2$$

$$\Rightarrow \lambda = -1$$

Substituting this into l,

$$\begin{pmatrix} 6 \\ 4 \\ 8 \end{pmatrix} - \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix} = \begin{pmatrix} 4 \\ 1 \\ 9 \end{pmatrix}$$

$\therefore N(4,1,9)$

11 a $$\frac{x-3}{5} = \frac{y+2}{3} = \frac{4-z}{1} = \lambda$$

$x = 3 + 5\lambda,\ y = -2 + 3\lambda,\ z = 4 - \lambda$

$4(3 + 5\lambda) + 3(-2 + 3\lambda) - 2(4 - \lambda) = -10$

$$\Rightarrow \lambda = -\frac{8}{31}$$

$$\therefore x = \frac{53}{31},\ y = -\frac{86}{31},\ z = \frac{132}{31}$$

$$\therefore P\left(\frac{53}{31}, -\frac{86}{31}, \frac{132}{31}\right)$$

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11 b $\mathbf{b} = \begin{pmatrix} 5 \\ 3 \\ -1 \end{pmatrix}$, $\mathbf{n} = \begin{pmatrix} 4 \\ 3 \\ -2 \end{pmatrix}$

$\mathbf{b} \cdot \mathbf{n} = \begin{pmatrix} 5 \\ 3 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} 4 \\ 3 \\ -2 \end{pmatrix} = 20 + 9 + 2 = 31$

$|\mathbf{b}| = \sqrt{5^2 + 3^2 + (-1)^2} = \sqrt{35}$

$|\mathbf{n}| = \sqrt{4^2 + 3^2 + (-2)^2} = \sqrt{29}$

$\therefore \sin\theta = \dfrac{31}{\sqrt{35}\sqrt{29}}$

$\Rightarrow \theta = 76.7^o$ (1d.p.)

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Exercise 9F

1 Let A be a general point on the first line and B be a general point on the second line,

then $\overrightarrow{AB} = \begin{pmatrix} -2 \\ +2 \\ 0 \end{pmatrix} + t\begin{pmatrix} -3 \\ -4 \\ 5 \end{pmatrix}$, where $t = \mu - \lambda$.

Find the minimum value of the quadratic by using calculus, or completion of the square.

Let the distance $AB = x$

then $x^2 = (-2-3t)^2 + (2-4t)^2 + (5t)^2 = 8 - 4t + 50t^2$

The minimum value of x^2 occurs when $t = \frac{1}{25}$

So $x^2 = 8 - \frac{4}{25} + \frac{50}{625}$

$= \frac{198}{25}$

$\therefore x = \frac{\sqrt{198}}{5}$ or 2.81 (3 s.f.)

2 Let the lines respectively be l_1 and l_2

Let A and B be general points on l_1 and l_2 respectively.

$\therefore \overrightarrow{AB} = \begin{pmatrix} 5+\lambda+4\mu \\ -3-\lambda-2\mu \\ 1+\lambda+3\mu \end{pmatrix}$

$\overrightarrow{AB}$ perpendicular to l_1

$\therefore \begin{pmatrix} 5+\lambda+4\mu \\ -3-\lambda-2\mu \\ 1+\lambda+3\mu \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix} = 0$

$\Rightarrow \lambda + 3\mu = -3$

$\overrightarrow{AB}$ perpendicular to l_2

$\begin{pmatrix} 5+\lambda+4\mu \\ -3-\lambda-2\mu \\ 1+\lambda+3\mu \end{pmatrix} \cdot \begin{pmatrix} 4 \\ -2 \\ 3 \end{pmatrix} = 0$

$\Rightarrow 9\lambda + 29\mu = -29$

Solving these simultaneous equations,

$\lambda = 0,\ \mu = -1$

$\therefore \overrightarrow{AB} = \begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix}$

$\Rightarrow |\overrightarrow{AB}| = \sqrt{1^2 + 1^2 + (-2)^2} = \sqrt{6}$

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3 a $\begin{pmatrix} 7+6\lambda \\ 3+2\lambda \\ 1-4\lambda \end{pmatrix} = \begin{pmatrix} -1 \\ 1-2\mu \\ 2 \end{pmatrix}$

i component $\Rightarrow \lambda = -\frac{4}{3}$

k component $\Rightarrow \lambda = -\frac{1}{4}$

So the equations aren't consistent

Therefore, l_1 and l_2 do not intersect

Let A and B be general points on l_1 and l_2 respectively.

$$\therefore \overrightarrow{AB} = \begin{pmatrix} -8-6\lambda \\ -2-2\lambda-2\mu \\ 1+4\lambda \end{pmatrix}$$

$\overrightarrow{AB}$ is perpendicular to l_1

$$\therefore \begin{pmatrix} -8-6\lambda \\ -2-2\lambda-2\mu \\ 1+4\lambda \end{pmatrix} \cdot \begin{pmatrix} 6 \\ 2 \\ -4 \end{pmatrix} = 0$$

$$\Rightarrow 14\lambda + \mu = -14$$

$\overrightarrow{AB}$ is perpendicular to l_2

$$\begin{pmatrix} -8-6\lambda \\ -2-2\lambda-2\mu \\ 1+4\lambda \end{pmatrix} \cdot \begin{pmatrix} 0 \\ -2 \\ 0 \end{pmatrix} = 0$$

$$\Rightarrow \lambda + \mu = -1$$

Solving these simultaneous equations,

$\lambda = -1,\ \mu = 0$

$$\therefore \overrightarrow{AB} = \begin{pmatrix} -2 \\ 0 \\ -3 \end{pmatrix}$$

$$\Rightarrow |\overrightarrow{AB}| = \sqrt{(-2)^2 + (-3)^2} = \sqrt{13} = 3.61 \text{ (3 s.f.)}$$

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3 b Assume that l_1 and l_2 meet:

$$\begin{pmatrix} 2+2\lambda \\ 1-2\lambda \\ -2+2\lambda \end{pmatrix} = \begin{pmatrix} 1+\mu \\ -1-\mu \\ 3+\mu \end{pmatrix}$$

i.e. $2+2\lambda = 1+\mu$ **(1)**

$1-2\lambda = -1-\mu$ **(2)**

$-2+2\lambda = 3+\mu$ **(3)**

Adding equations **(1)** and **(2)** gives $3 = 0$

This is a contradiction.

$\therefore$ Lines do not meet.

The lines are in fact parallel as $2\mathbf{i} - 2\mathbf{j} + 2\mathbf{k}$ is a multiple of $\mathbf{i} - \mathbf{j} + \mathbf{k}$.

The distance between them is found by considering A on line l_1 and B on line l_2.

Then $\overrightarrow{AB} = -\mathbf{i} - 2\mathbf{j} + 5\mathbf{k} + t(\mathbf{i} - \mathbf{j} + \mathbf{k})$

$$\left|\overrightarrow{AB}\right|^2 = x^2 = (-1+t)^2 + (-2-t)^2 + (5+t)^2$$
$$= 1 - 2t + t^2 + 4 + 4t + t^2 + 25 + 10t + t^2$$
$$= 30 + 12t + 3t^2$$

The minimum value of x^2 occurs when $\dfrac{\mathrm{d}(x^2)}{\mathrm{d}t} = 0$

$$\frac{\mathrm{d}(x)^2}{\mathrm{d}t} = 12 + 6t$$

When $\dfrac{\mathrm{d}(x)^2}{\mathrm{d}t} = 0, t = -2$

$$\therefore x^2 = 30 - 24 + 12$$
$$= 18$$
$$\therefore x = \sqrt{18} = 3\sqrt{2} \text{ or } 4.24 \text{ (3 s.f.)}$$

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3 c Assume that l_1 and l_2 meet. Then

$$\begin{pmatrix}1+2\lambda\\1+\lambda\\5-2\lambda\end{pmatrix}=\begin{pmatrix}-1+\mu\\-1+\mu\\2+\mu\end{pmatrix} \quad \textbf{(1)} \quad \textbf{(2)} \quad \textbf{(3)}$$

i.e. $1+2\lambda=-1+\mu$ **(1)**

$1+\lambda=-1+\mu$ **(2)**

$5-2\lambda=2+\mu$ **(3)**

Comparing equations **(1)** and **(2)** gives $1+2\lambda=1+\lambda$, so $\lambda=0$

Substituting $\lambda=0$ in **(1)** gives $\mu=2$

Substituting these values in (3) gives $5-2\times0=2+2$, which is a contradiction

$\therefore$ Lines do not meet.

Let A and B be general points on l_1 and l_2 respectively.

$$\therefore \overrightarrow{AB}=\begin{pmatrix}-2+\mu-2\lambda\\-2+\mu-\lambda\\-3+\mu+2\lambda\end{pmatrix}$$

$\overrightarrow{AB}$ is perpendicular to l_1

$$\therefore \begin{pmatrix}-2+\mu-2\lambda\\-2+\mu-\lambda\\-3+\mu+2\lambda\end{pmatrix}\cdot\begin{pmatrix}2\\1\\-2\end{pmatrix}=0$$

$\Rightarrow \mu-9\lambda=0$

$\overrightarrow{AB}$ is perpendicular to l_2

$$\begin{pmatrix}-2+\mu-2\lambda\\-2+\mu-\lambda\\-3+\mu+2\lambda\end{pmatrix}\cdot\begin{pmatrix}1\\1\\1\end{pmatrix}=0$$

$\Rightarrow -7+3\mu-\lambda=0$

Solving these simultaneous equations,

$\lambda=\frac{7}{26}$, $\mu=\frac{63}{26}$

$$\therefore \overrightarrow{AB}=\begin{pmatrix}-\frac{3}{26}\\\frac{4}{26}\\-\frac{1}{26}\end{pmatrix}$$

$$\Rightarrow |\overrightarrow{AB}|=\sqrt{\left(-\tfrac{3}{26}\right)^2+\left(\tfrac{4}{26}\right)^2+\left(-\tfrac{1}{26}\right)^2}=\frac{1}{\sqrt{26}}=0.196\ (3\ s.f.)$$

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4 Let A be the point $(4,1,-1)$ and B be the point $(3+2t,-1-t,2-t)$ which lies on the line.

Find the distance between (4, 1, –1) and (3 + 2*t*, –1 – *t*, 2 –*t*) at a point on the line.

Then $\overrightarrow{BA} = \mathbf{a} - \mathbf{b}$

$$= [4-(3+2t),1-(-1-t),-1-(2-t)]$$
$$= [1-2t,2+t,-3+t]$$
$$\therefore \left|\overrightarrow{BA}\right|^2 = (1-2t)^2+(2+t)^2+(-3+t)^2$$
$$= 6t^2-6t+14$$

$\left|\overrightarrow{BA}\right|$ is a minimum when $\left|\overrightarrow{BA}\right|^2$ is minimum

This minimum value can be found by calculus or completion of the square.

$$\left|\overrightarrow{BA}\right|^2 = 6(t^2-t)+14$$
$$= 6\left(t-\frac{1}{2}\right)^2+14-\frac{6}{4}$$

This is a minimum when $t=\frac{1}{2}$ and

$$\left|\overrightarrow{BA}\right|^2 = 14-1\frac{1}{2} = 12\frac{1}{2}$$
$$\therefore \left|\overrightarrow{BA}\right| = \sqrt{12\frac{1}{2}} = 3.54 \text{ (3 s.f.)}$$

5 a The distance from the origin to the plane $\mathbf{r}\cdot(6\mathbf{i}+6\mathbf{j}-7\mathbf{k})=55$ is $\frac{55}{\sqrt{6^2+6^2+(-7)^2}}$

$$= \frac{55}{\sqrt{121}}$$
$$= \frac{55}{11}$$
$$= 5$$

First find the distance from the origin to each plane, then subtract.

The distance from the origin to the plane $\mathbf{r}\cdot(6\mathbf{i}+6\mathbf{j}-7\mathbf{k})=22$ is $\frac{22}{\sqrt{6^2+6^2+(-7)^2}}$

$$= \frac{22}{11}$$
$$= 2$$

$\therefore$ The distance between the planes is $5-2=3$

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5 b Let $\mathbf{n} = (n_1, n_2, n_3)$,
Take the scalar product of $\mathbf{n}$ with each of the two direction vectors of either of the planes
(The first plane has been used in this solution)

$$\begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix} . \begin{pmatrix} 4 \\ 0 \\ 1 \end{pmatrix} = 4n_1 + n_3 = 0 \qquad \textbf{(1)}$$

$$\begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix} . \begin{pmatrix} 8 \\ 3 \\ 3 \end{pmatrix} = 8n_1 + 3n_2 + 3n_3 = 0 \qquad \textbf{(2)}$$

Since if two vectors are perpendicular, then their scalar product is zero.

Let $n_1 = k$
Substitute $n_1 = k$ into **(1)** to find $n_3 = -4k$
Substitute $n_1 = k$ and $n_3 = -4k$ into **(2)** to find $n_2 = \frac{4}{3}k$
So $\mathbf{n} = \left(k,\ \frac{4}{3}k,\ -4k \right)$
Let $k = 3$
(You could choose any value for k other than zero, 3 is convenient as it removes the fraction.)
$\mathbf{n} = (3,\ 4,\ -12)$
So the Cartesian form of the first plane is
$3x + 4y - 12z = 13$
To find the distance between the planes choose a point on the second plane and use
$\dfrac{|a\alpha + b\beta + c\gamma + d|}{\sqrt{a^2 + b^2 + c^2}}$
Choose (14, 2, 2) as the point on the second plane and substitute to find

$$\text{Distance between planes} = \frac{|3(14) + 4(2) + (-12)(2) + (-13)|}{\sqrt{3^2 + 4^2 + (-12)^2}}$$
$$= \frac{42 + 8 - 24 - 13}{\sqrt{169}}$$
$$= \frac{13}{13}$$
$$= 1$$

So the distance between the planes is 1.

6 a The length of the normal vector $10\mathbf{i} + 10\mathbf{j} + 23\mathbf{k}$ is $\sqrt{10^2 + 10^2 + 23^2} = \sqrt{729} = 27$
$\therefore \ \frac{1}{27}(10\mathbf{i} + 10\mathbf{j} + 23\mathbf{k})$ is a unit vector normal to the plane.
The plane has equation
$\mathbf{r} \cdot (10\mathbf{i} + 10\mathbf{j} + 23\mathbf{k}) = 81$
or $\mathbf{r} \cdot \frac{1}{27}(10\mathbf{i} + 10\mathbf{j} + 23\mathbf{k}) = \frac{81}{27} = 3$
$\therefore$ The perpendicular distance from the origin to the plane is 3.

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6 b A plane parallel to π through the point (–1, –1, 4) has equation

$$\mathbf{r}\cdot\frac{1}{27}(10\mathbf{i}+10\mathbf{j}+23\mathbf{k}) = (-\mathbf{i}-\mathbf{j}+4\mathbf{k})\cdot\frac{1}{27}(10\mathbf{i}+10\mathbf{j}+23\mathbf{k})$$
$$= \frac{-10}{27}-\frac{10}{27}+\frac{92}{27}$$
$$= \frac{72}{27}$$
$$= \frac{8}{3}$$

$\therefore$ The perpendicular distance from the origin to this new plane is $2\frac{2}{3}$

This distance between the planes is $3-2\frac{2}{3}=\frac{1}{3}$

$\therefore$ The perpendicular distance from the point $(-1,-1,4)$ to the plane π is $\frac{1}{3}$

c A plane parallel to π through the point (2, 1, 3) has equation

$$\mathbf{r}\cdot\frac{1}{24}(10\mathbf{i}+10\mathbf{j}+23\mathbf{k}) = (2\mathbf{i}+\mathbf{j}+3\mathbf{k})\cdot\frac{1}{27}(10\mathbf{i}+10\mathbf{j}+23\mathbf{k})$$
$$= \frac{20}{27}+\frac{10}{27}+\frac{69}{27}$$
$$= \frac{99}{27}$$
$$= \frac{11}{3}$$

$\therefore$ The perpendicular distance from the origin to this new plane is $3\frac{2}{3}$

$\therefore$ The distances between this plane and π is $3\frac{2}{3}-3=\frac{2}{3}$

$\therefore$ The perpendicular distance from $(2,1,3)$ to π is $\frac{2}{3}$

d A plane parallel to π through the point $(6,12,-9)$ has equation

$$\mathbf{r}\cdot\frac{1}{27}(10\mathbf{i}+10\mathbf{j}+23\mathbf{k}) = (6\mathbf{i}+12\mathbf{j}-9\mathbf{k})\cdot\frac{1}{27}(10\mathbf{i}+10\mathbf{j}+23\mathbf{k})$$
$$= \frac{60}{27}+\frac{120}{27}-\frac{207}{27}$$
$$= -\frac{27}{27}$$
$$= -1$$

$\therefore$ The perpendicular distance from the origin to this new plane is 1, in the opposite direction.

$\therefore$ The distance between this plane and π is $3-(-1)=4$

$\therefore$ The perpendicular distance from (2, 1, 3) to π is 4.

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7 Let **M** be the midpoint of the line segment joining P to its reflection in l.

Then, for some λ

$$\overrightarrow{PM} = \begin{pmatrix} -1+\lambda \\ -1+2\lambda \\ 1-\lambda \end{pmatrix}$$

$\overrightarrow{PM}$ must be perpendicular to l

$$\therefore \begin{pmatrix} -1+\lambda \\ -1+2\lambda \\ 1-\lambda \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} = 0$$

$$\Rightarrow \lambda = \frac{2}{3}$$

$$\therefore \overrightarrow{PM} = \begin{pmatrix} -\frac{1}{3} \\ \frac{1}{3} \\ \frac{1}{3} \end{pmatrix}$$

The reflection of P in l has position vector

$$\overrightarrow{OP} + 2\overrightarrow{PM} = \begin{pmatrix} 3 \\ 0 \\ 2 \end{pmatrix} + \begin{pmatrix} -\frac{2}{3} \\ \frac{2}{3} \\ \frac{2}{3} \end{pmatrix} = \begin{pmatrix} \frac{7}{3} \\ \frac{2}{3} \\ \frac{8}{3} \end{pmatrix}$$

$$\therefore \left(\frac{7}{3}, \frac{2}{3}, \frac{8}{3}\right)$$

8 a $$\frac{|1(-2)+0(1)+3(1)-5|}{\sqrt{(-2)^2+1^2+1^2}} = \frac{4}{\sqrt{6}} = \frac{2\sqrt{6}}{3}$$

b A unit normal vector to the plane is $\frac{1}{\sqrt{6}}\begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}$

The line segment along the shortest distance from P to the plane is normal to the plane,

and the distance is $\frac{2\sqrt{6}}{3}$

Therefore Q has position vector

$$\overrightarrow{OQ} = \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix} + 2\left(\frac{2\sqrt{6}}{3}\right)\left(\frac{1}{\sqrt{6}}\right)\begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -\frac{5}{3} \\ \frac{4}{3} \\ \frac{13}{3} \end{pmatrix}$$

$$\therefore Q\left(-\frac{5}{3}, \frac{4}{3}, \frac{13}{3}\right)$$

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9 a The kestrel's path in vector form is

$$\mathbf{r} = \begin{pmatrix} 3 \\ 5 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 9 \\ -5 \\ 1.2 \end{pmatrix},\ 0 \leqslant \lambda \leqslant 1$$

Let **G** be the position of the birdwatcher.

Let **H** be a general point in the path. Then,

$$\overrightarrow{GH} = \begin{pmatrix} -2+9\lambda \\ 1-5\lambda \\ -0.7+1.2\lambda \end{pmatrix}$$

The shortest distance between the kestrel and the birdwatcher will occur when $\overrightarrow{GH}$ is perpendicular to the path.

$$\therefore \begin{pmatrix} -2+9\lambda \\ 1-5\lambda \\ -0.7+1.2\lambda \end{pmatrix} \cdot \begin{pmatrix} 9 \\ -5 \\ 1.2 \end{pmatrix} = 0$$

$$\Rightarrow \lambda = \frac{35}{158}$$

$$\therefore \overrightarrow{GH} = \begin{pmatrix} -2+9\left(\frac{35}{158}\right) \\ 1-5\left(\frac{35}{158}\right) \\ -0.7+1.2\left(\frac{35}{158}\right) \end{pmatrix}$$

$$\therefore |\overrightarrow{GH}| = 0.45 \quad (2 \text{ d.p.})$$

$$\Rightarrow |\overrightarrow{GH}| < 0.5$$

So the birdwatcher will be able to spot the kestrel.

b In practice, the bird is very unlikely to fly in a straight line from A to B

10 a $$\frac{|4(3)-(-2)+8(4)-6|}{\sqrt{3^2+(-2)^2+4^2}} = \frac{40}{\sqrt{29}} = \frac{40\sqrt{29}}{29}$$

b $$\left(\lambda \begin{pmatrix} 2 \\ -2 \\ 3 \end{pmatrix} + \mu \begin{pmatrix} -3 \\ 0 \\ 3 \end{pmatrix}\right) \cdot \begin{pmatrix} 2 \\ 5 \\ 2 \end{pmatrix}$$

$$= \lambda \begin{pmatrix} 2 \\ -2 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 5 \\ 2 \end{pmatrix} + \mu \begin{pmatrix} -3 \\ 0 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 5 \\ 2 \end{pmatrix}$$

$$= \lambda(4-10+6) + \mu(-6+0+6) = 0$$

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10 c $\mathbf{n}_1 = \begin{pmatrix} 3 \\ -2 \\ 4 \end{pmatrix}, \mathbf{n}_2 = \begin{pmatrix} 2 \\ 5 \\ 2 \end{pmatrix}$

$\mathbf{n}_1 \cdot \mathbf{n}_2 = \begin{pmatrix} 3 \\ -2 \\ 4 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 5 \\ 2 \end{pmatrix} = 6 - 10 + 8 = 4$

$|\mathbf{n}_1| = \sqrt{29}, \quad |\mathbf{n}_2| = \sqrt{33}$

$\therefore \cos\theta = \dfrac{4}{\sqrt{29}\sqrt{33}}$

$\Rightarrow \theta = 82.6^o$ (1d.p.)

11 a A vector equation for l_1 is

$\mathbf{r} = \begin{pmatrix} -2 \\ 2 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -1 \\ -2 \end{pmatrix}$

Let M be a general point on l_1

$\overrightarrow{AM} = \begin{pmatrix} -5 + 2\lambda \\ 3 - \lambda \\ -3 - 2\lambda \end{pmatrix}$

$|\overrightarrow{AM}|$ is minimum when $\overrightarrow{AM}$ is perpendicular to l_1

$\therefore \begin{pmatrix} -5 + 2\lambda \\ 3 - \lambda \\ -3 - 2\lambda \end{pmatrix} \cdot \begin{pmatrix} 2 \\ -1 \\ -2 \end{pmatrix} = 0$

$\Rightarrow \lambda = \dfrac{7}{9}$

$\therefore \overrightarrow{AM} = \dfrac{1}{9} \begin{pmatrix} -31 \\ 20 \\ -41 \end{pmatrix}$

$\Rightarrow |\overrightarrow{AM}| = \dfrac{13\sqrt{2}}{3}$

b A vector equation is

$\begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} -31 \\ 20 \\ -41 \end{pmatrix}$

So a cartesian equation is

$\dfrac{x-3}{-31} = \dfrac{y+1}{20} = \dfrac{z-2}{-41}$

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12 First find the intersection between l_1 and the plane.

$$\begin{pmatrix} 2+4\lambda \\ -1-3\lambda \\ 2+4\lambda \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix} = 1+3\lambda = 4$$

$\Rightarrow \lambda = 1$

$\therefore (6, -4, 6)$

Choose any other point along l_1, and reflect it in the plane to obtain e.g. $\begin{pmatrix} 4 \\ -3 \\ 0 \end{pmatrix}$

$$r = \begin{pmatrix} 4 \\ -3 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ -1 \\ 6 \end{pmatrix}$$

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9 Mixed Exercise

1 a $\overrightarrow{AB} = \begin{pmatrix}1\\2\\2\end{pmatrix} - \begin{pmatrix}1\\-1\\3\end{pmatrix} = \begin{pmatrix}0\\3\\-1\end{pmatrix}$

$\therefore \mathbf{r} = \begin{pmatrix}1\\-1\\3\end{pmatrix} + \lambda\begin{pmatrix}0\\3\\-1\end{pmatrix} = \mathbf{i} - \mathbf{j} + 3\mathbf{k} + \lambda(3\mathbf{j} - \mathbf{k})$

b $\mathbf{c} = \overrightarrow{OA} + \frac{2}{3}\overrightarrow{AB}$

$= \begin{pmatrix}1\\-1\\3\end{pmatrix} + \frac{2}{3}\begin{pmatrix}0\\3\\-1\end{pmatrix} = \begin{pmatrix}1\\1\\\frac{7}{3}\end{pmatrix}$

$= \mathbf{i} + \mathbf{j} + \frac{7}{3}\mathbf{k}$

2 $A(7,-1,2)$ and $B(-1,3,8)$

$\overrightarrow{AB} = \begin{pmatrix}-8\\4\\6\end{pmatrix} = -2\begin{pmatrix}4\\-2\\-3\end{pmatrix}$

$\therefore$ **A** vector equation is

$\mathbf{r} = \begin{pmatrix}7\\-1\\2\end{pmatrix} + \lambda\begin{pmatrix}4\\-2\\-3\end{pmatrix}$

$\therefore \frac{x-7}{4} = \frac{y+1}{-2} = \frac{z-2}{-3}$

3 $(2\mathbf{i} + 3\mathbf{j} - 4\mathbf{k}) + \lambda(2\mathbf{j} + 3\mathbf{k})$

4 a $\frac{x-3}{2} = \frac{y+2}{1} = \frac{z-1}{-1}$

b $\frac{0-3}{2} = \frac{a+2}{1} = \frac{b-1}{-1}$

$\Rightarrow a = -\frac{7}{2},\ b = \frac{5}{2}$

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5 Use the equation $\mathbf{r} = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ 1 \\ -2 \end{pmatrix}$

When $\lambda = 2$, $\mathbf{r} = \begin{pmatrix} 7 \\ 4 \\ -5 \end{pmatrix}$, so the point $(7, 4, -5)$ lies on l

$\begin{pmatrix} 9 \\ 3 \\ -6 \end{pmatrix} = 3\begin{pmatrix} 3 \\ 1 \\ -2 \end{pmatrix}$ so these two vectors are parallel.

So an alternative form of the equation is $\mathbf{r} = \begin{pmatrix} 7 \\ 4 \\ -5 \end{pmatrix} + \lambda \begin{pmatrix} 9 \\ 3 \\ -6 \end{pmatrix}$

6 a $\mathbf{r} = \begin{pmatrix} -2 \\ 2 \\ -3 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 3 \\ 4 \end{pmatrix}$

b Set $\lambda = 2$

7 a $\overrightarrow{AB} = \begin{pmatrix} -1 \\ 4 \\ -3 \end{pmatrix}$, $\overrightarrow{AC} = \begin{pmatrix} 2 \\ 3 \\ 3 \end{pmatrix}$

$\therefore \mathbf{r} = \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 4 \\ -3 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ 3 \\ 3 \end{pmatrix}$

b $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 4 \\ -3 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ 3 \\ 3 \end{pmatrix}$

$x - 2 = -\lambda + 2\mu \quad (1)$

$y + 1 = 4\lambda + 3\mu \quad (2)$

$z - 2 = -3\lambda + 3\mu \quad (3)$

$(3) - (2)$: $z - y - 3 = -7\lambda \Rightarrow \lambda = \dfrac{y - z + 3}{7}$

$\Rightarrow \mu = \dfrac{3y + 4z - 5}{21}$

Into (1):

$x - 2 = -\dfrac{y + 3 - z}{7} + 2\left(\dfrac{3y + 4z - 5}{21}\right)$

$\Rightarrow 21(x - 2) = -3(y + 3 - z) + 2(3y + 4z - 5)$

$\Rightarrow 21x - 3y - 11z = 23$

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8 Three points on the plane are $A(6,0,0)$, $B(8,-3,0)$ and $C(10,0,3)$

$$\overrightarrow{AB} = \begin{pmatrix} 2 \\ -3 \\ 0 \end{pmatrix}, \quad \overrightarrow{AC} = \begin{pmatrix} 4 \\ 0 \\ 3 \end{pmatrix}$$

$$\therefore \mathbf{r} = \begin{pmatrix} 6 \\ 0 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -3 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} 4 \\ 0 \\ 3 \end{pmatrix}$$

9 a $\overrightarrow{ML} = \begin{pmatrix} 4 \\ 7 \\ 7 \end{pmatrix} - \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \\ 5 \end{pmatrix}$

$$\overrightarrow{MN} = \begin{pmatrix} 2 \\ 4 \\ 6 \end{pmatrix} - \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 4 \end{pmatrix}$$

b $\dfrac{\overrightarrow{ML}.\overrightarrow{MN}}{\left|\overrightarrow{ML}\right|\left|\overrightarrow{MN}\right|} = \dfrac{27}{5\sqrt{2}\ 3\sqrt{2}} = \dfrac{9}{10}$

10 a $\mathbf{a} = \begin{pmatrix} 9 \\ -2 \\ 1 \end{pmatrix}, \mathbf{b} = \begin{pmatrix} 6 \\ 2 \\ 6 \end{pmatrix}, \mathbf{c} = \begin{pmatrix} 3 \\ p \\ q \end{pmatrix}$

$$\mathbf{b} - \mathbf{a} = \begin{pmatrix} 6 \\ 2 \\ 6 \end{pmatrix} - \begin{pmatrix} 9 \\ -2 \\ 1 \end{pmatrix} = \begin{pmatrix} -3 \\ 4 \\ 5 \end{pmatrix}$$

Equation of l:

$$\mathbf{r} = \begin{pmatrix} 9 \\ -2 \\ 1 \end{pmatrix} + \mu \begin{pmatrix} -3 \\ 4 \\ 5 \end{pmatrix}$$

b Since C lies on l,

$$\begin{pmatrix} 3 \\ p \\ q \end{pmatrix} = \begin{pmatrix} 9 \\ -2 \\ 1 \end{pmatrix} + \mu \begin{pmatrix} -3 \\ 4 \\ 5 \end{pmatrix}$$

$3 = 9 - 3\mu$

$3\mu = 6$

$\mu = 2$

So $p = -2 + 4\mu = 6$ and $q = 1 + 5\mu = 11$

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10 c $\cos\theta = \dfrac{\mathbf{OC}.\mathbf{AB}}{|\mathbf{OC}||\mathbf{AB}|}$

$\mathbf{OC}.\ \mathbf{AB} = \begin{pmatrix}3\\6\\11\end{pmatrix}\cdot\begin{pmatrix}-3\\4\\5\end{pmatrix} = -9+24+55 = 70$

$|\mathbf{OC}| = \sqrt{3^2+6^2+11^2} = \sqrt{166}$

$|\mathbf{AB}| = \sqrt{(-3)^2+4^2+5^2} = \sqrt{50}$

$\cos\theta = \dfrac{70}{\sqrt{166}\sqrt{50}}$

$\theta = 39.8°$ (1 d.p.)

d If **OD** and **AB** are perpendicular, $\mathbf{d}\ .\ (\mathbf{b} - \mathbf{a}) = 0$

Since **d** lies on **AB**, use $\mathbf{d} = \begin{pmatrix}9-3\mu\\-2+4\mu\\1+5\mu\end{pmatrix}$

$\begin{pmatrix}9-3\mu\\-2+4\mu\\1+5\mu\end{pmatrix}\cdot\begin{pmatrix}-3\\4\\5\end{pmatrix} = 0$

$-3(9-3\mu)+4(-2+4\mu)+5(1+5\mu) = 0$

$-27+9\mu-8+16\mu+5+25\mu = 0$

$50\mu = 30$

$\mu = \dfrac{3}{5}$

$\mathbf{d} = \begin{pmatrix}9-\frac{9}{5}\\-2+\frac{12}{5}\\1+3\end{pmatrix} = \frac{36}{5}\mathbf{i}+\frac{2}{5}\mathbf{j}+4\mathbf{k}$

11 a $\overrightarrow{AB} = \begin{pmatrix}5\\0\\-3\end{pmatrix}-\begin{pmatrix}1\\2\\-3\end{pmatrix} = \begin{pmatrix}4\\-2\\0\end{pmatrix}$

$\therefore\ \mathbf{r} = \begin{pmatrix}1\\2\\-3\end{pmatrix}+\mu\begin{pmatrix}4\\-2\\0\end{pmatrix}$

b Set $\mu = -3$

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11 c Let $\mathbf{a} = \begin{pmatrix} 4 \\ -2 \\ 0 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix}$

$\mathbf{a} \cdot \mathbf{b} = \begin{pmatrix} 4 \\ -2 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix} = 4 + 4 + 0 = 8$

$|\mathbf{a}| = \sqrt{4^2 + (-2)^2} = \sqrt{20}$

$|\mathbf{b}| = \sqrt{1^2 + (-2)^2 + 2^2} = 3$

$\therefore \cos\theta = \dfrac{8}{3\sqrt{20}}$

$\Rightarrow \theta = 53.4^o$ (1.d.p.)

d Let G be a general point on l_1

$\overrightarrow{\mathbf{CG}} = \begin{pmatrix} 1+4\mu \\ -2-2\mu \\ 2 \end{pmatrix}$

When **C** is closest to l_1, $\overrightarrow{\mathbf{CG}}$ is perpendicular to l_1.

$\therefore \begin{pmatrix} 1+4\mu \\ -2-2\mu \\ 2 \end{pmatrix} \cdot \begin{pmatrix} 4 \\ -2 \\ 0 \end{pmatrix} = 0$

$\Rightarrow \mu = -\dfrac{2}{5}$

$\therefore \overrightarrow{\mathbf{CG}} = \begin{pmatrix} -\frac{3}{5} \\ -\frac{6}{5} \\ 2 \end{pmatrix}$

$\Rightarrow |\overrightarrow{\mathbf{CG}}| = \sqrt{\left(-\frac{3}{5}\right)^2 + \left(-\frac{6}{5}\right)^2 + 2^2} = \dfrac{\sqrt{145}}{5}$

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12 a Line l_1: $\mathbf{r} = \begin{pmatrix} 3 \\ 4 \\ -5 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix}$

Line l_2: $\mathbf{r} = \begin{pmatrix} 9 \\ 1 \\ -2 \end{pmatrix} + \mu \begin{pmatrix} 4 \\ 1 \\ -1 \end{pmatrix}$

Using the direction vectors:

$\begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} 4 \\ 1 \\ -1 \end{pmatrix} = 4 - 2 - 2 = 0$

Since the scalar product is zero, the directions are perpendicular.

b At an intersection point: $\begin{pmatrix} 3+\lambda \\ 4-2\lambda \\ -5+2\lambda \end{pmatrix} = \begin{pmatrix} 9+4\mu \\ 1+\mu \\ -2-\mu \end{pmatrix}$

$3 + \lambda = 9 + 4\mu$ $(\times 2)$

$4 - 2\lambda = 1 + \mu$

$6 + 2\lambda = 18 + 8\mu$

$4 - 2\lambda = 1 + \mu$

Adding: $10 = 19 + 9\mu$

$\Rightarrow 9\mu = -9$

$\Rightarrow \mu = -1$

$3 + \lambda = 9 - 4$

$\Rightarrow \lambda = 2$

Intersection point: $\begin{pmatrix} 3+\lambda \\ 4-2\lambda \\ -5+2\lambda \end{pmatrix} = \begin{pmatrix} 5 \\ 0 \\ -1 \end{pmatrix}$

Position vector of **A** is a = 5**i** – **k**.

c Position vector of **B**: $\mathbf{b} = 10\mathbf{j} - 11\mathbf{k} = \begin{pmatrix} 0 \\ 10 \\ -11 \end{pmatrix}$

For l_1, to give zero as the x component, $\lambda = -3$.

$\mathbf{r} = \begin{pmatrix} 3 \\ 4 \\ -5 \end{pmatrix} - 3\begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 10 \\ -11 \end{pmatrix}$

So **B** lies on l_1.

For l_2, to give – 11 as the z component, $\mu = 9$.

$\mathbf{r} = \begin{pmatrix} 9 \\ 1 \\ -2 \end{pmatrix} + 9\begin{pmatrix} 4 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 45 \\ 10 \\ -11 \end{pmatrix}$

So **B** does not lie on l_2.

So Only one of the submarines passes through **B**.

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12 d $|\mathbf{AB}| = \sqrt{(0-5)^2 + (10-0)^2 + [-11-(-1)]^2}$

$= \sqrt{(-5)^2 + 10^2 + (-10)^2}$

$= \sqrt{225} = 15$

Since 1 unit represents 100 m, the distance **AB** is $15 \times 100 = 1500$ m $= 1.5$ km.

13 Let A and B be general points on l_1 and l_2 respectively.

$$\therefore \overrightarrow{AB} = \begin{pmatrix} 6+t \\ -2-2t-s \\ t \end{pmatrix}$$

$\overrightarrow{AB}$ is perpendicular to l_1

$$\therefore \begin{pmatrix} 6+t \\ -2-2t-s \\ t \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = 0$$

$\Rightarrow -2-2t-s = 0$

$\overrightarrow{AB}$ is perpendicular to l_2

$$\begin{pmatrix} 6+t \\ -2-2t-s \\ t \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} = 0$$

$\Rightarrow 10+6t+2s = 0$

Solving these simultaneous equations:

$s = 4,\ \mathrm{t} = -3$

$$\therefore \overrightarrow{AB} = \begin{pmatrix} 3 \\ 0 \\ -3 \end{pmatrix}$$

$\Rightarrow |\overrightarrow{AB}| = \sqrt{3^2 + (-3)^2} = \sqrt{18} = 3\sqrt{2} = 4.24$ (3 s.f.)

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14 Let A and B be general points on l_1 and l_2 respectively.

$$\therefore \overrightarrow{AB} = \begin{pmatrix} 6+t-3s \\ -2+2t+s \\ -s \end{pmatrix}$$

$\overrightarrow{AB}$ is perpendicular to l_1

$$\therefore \begin{pmatrix} 6+t-3s \\ -2+2t+s \\ -s \end{pmatrix} \cdot \begin{pmatrix} 3 \\ -1 \\ 1 \end{pmatrix} = 0$$

$$\Rightarrow 20+t-11s=0$$

$\overrightarrow{AB}$ is perpendicular to l_2

$$\begin{pmatrix} 6+t-3s \\ -2+2t+s \\ -s \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} = 0$$

$$\Rightarrow 2+5t-s=0$$

Solving these simultaneous equations:

$$s=\frac{49}{27},\ \mathrm{t}=-\frac{1}{27}$$

$$\therefore \overrightarrow{AB} = \begin{pmatrix} \frac{14}{27} \\ -\frac{7}{27} \\ -\frac{49}{27} \end{pmatrix}$$

$$\Rightarrow |\overrightarrow{AB}| = \sqrt{\left(\tfrac{14}{27}\right)^2+\left(-\tfrac{7}{27}\right)^2+\left(-\tfrac{49}{27}\right)^2} = \sqrt{\frac{2646}{729}} = \sqrt{\frac{294}{81}} = \frac{7\sqrt{6}}{9} = 1.91 \text{ (3 s.f.)}$$

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15 a $\mathbf{r} = \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix}$ and $\mathbf{a} = \begin{pmatrix} 4 \\ 3 \\ 1 \end{pmatrix}$

When $\lambda = 0$, $\mathbf{r} = \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}$

When $\lambda = 1$, $\mathbf{r} = \begin{pmatrix} 3 \\ 1 \\ -3 \end{pmatrix}$

So the position vectors $\begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}$, $\begin{pmatrix} 3 \\ 1 \\ -3 \end{pmatrix}$ and $\begin{pmatrix} 4 \\ 3 \\ 1 \end{pmatrix}$ all lie on the plane

Suppose the plane has equation $ax + by + cz = 1$

Substituting each of the coordinates into this equation gives:

$a + b - 2c = 1$

$3a + b - 3c = 1$

$4a + 3b + c = 1$

Solving these equations simultaneously gives:

$a = -\frac{2}{15}, b = \frac{9}{15}, c = -\frac{4}{15}$

Therefore the equation of the plane is

$-\frac{2}{15}x + \frac{9}{15}y - \frac{4}{15}z = 1$

or $2x - 9y + 4z = -15$

So the required equation is $\mathbf{r}.\begin{pmatrix} 2 \\ -9 \\ 4 \end{pmatrix} = -15$

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15 b $\mathbf{r} = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix}$ and $\mathbf{a} = \begin{pmatrix} 3 \\ 5 \\ 1 \end{pmatrix}$

When $\lambda = 0$, $\mathbf{r} = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$

When $\lambda = 1$, $\mathbf{r} = \begin{pmatrix} 3 \\ 3 \\ -1 \end{pmatrix}$

So the position vectors $\begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$, $\begin{pmatrix} 3 \\ 3 \\ -1 \end{pmatrix}$ and $\begin{pmatrix} 3 \\ 5 \\ 1 \end{pmatrix}$ all lie on the plane

Suppose the plane has equation $ax + by + cz = 1$

Substituting each of the coordinates into this equation gives:

$a + 2b + 2c = 1$

$3a + 3b - c = 1$

$3a + 5b + c = 1$

Solving these equations simultaneously gives:

$a = 1,\ b = -\frac{1}{2}, c = \frac{1}{2}$

Therefore the equation of the plane is

$x - \frac{1}{2}y + \frac{1}{2}z = 1$

or $2x - y + z = 2$

So the required equation is $\mathbf{r}.\begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} = 2$

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15 c $\mathbf{r} = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$ and $\mathbf{a} = \begin{pmatrix} 7 \\ 8 \\ 6 \end{pmatrix}$

When $\lambda = 0$, $\mathbf{r} = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$

When $\lambda = 1$, $\mathbf{r} = \begin{pmatrix} 3 \\ 1 \\ 3 \end{pmatrix}$

So the position vectors $\begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$, $\begin{pmatrix} 3 \\ 1 \\ 3 \end{pmatrix}$ and $\begin{pmatrix} 7 \\ 8 \\ 6 \end{pmatrix}$ all lie on the plane

Suppose the plane has equation $ax + by + cz = 1$

Substituting each of the coordinates into this equation gives:

$2a - b + c = 1$

$3a + b + 3c = 1$

$7a + 8b + 6c = 1$

Solving these equations simultaneously gives:

$a = \frac{8}{22}, b = -\frac{5}{22}, c = \frac{1}{22}$

Therefore the equation of the plane is

$\frac{8}{22}x - \frac{5}{22}y + \frac{1}{22}z = 1$

or $8x - 5y + z = 22$

So the required equation is $\mathbf{r}.\begin{pmatrix} 8 \\ -5 \\ 1 \end{pmatrix} = 22$

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16 The line is in the direction $3\mathbf{i}+\mathbf{j}+2\mathbf{k}$. This lies in the plane.

$(2, -4, 1)$ is a point on the line. This also lies in the plane, as does the point $(1, 1, 1)$.

$\therefore \begin{pmatrix}2\\-4\\1\end{pmatrix}-\begin{pmatrix}1\\1\\1\end{pmatrix}=\begin{pmatrix}1\\-5\\0\end{pmatrix}$ is a direction in the plane.

First obtain the equation of the plane in the form, $\mathbf{r}\cdot\mathbf{n}=\mathbf{a}\cdot\mathbf{n}$, then convert to Cartesian form.

Let $\mathbf{n}=\begin{pmatrix}a\\b\\c\end{pmatrix}$ be perpendicular to the plane

So $\begin{pmatrix}3\\1\\2\end{pmatrix}.\begin{pmatrix}\mathbf{a}\\\mathbf{b}\\\mathbf{c}\end{pmatrix}=0$ and $\begin{pmatrix}1\\-5\\0\end{pmatrix}.\begin{pmatrix}\mathbf{a}\\\mathbf{b}\\\mathbf{c}\end{pmatrix}=0$

Therefore $3a+b+2c=0$ and $a-5b=0$

Choosing $a=5$ gives $b=1$ and $c=-8$

Therefore a normal vector is given by $\begin{pmatrix}5\\1\\-8\end{pmatrix}$

$\therefore$ The equation of the plane is

$\mathbf{r}\cdot(5\mathbf{i}+\mathbf{j}-8\mathbf{k})=(\mathbf{i}+\mathbf{j}+\mathbf{k})\cdot(5\mathbf{i}+\mathbf{j}-8\mathbf{k})$

i.e: $5x+y-8z=-2$

This is a Cartesian equation of the plane.

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17 a $\overrightarrow{\mathbf{AB}} = \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}$ and $\overrightarrow{\mathbf{AC}} = \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}$

Let $\mathbf{n} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$ be perpendicular to the plane

Then $\mathbf{n}$ is perpendicular to both $\begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}$

So $\begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix} . \begin{pmatrix} \mathbf{a} \\ \mathbf{b} \\ \mathbf{c} \end{pmatrix} = 0$ and $\begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} . \begin{pmatrix} \mathbf{a} \\ \mathbf{b} \\ \mathbf{c} \end{pmatrix} = 0$

Therefore $2a - 2b + c = 0$ and $a + b - 2c = 0$

Choosing $a = 1$ gives $b - 2c = -1$ and $-2b + c = -2$

Solving simultaneously gives $b = \frac{5}{3}$ and $c = \frac{4}{3}$

Therefore a normal vector is given by $\begin{pmatrix} 1 \\ \frac{5}{3} \\ \frac{4}{3} \end{pmatrix}$, or $\begin{pmatrix} 3 \\ 5 \\ 4 \end{pmatrix}$

Therefore a unit vector normal to the plane is $\frac{1}{\sqrt{3^2 + 5^2 + 4^2}}(3\mathbf{i} + 5\mathbf{j} + 4\mathbf{k})$

$= \frac{1}{\sqrt{50}}(3\mathbf{i} + 5\mathbf{j} + 4\mathbf{k})$

b The equation of the plane may be written as

$$\begin{aligned} \mathbf{r} \cdot (3\mathbf{i} + 5\mathbf{j} + 4\mathbf{k}) &= (\mathbf{i} + 3\mathbf{j} + 3\mathbf{k}) \cdot (3\mathbf{i} + 5\mathbf{j} + 4\mathbf{k}) \\ &= 3 + 15 + 12 \\ &= 30 \end{aligned}$$

i.e. $3x + 5y + 4z = 30$

c The perpendicular distance from the origin to the plane is

$$\frac{30}{\sqrt{3^2 + 5^2 + 4^2}} = \frac{30}{\sqrt{50}} = \frac{30\sqrt{50}}{50} = 3\sqrt{2}.$$

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18 a The plane with vector equation
$\mathbf{r} = \mathbf{i} + s\mathbf{j} + t(\mathbf{i} - \mathbf{k})$
is perpendicular to $\mathbf{i} + \mathbf{k}$, as $(\mathbf{i} + \mathbf{k}) \cdot \mathbf{j} = 0$ and $(\mathbf{i} + \mathbf{k}) \cdot (\mathbf{i} - \mathbf{k}) = 1 - 1 = 0$
The plane also has equation
$\mathbf{r} \cdot (\mathbf{i} + \mathbf{k}) - \mathbf{i} \cdot (\mathbf{i} + \mathbf{k})$, as $\mathbf{i}$ is the position vector of a point on the plane.
i.e. $\mathbf{r} \cdot (\mathbf{i} + \mathbf{k}) = 1$

b The perpendicular distance from the origin to this plane is $\dfrac{1}{\sqrt{1^2 + 1^2}} = \dfrac{1}{\sqrt{2}} = \dfrac{\sqrt{2}}{2}$ or 0.707 (3 s.f.)

c The Cartesian form of the equation of the plane is $x + z = 1$.

19 a $\overrightarrow{\mathbf{AB}} = \overrightarrow{\mathbf{OB}} - \overrightarrow{\mathbf{OA}} = (5\mathbf{i} - 2\mathbf{j} + \mathbf{k}) - (\mathbf{i} + \mathbf{j} + \mathbf{k})$
$= 4\mathbf{i} - 3\mathbf{j}$
$\overrightarrow{\mathbf{AC}} = \overrightarrow{\mathbf{OC}} - \overrightarrow{\mathbf{OA}} = (3\mathbf{i} + 2\mathbf{j} + 6\mathbf{k}) - (\mathbf{i} + \mathbf{j} + \mathbf{k})$
$= 2\mathbf{i} + \mathbf{j} + 5\mathbf{k}$

$\overrightarrow{\mathbf{AB}} = \begin{pmatrix} 4 \\ -3 \\ 0 \end{pmatrix}$ and $\overrightarrow{\mathbf{AC}} = \begin{pmatrix} 2 \\ 1 \\ 5 \end{pmatrix}$

Let $\mathbf{n} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$ be perpendicular to the plane

So $\begin{pmatrix} 4 \\ -3 \\ 0 \end{pmatrix} . \begin{pmatrix} \mathbf{a} \\ \mathbf{b} \\ \mathbf{c} \end{pmatrix} = 0$ and $\begin{pmatrix} 2 \\ 1 \\ 5 \end{pmatrix} . \begin{pmatrix} \mathbf{a} \\ \mathbf{b} \\ \mathbf{c} \end{pmatrix} = 0$

Therefore $4a - 3b = 0$ and $2a + b + 5c = 0$

Choosing $a = 3$ gives $b = 4$ and $c = -2$

Therefore a normal vector is given by $\begin{pmatrix} 3 \\ 4 \\ -2 \end{pmatrix}$

Or any multiple of $3\mathbf{i} + 4\mathbf{j} - 2\mathbf{k}$

b An equation of the plane containing **A**, **B** and **C** is
$\mathbf{r} \cdot (3\mathbf{i} + 4\mathbf{j} - 2\mathbf{k}) = (\mathbf{i} + \mathbf{j} + \mathbf{k}) \cdot (3\mathbf{i} + 4\mathbf{j} - 2\mathbf{k})$
i.e. $3x + 4y - 2z - 5 = 0$

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20 a $(2\mathbf{i}-2\mathbf{j}+3\mathbf{k})\cdot(\mathbf{i}+5\mathbf{j}+3\mathbf{k}) = 2-10+9$
$= 1$
$\therefore$ (2, –2, 3) lies on the plane Π_2

b $(2\mathbf{i}-\mathbf{j}+\mathbf{k})\cdot(\mathbf{i}+5\mathbf{j}+3\mathbf{k}) = 2-5+3$
$= 0$
$\therefore$ the normal to plane Π_1 is perpendicular to the normal to plane Π_2.
$\therefore$ Π_1 is perpendicular to Π_2.

c $\mathbf{r} = 2\mathbf{i}-2\mathbf{j}+3\mathbf{k}+\lambda(2\mathbf{i}-\mathbf{j}+\mathbf{k})$

d This line meets the plane Π_1 when
$\left[(2+2\lambda)\mathbf{i}+(-2-\lambda)\mathbf{j}+(3+\lambda)\mathbf{k}\right]\cdot(2\mathbf{i}-\mathbf{j}+\mathbf{k}) = 0$
i.e. $4+4\lambda+2+\lambda+3+\lambda = 0$
i.e. $6\lambda+9 = 0$
$\therefore \lambda = -\frac{3}{2}$
Substitute $\therefore \lambda = -\frac{3}{2}$ into the equation of the line: then $\mathbf{r} = -\mathbf{i}-\frac{1}{2}\mathbf{j}+\frac{3}{2}\mathbf{k}$
i.e. The line meets Π_1 at the point $\left(-1, -\frac{1}{2}, \frac{3}{2}\right)$

e The distance required is
$$\sqrt{(2-(-1))^2+\left(-2-\left(-\frac{1}{2}\right)\right)^2+\left(3-\frac{3}{2}\right)^2} = \sqrt{9+2\frac{1}{4}+2\frac{1}{4}} = \sqrt{13\frac{1}{2}}$$
$= 3.67$ (3 s.f.)

21 a $l_1 : \mathbf{r} = \begin{pmatrix}1\\1\\0\end{pmatrix}+\lambda\begin{pmatrix}2\\1\\-2\end{pmatrix} = \begin{pmatrix}1+2\lambda\\1+\lambda\\-2\lambda\end{pmatrix}$ and $l_2 : \mathbf{r} = \begin{pmatrix}1\\4\\-4\end{pmatrix}+\mu\begin{pmatrix}-3\\0\\1\end{pmatrix} = \begin{pmatrix}1-3\mu\\4\\-4+\mu\end{pmatrix}$

If l_1 and l_2 intersect, then $\begin{pmatrix}1+2\lambda\\1+\lambda\\-2\lambda\end{pmatrix} = \begin{pmatrix}1-3\mu\\4\\-4+\mu\end{pmatrix}$

$1+2\lambda = 1-3\mu$ **(1)**
$1+\lambda = 4$ **(2)**

Equation **(2)** gives $\lambda = 3$
Substituting into **(1)** gives $7 = 1-3\mu$, so $\mu = -2$
Check for consistency: $-2\lambda = -6$ and $-4+\mu = -6$
$2+4\lambda = -5+2\mu$, so these equations are consistent.
Therefore l_1 and l_2 intersect.

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21 b Substitute $\lambda = 3$ into line l_1, so $\mathbf{r} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + 3\begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix} = \begin{pmatrix} 7 \\ 4 \\ -6 \end{pmatrix}$

Therefore the position vector of their point of intersection is $\begin{pmatrix} 7 \\ 4 \\ -6 \end{pmatrix}$

c The cosine of the acute angle θ between the lines is the cosine of the acute angle between their respective direction vectors.

$$\begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix} . \begin{pmatrix} -3 \\ 0 \\ 1 \end{pmatrix} = (2 \times -3) + (1 \times 0) + (-2 \times 1) = -8$$

$$\left| \begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix} \right| = \sqrt{2^2 + 1^2 + (-2)^2} = \sqrt{9} = 3 \text{ and } \left| \begin{pmatrix} -3 \\ 0 \\ 1 \end{pmatrix} \right| = \sqrt{(-3)^2 + 0^2 + 1^2} = \sqrt{10}$$

So $\cos\theta = \left| \frac{-8}{3\sqrt{10}} \right| = \frac{8}{3\sqrt{10}}$

Therefore $\cos\theta = \frac{8}{3\sqrt{10}} \times \frac{\sqrt{10}}{\sqrt{10}} = \frac{8\sqrt{10}}{30} = \frac{4\sqrt{10}}{15}$, as required.

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22 a $\begin{pmatrix}6\\8\\5\end{pmatrix}+\lambda\begin{pmatrix}1\\-1\\1\end{pmatrix}=\begin{pmatrix}3\\a\\2\end{pmatrix}$

$\lambda=-3 \Rightarrow a=11$

$\begin{pmatrix}6\\8\\5\end{pmatrix}+\lambda\begin{pmatrix}1\\-1\\1\end{pmatrix}=\begin{pmatrix}8\\6\\b\end{pmatrix}$

$\lambda=2 \Rightarrow b=7$

b $\left(\begin{pmatrix}6\\8\\5\end{pmatrix}+\lambda\begin{pmatrix}1\\-1\\1\end{pmatrix}\right)\cdot\begin{pmatrix}1\\-1\\1\end{pmatrix}=0$

$\Rightarrow \begin{pmatrix}6\\8\\5\end{pmatrix}\cdot\begin{pmatrix}1\\-1\\1\end{pmatrix}+\lambda\begin{pmatrix}1\\-1\\1\end{pmatrix}\cdot\begin{pmatrix}1\\-1\\1\end{pmatrix}=0$

$\Rightarrow 6-8+5+\lambda(1+1+1)=0$

$\Rightarrow \lambda=-1$

$\therefore$ Position vector of **P** is $\begin{pmatrix}6\\8\\5\end{pmatrix}-\begin{pmatrix}1\\-1\\1\end{pmatrix}=\begin{pmatrix}5\\9\\4\end{pmatrix}$

$\therefore \mathbf{P}(5,9,4)$

c $\sqrt{5^2+9^2+4^2}=\sqrt{122}$

23 a $\overrightarrow{\mathbf{AB}}=\begin{pmatrix}5\\2\\6\end{pmatrix}-\begin{pmatrix}6\\3\\4\end{pmatrix}=\begin{pmatrix}-1\\-1\\2\end{pmatrix}$

b $\mathbf{r}=\begin{pmatrix}6\\3\\4\end{pmatrix}+\lambda\begin{pmatrix}-1\\-1\\2\end{pmatrix}$

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23 c $\overrightarrow{\mathbf{CP}} = \begin{pmatrix} 2-\lambda \\ -7-\lambda \\ 2+2\lambda \end{pmatrix}$

$\begin{pmatrix} 2-\lambda \\ -7-\lambda \\ 2+2\lambda \end{pmatrix} \cdot \begin{pmatrix} -1 \\ -1 \\ 2 \end{pmatrix} = 6\lambda + 9 = 0$

$\Rightarrow \lambda = -\frac{3}{2}$

$\therefore$ The position vector of **P** is

$\begin{pmatrix} 6 \\ 3 \\ 4 \end{pmatrix} - \frac{3}{2}\begin{pmatrix} -1 \\ -1 \\ 2 \end{pmatrix} = \begin{pmatrix} \frac{15}{2} \\ \frac{9}{2} \\ 1 \end{pmatrix}$

$\therefore \mathbf{P}\left(\frac{15}{2}, \frac{9}{2}, 1\right)$ or $\mathbf{P}(7.5, 4.5, 1)$

24 a $\begin{pmatrix} 3+2\lambda \\ -2+\lambda \\ 4-\lambda \end{pmatrix} = \begin{pmatrix} 1+\mu \\ 12-2\mu \\ 8-\mu \end{pmatrix}$

j and **k** components $\Rightarrow \lambda = 2,\ \mu = 6$

Check **i** component: $3+2(2) = 7 = 1+6$

So the equations are consistent

Therefore the lines meet, and they meet at

$\begin{pmatrix} 3+2(2) \\ -2+2 \\ 4-2 \end{pmatrix} = \begin{pmatrix} 7 \\ 0 \\ 2 \end{pmatrix} \quad \therefore A(7, 0, 2)$

b Let $\mathbf{a} = \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 1 \\ -2 \\ -1 \end{pmatrix}$

$\mathbf{a} \cdot \mathbf{b} = \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -2 \\ -1 \end{pmatrix} = 2 - 2 + 1 = 1$

$|\mathbf{a}| = \sqrt{2^2 + 1^2 + (-1)^2} = \sqrt{6}$

$|\mathbf{b}| = \sqrt{6}$

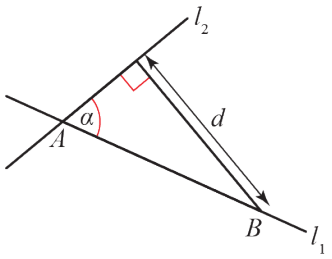
$\therefore \cos\theta = \frac{1}{6}$

$\Rightarrow \theta = 80.4^o$ (1 d.p.)

c Set $\lambda = 2$

24 d The shortest distance of **B** to the line l_2 is given by $\mathbf{d} = \left|\overrightarrow{\mathbf{BA}}\right| \sin\alpha$

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$$\overrightarrow{\mathbf{OA}} = \begin{pmatrix} 7 \\ 0 \\ 2 \end{pmatrix} \text{ and } \overrightarrow{\mathbf{OB}} = \begin{pmatrix} 5 \\ -1 \\ 3 \end{pmatrix}$$

$$\overrightarrow{\mathbf{BA}} = \overrightarrow{\mathbf{OA}} - \overrightarrow{\mathbf{OB}} = \begin{pmatrix} 7 \\ 0 \\ 2 \end{pmatrix} - \begin{pmatrix} 5 \\ -1 \\ 3 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}$$

$$\left|\overrightarrow{\mathbf{BA}}\right| = \sqrt{2^2 + 1^2 + (-1)^2} = \sqrt{6}$$

So $\mathbf{d} = \sqrt{6}\sin 80.4^\circ = 2.42$

25 a $\overrightarrow{AP} = \mathbf{a} = \begin{pmatrix} 3 \\ 0 \\ -3 \end{pmatrix}$

$$\mathbf{n} = \begin{pmatrix} 2 \\ -2 \\ 3 \end{pmatrix}$$

$$\mathbf{a} \cdot \mathbf{n} = \begin{pmatrix} 3 \\ 0 \\ -3 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ -2 \\ 3 \end{pmatrix} = -3$$

$$|\mathbf{a}| = \sqrt{3^2 + (-3)^2} = \sqrt{18}$$

$$|\mathbf{n}| = \sqrt{2^2 + (-2)^2 + 3^2} = \sqrt{17}$$

$$\therefore \sin\theta = \frac{|-3|}{\sqrt{18}\sqrt{17}} = \frac{3}{\sqrt{18}\sqrt{17}}$$

$\Rightarrow \theta = 10^o$ to the nearest degree

b $\dfrac{|1(2) + 2(-2) + 2(3) - 1|}{\sqrt{2^2 + (-2)^2 + 3^2}} = \dfrac{3}{\sqrt{17}} = \dfrac{3\sqrt{17}}{17}$

26 a Let l_1 denote the path of the first aeroplane, and l_2 the second.

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$$l_1: \mathbf{r} = \begin{pmatrix} 120 \\ -80 \\ 13 \end{pmatrix} + \lambda \begin{pmatrix} 80 \\ 100 \\ -8 \end{pmatrix}$$

$$l_2: \mathbf{r} = \begin{pmatrix} -20 \\ 35 \\ 5 \end{pmatrix} + \mu \begin{pmatrix} 10 \\ -2 \\ 0.1 \end{pmatrix}$$

$$\begin{pmatrix} 120+80\lambda \\ -80+100\lambda \\ 13-8\lambda \end{pmatrix} = \begin{pmatrix} -20+10\mu \\ 35-2\mu \\ 5+0.1\mu \end{pmatrix}$$

i and **k** components $\Rightarrow \lambda = \frac{3}{4},\ \mu = 20$

Check **j** component: $-80+100\left(\frac{3}{4}\right) = -5 = 35-2(20)$

So the equations are consistent

Therefore the paths intersect, and they intersect at

$$\begin{pmatrix} 120 \\ -80 \\ 13 \end{pmatrix} + \frac{3}{4}\begin{pmatrix} 80 \\ 100 \\ -8 \end{pmatrix} = \begin{pmatrix} 180 \\ -5 \\ 7 \end{pmatrix} \Rightarrow (180, -5, 7)$$

b The planes pass through the same point, but not necessarily at the same time.

Challenge

1 a $-2(-2x + y - 3z) = (-2) \times (-5)$ gives $4x - 2y + 6z = 10$

b matrix **A** is singular if det **A** = 0

$4(c + 3b) + 2(-2c + 3a) + 6(-2b - a) = 6a - 6a + 12b - 12b + 4c - 4c = 0$

c i $a = 2n$, $b = -n$, $c = 30$ where $n \in \mathbb{R}$, $n \neq 3$

ii $a = 6$, $b = -3$ and $c = 9$

Challenge

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2 Coordinates $A(4,-4,5), B(0,4,1)$ and $C(0,0,5)$ lie on the circumference of the circle.

Let O be the centre of the circle.

Then A, B, C and O lie on the same plane.

Without loss of generality, suppose the plane has equation $ax+by+cz=1$

Then substituting point A gives: $4a-4b+5c=1$ **(1)**

substituting point B gives: $4b+c=1$ **(2)**

substituting point C gives: $5c=1$ **(3)**

From equation **(3)**, $c=\frac{1}{5}$

Substituting into **(2)** gives $4b+\frac{1}{5}=1$

$4b=\frac{4}{5}$

$b=\frac{1}{5}$

Substituting b and c in equation **(1)** gives $4a-\frac{4}{5}+\frac{5}{5}=1$

$4a=\frac{4}{5}$

$a=\frac{1}{5}$

So the equation of the plane is $\frac{x}{5}+\frac{y}{5}+\frac{z}{5}=1$

Or $x+y+z=5$ **(4)**

The centre of the circle lies on the intersection of any two perpendicular bisectors chosen from AB, BC or AC.

Each perpendicular bisector must also lie on the plane $x+y+z=5$

To find the equation of the perpendicular bisector of **AB**:

The mid-point of **AB** is $\mathbf{M}\left(\frac{4+0}{2},\frac{-4+4}{2},\frac{5+1}{2}\right)=\mathbf{M}(2,0,3)$

Therefore the equation of the line **MO** may be written as $\mathbf{r}=\begin{pmatrix}2\\0\\3\end{pmatrix}+\lambda\begin{pmatrix}\alpha\\\beta\\\gamma\end{pmatrix}$

$\overrightarrow{\mathbf{AB}}=\begin{pmatrix}-4\\8\\4\end{pmatrix}$ and $\overrightarrow{\mathbf{AB}}$ is perpendicular to the direction of **MO**, so $\overrightarrow{\mathbf{AB}}\cdot\begin{pmatrix}\alpha\\\beta\\\gamma\end{pmatrix}=0$

i.e. $\begin{pmatrix}-4\\8\\4\end{pmatrix}\cdot\begin{pmatrix}\alpha\\\beta\\\gamma\end{pmatrix}=0$, so $-4\alpha+8\beta-4\gamma=0$ **(5)**

The line $\mathbf{r}=\begin{pmatrix}2\\0\\3\end{pmatrix}+\lambda\begin{pmatrix}\alpha\\\beta\\\gamma\end{pmatrix}$ lies on the plane $x+y+z=5$ for all values of λ.

Choosing $\lambda=1$, you have $\mathbf{r}=\begin{pmatrix}2+\alpha\\\beta\\3+\gamma\end{pmatrix}$

Challenge
2 continued

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Solving $\mathbf{r} = \begin{pmatrix} 2+\alpha \\ \beta \\ 3+\gamma \end{pmatrix}$ and $x+y+z=5$ simultaneously gives $(2+\alpha)+\beta+(3+\gamma)=5,$ or

$\alpha+\beta+\gamma=0$ **(6)**

Equation **(6)** is equivalent to $4\alpha+4\beta+4\gamma=0$

Adding this to equation **(5)** gives $12\beta=0,$ so $\beta=0$ and $\alpha=-\gamma$

Therefore $\mathbf{r} = \begin{pmatrix} 2+\alpha \\ \beta \\ 3+\gamma \end{pmatrix} = \begin{pmatrix} 2+\alpha \\ 0 \\ 3-\alpha \end{pmatrix}$

Now to find the equation of the perpendicular bisector of **BC**:

The mid-point of **BC** is $\mathbf{M}_1\left(\frac{0+0}{2}, \frac{4+0}{2}, \frac{1+5}{2}\right) = \mathbf{M}_1(0,2,3)$

Therefore the equation of the line $\mathbf{M}_1\mathbf{O}$ may be written as $\mathbf{r} = \begin{pmatrix} 0 \\ 2 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} \alpha_1 \\ \beta_1 \\ \gamma_1 \end{pmatrix}$

$\overrightarrow{\mathbf{BC}} = \begin{pmatrix} 0 \\ -4 \\ 4 \end{pmatrix}$ and $\overrightarrow{\mathbf{BC}}$ is perpendicular to the direction of $\mathbf{M}_1\mathbf{O}$, so $\overrightarrow{\mathbf{BC}} \cdot \begin{pmatrix} \alpha_1 \\ \beta_1 \\ \gamma_1 \end{pmatrix} = 0$

i.e. $\begin{pmatrix} 0 \\ -4 \\ 4 \end{pmatrix} \cdot \begin{pmatrix} \alpha_1 \\ \beta_1 \\ \gamma_1 \end{pmatrix} = 0,$ so $-4\beta_1+4\gamma_1=0,$ or $\beta_1=\gamma_1$ **(7)**

The line $\mathbf{r} = \begin{pmatrix} 0 \\ 2 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} \alpha_1 \\ \beta_1 \\ \gamma_1 \end{pmatrix}$ lies on the plane $x+y+z=5$ for all values of λ.

Choosing $\lambda=1,$ you have $\mathbf{r} = \begin{pmatrix} \alpha_1 \\ 2+\beta_1 \\ 3+\gamma_1 \end{pmatrix}$

Solving $\mathbf{r} = \begin{pmatrix} \alpha_1 \\ 2+\beta_1 \\ 3+\gamma_1 \end{pmatrix}$ and $x+y+z=5$ simultaneously gives $\alpha_1+(2+\beta_1)+(3+\gamma_1)=5,$ or

$\alpha_1+\beta_1+\gamma_1=0$ **(8)**

Solving equations **(7)** and **(8)** simultaneously gives $\alpha_1=-\beta_1-\gamma_1=-\gamma_1-\gamma_1=-2\gamma_1$

Therefore $\mathbf{r} = \begin{pmatrix} \alpha_1 \\ 2+\beta_1 \\ 3+\gamma_1 \end{pmatrix} = \begin{pmatrix} -2\gamma_1 \\ 2+\gamma_1 \\ 3+\gamma_1 \end{pmatrix}$

Challenge
2 continued

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Now consider the equations of the two perpendicular bisectors:

$\mathbf{r} = \begin{pmatrix} 2+\alpha \\ 0 \\ 3-\alpha \end{pmatrix}$ and $\mathbf{r} = \begin{pmatrix} -2\gamma_1 \\ 2+\gamma_1 \\ 3+\gamma_1 \end{pmatrix}$

These meet at the centre of the circle, O.

Equating the **j** components gives $0 = 2+\gamma_1$, so $\gamma_1 = -2$

Therefore the two bisectors must meet at the position vector $\mathbf{r} = \begin{pmatrix} -2(-2) \\ 2+(-2) \\ 3+(-2) \end{pmatrix} = \begin{pmatrix} 4 \\ 0 \\ 1 \end{pmatrix}$

The centre of the circle is therefore $O(4,0,1)$

The radius is the distance between O and any of the points A, B or C

Consider the point $C(0,0,5)$

Then $OC = \sqrt{(0-4)^2+(0-0)^2+(5-1)^2} = \sqrt{32} = 4\sqrt{2}$

Therefore the circle has radius $4\sqrt{2}$

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Review Exercise 2

1 a **A** has 3 columns and **B** has 2 rows.

The number of columns in **A** is not the same as the number of rows in **B**.

Therefore, the product **AB** does not exist.

b **B** has 2 columns and **A** has 2 rows.

The number of columns in **B** is the same as the number of rows in **A**.

Therefore, the product **BA** exists:

$$\mathbf{BA} = \begin{pmatrix} q & 0 \\ 3 & -1 \end{pmatrix} \times \begin{pmatrix} 3 & 2 & p \\ 0 & 2 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} 3q & 2q & pq \\ 9 & 4 & 3p+1 \end{pmatrix}$$

c **BA** has 3 columns and **C** has 3 rows.

The number of columns in **BA** is the same as the number of rows in **A**.

Therefore, the product **BAC** exists:

$$\mathbf{BAC} = \begin{pmatrix} 3q & 2q & pq \\ 9 & 4 & 3p+1 \end{pmatrix} \times \begin{pmatrix} 4 \\ -3 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} 12q - 6q + pq \\ 36 - 12 + 3p + 1 \end{pmatrix}$$

$$= \begin{pmatrix} 6q + pq \\ 3p + 25 \end{pmatrix}$$

d **C** has 1 column and **B** has 2 rows.

The number of columns in **C** is not the same as the number of rows in **B**.

Therefore, the product **CBA** does not exist.

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2 $\mathbf{M}^2 = \begin{pmatrix} 0 & 3 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} 0 & 3 \\ -1 & 2 \end{pmatrix}$

$= \begin{pmatrix} 0\times0+3\times(-1) & 0\times3+3\times2 \\ (-1)\times0+2\times(-1) & (-1)\times3+2\times2 \end{pmatrix}$

$= \begin{pmatrix} 0-3 & 0+6 \\ 0-2 & -3+4 \end{pmatrix} = \begin{pmatrix} -3 & 6 \\ -2 & 1 \end{pmatrix}$

$\mathbf{M}^2$ is quite complicated to work out and it is sensible to calculate this before working out $\mathbf{M}^2 + a\mathbf{M} + b\mathbf{I}$

Then consider $\mathbf{M}^2 + a\mathbf{M} + b\mathbf{I} = \mathbf{O}$

$$\begin{pmatrix} -3 & 6 \\ -2 & 1 \end{pmatrix} + a\begin{pmatrix} 0 & 3 \\ -1 & 2 \end{pmatrix} + b\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} -3 & 6 \\ -2 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 3a \\ -a & 2a \end{pmatrix} + \begin{pmatrix} b & 0 \\ 0 & b \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} -3+b & 6+3a \\ -2-a & 1+2a+b \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

Equating the top left elements

$-3 + b = 0 \Rightarrow b = 3$

Equating the top right elements

$6 + 3a = 0 \Rightarrow a = -2$

$a = -2,\ b = 3$

There are four elements which could be equated but you only need to equate two of them to find a and b. You could use the others to check your working. For example; if $a = -2$, $b = 3$ then $1 + 2a + b = 1 - 4 + 3$ which does equal 0.

3 $\mathbf{A}^2 = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a^2+bc & ab+bd \\ ac+cd & bc+d^2 \end{pmatrix}$

$\mathbf{A}^2 - (a+d)\mathbf{A}$

$= \begin{pmatrix} a^2+bc & ab+bd \\ ac+cd & bc+d^2 \end{pmatrix} - \begin{pmatrix} (a+d)a & (a+d)b \\ (a+d)c & (a+d)d \end{pmatrix}$

$= \begin{pmatrix} a^2+bc-a^2-ad & ab+bd-ab-bd \\ ac+cd-ac-cd & bc+d^2-ad-d^2 \end{pmatrix}$

$= \begin{pmatrix} bc-ad & 0 \\ 0 & bc-ad \end{pmatrix} = \lambda\mathbf{I} = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}$

$$\mathbf{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \text{ so}$$

$$\lambda\mathbf{I} = \lambda\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}.$$

You can write down the results of simple calculations like this without showing all of the working.

Equating the top left (or bottom right elements)

$\lambda = bc - ad$

Note that $\lambda = -\det(\mathbf{A})$.

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4 $\mathbf{A}^2 = \begin{pmatrix} 1 & 2 & b \\ 3 & 0 & 1 \\ a & -1 & 2 \end{pmatrix} \times \begin{pmatrix} 1 & 2 & b \\ 3 & 0 & 1 \\ a & -1 & 2 \end{pmatrix}$

$$= \begin{pmatrix} 1\times1+2\times3+b\times a & 1\times2+2\times0+b\times(-1) & 1\times b+2\times1+b\times2 \\ 3\times1+0\times3+1\times a & 3\times2+0\times0+1\times(-1) & 3\times b+0\times1+1\times2 \\ a\times1+(-1)\times3+2\times a & a\times2+(-1)\times0+2\times(-1) & a\times b+(-1)\times1+2\times2 \end{pmatrix}$$

$$= \begin{pmatrix} ab+7 & 2-b & 3b+2 \\ 3+a & 5 & 3b+2 \\ 3a-3 & 2a-2 & ab+3 \end{pmatrix}$$

Compare corresponding elements to given $\mathbf{A}^2$:

$$\begin{pmatrix} ab+7 & 2-b & 3b+2 \\ 3+a & 5 & 3b+2 \\ 3a-3 & 2a-2 & ab+3 \end{pmatrix} = \begin{pmatrix} 3 & 3 & -1 \\ 7 & 5 & -1 \\ 9 & 6 & -1 \end{pmatrix}$$

$2a-2=6$ $\qquad$ $2-b=3$

$2a=8$ $\qquad$ $\Rightarrow b=-1$

$\Rightarrow a=4$

5 a $\det(\mathbf{A}) = 2\times(-1)-3\times p = -2-3p$

If $\mathbf{A}$ is singular, $\det(\mathbf{A}) = 0$.

$-2-3p=0 \Rightarrow 3p=-2 \Rightarrow p=-\frac{2}{3}$

> You need to know that, if $\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then $\det(\mathbf{A}) = ad-bc$.

b As in part (a), $\det(\mathbf{A}) = -2-3p$

$-2-3p=4 \Rightarrow -3p=6 \Rightarrow p=-2$

c $\mathbf{A}^2 = \begin{pmatrix} 2 & 3 \\ -2 & -1 \end{pmatrix}\begin{pmatrix} 2 & 3 \\ -2 & -1 \end{pmatrix}$

$$= \begin{pmatrix} 4-6 & 6-3 \\ -4+2 & -6+1 \end{pmatrix} = \begin{pmatrix} -2 & 3 \\ -2 & -5 \end{pmatrix}$$

$$\mathbf{A}^2 - \mathbf{A} = \begin{pmatrix} -2 & 3 \\ -2 & -5 \end{pmatrix} - \begin{pmatrix} 2 & 3 \\ -2 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} -4 & 0 \\ 0 & -4 \end{pmatrix} = -4\mathbf{I}$$

This is the required result with $k = -4$.

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6 a For a matrix $\mathbf{M} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then $\mathbf{M}^{-1} = \frac{1}{\det(\mathbf{M})}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$.

Here, $\det(\mathbf{A}) = 4\times 2 - (-1)\times(-6) = 8 - 6 = 2$

$$\mathbf{A}^{-1} = \frac{1}{2}\begin{pmatrix} 2 & 1 \\ 6 & 4 \end{pmatrix} = \begin{pmatrix} 1 & \frac{1}{2} \\ 3 & 2 \end{pmatrix}$$

You need to remember this property of the inverse of matrices. The order of **A** and **B** is reversed in this formula.

b $(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$

$$= \begin{pmatrix} 2 & 0 \\ 3 & p \end{pmatrix}\begin{pmatrix} 1 & \frac{1}{2} \\ 3 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} 2 & 1 \\ 3p+3 & 2p+\frac{3}{2} \end{pmatrix}$$

c $(\mathbf{AB})(\mathbf{AB})^{-1} = \mathbf{I}$

$$\begin{pmatrix} -1 & 2 \\ 3 & -4 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 3p+3 & 2p+\frac{3}{2} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

The product of any matrix and its inverse is **I**. This applies to a product matrix, **AB** in this case, as well as to a matrix such as **A**.

Equating the upper left elements

$$-1\times 2 + 2(3p+3) = 1$$
$$-2 + 6p + 6 = 1$$
$$6p = -3$$
$$p = -\frac{1}{2}$$

Finding all four of the elements of the product matrix of the left hand side of this equation would be lengthy. To find p, you only need one equation, so you only need to consider one element. Here the upper left hand element has been used but you could choose any of the four elements.

7 a
$$\det \mathbf{A} = k\begin{vmatrix} -1 & k \\ 1 & 0 \end{vmatrix} - 1\begin{vmatrix} 0 & k \\ 9 & 0 \end{vmatrix} + (-2)\begin{vmatrix} 0 & -1 \\ 9 & 1 \end{vmatrix}$$

$$= k(-k) - 1\times(-9k) + (-2)\times 9$$
$$= -k^2 + 9k - 18 = 0$$
$$k^2 - 9k + 18 = (k-3)(k-6) = 0$$
$$k = 3, 6$$

The 2 × 2 determinants are worked out using the formula $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$, which you learnt in book FP1.

A singular matrix is a matrix without an inverse. The determinant of a singular matrix is 0.

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7 b The matrix of the minors, **M** say, is given by

$$\mathbf{M}=\begin{pmatrix}\begin{vmatrix}-1 & k\\ 1 & 0\end{vmatrix} & \begin{vmatrix}0 & k\\ 9 & 0\end{vmatrix} & \begin{vmatrix}0 & -1\\ 9 & 1\end{vmatrix}\\ \begin{vmatrix}1 & -2\\ 1 & 0\end{vmatrix} & \begin{vmatrix}k & -2\\ 9 & 0\end{vmatrix} & \begin{vmatrix}k & 1\\ 9 & 1\end{vmatrix}\\ \begin{vmatrix}1 & -2\\ -1 & k\end{vmatrix} & \begin{vmatrix}k & -2\\ 0 & k\end{vmatrix} & \begin{vmatrix}k & 1\\ 0 & -1\end{vmatrix}\end{pmatrix}$$

$$=\begin{pmatrix}-k & -9k & 9\\ 2 & 18 & k-9\\ k-2 & k^2 & -k\end{pmatrix}$$

As you have worked out the determinant of **A** in part **a**, the remaining steps for working out an inverse of a 3 × 3 matrix are:
1 Work out the matrix of the minors.
2 Obtain the matrix of cofactors by adjusting the signs of the minors using the pattern

$$\begin{pmatrix}+ & - & +\\ - & + & -\\ + & - & +\end{pmatrix}$$

3 Transpose the matrix of the cofactors.
4 Divide the transpose of the matrix of cofactors by the determinant of the matrix.

The matrix of the cofactors, **C** say, is given by

$$\mathbf{C}=\begin{pmatrix}-k & 9k & 9\\ -2 & 18 & -k+9\\ k-2 & -k^2 & -k\end{pmatrix}$$

The transpose of the matrix of the cofactors is given by

$$\mathbf{C}^{\mathrm{T}}=\begin{pmatrix}-k & -2 & k-2\\ 9k & 18 & -k^2\\ 9 & -k+9 & -k\end{pmatrix}$$

The inverse of **A** is given by

$$\mathbf{A}^{-1}=\frac{1}{\det\mathbf{A}}\mathbf{C}^{\mathrm{T}}$$

$$=\frac{1}{-k^2+9k-18}\begin{pmatrix}-k & -2 & k-2\\ 9k & 18 & -k^2\\ 9 & -k+9 & -k\end{pmatrix}$$

You have worked out the determinant of **A** in part **a**. It is perfectly acceptable to leave your answer in this form. You do not have to divide every individual term in the matrix by $-k^2+9k-18$.

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8 $\mathbf{A} = \begin{pmatrix} 2p & p & 2 \\ 3 & 0 & 0 \\ -1 & 1 & -1 \end{pmatrix}$

$$\det \mathbf{A} = 2p\begin{vmatrix} 0 & 0 \\ 1 & -1 \end{vmatrix} - p\begin{vmatrix} 3 & 0 \\ -1 & -1 \end{vmatrix} + 2\begin{vmatrix} 3 & 0 \\ -1 & 1 \end{vmatrix}$$
$$= 2p(0-0) - p(-3-0) + 2(3-0)$$
$$= 3p + 6 = 3(p+2)$$

$$\mathbf{M} = \begin{pmatrix} \begin{vmatrix} 0 & 0 \\ 1 & -1 \end{vmatrix} & \begin{vmatrix} 3 & 0 \\ -1 & -1 \end{vmatrix} & \begin{vmatrix} 3 & 0 \\ -1 & 1 \end{vmatrix} \\ \begin{vmatrix} p & 2 \\ 1 & -1 \end{vmatrix} & \begin{vmatrix} 2p & 2 \\ -1 & -1 \end{vmatrix} & \begin{vmatrix} 2p & p \\ -1 & 1 \end{vmatrix} \\ \begin{vmatrix} p & 2 \\ 0 & 0 \end{vmatrix} & \begin{vmatrix} 2p & 2 \\ 3 & 0 \end{vmatrix} & \begin{vmatrix} 2p & p \\ 3 & 0 \end{vmatrix} \end{pmatrix}$$
$$= \begin{pmatrix} 0 & -3 & 3 \\ -p-2 & -2p+2 & 3p \\ 0 & -6 & -3p \end{pmatrix}$$

$$\mathbf{C} = \begin{pmatrix} 0 & 3 & 3 \\ p+2 & -2p+2 & -3p \\ 0 & 6 & -3p \end{pmatrix}$$

$$\mathbf{C}^{\mathrm{T}} = \begin{pmatrix} 0 & p+2 & 0 \\ 3 & -2p+2 & 6 \\ 3 & -3p & -3p \end{pmatrix}$$

$$\mathbf{A}^{-1} = \frac{1}{3(p+2)}\begin{pmatrix} 0 & p+2 & 0 \\ 3 & -2p+2 & 6 \\ 3 & -3p & -3p \end{pmatrix}$$

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9 a $\det(\mathbf{A}) = 4p \times q - (-q) \times (-3p)$

$= 4pq - 3pq = pq$

$$\mathbf{A}^{-1} = \frac{1}{pq}\begin{pmatrix} q & q \\ 3p & 4p \end{pmatrix}$$

In your final answer, you could multiply each term in the matrix by $\frac{1}{pq}$, which would give

$$\mathbf{A}^{-1} = \begin{pmatrix} \frac{1}{p} & \frac{1}{p} \\ \frac{3}{q} & \frac{4}{q} \end{pmatrix}$$

b $\mathbf{AX} = \begin{pmatrix} 2p & 3q \\ -p & q \end{pmatrix}$

Multiply both sides on the left by $\mathbf{A}^{-1}$

$$\mathbf{A}^{-1}\mathbf{AX} = \mathbf{A}^{-1}\begin{pmatrix} 2p & 3q \\ -p & q \end{pmatrix}$$

$$\mathbf{X} = \frac{1}{pq}\begin{pmatrix} q & q \\ 3p & 4p \end{pmatrix}\begin{pmatrix} 2p & 3q \\ -p & q \end{pmatrix}$$

$$= \frac{1}{pq}\begin{pmatrix} 2pq - pq & 3q^2 + q^2 \\ 6p^2 - 4p^2 & 9pq + 4pq \end{pmatrix}$$

$$= \frac{1}{pq}\begin{pmatrix} pq & 4q^2 \\ 2p^2 & 13pq \end{pmatrix}$$

It is important to multiply by $\mathbf{A}^{-1}$ on the correct side of the expression. As shown here, multiplying on the left of $\mathbf{AX}$, you get $\mathbf{A}^{-1}\mathbf{AX} = (\mathbf{A}^{-1}\mathbf{A})\mathbf{X} = \mathbf{IX} = \mathbf{X}$, which is what you are asked to find.

If instead you multiplied both sides on the right by $\mathbf{A}^{-1}$ you would get $\mathbf{AXA}^{-1}$, which does not simplify, and no further progress can be made.

Working out $\begin{pmatrix} 2p & 3q \\ -p & q \end{pmatrix}\frac{1}{pq}\begin{pmatrix} q & q \\ 3p & 4p \end{pmatrix}$ instead of $\frac{1}{pq}\begin{pmatrix} q & q \\ 3p & 4p \end{pmatrix}\begin{pmatrix} 2p & 3q \\ -p & q \end{pmatrix}$ is a common error.

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10 Write the system of equations using matrices:

$$\begin{pmatrix} 1 & 1 & 1 \\ 2 & -1 & -2 \\ 3 & -2 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \\ -1 \end{pmatrix} \qquad (*)$$

Find the inverse of the left-hand matrix, **A**:

$$\det \begin{vmatrix} 1 & 1 & 1 \\ 2 & -1 & -2 \\ 3 & -2 & 1 \end{vmatrix} = -14$$

$$\mathbf{M} = \begin{pmatrix} \begin{vmatrix} -1 & -2 \\ -2 & 1 \end{vmatrix} & \begin{vmatrix} 2 & -2 \\ 3 & 1 \end{vmatrix} & \begin{vmatrix} 2 & -1 \\ 3 & -2 \end{vmatrix} \\ \begin{vmatrix} 1 & 1 \\ -2 & 1 \end{vmatrix} & \begin{vmatrix} 1 & 1 \\ 3 & 1 \end{vmatrix} & \begin{vmatrix} 1 & 1 \\ 3 & -2 \end{vmatrix} \\ \begin{vmatrix} 1 & 1 \\ -1 & -2 \end{vmatrix} & \begin{vmatrix} 1 & 1 \\ 2 & -2 \end{vmatrix} & \begin{vmatrix} 1 & 1 \\ 2 & -1 \end{vmatrix} \end{pmatrix}$$

$$= \begin{pmatrix} -5 & 8 & -1 \\ 3 & -2 & -5 \\ -1 & -4 & -3 \end{pmatrix}$$

$$\mathbf{C} = \begin{pmatrix} -5 & -8 & -1 \\ -3 & -2 & 5 \\ -1 & 4 & -3 \end{pmatrix}$$

$$\mathbf{C}^{\mathrm{T}} = \begin{pmatrix} -5 & -3 & -1 \\ -8 & -2 & 4 \\ -1 & 5 & -3 \end{pmatrix}$$

$$\mathbf{A}^{-1} = -\frac{1}{14} \begin{pmatrix} -5 & -3 & -1 \\ -8 & -2 & 4 \\ -1 & 5 & -3 \end{pmatrix}$$

Multiplying both sides of the matrix equation (*) on the left by $\mathbf{A}^{-1}$ gives

$$\mathbf{A}^{-1}\mathbf{A} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \mathbf{A}^{-1} \begin{pmatrix} 3 \\ 0 \\ -1 \end{pmatrix}$$

$$\mathbf{I} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = -\frac{1}{14} \begin{pmatrix} -5 & -3 & -1 \\ -8 & -2 & 4 \\ -1 & 5 & -3 \end{pmatrix} \begin{pmatrix} 3 \\ 0 \\ -1 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = -\frac{1}{14} \begin{pmatrix} -14 \\ -28 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$$

The single point of intersection is (1, 2, 0).

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11 Let w be the initial number of woolly llamas, c the initial number of classic llamas and S the initial number of Suri llamas.

Flock initially has 2810:

$\Rightarrow w+c+s=2810$ (1)

160 more woolly llamas than classic:

$\Rightarrow w-c=160$ (2)

Given increase after one year:

$1.05w+1.03c+0.96S=2810+46=2856$ (3)

Write equations (1), (2) and (3) using matrices:

$$\begin{pmatrix}1 & 1 & 1\\ 1 & -1 & 0\\ 1.05 & 1.03 & 0.96\end{pmatrix}\begin{pmatrix}w\\ c\\ S\end{pmatrix}=\begin{pmatrix}2810\\ 160\\ 2856\end{pmatrix} \quad (*)$$

Find the inverse of the left-hand matrix, $\mathbf{A}$:

$$\det\begin{vmatrix}1 & 1 & 1\\ 1 & -1 & 0\\ 1.05 & 1.03 & 0.96\end{vmatrix}=0.16$$

$$\mathbf{M}=\begin{pmatrix}\begin{vmatrix}-1 & 0\\ 1.03 & 0.96\end{vmatrix} & \begin{vmatrix}1 & 0\\ 1.05 & 0.96\end{vmatrix} & \begin{vmatrix}1 & -1\\ 1.05 & 1.03\end{vmatrix}\\ \begin{vmatrix}1 & 1\\ 1.03 & 0.96\end{vmatrix} & \begin{vmatrix}1 & 1\\ 1.05 & 0.96\end{vmatrix} & \begin{vmatrix}1 & 1\\ 1.05 & 1.03\end{vmatrix}\\ \begin{vmatrix}1 & 1\\ -1 & 0\end{vmatrix} & \begin{vmatrix}1 & 1\\ 1 & 0\end{vmatrix} & \begin{vmatrix}1 & 1\\ 1 & -1\end{vmatrix}\end{pmatrix}$$

$$=\begin{pmatrix}-0.96 & 0.96 & 2.08\\ -0.07 & -0.09 & -0.02\\ 1 & -1 & -2\end{pmatrix}$$

$$\mathbf{C}=\begin{pmatrix}-0.96 & -0.96 & 2.08\\ 0.07 & -0.09 & 0.02\\ 1 & 1 & -2\end{pmatrix}$$

$$\mathbf{C}^{\mathrm{T}}=\begin{pmatrix}-0.96 & 0.07 & 1\\ -0.96 & -0.09 & 1\\ 2.08 & 0.02 & -2\end{pmatrix}$$

$$\mathbf{A}^{-1}=\frac{1}{0.16}\begin{pmatrix}-0.96 & 0.07 & 1\\ -0.96 & -0.09 & 1\\ 2.08 & 0.02 & -2\end{pmatrix}$$

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11 (cont.)

Multiplying both sides of the matrix equation (*) on the left by $\mathbf{A}^{-1}$ gives

$$\mathbf{A}^{-1}\mathbf{A}\begin{pmatrix} w \\ c \\ S \end{pmatrix} = \mathbf{A}^{-1}\begin{pmatrix} 2810 \\ 160 \\ 2856 \end{pmatrix}$$

$$\mathbf{I}\begin{pmatrix} w \\ c \\ S \end{pmatrix} = \frac{1}{0.16}\begin{pmatrix} -0.96 & 0.07 & 1 \\ -0.96 & -0.09 & 1 \\ 2.08 & 0.02 & -2 \end{pmatrix}\begin{pmatrix} 2810 \\ 160 \\ 2856 \end{pmatrix}$$

$$\begin{pmatrix} w \\ c \\ S \end{pmatrix} = \frac{1}{0.16}\begin{pmatrix} 169.6 \\ 144 \\ 136 \end{pmatrix} = \begin{pmatrix} 1060 \\ 900 \\ 850 \end{pmatrix}$$

Initially there were 1060 woolly llamas, 900 classic llamas and 850 Suri llamas.

12 a $\begin{pmatrix} 1 & -2 & -p \\ 2 & p & 5 \\ 1 & 3 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -2 \\ p \\ -p \end{pmatrix}$

planes do not meet, so $\begin{pmatrix} 1 & -2 & -p \\ 2 & p & 5 \\ 1 & 3 & -2 \end{pmatrix}$ has no inverse

Hence $\det\begin{vmatrix} 1 & -2 & -p \\ 2 & p & 5 \\ 1 & 3 & -2 \end{vmatrix} = 0$

$0 = 1(-2p - 15) - (-2)(-4 - 5) - p(6 - p)$

$0 = -2p - 15 - 18 - 6p + p^2$

$0 = p^2 - 8p - 33$

$0 = (p + 3)(p - 11)$

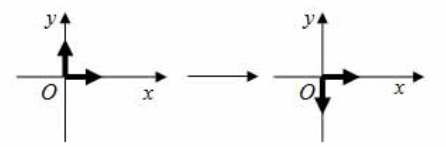
$p = -3$ or $p = 11$

b $p = -3$ the system is consistent and the planes form a sheaf.

$p = 11$ the system is inconsistent and the planes form a prism.

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13 a Reflection in the x axis transforms



$\begin{pmatrix}1\\0\end{pmatrix} \to \begin{pmatrix}1\\0\end{pmatrix}$

$\begin{pmatrix}0\\1\end{pmatrix} \to \begin{pmatrix}0\\-1\end{pmatrix}$

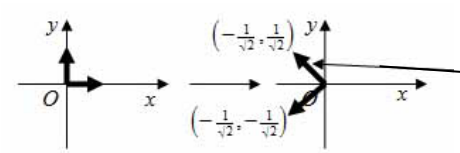
(1, 0) lies on the x-axis and so is not changed by reflection in the x-axis.

So

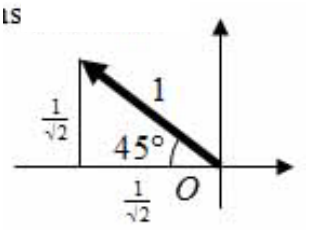
$$\mathbf{A} = \begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}$$

Unless the question states otherwise, it is acceptable to write down a simple matrix like this without showing your working.

Rotation of +135° about (0,0) transforms



The geometry of the vector to which $\begin{pmatrix}1\\0\end{pmatrix}$ is transformed is



$\begin{pmatrix}1\\0\end{pmatrix} \to \begin{pmatrix}-\frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}}\end{pmatrix}$

$\begin{pmatrix}0\\1\end{pmatrix} \to \begin{pmatrix}-\frac{1}{\sqrt{2}}\\ -\frac{1}{\sqrt{2}}\end{pmatrix}$

So

Arrows have been added so that you can see where the columns in **B** come from.

$$\mathbf{B} = \begin{pmatrix}-\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}}\end{pmatrix}$$

$$\mathbf{C} = \mathbf{AB} = \begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}\begin{pmatrix}-\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}}\end{pmatrix}$$

$$= \begin{pmatrix}-\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}}\\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\end{pmatrix}$$

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13 b $\mathbf{C}^2 = \begin{pmatrix} -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}\begin{pmatrix} -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$

$= \begin{pmatrix} \left(-\frac{1}{\sqrt{2}}\right)\left(-\frac{1}{\sqrt{2}}\right)+\left(-\frac{1}{\sqrt{2}}\right)\left(-\frac{1}{\sqrt{2}}\right) & \left(-\frac{1}{\sqrt{2}}\right)\left(-\frac{1}{\sqrt{2}}\right)+\left(-\frac{1}{\sqrt{2}}\right)\left(\frac{1}{\sqrt{2}}\right) \\ \left(-\frac{1}{\sqrt{2}}\right)\left(-\frac{1}{\sqrt{2}}\right)+\left(\frac{1}{\sqrt{2}}\right)\left(-\frac{1}{\sqrt{2}}\right) & \left(-\frac{1}{\sqrt{2}}\right)\left(-\frac{1}{\sqrt{2}}\right)+\left(\frac{1}{\sqrt{2}}\right)\left(\frac{1}{\sqrt{2}}\right) \end{pmatrix}$

$= \begin{pmatrix} \frac{1}{2}+\frac{1}{2} & \frac{1}{2}-\frac{1}{2} \\ \frac{1}{2}-\frac{1}{2} & \frac{1}{2}+\frac{1}{2} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \mathbf{I}$, as required.

As an example of the calculations;

$-\frac{1}{\sqrt{2}} \times -\frac{1}{\sqrt{2}} = +\frac{1 \times 1}{\sqrt{2} \times \sqrt{2}} = \frac{1}{2}$

Alternatively, to make the calculation simpler you could initially write

$\mathbf{C} = \frac{1}{\sqrt{2}}\begin{pmatrix} -1 & -1 \\ -1 & 1 \end{pmatrix}$ and square this expression.

14 a $\mathbf{M}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$

$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} a \\ c \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$

Equating the elements:

$a = 3, c = 2$

Also,

$\mathbf{M}\begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

Using $a = 3, c = 2$ from above,

$\begin{pmatrix} 3 & b \\ 2 & d \end{pmatrix}\begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 6+b \\ 4+d \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

Equating the upper elements

$6 + b = 2 \Rightarrow b = -4$

Equating the lower elements

$4 + d = 1 \Rightarrow d = -3$

$a = 3, b = -4, c = 2, d = -3$

In questions about transformations, you need to write the coordinates of points as column vectors. For example, the coordinate (1, 0) is written as the column vector $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

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14 b $\mathbf{M} = \begin{pmatrix} 3 & -4 \\ 2 & -3 \end{pmatrix}$

$$\mathbf{M}^2 = \begin{pmatrix} 3 & -4 \\ 2 & -3 \end{pmatrix}\begin{pmatrix} 3 & -4 \\ 2 & -3 \end{pmatrix} = \begin{pmatrix} 9-8 & -12+12 \\ 6-6 & -8+9 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \mathbf{I}, \text{as required}$$

The matrix **M** is its own inverse. This follows from the result in part **b**. In more detail:

$$\mathbf{M}^2 = \mathbf{I}$$
$$\mathbf{MM} = \mathbf{I}$$
$$\mathbf{M}^{-1}(\mathbf{MM}) = \mathbf{M}^{-1}\mathbf{I}$$
$$(\mathbf{M}^{-1}\mathbf{M})\mathbf{M} = \mathbf{M}^{-1}$$
$$\mathbf{IM} = \mathbf{M}^{-1}$$
$$\mathbf{M} = \mathbf{M}^{-1}$$

c As $\mathrm{M}^2 = \mathbf{I}, \mathbf{M}^{-1}\mathbf{M}^2 = \mathbf{M}^{-1}\mathbf{I}$

$$\mathbf{M} = \mathbf{M}^{-1}$$

$$\mathbf{M}\begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} 8 \\ -3 \end{pmatrix}$$

$$\mathbf{M}^{-1}\mathbf{M}\begin{pmatrix} p \\ q \end{pmatrix} = \mathbf{M}^{-1}\begin{pmatrix} 8 \\ -3 \end{pmatrix}$$

$$\mathbf{I}\begin{pmatrix} p \\ q \end{pmatrix} = \mathbf{M}\begin{pmatrix} 8 \\ -3 \end{pmatrix}$$

In this question, as **M** is its own inverse, you can replace $\mathbf{M}^{-1}$ by $\mathbf{M}$.

$$\begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} 3 & -4 \\ 2 & -3 \end{pmatrix}\begin{pmatrix} 8 \\ -3 \end{pmatrix} = \begin{pmatrix} 24+12 \\ 16+9 \end{pmatrix} = \begin{pmatrix} 36 \\ 25 \end{pmatrix}$$

Hence $p = 36, q = 25$

15 a $\begin{pmatrix} x \\ y \end{pmatrix} \rightarrow \begin{pmatrix} 2y - x \\ 3y \end{pmatrix} = \begin{pmatrix} -1x + 2y \\ 0x + 3y \end{pmatrix}$

$$= \begin{pmatrix} -1 & 2 \\ 0 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

So $\mathbf{C} = \begin{pmatrix} -1 & 2 \\ 0 & 3 \end{pmatrix}$

b

$\begin{pmatrix} -1 & 2 \\ 0 & 3 \end{pmatrix}\begin{pmatrix} x \\ 2x \end{pmatrix} = \begin{pmatrix} 3x \\ 6x \end{pmatrix}$; $6x = 2(3x)$ so the point satisfies the equation of the original line.

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15 c $\det(\mathbf{C}) = -1\times 3 - 2\times 0 = -3$

$$\mathbf{C}^{-1} = \frac{1}{-3}\begin{pmatrix} 3 & -2 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} -1 & \frac{2}{3} \\ 0 & \frac{1}{3} \end{pmatrix}$$

You are given the results of transforming the points by T and are asked to find the original points. You are "working backwards" to the original points and so you will need the inverse matrix.

Let the coordinates of A, B and C be

$(x_A, y_A), (x_B, y_B)$ and (x_C, y_C) respectively.

$$\mathbf{C}\begin{pmatrix} x_A & x_B & x_C \\ y_A & y_B & y_C \end{pmatrix} = \begin{pmatrix} 0 & 10 & 10 \\ 3 & 15 & 12 \end{pmatrix}$$

$$\mathbf{C}^{-1}\mathbf{C}\begin{pmatrix} x_A & x_B & x_C \\ y_A & y_B & y_C \end{pmatrix} = \mathbf{C}^{-1}\begin{pmatrix} 0 & 10 & 10 \\ 3 & 15 & 12 \end{pmatrix}$$

$$\begin{pmatrix} x_A & x_B & x_C \\ y_A & y_B & y_C \end{pmatrix} = \begin{pmatrix} -1 & \frac{2}{3} \\ 0 & \frac{1}{3} \end{pmatrix}\begin{pmatrix} 0 & 10 & 10 \\ 3 & 15 & 12 \end{pmatrix}$$

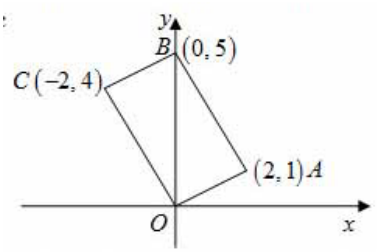
$$= \begin{pmatrix} 2 & -10+10 & -10+8 \\ 1 & 5 & 4 \end{pmatrix}$$

$$= \begin{pmatrix} 2 & 0 & -2 \\ 1 & 5 & 4 \end{pmatrix}$$

Hence $A:(2,1)$, $B:(0,5)$, $C:(-2,4)$

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15 d



Considering the gradients of the sides

$$m_{OA} = \frac{1}{2};\ m_{CB} = \frac{5-4}{0-(-2)} = \frac{1}{2}$$

Using the properties of quadrilaterals you learnt for GCSE, there are many alternative ways of showing that *OABC* is a rectangle. This is just one of many possibilities, using the result you learnt in the C1 module that the gradient of the line joining (x_1, y_1) to (x_2, y_2) is given by $m = \frac{y_2 - y_1}{x_2 - x_1}$.

So OA is parallel to CB.

$$m_{OC} = \frac{4-0}{-2-0} = -2;\ \ m_{AB} = \frac{5-1}{0-2} = \frac{4}{-2} = -2$$

So *OC* is parallel to *AB*.
The opposite sides of *OABC* are parallel to each other and so *OABC* is a parallelogram.
To show that *OABC* is specifically a rectangle (not just any parallelogram), we must show that one interior angle is a right-angle:
Consider the angle *AOC*

$$m_{OA} \times m_{OC} = \frac{1}{2} \times -2 = -1.$$

So *OA* is perpendicular to *OC,* and hence *AOC* is a right angle.
So the parallelogram *OABC* contains a right angle, and hence *OABC* is a rectangle.

16 a

$$\begin{aligned}\det \mathbf{A} &= 3\begin{vmatrix}1 & 1\\ 3 & u\end{vmatrix} - 1\begin{vmatrix}1 & 1\\ 5 & u\end{vmatrix} + (-1)\begin{vmatrix}1 & 1\\ 5 & 3\end{vmatrix}\\ &= 3(u-3) - 1(u-5) - 1\times(-2)\\ &= 3u - 9 - u + 5 + 2\\ &= 2u - 2\\ &= 2(u-1), \text{as required}\end{aligned}$$

Each 2×2 determinant is evaluated using the formula $\begin{vmatrix}a & b\\ c & d\end{vmatrix} = ad - bc.$

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16 b The matrix of the minors, **M** say, is given by

$$\mathbf{M} = \begin{pmatrix} \begin{vmatrix} 1 & 1 \\ 3 & u \end{vmatrix} & \begin{vmatrix} 1 & 1 \\ 5 & u \end{vmatrix} & \begin{vmatrix} 1 & 1 \\ 5 & 3 \end{vmatrix} \\ \begin{vmatrix} 1 & -1 \\ 3 & u \end{vmatrix} & \begin{vmatrix} 3 & -1 \\ 5 & u \end{vmatrix} & \begin{vmatrix} 3 & 1 \\ 5 & 3 \end{vmatrix} \\ \begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix} & \begin{vmatrix} 3 & -1 \\ 1 & 1 \end{vmatrix} & \begin{vmatrix} 3 & 1 \\ 1 & 1 \end{vmatrix} \end{pmatrix}$$

$$= \begin{pmatrix} u-3 & u-5 & -2 \\ u+3 & 3u+5 & 4 \\ 2 & 4 & 2 \end{pmatrix}$$

The minor of an element is found by deleting the row and the column in which the element lies.
For example, to find the minor of b in $\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$, delete the row and column through b $\begin{pmatrix} \cancel{a} & \cancel{b} & \cancel{c} \\ d & e & f \\ g & h & i \end{pmatrix}$.
The minor is the determinant of the elements left, that is $\begin{vmatrix} d & f \\ g & i \end{vmatrix}$.

The matrix of the cofactors, **C** say, is given by

$$\mathbf{C} = \begin{pmatrix} u-3 & -u+5 & 2 \\ -u-3 & 3u+5 & -4 \\ 2 & -4 & 2 \end{pmatrix}$$

The transpose of the matrix of the cofactors is given by

$$\mathbf{C}^{\mathrm{T}} = \begin{pmatrix} u-3 & -u-3 & 2 \\ -u+5 & 3u+5 & -4 \\ -2 & -4 & 2 \end{pmatrix}$$

The inverse of **A** is given by

$$\mathbf{A}^{-1} = \frac{1}{\det \mathbf{A}} \mathbf{C}^{\mathrm{T}}$$

$$= \frac{1}{2(u-1)} \begin{pmatrix} u-3 & -u-3 & 2 \\ -u+5 & 3u+5 & -4 \\ -2 & -4 & 2 \end{pmatrix}$$

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16 c Substituting $u = 6$, $\mathbf{A}^{-1}$ becomes:

$$\mathbf{A}^{-1} = \frac{1}{10}\begin{pmatrix} 3 & -9 & 2 \\ -1 & 23 & -4 \\ -2 & -4 & 2 \end{pmatrix}$$

The matrix you're given in part **c** is the matrix **A**, used in parts **a** and **b**, with $u = 6$.

To find the object vector when you are given the image vector, you will need the inverse matrix, $\mathbf{A}^{-1}$, with $u = 6$.

$$\mathbf{A}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \\ 6 \end{pmatrix}$$

This equation expresses the information that the image of $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$, under the transformation represented by the matrix **A**, is $\begin{pmatrix} 3 \\ 1 \\ 6 \end{pmatrix}$.

Multiplying both sides on the left by $\mathbf{A}^{-1}$

$$\mathbf{A}^{-1}\mathbf{A}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \mathbf{A}^{-1}\begin{pmatrix} 3 \\ 1 \\ 6 \end{pmatrix}$$

Hence, as $\mathrm{A}\mathrm{A}^{-1} = \mathbf{I}$,

$$\mathbf{I}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \frac{1}{10}\begin{pmatrix} 3 & -9 & 2 \\ -1 & 23 & -4 \\ -2 & -4 & 2 \end{pmatrix}\begin{pmatrix} 3 \\ 1 \\ 6 \end{pmatrix}$$

$$= \frac{1}{10}\begin{pmatrix} 9-9+12 \\ -3+23-24 \\ -6-4+12 \end{pmatrix} = \frac{1}{10}\begin{pmatrix} 12 \\ -4 \\ 2 \end{pmatrix} = \begin{pmatrix} 1.2 \\ -0.4 \\ 0.2 \end{pmatrix}$$

$\therefore a = 1.2,\ b = -0.4,\ c = 0.2$

17 a

$$\mathbf{MM} = \mathbf{I}$$

By definition, $\mathbf{M}\mathbf{M}^{-1} = \mathbf{I}$.
As you have been given that $\mathbf{M} = \mathbf{M}^{-1}$, it follows that $\mathbf{MM} = \mathbf{I}$. The matrix is self-inverse.

$$\begin{pmatrix} 3 & a & 0 \\ 2 & b & 0 \\ c & 0 & 1 \end{pmatrix}\begin{pmatrix} 3 & a & 0 \\ 2 & b & 0 \\ c & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 9+2a & 3a+ab & 0 \\ 6+2b & 2a+b^2 & 0 \\ 4c & ac & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

If two matrices are equal, then all of the corresponding elements in the matrices must be equal. Potentially, there are 9 equations here. However, since this question has only 3 unknowns (a, b and c), you need to pick out 3 equations which would be most convenient to solve for a, b and c.

Equating the first elements in the first row
$9 + 2a = 1 \Rightarrow a = -4$
Equating the first elements in the second row
$6 + 2b = 0 \Rightarrow b = -3$
Equating the first elements in the third row
$4c = 0 \Rightarrow c = 0$

17 b Using the values of a, b, and c found in part **a**

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$$\mathbf{M} = \begin{pmatrix} 3 & -4 & 0 \\ 2 & -3 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\det \mathbf{M} = 3\begin{vmatrix} -3 & 0 \\ 0 & 1 \end{vmatrix} - (-4)\begin{vmatrix} 2 & 0 \\ 1 & 1 \end{vmatrix} + 0\begin{vmatrix} 2 & -3 \\ 0 & 0 \end{vmatrix}$$

$$= 3\times(-3) + 4\times 2 = -1$$

c Let a point which is invariant under R have position vector $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$.

$$\begin{pmatrix} 3 & -4 & 0 \\ 2 & -3 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

The vector of an invariant point is unchanged when multiplied by the matrix representing the transformation.

$$\begin{pmatrix} 3x-4y \\ 2x-3y \\ z \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

Equating the top elements
$3x-4y = x \Rightarrow 2x-4y = 0 \Rightarrow x = 2y$

Equating the middle elements
$2x-3y = y \Rightarrow 2x-4y = 0 \Rightarrow x = 2y$

Equating the top elements and the middle elements gives the same equation, $x = 2y$.
Equating the bottom elements gives $z = z$. This is an identity, always satisfied, and gives you no further information.
So all points satisfying $x = 2y$ remain invariant under R

An equation satisfied by all the invariant points is $x = 2y$.

Because the transformation R is 3-dimensional, $x = 2y$ represents a plane of points.

18 a $\det(\mathbf{A}) = k\times 2k - (k-1)\times(-3)$

$$= 2k^2 + 3k - 3$$

b The triangle has been enlarged by a factor of

$$\frac{198}{18} = 11$$

So $\det(\mathbf{A}) = 11$

$$2k^2 + 3k - 3 = 11$$

$$2k^2 + 3k - 14 = 0$$

$$(2k+7)(k-2) = 0$$

$$k = -\frac{7}{2}, 2$$

The determinant is the area scale factor in transformations. This is equivalent to $\frac{\text{area of image}}{\text{area of object}} = \det(\mathbf{A})$. So the scale factor is part **b** must equal the determinant in part **a**.

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19 a $\det\mathbf{M} = \begin{vmatrix} \frac{3}{\sqrt{2}} & -\frac{3}{\sqrt{2}} \\ \frac{3}{\sqrt{2}} & \frac{3}{\sqrt{2}} \end{vmatrix}$

For a linear transformation represented by matrix **M**, det **M** represents the scale factor for the change in area.
Hence $\sqrt{\det \mathbf{M}}$ represents the linear scale factor for the enlargement.

$$= \frac{3}{\sqrt{2}} \times \frac{3}{\sqrt{2}} - \left(-\frac{3}{\sqrt{2}}\right) \times \frac{3}{\sqrt{2}}$$

$$= \frac{9}{2} + \frac{9}{2} = 9$$

Area scale factor $= 9$

Scale factor $= \sqrt{9} = 3$

b $\begin{pmatrix} \frac{3}{\sqrt{2}} & -\frac{3}{\sqrt{2}} \\ \frac{3}{\sqrt{2}} & \frac{3}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}$

$$= \begin{pmatrix} 3\cos\theta & -3\sin\theta \\ 3\sin\theta & 3\cos\theta \end{pmatrix}$$

$$3\cos\theta = \frac{3}{\sqrt{2}}$$

$$\cos\theta = \frac{3}{3\sqrt{2}} = \frac{1}{\sqrt{2}}$$

$\theta = 45^\circ$ anti-clockwise about $(0,0)$

Checking using the elements gives **M**.

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19 c Let the coordinates of P be (x, y)

Applying the matrix $\mathbf{M}$ to $\begin{pmatrix} x \\ y \end{pmatrix}$ gives $\begin{pmatrix} p \\ q \end{pmatrix}$

$$\mathbf{M}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} p \\ q \end{pmatrix}$$

Left multiply by the inverse $\mathbf{M}^{-1}$

$$\mathbf{M}^{-1}\mathbf{M}\begin{pmatrix} x \\ y \end{pmatrix} = \mathbf{M}^{-1}\begin{pmatrix} p \\ q \end{pmatrix}$$

$$\mathbf{I}\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{9}\begin{pmatrix} \frac{3}{\sqrt{2}} & \frac{3}{\sqrt{2}} \\ -\frac{3}{\sqrt{2}} & \frac{3}{\sqrt{2}} \end{pmatrix}\begin{pmatrix} p \\ q \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{3p}{9\sqrt{2}} + \frac{3q}{9\sqrt{2}} \\ -\frac{3p}{9\sqrt{2}} + \frac{3q}{9\sqrt{2}} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{p+q}{3\sqrt{2}} \\ \frac{-p+q}{3\sqrt{2}} \end{pmatrix}$$

The coordinates of P are $\left(\frac{p+q}{3\sqrt{2}}, \frac{-p+q}{3\sqrt{2}}\right)$

20 Take a general point on the line, $(y, k - \frac{1}{2}x)$ and transform it by T

$$\begin{pmatrix} 0 & 2 \\ \frac{1}{2} & 0 \end{pmatrix}\begin{pmatrix} x \\ k - \frac{1}{2}x \end{pmatrix} = \begin{pmatrix} 2k - x \\ \frac{1}{2}x \end{pmatrix}$$

$k - \frac{1}{2}(2k - x) = k - k + \frac{1}{2}x = \frac{1}{2}x$ k – 1 so the transformed point satisfies the equation of the original line.

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21 $\sum_{r=1}^{n} r(r+3) = \frac{1}{3}n(n+1)(n+5)$

Let $n = 1$.

The left-hand side becomes

$$\sum_{r=1}^{1} r(r+3) = 1(1+3) = 4$$

$\sum_{r=1}^{1} r(r+3)$ consists of just one term.
That is $r(r+3)$ with 1 substituted for r.

The right-hand side becomes

$$\frac{1}{3} \times 1(1+1)(1+5) = \frac{1}{3} \times 2 \times 6 = 4$$

The left-hand side and the right-hand side are equal and so the summation is true for $n = 1$.

Assume the summation is true for $n = k$.

That is $\sum_{r=1}^{k} r(r+3) = \frac{1}{3}k(k+1)(k+5)\ldots\ \ldots *$

This is often called the **induction hypothesis**.

$$\sum_{r=1}^{k+1} r(r+3) = \sum_{r=1}^{k} r(r+3) + (k+1)(k+4)$$

The sum from 1 to $k + 1$ is the sum from 1 to k plus one extra term.
In this case, the extra term is found by replacing each r in $r(r+3)$ by $k+1$.

$$= \frac{1}{3}k(k+1)(k+5) + (k+1)(k+4), \text{using} *$$

$$= \frac{1}{3}k(k+1)(k+5) + \frac{3}{3}(k+1)(k+4)$$

$$= \frac{1}{3}(k+1)[k(k+5) + 3(k+4)]$$

Multiplying out the brackets would give you an awkward cubic expression which would be difficult to factorise. You should try to simplify the working by looking for any common factors and taking them outside a bracket. Here $k+1$ is a common factor.

$$= \frac{1}{3}(k+1)[k^2 + 5k + 3k + 12]$$

$$= \frac{1}{3}(k+1)[k^2 + 8k + 12]$$

$$= \frac{1}{3}(k+1)(k+2)(k+6)$$

$$= \frac{1}{3}(k+1)((k+1)+1)((k+1)+5)$$

This expression is $\frac{1}{3}n(n+1)(n+5)$ with each n replaced by $k+1$.

This is the result obtained by substituting $n = k+1$ into the right-hand side of the summation and so the summation is true for $n = k+1$.

The summation is true for $n = 1$, and, if it is true for $n = k$, then it is true for $n = k+1$.

By mathematical induction the summation is true for all positive integers n.

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22 $\sum_{r=1}^{n}(2r-1)^2 = \frac{1}{3}n(2n-1)(2n+1)$

Let $n = 1$.

The left-hand side becomes

$$\sum_{r=1}^{1}(2r-1)^2 = (2-1)^2 = 1^2 = 1$$

$\sum_{r=1}^{1}(2r-1)^2$ consists of just one term.
That is $(2r-1)^2$ with 1 substituted for r.

The right-hand side becomes

$$\frac{1}{3}\times 1(2-1)(2+1) = \frac{1}{3}\times 1\times 1\times 3 = 1$$

The left-hand side and the right-hand side are equal and so the summation is true for $n = 1$.

Assume the summation is true for $n = k$.

That is $\sum_{r=1}^{k}(2r-1)^2 = \frac{1}{3}k(2k-1)(2k+1)\ldots\ldots *$

The sum from 1 to $k+1$ is the sum from 1 to k plus one extra term. In this case, the extra term is found by replacing the r in $(2r-1)^2$ by $k+1$.

$$\sum_{r=1}^{k+1}(2r-1)^2 = \sum_{r=1}^{k}(2r-1)^2 + (2(k+1)-1)^2$$

$$= \sum_{r=1}^{k}(2r-1)^2 + (2k+1)^2$$

$$= \frac{1}{3}k(2k-1)(2k+1) + \frac{3}{3}(2k+1)^2, \text{ using } *$$

$$= \frac{1}{3}(2k+1)[k(2k-1) + 3(2k+1)]$$

$$= \frac{1}{3}(2k+1)[2k^2 + 5k + 3]$$

$$= \frac{1}{3}(2k+1)(k+1)(2k+3)$$

$$= \frac{1}{3}(k+1)(2(k+1)-1)(2(k+1)+1)$$

Multiplying out the brackets would give you an awkward cubic expression which would be difficult to factorise. Look for any common factors and take them outside a bracket. Here $(2k+1)$ is a common factor.

This expression is $\frac{1}{3}n(2n-1)(2n+1)$ with each n replaced by $k+1$.

This is the result obtained by substituting $n = k+1$ into the right-hand side of the summation and so the summation is true for $n = k+1$.

The summation is true for $n = 1$, and, if it is true for $n = k$, then it is true for $n = k+1$.

By mathematical induction the summation is true for all positive integers n.

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23 $\sum_{r=1}^{n} a_r = \sum_{r=1}^{n} r(r+1)(2r+1) = \frac{1}{2}n(n+1)^2(n+2)$

All inductions need to be shown to be true for a small number, usually 1.

Let $n = 1$.

The left-hand side becomes

$$\sum_{r=1}^{1} r(r+1)(2r+1) = 1 \times 2 \times 3 = 6$$

The right-hand side becomes

$$\frac{1}{2} \times 1 \times 2^2 \times 3 = 6$$

The left-hand side and the right-hand side are equal and so the summation is true for $n = 1$.

Assume the summation is true for $n = k$.

That is $\sum_{r=1}^{k} r(r+1)(2r+1) = \frac{1}{2}k(k+1)^2(k+2)$ *

$$\sum_{r=1}^{k+1} r(r+1)(2r+1) = \sum_{r=1}^{k} r(r+1)(2r+1) + (k+1)(k+2)(2k+3)$$

$$= \frac{1}{2}k(k+1)^2(k+2) + \frac{2}{2}(k+1)(k+2)(2k+3), \text{using} \quad *$$

$$= \frac{1}{2}(k+1)(k+2)[k(k+1) + 2(2k+3)]$$

$$= \frac{1}{2}(k+1)(k+2)[k^2 + 5k + 6]$$

$$= \frac{1}{2}(k+1)(k+2)(k+2)(k+3)$$

$$= \frac{1}{2}(k+1)(k+2)^2(k+3)$$

$$= \frac{1}{2}(k+1)((k+1)+1)^2((k+1)+2)$$

Fractions need to be expressed to the same denominator before factorising. The form of the answer shows that you need to have $\frac{1}{2}$ as a common factor and it helps you to write $\frac{2}{2}$ before the second term on the right-hand side of the summation.

This expression is $\frac{1}{2}n(n+1)^2(2n+1)$ with each n replaced by $k+1$.

This is the result obtained by substituting $n = k+1$ into the right-hand side of the summation and so the summation is true for $n = k+1$.

The summation is true for $n = 1$, and if it is true for $n = k$, then it is true for $n = k+1$.

By mathematical induction the summation is true for all positive integers n.

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24 $\sum_{r=1}^{n} r^2(r-1) = \frac{1}{12}n(n-1)(n+1)(3n+2)$

Let $n = 1$.

The left-hand side becomes

$$\sum_{r=1}^{1} r^2(r-1) = 1^2 \times (1-1) = 0$$

$\sum_{r=1}^{1} r^2(r-1)$ consists of just one term. That is $r^2(r-1)$ with 1 substituted for r. In this case, because of the bracket, this clearly gives 0.

The right-hand side becomes

$$\frac{1}{12} \times 1 \times (1-1) \times (1+1) \times (3+2)$$

$$= \frac{1}{12} \times 1 \times 0 \times 2 \times 5 = 0$$

The left-hand side and the right-hand side are equal and so the summation is true for $n = 1$.

Assume the summation is true for $n = k$.

That is $\sum_{r=1}^{k} r^2(r-1) = \frac{1}{12}k(k-1)(k+1)(3k+2) \ldots\ldots *$

$$\sum_{r=1}^{k+1} r^2(r-1) = \sum_{r=1}^{k} r^2(r-1) + (k+1)^2(k+1-1)$$

The common factors in these two terms are $\frac{1}{12}$, k and $(k+1)$.

$$= \frac{1}{12}k(k-1)(k+1)(3k+2) + \frac{12}{12}k(k+1)^2, \text{using} *$$

$$= \frac{1}{12}k(k+1)[(k-1)(3k+2) + 12(k+1)]$$

$$= \frac{1}{12}k(k+1)[3k^2 - k - 2 + 12k + 12]$$

$$= \frac{1}{12}k(k+1)[3k^2 + 11k + 10]$$

$$= \frac{1}{12}(k+1)k(k+2)(3k+5)$$

$$= \frac{1}{12}(k+1)((k+1)-1)((k+1)+1)(3(k+1)+2)$$

Rearrange this expression so that it is the right-hand side of the summation with n replaced by $k+1$.

This is the result obtained by substituting $n = k+1$ into the right-hand side of the summation and so the summation is true for $n = k+1$.

The summation is true for $n = 1$, and, if it is true for $n = k$, then it is true for $n = k+1$.

By mathematical induction the summation is true for all positive integers n.

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25 a $f(n) = 3^{4n} + 2^{4n+2}$

$$f(k+1) - f(k) = 3^{4(k+1)} + 2^{4(k+1)+2} - (3^{4k} + 2^{4k+2})$$
$$= 3^{4k+4} - 3^{4k} + 2^{4k+6} - 2^{4k+2}$$
$$= 3^{4k}(3^4 - 1) + 2^{4k}(2^6 - 2^2)$$
$$= 3^{4k} \times 80 + 2^{4k} \times 60$$
$$= 3^{4k-1} \times 3 \times 80 + 2^{4k} \times 60$$
$$= 240 \times 3^{4k-1} + 60 \times 2^{4k}$$
$$= 15(16 \times 3^{4k-1} + 4 \times 2^{4k}) \quad *$$

For all $k \in \mathbb{Z}^+$, $(16 \times 3^{4k-1} + 4 \times 2^{4k})$ is an integer, and, hence, $f(k+1) - f(k)$ is divisible by 15.

At this stage $f(k+1) - f(k)$ is clearly divisible by 10 (and 20), but since 80 is not a multiple of 15, to obtain that the expression is divisible by 15 you have to remove a further factor of 3 by writing 3^{4k} as $3^{4k-1} \times 3^1$.

This shows that 15 is a factor of $f(k+1) - f(k)$ and this is the equivalent to showing that $f(k+1) - f(k)$ is exactly divisible by 15. Note that the result would not be true for negative integers k as, for example, 4×2^{4k} would be a fraction less than one.

b Let $n = 1$

$f(1) = 3^4 + 2^6 = 81 + 64 = 145 = 5 \times 29$

So $f(n)$ is divisible by 5 for $n = 1$.

Assume that $f(k)$ is divisible by 5.

It would follow that $f(k) = 5m$, where m is an integer.

From *

$$f(k+1) = f(k) + 15(16 \times 3^{4k-1} + 4 \times 2^{4k})$$
$$= 5m + 15(16 \times 3^{4k-1} + 4 \times 2^{4k})$$
$$= 5(m + 3(16 \times 3^{4k-1} + 4 \times 2^{4k}))$$

So $f(k+1)$ is divisible by 5.

$f(n)$ is divisible by 5 for $n = 1$, and, if it is divisible by 5 for $n = k$, then it divisible by 5 for $n = k + 1$.

By mathematical induction, $f(n)$ is divisible by 5 for all $n \in \mathbb{Z}^+$.

An expression which is divisible by 15 is certainly divisible by 5, which is all that is required for part **b**.

If both $f(k)$ and $15(16 \times 3^{4k-1} + 4 \times 2^{4k})$ are divisible by 5, then their sum, $f(k+1)$ is divisible by 5.

Although $f(k+1) - f(k)$ is divisible by 15, $f(n)$ is never divisible by 15 for any n. After completing part **a**, you might misread the question for part **b** and attempt to prove that 15 was a factor of $f(n)$. It is always necessary to read questions carefully.

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26 a $f(n) = 24 \times 2^{4n} + 3^{4n}$

$f(n+1) - f(n)$

$= 24 \times 2^{4(n+1)} + 3^{4(n+1)} - 24 \times 2^{4n} - 3^{4n}$

This is an acceptable answer for part **a**. However, reading ahead, the question concerns divisibility by 5. So it is sensible to further work on this expression and show that it is divisible by 5.

b $f(n+1) - f(n)$

$= 24 \times 2^{4n+4} - 24 \times 2^{4n} + 3^{4n+4} - 3^{4n}$

$= 24 \times 2^{4n}(2^4 - 1) + 3^{4n}(3^4 - 1)$

$= 24 \times 2^{4n} \times 15 + 3^{4n} \times 80$

$= 5(72 \times 2^{4n} + 16 \times 3^{4n}) \ldots *$

In the middle of a question it is easy to forget that, in all inductions, you need to show that the result is true for a small number. This is usually 1 but this question asks you to show a result is true for all non-negative integers and 0 is a non-negative integer, so you should begin with 0.

Let $n = 0$

$f(0) = 24 \times 2^0 + 3^0 = 24 + 1 = 25$

So $f(n)$ is divisible by 5 for $n = 0$.

Assume that f (k) is divisible by 5. It would follow that $f(k) = 5m$, where m is an integer.

From*, substituting $n = k$ and rearranging,

$f(k+1) = f(k) + 5(72 \times 2^{4n} + 16 \times 3^{4n})$

$= 5m + 5(72 \times 2^{4n} + 16 \times 3^{4n})$

$= 5(m + 72 \times 2^{4n} + 16 \times 3^{4n})$

So $f(k+1)$ is divisible by 5.

$f(n)$ is divisible by 5 for $n = 0$, and, if it is divisible by 5 for $n = k$, then it divisible by 5 for $n = k + 1$.

By mathematical induction, $f(n)$ is divisible by 5 for all non-negative integers n.

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27 Let $f(n) = 7^n + 4^n + 1$

The question gives no label to the function $7^n + 4^n + 1$. Since you are going to have to refer to this function a number of times in your solution, it helps if you call it f(n).

Let $n = 1$

$f(1) = 7^1 + 4^1 + 1 = 12$

12 is divisible by 6, so $f(n)$ is divisible by 6 for $n = 1$.

Consider $f(k+1) - f(k)$

This question gives you no hint to help you. With divisibility questions, it often helps to consider $f(k+1) - f(k)$ and try and show that this divides by the appropriate number, here 6. It does not always work and there are other methods which often work just as well or better.

$$\begin{aligned} f(k+1) - f(k) &= 7^{k+1} + 4^{k+1} + 1 - (7^k + 4^k + 1) \\ &= 7^{k+1} - 7^k + 4^{k+1} - 4^k \\ &= 7^k(7-1) + 4^k(4-1) \\ &= 6 \times 7^k + 3 \times 4^k \\ &= 6 \times 7^k + 3 \times 4 \times 4^{k-1} \\ &= 6(7^k + 2 \times 4^{k-1}) \ldots \ * \end{aligned}$$

So 6 is a factor of $f(k+1) - f(k)$.

Assume that f(k) is divisible by 6.

It would follow that $f(k) = 6m$, where m is an integer.

From *

If both f(k) and $6(7^k + 2 \times 4^{k-1})$ are divisible by 6, then their sum, $f(k+1)$ is divisible by 6. You could write this down instead of the working shown here.

$$\begin{aligned} f(k+1) &= f(k) + 6(7^k + 2 \times 4^{k-1}) \\ &= 6m + 6(7^k + 2 \times 4^{k-1}) \\ &= 6(m + 7^k + 2 \times 4^{k-1}) \end{aligned}$$

So $f(k+1)$ is divisible by 6.

f(n) is divisible by 6 for $n = 1$, and, if it is divisible by 6 for $n = k$, then it divisible by 6 for $n = k + 1$.

By mathematical induction, f(n) is divisible by 6 for all positive integers n.

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28 Let $f(n) = 4^n + 6n - 1$

Let $n = 1$

$f(1) = 4^1 + 6 - 1 = 9$

So f(n) is divisible by 9 for $n = 1$.

Assume that f(k) is divisible by 9,

Then, for some integer m,

$f(k) = 4^k + 6k - 1 = 9m$

Rearranging

$4^k = 9m - 6k + 1 \ldots *$

$$\begin{aligned} f(k+1) &= 4^{k+1} + 6(k+1) - 1 \\ &= 4 \times 4^k + 6k + 5 \\ &= 4 \times (9m - 6k + 1) + 6k + 5 \\ &= 36m - 24k + 4 + 6k + 5 \\ &= 36m - 18k + 9 \\ &= 9(4m - 2k + 1) \end{aligned}$$

This is divisible by 9.

f(n) is divisible by 9 for $n = 1$, and, if it is divisible by 9 for $n = k$, then it divisible by 9 for $n = k + 1$.

By mathematical induction, f(n) is divisible by 9 for all $n \in \mathbb{Z}^+$.

The question gives no label to the function $4^n + 6n - 1$. Since you are going to have to refer to this function a number of times in your solution, it helps if you call it f(n).

With divisibility questions, it often helps to consider $f(k+1) - f(k)$ and try and show that this divides by the appropriate number, here 9. This will work in this question if you choose to do it. However the method shown here is, for this question, a neat one and you need to be aware of various alternative methods. No particular method works every time.

Here you substitute the expression for 4^k in * for the 4^k in your expression for $f(k+1)$

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29 Let $f(n) = 3^{4n-1} + 2^{4n-1} + 5$

Let $n = 1$

$f(1) = 3^3 + 2^3 + 5 = 27 + 8 + 5 = 40 = 10 \times 4$

So f (n) is divisible by 10 for $n = 1$.

Consider $f(k+1) - f(k)$

$$f(k+1) - f(k)$$
$$= 3^{4k+3} + 2^{4k+3} - 5 - (3^{4k-1} + 2^{4k-1} - 5)$$
$$= 3^{4k+3} - 3^{4k-1} + 2^{4k+3} - 2^{4k-1}$$
$$= 3^{4k-1}(3^4 - 1) + 2^{4k-3}(2^6 - 2^2)$$
$$= 3^{4k-1} \times 80 + 2^{4k-3} \times 30$$
$$= 10(8 \times 3^{4k-1} + 3 \times 2^{4k-3}) \ldots *$$

When you replace n by $k+1$ in, for example, 3^{4n-1} you get $3^{4(k+1)-1} = 3^{4k+4-1} = 3^{4k+3}$.

If we were to simplify $2^{4k+3} - 2^{4k-1}$ as far as possible, we could write $2^{4k-1}\left(2^4 - 1\right)$ in the next line.

However, $2^4 - 1 = 15$ which is not divisible by 10, so we only simplify the expression to $2^{4k-3}(2^6 - 2^2)$ which is divisible by 10.

Assume that f (k) is divisible by 10.

It would follow that $f(k) = 10m$, where m is an integer.

From *

$$f(k+1) = f(k) + 10(8 \times 3^{4k-1} + 3 \times 2^{4k-3})$$
$$= 10m + 10(8 \times 3^{4k-1} + 3 \times 2^{4k-3})$$
$$= 10(m + (8 \times 3^{4k-1} + 3 \times 2^{4k-3}))$$

If both f (k) and $10(8 \times 3^{4k-1} + 3 \times 2^{4k-3})$ are divisible by 10, then their sum, f $(k+1)$ is divisible by 10. If you preferred, you could write this down instead of the working shown here.

So $f(k+1)$ is divisible by 10.

f(n) is divisible by 10 for $n = 1$, and, if it is divisible by 10 for $n = k$, then it divisible by 10 for $n = k+1$.

By mathematical induction, f(n) is divisible by 10 for all positive integers n.

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30 $\mathbf{A}^n = \begin{pmatrix} 1 & (2^n - 1)c \\ 0 & 2^n \end{pmatrix}$

Let $n = 1$

$\mathbf{A}^1 = \begin{pmatrix} 1 & (2^1 - 1)c \\ 0 & 2^1 \end{pmatrix} = \begin{pmatrix} 1 & c \\ 0 & 2 \end{pmatrix}$

You need to begin by showing the result is true for $n = 1$. You substitute $n = 1$ into the printed expression for $\mathbf{A}^n$ and check that you get the matrix **A** as given in the question.

This is **A**, as defined in the question, so the result is true for $n = 1$.

Assume the result is true for $n = k$.

That is $\mathbf{A}^k = \begin{pmatrix} 1 & (2^k - 1)c \\ 0 & 2^k \end{pmatrix} \ldots\ldots *$

$\mathbf{A}^{k+1} = \mathbf{A}^k.\mathbf{A}$

$= \begin{pmatrix} 1 & (2^k - 1)c \\ 0 & 2^k \end{pmatrix}\begin{pmatrix} 1 & c \\ 0 & 2 \end{pmatrix}$

$= \begin{pmatrix} 1 & c + 2(2^k - 1)c \\ 0 & 2 \times 2^k \end{pmatrix}$

$= \begin{pmatrix} 1 & c + 2^{k+1}c - 2c \\ 0 & 2^{k+1} \end{pmatrix}$

$= \begin{pmatrix} 1 & (2^{k+1} - 1)c \\ 0 & 2^{k+1} \end{pmatrix}$

Keep in mind as you multiply out the matrices that you are aiming at the expression $\mathbf{A}^n = \begin{pmatrix} 1 & (2^n - 1)c \\ 0 & 2^n \end{pmatrix}$ with each n replaced by $k + 1$.

$2 \times 2^k = 2^1 \times 2^k = 2^{k+1}$ by one of the laws of indices.
You use this twice.

This is the result obtained by substituting $n = k + 1$ into the result $\mathbf{A}^n = \begin{pmatrix} 1 & (2^n - 1)c \\ 0 & 2^n \end{pmatrix}$ and so the result is true for $n = k + 1$.

The result is true for $n = 1$, and, if it is true for $n = k$, then it is true for $n = k + 1$.

By mathematical induction the result is true for all positive integers n.

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31 $\mathbf{A}^n = \begin{pmatrix} 2n+1 & n \\ -4n & -2n+1 \end{pmatrix}$

You need to begin by showing the result is true for $n = 1$. You substitute $n = 1$ into the given expression for $\mathbf{A}^n$ and check that you get the matrix $\mathbf{A}$, as given in the question.

Let $n = 1$

$$\mathbf{A}^1 = \begin{pmatrix} 2+1 & 1 \\ -4 & -2+1 \end{pmatrix} = \begin{pmatrix} 3 & 1 \\ -4 & -1 \end{pmatrix}$$

This is $\mathbf{A}$, as defined in the question, so the result is true for $n = 1$.

Assume the result is true for $n = k$.

The **induction hypothesis** is the result you are asked to prove with n replaced by k.

That is $\mathbf{A}^k = \begin{pmatrix} 2k+1 & k \\ -4k & -2k+1 \end{pmatrix}$

$\mathbf{A}^{k+1}$ is the matrix $\mathbf{A}$, multiplied by itself k times, multiplied by $\mathbf{A}$ one more time.
$\mathbf{A}^{k+1} = \mathbf{A}^k.\mathbf{A}^1 = \mathbf{A}^k.\mathbf{A}$. This is one of the index laws applied to matrices

$$\mathbf{A}^{k+1} = \mathbf{A}^k.\mathbf{A}$$

$$= \begin{pmatrix} 2k+1 & k \\ -4k & -2k+1 \end{pmatrix}\begin{pmatrix} 3 & 1 \\ -4 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} 3(2k+1)-4k & 2k+1-k \\ -12k-4(-2k+1) & -4k-(-2k+1) \end{pmatrix}$$

$$= \begin{pmatrix} 2k+3 & k+1 \\ -4k-4 & -2k-1 \end{pmatrix}$$

$$= \begin{pmatrix} 2(k+1)+1 & k+1 \\ -4(k+1) & -2(k+1)+1 \end{pmatrix}$$

This is the result obtained by substituting $n = k+1$ into the result $\mathbf{A}^n = \begin{pmatrix} 2n+1 & n \\ -4n & -2n+1 \end{pmatrix}$ and so the result is true for $n = k+1$.

The result is true for n = 1, and, if it is true for $n = k$, then it is true for $n = k + 1$.

By mathematical induction the result is true for all positive integers n.

32 a He has not shown the general statement to be true for $k = 1$.

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32 b Let $f(n) = 2^{2n} - 1$, where $n \in \mathbb{Z}^+$

$f(1) = 2^{2(1)} - 1 = 3$, which is divisible by 3.

$f(n)$ is divisible by 3 when $n = 1$.

Assume true for $n = k$, so that $f(k) = 2^{2k} - 1$ is divisible by 3.

$$\begin{aligned} f(k+1) &= 2^{2(k+1)} - 1 \\ &= 2^{2k} \times 2^2 - 1 \\ &= 4(2^{2k}) - 1 \\ &= 4(f(k) + 1) - 1 \\ &= 4f(k) + 3 \end{aligned}$$

Therefore $f(n)$ is divisible by 3 when $n = k + 1$

If $f(n)$ is divisible by 3 when $n = k$, then it has been shown that $f(n)$ is also divisible by 3 when $n = k + 1$.

As $f(n)$ is divisible by 3 when $n = 1$, $f(n)$ is also divisible by 3 for all $n \in \mathbb{Z}^+$ by mathematical induction.

33 A general point on the line has position vector $\begin{pmatrix} 2-\lambda \\ -1-2\lambda \\ 3+3\lambda \end{pmatrix}$.

Let $A = \begin{pmatrix} x_A \\ y_A \\ z_A \end{pmatrix}$ and $B = \begin{pmatrix} x_B \\ y_B \\ z_B \end{pmatrix}$

For A, when $\lambda = 4$:

$x_A = 2 - 4 \Rightarrow x_A = -2$

$y_A = -1 - 2(4) \Rightarrow y_A = -9$

$z_A = 3 + 3(4) \Rightarrow z_A = 15$

For B, when $\lambda = -1$:

$x_B = 2 + 1 \Rightarrow x_B = 3$

$y_B = -1 - 2(-1) \Rightarrow y_B = 1$

$z_B = 3 + 3(-1) \Rightarrow z_B = 0$

The distance AB is given by

$$\begin{aligned} \left|\overrightarrow{AB}\right| &= \sqrt{\left(3-(-2)\right)^2 + \left(1-(-9)\right)^2 + (0-15)^2)} \\ &= \sqrt{350} = 5\sqrt{14} \end{aligned}$$

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34 Line through P, Q and R has direction vector $\overrightarrow{PQ} = \begin{pmatrix} a-1 \\ 4 \\ 5 \end{pmatrix}$

You could write any alternative form of the direction vector, e.g. $\overrightarrow{PR}$ or $\overrightarrow{QR}$

So the equation of the line through all 3 points can be written as $\mathbf{r} = \begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} a-1 \\ 4 \\ 5 \end{pmatrix}$

You could write any alternative form of the line, e.g. $\mathbf{r} = \vec{R} + \lambda\overrightarrow{PR}$ or $\mathbf{r} = \vec{Q} + \lambda\overrightarrow{QR}$

Now R is a point on the line, so for some λ it is true that:

$$\begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} a-1 \\ 4 \\ 5 \end{pmatrix} = \begin{pmatrix} 5 \\ 7 \\ b \end{pmatrix}$$

Equate your expression for the line with the point you haven't yet used.

Equate y-components to find λ: $-1 + 4\lambda = 7$

$$4\lambda = 8 \Rightarrow \lambda = 2$$

Use the x-component of $\mathbf{r}$ to find a:

$$1 + 2(a-1) = 5$$
$$2a - 1 = 5$$
$$2a = 6 \Rightarrow a = 3$$

Use the z-component of $\mathbf{r}$ to find b:

$$3 + 2 \times 5 = b$$
$$b = 13$$

The vector equation of the line is given by

$$\mathbf{r} = \begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 4 \\ 5 \end{pmatrix}$$

35 a $\overrightarrow{AB} = \begin{pmatrix} -2 \\ 2 \\ -2 \end{pmatrix}$

$$\overrightarrow{AC} = \begin{pmatrix} 1 \\ -1 \\ -4 \end{pmatrix}$$

Let $\mathbf{n} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$ be perpendicular to the plane

Then $\mathbf{n}$ is perpendicular to both $\begin{pmatrix} -2 \\ 2 \\ -2 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ -1 \\ -4 \end{pmatrix}$

So $\begin{pmatrix} -2 \\ 2 \\ -2 \end{pmatrix} . \begin{pmatrix} \mathbf{a} \\ \mathbf{b} \\ \mathbf{c} \end{pmatrix} = 0$ and $\begin{pmatrix} 1 \\ -1 \\ -4 \end{pmatrix} . \begin{pmatrix} \mathbf{a} \\ \mathbf{b} \\ \mathbf{c} \end{pmatrix} = 0$

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Therefore $-2a+2b-c=0$ and $a-b-4c=0$

Choosing $c=0$ gives $a=b$

Therefore, choosing $a=1$ gives a normal vector of $\begin{pmatrix}1\\1\\0\end{pmatrix}$

The equation of the plane is **r.n = a.n**
Since $B(1,1,2)$ lies on the plane, you have

$$\begin{pmatrix}x\\y\\z\end{pmatrix}.\begin{pmatrix}1\\1\\0\end{pmatrix}=\begin{pmatrix}1\\1\\2\end{pmatrix}.\begin{pmatrix}1\\1\\0\end{pmatrix}$$

or $x+y=2$

b A normal vector is given by

$$\begin{pmatrix}1\\1\\0\end{pmatrix}$$

c $\frac{(2)}{2}+\frac{k}{2}=1$

$1+\frac{k}{2}=1$

$2+k=2 \Rightarrow k=0$

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36 a Assuming that the lines do intersect:

$$\begin{pmatrix}11+4\lambda\\5+2\lambda\\6+4\lambda\end{pmatrix}=\begin{pmatrix}24+7\mu\\4+\mu\\13+5\mu\end{pmatrix} \quad *$$

You can write the equations of the lines in column vector form and equate them.

Rearranging gives:

$$4\lambda-7\mu=13 \quad \textbf{(1)}$$
$$2\lambda-\mu=-1 \quad \textbf{(2)}$$
$$4\lambda-5\mu=7 \quad \textbf{(3)}$$

Equate the x, y and z components.

Solve these simultaneous equations.

(1) – **(3)** gives

$$-2\mu=6$$
$$\therefore \mu=-3$$

Solve equations **(1)** and **(3)** simultaneously.

Substitute into **(1)** to give

$$4\lambda+21=13 \Rightarrow \lambda=-2$$

Check that $\mu=-3$ and $\lambda=-2$ satisfy equation (2):

$$2(-2)-(-3)=-4+3=-1$$

So l_1 and l_2 intersect.

b Substituting $\mu=-3$ and $\lambda=-2$ into * gives the coordinates of the point of intersection

$$\begin{pmatrix}11-8\\5-4\\6-8\end{pmatrix}=\begin{pmatrix}24-21\\4-3\\13-15\end{pmatrix}=\begin{pmatrix}3\\1\\-2\end{pmatrix}$$

Substituting λ or μ will give the point of intersection.

$\therefore (3,1,-2)$ is point of intersection.

c The directions of the lines are

$4\mathbf{i}+2\mathbf{j}+4\mathbf{k}$ and $7\mathbf{i}+\mathbf{j}+5\mathbf{k}$

$$\cos\theta=\frac{(4\mathbf{i}+2\mathbf{j}+4\mathbf{k})\cdot(7\mathbf{i}+\mathbf{j}+5\mathbf{k})}{\sqrt{4^2+2^2+4^2}\sqrt{7^2+1^2+5^2}}$$
$$=\frac{28+2+20}{\sqrt{36}\sqrt{75}}$$
$$=\frac{50}{6\times5\sqrt{3}}$$
$$=\frac{5}{3\sqrt{3}}$$
$$=\frac{5}{9}\sqrt{3}$$

Use $\cos\theta=\dfrac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}||\mathbf{b}|}$, where $\mathbf{a}$ and $\mathbf{b}$ are the direction vectors of the lines.

Simplify the surds.

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37 a The line can be written as $\mathbf{r} = \begin{pmatrix} 8+\lambda \\ 12+\lambda \\ 14-\lambda \end{pmatrix}$

You can write the line equation in this form.

Since A lies on the line, there is a value of λ such that $\begin{pmatrix} 4 \\ 8 \\ a \end{pmatrix} = \begin{pmatrix} 8+\lambda \\ 12+\lambda \\ 14-\lambda \end{pmatrix}$

use $4 = 8+\lambda$ *or* $8 = 12+\lambda$
$\therefore \lambda = -4$

Equate the x or y coordinates to find λ.

Equate z-coordinates with $\lambda = -4$ to give $a = 14-(-4) = 18$

Similarly, since B lies on the line, there is a value of λ such that $\begin{pmatrix} b \\ 13 \\ 13 \end{pmatrix} = \begin{pmatrix} 8+\lambda \\ 12+\lambda \\ 14-\lambda \end{pmatrix}$

Use $13 = 12+\lambda$ or $13 = 14-\lambda$
$\therefore \lambda = 1$

Use the y or z coordinates to find λ.

Equate x-coordinates with $\lambda = 1$ to give $b = 8+1 = 9$

b Direction $\overrightarrow{OP}$ is $\begin{pmatrix} 8+\lambda \\ 12+\lambda \\ 14-\lambda \end{pmatrix}$ for some λ

and direction of l_1 is $\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$

This is obtained from the equation of l_1.

These are perpendicular

$\therefore \begin{pmatrix} 8+\lambda \\ 12+\lambda \\ 14-\lambda \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} = 0$

Use the condition for perpendicular lines with direction vectors **c** and **d**, $\mathbf{c} \cdot \mathbf{d} = 0$.

$\therefore 8+\lambda+12+\lambda-(14-\lambda) = 0$
$\therefore 3\lambda + 6 = 0$
$\Rightarrow \lambda = -2$
$\therefore$ Point P is at $(6, 10, 16)$

Substitute the value of λ into the line equation to give the coordinates of P.

c Distance $OP = \sqrt{6^2+10^2+16^2}$
$= \sqrt{392}$
$= 14\sqrt{2}$

Use the formula for magnitude of a vector.

Simplify the surd using $\sqrt{392} = \sqrt{196}\sqrt{2}$.

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38 a $a(4\mathbf{i}+\mathbf{j}+2\mathbf{k})\cdot(\mathbf{i}-5\mathbf{j}+3\mathbf{k}) = a(4\times1+1\times(-5)+2\times3)$
$= a(4-5+6) = 5a$

For A to lie on the plane with equation $\mathbf{r}.\mathbf{n} = 5a$, when $\mathbf{r}$ is replaced by the position vector of A, $\mathbf{r}.\mathbf{n}$ must give $5a$.

Hence A lies in the plane Π, as required.

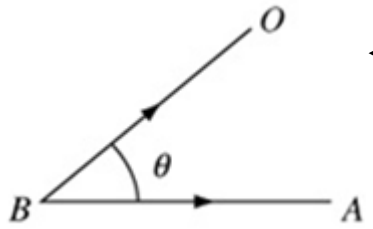
b $\overrightarrow{BA} = a(4\mathbf{i}+\mathbf{j}+2\mathbf{k}) - a(2\mathbf{i}+11\mathbf{j}-4\mathbf{k})$
$= a(2\mathbf{i}-10\mathbf{j}+6\mathbf{k})$
$\overrightarrow{BA} = 2a(\mathbf{i}-5\mathbf{j}+3\mathbf{k})$

$\overrightarrow{BA}$ is parallel to the vector $\mathbf{i}-5\mathbf{j}+3\mathbf{k}$, which is perpendicular to the plane Π.

This is because the equation of the plane is $\mathbf{r}.(\mathbf{i}-5\mathbf{j}+3\mathbf{k}) = 5a$ where, by definition, the vector $(\mathbf{i}-5\mathbf{j}+3\mathbf{k})$ is perpendicular to the plane.

Hence $\overrightarrow{BA}$ is perpendicular to the plane Π, as required.

c



The angle OBA is the angle between BO and BA. Both these line segments have a definite direction and so you must use the scalar product $\overrightarrow{BO}.\overrightarrow{BA}$ to find θ. If you used $\overrightarrow{OB}.\overrightarrow{BA}$, you would obtain the supplementary angle $(180° - \theta)$, which is not the correct answer.

Let $\angle OAB = \theta$

$|\overrightarrow{BO}| = a\sqrt{((-2^2)+(-11)^2+4^2)} = a\sqrt{(141)}$
$|\overrightarrow{BA}| = a\sqrt{(2^2+(-10)^2+6^2)} = a\sqrt{(140)}$
$\overrightarrow{BO}.\overrightarrow{BA} = a(-2\mathbf{i}-11\mathbf{j}+4\mathbf{k}).a(2\mathbf{i}-10\mathbf{j}+6\mathbf{k})$

$$\cos\theta = \frac{\overrightarrow{BO}.\overrightarrow{BA}}{|\overrightarrow{BO}||\overrightarrow{BA}|}$$
$$= \frac{a^2((-2)\times2+(-11)\times(-10)+4\times6)}{a\sqrt{(141)}\times a\sqrt{(140)}}$$
$$= \frac{a^2(-4+110+24)}{a^2\sqrt{(141)}\sqrt{(140)}}$$
$$= \frac{130}{\sqrt{(141)}\sqrt{(140)}}$$
$$= 0.925272\ldots$$
$$\theta = 22.3°$$

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39 a $\mathbf{r}\cdot\begin{pmatrix}-1\\2\\1\end{pmatrix}=k$ where $k=\mathbf{a}\cdot\begin{pmatrix}-1\\2\\1\end{pmatrix}$ for any point in the plane with position vector $\mathbf{a}$

Since $\begin{pmatrix}3\\4\\-1\end{pmatrix}$ is in the plane, $k=\begin{pmatrix}3\\-4\\1\end{pmatrix}\cdot\begin{pmatrix}-1\\2\\1\end{pmatrix}=-10$

b Letting $\mathbf{r}=\begin{pmatrix}x\\y\\z\end{pmatrix}$, we can write the plane as $\begin{pmatrix}x\\y\\z\end{pmatrix}\cdot\begin{pmatrix}-1\\2\\1\end{pmatrix}=-10$

So a cartesian equation of the plane is

$-x+2y+z=-10$

c Equation of l is $\mathbf{r}=\begin{pmatrix}4\\-3\\2\end{pmatrix}+t\begin{pmatrix}-1\\2\\1\end{pmatrix}$

At intersection

$$\begin{pmatrix}4-t\\-3+2t\\2+t\end{pmatrix}\cdot\begin{pmatrix}-1\\2\\1\end{pmatrix}=-10$$

$$-4+t-6+4t+2+t=-10$$

$$6t+2=0$$

$$6t=-2\Rightarrow t=-\frac{1}{3}$$

Then coordiantes of N are

$$4-\left(-\frac{1}{3}\right)=\frac{13}{3}$$

$$-3+2\left(-\frac{1}{3}\right)=-\frac{11}{3}$$

$$2+\left(-\frac{1}{3}\right)=\frac{5}{3}$$

$\therefore\ N\left(\frac{13}{3},\ -\frac{11}{3},\ \frac{5}{3}\right)$

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40 a For the line l, first express each of x, y and z in terms of a third variable λ:

$$\frac{x-1}{2}=\lambda \Rightarrow x=2\lambda+1$$

$$\frac{y+3}{1}=\lambda \Rightarrow y=\lambda-3$$

$$\frac{2-z}{3}=\lambda \Rightarrow z=2-3\lambda$$

Now substitute each one into the cartesian equation of the plane:

$$2(2\lambda+1)+1(\lambda-3)-1(2-3\lambda)=5$$

$$8\lambda-3=5$$

$$8\lambda=8$$

$$\lambda=1$$

$$x=2(1)+1=3$$

$$y=(1)-3=-2$$

$$z=2-3(1)=-1$$

Position vector of P $\begin{pmatrix}3\\-2\\-1\end{pmatrix}$

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40 b First, write both the plane and the line in vector form:

Plane: $2x+y-z=5$

$$\Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} = 5$$

$$\Rightarrow \mathbf{r} \cdot \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} = 5 \text{ where } \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} \text{ is normal to the plane}$$

Line: From part **a,** $\dfrac{x-1}{2} = \lambda \Rightarrow x = 2\lambda + 1$

$$\frac{y+3}{1} = \lambda \Rightarrow y = \lambda - 3$$

$$\frac{2-z}{3} = \lambda \Rightarrow z = 2 - 3\lambda$$

Hence $\mathbf{r} = (2\lambda+1)\mathbf{i} + (\lambda-3)\mathbf{j} + (2-3\lambda)\mathbf{k}$

$$\mathbf{r} = \begin{pmatrix} 1 \\ -3 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix}$$

Let θ be the angle between the line and the *normal to the plane*:

$$\cos\theta = \frac{(2\mathbf{i}+\mathbf{j}-3\mathbf{k})\cdot(2\mathbf{i}+\mathbf{j}-\mathbf{k})}{\sqrt{2^2+1^2+(-3)^2}\sqrt{2^2+1^2+(-1)^2}}$$

$$= \frac{4+1+3}{\sqrt{14}\sqrt{6}}$$

$$= \frac{8}{\sqrt{84}}$$

$$\theta = 0.509^c$$

Then the angle between the line and the plane is: $\dfrac{\pi}{2} - \theta = 1.061^c$

41 a $\mathbf{a} = \mathbf{i} - \mathbf{j} + 3\mathbf{k}, \mathbf{b} = 4\mathbf{i} + 3\mathbf{j} - 2\mathbf{k}$

$\therefore \overrightarrow{AB} = \mathbf{b} - \mathbf{a}$ ← Use the triangle law.

$= 3\mathbf{i} + 4\mathbf{j} - 5\mathbf{k}$

$\therefore \left|\overrightarrow{AB}\right| = \sqrt{3^2+4^2+(-5)^2}$ ← Use the formula for the magnitude of a vector.

$= \sqrt{50}$ or $5\sqrt{2}$ or 7.07

b $\mathbf{r} = \mathbf{i} - \mathbf{j} + 3\mathbf{k} + \lambda(3\mathbf{i} + 4\mathbf{j} - 5\mathbf{k})$

or

$\mathbf{r} = 4\mathbf{i} - 3\mathbf{j} - 2\mathbf{k} + \mu(3\mathbf{i} + 4\mathbf{j} - 5\mathbf{k})$

There are other forms of this equation, but these two are the simplest.

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41 c If $\mathbf{r} = 6\mathbf{i} + 4\mathbf{j} - 3\mathbf{k} + \mu(2\mathbf{i} + \mathbf{j} - \mathbf{k})$ passes through $4\mathbf{i} + 3\mathbf{j} - 2\mathbf{k}$

then $6 + 2\mu = 4$

$4 + \mu = 3$

$-3 - \mu = -2$

Equate x, y and z components.

As $\mu = -1$ satisfies all three equations, the line passes through B as required.

Solve for μ and check that μ satisfies all three equations.

d The lines have directions

$3\mathbf{i} + 4\mathbf{j} - 5\mathbf{k}$ and $2\mathbf{i} + \mathbf{j} - \mathbf{k}$

If the angle between the lines is θ then

$$\cos\theta = \frac{(3\mathbf{i} + 4\mathbf{j} - 5\mathbf{k}) \cdot (2\mathbf{i} + \mathbf{j} - \mathbf{k})}{|3\mathbf{i} + 4\mathbf{j} - 5\mathbf{k}||2\mathbf{i} + \mathbf{j} - \mathbf{k}|}$$

$$= \frac{3 \times 2 + 4 \times 1 + (-5) \times (-1)}{\sqrt{50}\sqrt{2^2 + 1^2 + (-1)^2}}$$

$$= \frac{15}{\sqrt{50}\sqrt{6}}$$

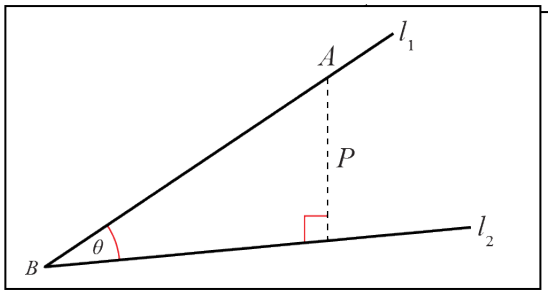
Use $\cos\theta = \dfrac{\mathbf{c} \cdot \mathbf{d}}{|\mathbf{c}||\mathbf{d}|}$ where $\mathbf{c}$ and $\mathbf{d}$ are the directions of the lines.

$$\therefore \cos\theta = \frac{\sqrt{3}}{2}$$

This answer is acute. If your answer is obtuse, subtract it from 180°.

and $\theta = 30°$ or $\dfrac{\pi}{6}$

e



Draw a diagram showing l_1, l_2 with common point B.

The shortest distance is the perpendicular distance.

The shortest distance from point A to the line l_2 is

$$|\overrightarrow{AB}| \sin\theta = 5\sqrt{2} \times \frac{1}{2}$$

$$= \frac{5\sqrt{2}}{2}$$

Use trigonometry $\sin\theta = \dfrac{P}{|\overrightarrow{AB}|}$.

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42 a Substitute the coordinates of point P into the equation of the plane:

$-3(-1)+2(2)+3(-3)+7=5$

Distance between P and Π:

$$\frac{5}{\sqrt{(-3)^2+2^2+3^2}}=\frac{5}{\sqrt{22}}=1.066$$

b A perpendicular vector to Π is

$\mathbf{n}=-3\mathbf{i}+2\mathbf{j}+3\mathbf{k}$

Let Q have coordinates (x_1, y_1, z_1).

Let M be the midpoint of PQ.

A vector equation of the line l through P, M and Q is $\mathbf{r}=\begin{pmatrix}-1\\2\\-3\end{pmatrix}+\lambda\begin{pmatrix}-3\\2\\3\end{pmatrix}$

M lies on this line so has position vector $\mathbf{r}_M=\begin{pmatrix}-1-3\lambda\\2+2\lambda\\-3+3\lambda\end{pmatrix}$ for some λ

M also lies on Π, so $\begin{pmatrix}-1-3\lambda\\2+2\lambda\\-3+3\lambda\end{pmatrix}\cdot\begin{pmatrix}-3\\2\\3\end{pmatrix}=-7$

$$3+9\lambda+4+4\lambda-9+9\lambda=-7$$
$$22\lambda-2=-7$$
$$22\lambda=-5$$
$$\lambda=-\frac{5}{22}$$

$\therefore$ M has position vector $\mathbf{r}_M=\begin{pmatrix}-1\\2\\-3\end{pmatrix}-\frac{5}{22}\begin{pmatrix}-3\\2\\3\end{pmatrix}$

P is the initial point in the equation of l, so because M has position vector $\mathbf{r}_M=\begin{pmatrix}-1\\2\\-3\end{pmatrix}-\frac{5}{22}\begin{pmatrix}-3\\2\\3\end{pmatrix}$

then Q has position vector

$$\mathbf{r}_Q=\begin{pmatrix}-1\\2\\-3\end{pmatrix}+2\times\left(-\frac{5}{22}\right)\begin{pmatrix}-3\\2\\3\end{pmatrix}=\begin{pmatrix}-1+\frac{15}{11}\\2-\frac{10}{11}\\-3-\frac{15}{11}\end{pmatrix}$$

Point Q has coordinates $\left(\frac{4}{11}, \frac{12}{11}, -\frac{48}{11}\right)$

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43 a For the shark

$$\mathbf{r}_s = \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} -2-2 \\ 11-3 \\ 11-(-1) \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} -4 \\ 8 \\ 12 \end{pmatrix}$$

For the flounder

$$\mathbf{r}_f = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} + \mu \begin{pmatrix} -2 \\ -1 \\ 3 \end{pmatrix}$$

$$2-4\lambda = 2-2\mu \quad \textbf{(1)}$$
$$3+8\lambda = -\mu \quad \textbf{(2)}$$
$$-1+12\lambda = 1+3\mu \quad \textbf{(3)}$$

Substitute **(2)** into **(1)**:

$$2-4\lambda = 2-2(-3-8\lambda)$$
$$2-4\lambda = 2+6+16\lambda$$
$$20\lambda = -6$$
$$\lambda = -\frac{3}{10}$$

Substitute $\lambda = -\frac{3}{10}$ into **(2)**

$$3+8\left(-\frac{3}{10}\right) = -\mu$$
$$\mu = \frac{3}{5}$$

Check in **(3)**

$$\text{LHS} = -1+12\left(-\frac{3}{10}\right) = -\frac{23}{5}$$
$$\text{RHS} = 1+3\left(\frac{3}{5}\right) = \frac{14}{5}$$
$$\text{LHS} \neq \text{RHS}$$

Hence, the shark will never catch the flounder as their paths do not intersect.

b Unlikely that the shark will not adjust course to intercept flounder.

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Challenge

1 A reflection in the plane $x = 0$ is represented by the matrix $\mathbf{A} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

Rotation, angle θ, about the y-axis is represented by the matrix $\begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix}$

Now $\sin 270 = -1$ and $\cos 270 = 0$

Hence, rotation angle 270° about the y-axis is represented by the matrix $\mathbf{B} = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$

A reflection in the plane $y = 0$ is represented by the matrix $\mathbf{C} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

So the combination of these three transformations is given by the matrix product $\mathbf{CBA}$:

$$\mathbf{CBA} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}\begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{pmatrix}$$

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Challenge

2 Let centre P have position vector: $\mathbf{r}_p = \mathbf{c} + \lambda(\mathbf{c}-\mathbf{b}) + \mu(\mathbf{c}-\mathbf{a})$

$$= \begin{pmatrix}1\\1\\1\end{pmatrix} + \lambda\begin{pmatrix}2\\2\\-2\end{pmatrix} + \mu\begin{pmatrix}3\\4\\1\end{pmatrix}$$

$\left|\overrightarrow{AP}\right|^2 = \left|\overrightarrow{CP}\right|^2$ since $\overrightarrow{AP}$ and $\overrightarrow{CP}$ are both radii of the circle which passes through A, B and C

$$(3+(2\lambda+3\mu))^2 + (4+(2\lambda+4\mu))^2 + (1+(-2\lambda+\mu))^2$$

$$= (2\lambda+3\mu)^2 + (2\lambda+4\mu)^2 + (-2\lambda+\mu)^2$$

$$26 + 6(2\lambda+3\mu) + 8(2\lambda+4\mu) + 2(-2\lambda+\mu) = 0$$

$$26 + 24\lambda + 52\mu = 0 \qquad \textbf{(1)}$$

$\left|\overrightarrow{BP}\right|^2 = \left|\overrightarrow{CP}\right|^2$ since $\overrightarrow{BP}$ and $\overrightarrow{CP}$ are both radii of the circle which passes through A, B and C

$$(2+(2\lambda+3\mu))^2 + (2+(2\lambda+4\mu))^2 + (-2+(-2\lambda+\mu))^2$$

$$= (2\lambda+3\mu)^2 + (2\lambda+4\mu)^2 + (-2\lambda+\mu)^2$$

$$12 + 4(2\lambda+3\mu) + 4(2\lambda+4\mu) - 4(-2\lambda+\mu) = 0$$

$$12 + 24\lambda + 24\mu = 0 \qquad \textbf{(2)}$$

(1) – (2): $14 + 28\mu = 0$

$$28\mu = -14 \Rightarrow \mu = -\frac{14}{28} = -\frac{1}{2}$$

Substitute $\mu = -\frac{1}{2}$ into **(2)**:

$$12 + 24\lambda + 24\left(-\frac{1}{2}\right) = 0$$

$$24\lambda = 0 \Rightarrow \lambda = 0$$

So the position vector of centre P is

$$\mathbf{r}_p = \begin{pmatrix}1\\1\\1\end{pmatrix} - \frac{1}{2}\begin{pmatrix}3\\4\\1\end{pmatrix} = \begin{pmatrix}-\frac{1}{2}\\-1\\\frac{1}{2}\end{pmatrix}$$

The coordinates of the centre of the circle are $\left(-\frac{1}{2},\ -1,\ \frac{1}{2}\right)$.

The point P is equidistant to each point A, B and C (since A, B and C lie on circumference of circle centre P). Considering point C, the radius is given by

$$\left|\overrightarrow{PC}\right| = \sqrt{\left(1-\left(-\frac{1}{2}\right)\right)^2 + (1-(-1))^2 + \left(1-\frac{1}{2}\right)^2}$$

$$= \sqrt{\frac{26}{4}}$$

$$= \sqrt{\frac{13}{2}}$$

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Challenge

3 We will prove the result by induction.

$n=1:\quad 2(1)\le r\le \frac{1}{2}\left(1^2+1+2\right)$

$2\le r\le 2\Rightarrow r=2$

It is certainly true that a single line divides a plane into two regions.

Hence the statement is true for $n=1$.

Assume that the statement is true for $n=k$:

i.e. if k non-parallel lines divide the plane into r regions, then

$2k\le r\le \frac{1}{2}\left(k^2+k+2\right)$

Now consider $k+1$ non-parallel lines

For the lower bound: when $n=k$, the plane is divided into $2k$ regions when all k lines intersect at a single point

When $n=k+1$ and all $k+1$ lines intersect at a single point there are two more regions created.

$\therefore\ r_{k+1}\geqslant 2k+2=2(k+1)$

For the upper bound: when $n=k$, the plane is divided into $\frac{1}{2}\left(k^2+k+2\right)$ regions when only two lines intersect at any point

When $n=k+1$ and only two lines intersect at any point, $k+1$ more regions are created

$$\therefore r_{k+1}\leqslant \frac{1}{2}\left(k^2+k+2\right)+\left(k+1\right)$$
$$=\frac{1}{2}\left(k^2+3k+4\right)$$
$$=\frac{1}{2}\left(\left(k+1\right)^2+\left(k+1\right)+2\right)$$

Hence $2(k+1)\leqslant r_{k+1}\leqslant \frac{1}{2}\left(\left(k+1\right)^2+\left(k+1\right)+2\right)$

If the statement holds for $n=k$, it holds for $n=k+1$.

$\therefore$ The statement holds for all positive integers, n.

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Exam-style Practice

1 a A general point on line l_1 is

$$\begin{pmatrix}-3\\0\\5\end{pmatrix}+\lambda\begin{pmatrix}5\\-1\\1\end{pmatrix}=\begin{pmatrix}-3+5\lambda\\-\lambda\\5+\lambda\end{pmatrix}$$

A general point on line l_2 is

$$\begin{pmatrix}10\\-1\\15\end{pmatrix}+\mu\begin{pmatrix}6\\-2\\4\end{pmatrix}=\begin{pmatrix}10+6\mu\\-1-2\mu\\15+4\mu\end{pmatrix}$$

If l_1 and l_2 intersect, then there are unique values of λ and μ such that

$-3+5\lambda=10+6\mu$ **(1)**

$-\lambda=-1-2\mu$ **(2)**

$5+\lambda=15+4\mu$ **(3)**

Substituting **(2)** into **(3)** gives

$5+(1+2\mu)=15+4\mu$

$-9=2\mu$

$\mu=-\frac{2}{9}$

Substituting $\mu=-\frac{2}{9}$ back into **(2)** gives

$\lambda=1+2\left(-\frac{2}{9}\right)=\frac{5}{9}$

Determine if $\mu=-\frac{2}{9}$ and $\lambda=\frac{5}{9}$ are consistent with **(1)**:

$\text{LHS}=-3+5\left(\frac{5}{9}\right)=-\frac{2}{9}$

$\text{RHS}=10+6\left(-\frac{2}{9}\right)=\frac{26}{3}$

$-\frac{2}{9}\neq\frac{26}{3}$ so the system of equations is not consistent, and hence l_1 and l_2 do not meet.

2 a $\det\begin{vmatrix}2&k&3\\1&-3&1\\3&-1&2\end{vmatrix}=2\begin{vmatrix}-3&1\\-1&2\end{vmatrix}-k\begin{vmatrix}1&1\\3&2\end{vmatrix}+3\begin{vmatrix}1&-3\\3&-1\end{vmatrix}$

$=2(-6+1)-k(2-3)+3(-1+9)$

$=-10+k+24=k+14$

b The three planes do not meet at a single point, so

$\det\begin{vmatrix}2&k&3\\1&-3&1\\3&-1&2\end{vmatrix}=0$

$k+14=0$

$k=-14$

The three equations then become:

$2x-14y+3z=1$ **(1)**

$x-3y+z=-2$ **(2)**

$3x-y+2z=3$ **(3)**

Determine whether this system has any solutions:

$\mathbf{(3)}-2\times\mathbf{(2)}:\ x+5y=7$ **(4)**

$3\times(\mathbf{2})-1:\quad x+5y=-7$ **(5)**

Equations **(4)** and **(5)** are inconsistent so the system is inconsistent and has no solutions. The planes form a prism.

3 Let $w=2x-1\ \Rightarrow x=\frac{w+1}{2}$

Substituting $x=\frac{w+1}{2}$ in the cubic equation:

$2\left(\frac{w+1}{2}\right)^3-3\left(\frac{w+1}{2}\right)^2-7\left(\frac{w+1}{2}\right)-1=0$

$2\left(\frac{w^3+3w^2+3w+1}{8}\right)-3\left(\frac{w^2+2w+1}{4}\right)-7\left(\frac{w+1}{2}\right)-1=0$

$w^3-17w-20=0$

$p=1,\ q=0,\ r=-17,\ s=-20$

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4 a $n=1$: $\text{LHS}=\sum_{r=1}^{1} r^3 = 1^3 = 1$

$$\text{RHS}=\frac{1}{4}(1)^2\left((1)+1\right)^2=\frac{4}{4}=1$$

As LHS = RHS, the summation formula is true for $n = 1$.

Assume that the summation formula is true for $n = k$:

$$\sum_{r=1}^{k} r^3 = \frac{1}{4}(k)^2(k+1)^2$$

With $n = k + 1$ terms the summation formula becomes:

$$\sum_{r=1}^{k+1} r^3 = \sum_{r=1}^{k} r^3 + (k+1)^3$$

$$= \frac{1}{4}k^2(k+1)^2 + (k+1)^3$$

$$= \frac{1}{4}(k+1)^2\left(k^2+4(k+1)\right)$$

$$= \frac{1}{4}(k+1)^2\left(k^2+4k+4\right)$$

$$= \frac{1}{4}(k+1)^2(k+2)^2$$

$$= \frac{1}{4}(k+1)^2\left[(k+1)+1\right]^2$$

Therefore, the summation formula is true when $n = k+1$. If the summation formula is true for $n = k$ then it is shown to be true for $n = k + 1$. As the result is true for $n = 1$, it is now also true for all $n \in \mathbb{Z}^+$ by mathematical induction.

b $\sum_{r=1}^{n} 2r(r+1) = 2\sum_{r=1}^{n} r^2 + 2\sum_{r=1}^{n} r$

$$= 2\times\frac{1}{6}n(n+1)(2n+1) + 2\times\frac{1}{2}n(n+1)$$

$$= 2n(n+1)\left(\frac{1}{6}(2n+1)+\frac{1}{2}\right)$$

$$= 2n(n+1)\left(\frac{n+2}{3}\right)$$

$$= \frac{2}{3}n(n+1)(n+2)$$

c $\sum_{r=1}^{n} 2r(r+1) = 4\sum_{r=1}^{n} r^3$

$\Rightarrow \frac{2}{3}n(n+1)(n+2) = n^2(n+1)^2$ for positive integers n, by parts **a** and **b**

$\Rightarrow 3n^4+4n^3-3n^2-4n=0$

$\Rightarrow n(n-1)(n+1)(3n+4)=0$

So $n = 0, 1, -1$ or $-\frac{4}{3}$

However, the result only holds for positive integers, so $n = 1$

5 a $(3+2\text{i})$ and $(3-2\text{i})$ are complex conjugate roots of $\text{f}(z)=0$

So $(z-(3+2\text{i}))(z-(3-2\text{i}))$ is a factor of $\text{f}(z)$.

$(z-(3+2\text{i}))(z-(3-2\text{i}))$

$= z^2-(3+2\text{i}+3-2\text{i})z+(3+2\text{i})(3-2\text{i})$

$= z^2-6z+3^2+6\text{i}-6\text{i}-(2i)^2$

$= z^2-6z+9-(-4)$

$= z^2-6z+13$

So $(z^2-6z+13)$ is a factor of f(z).

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5 b $(z^2-6z+13)(az^2+bz+c)$

$=z^4-14z^3+78z^2+kz+221$

Equate coefficients of z^4:

$a=1$

Equate coefficients of z^3:

The z^3 terms on the LHS are $z^2\times bz$ and $-6z\times az^2$, so

$bz^3-6az^3=-14z^3$

$b-6a=-14$

$b-6\times(1)=-14$

$b=-8$

Equate constant terms:

$13c=221$

$c=17$

To find k, equate coefficients of z:

The z terms on the LHS are $-6z\times c$ and $13\times bz$,

so $-6cz+13bz=kz$

$-6c+13b=k$

$-6\times(17)+13\times(-8)=k$

$k=-206$

c $(z^2-6z+13)(z^2-8z+17)$

$=z^4-14z^3+78z^2-206z+221$

Solving $(z^2-8z+17)=0$

$(z-4)^2+1=0$

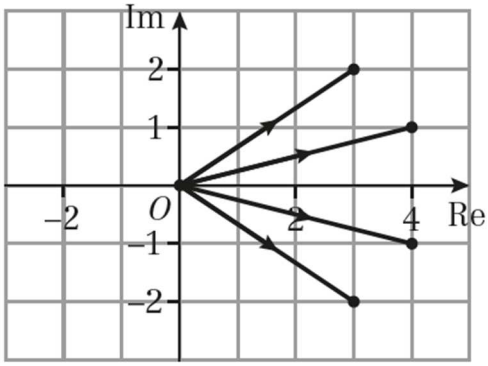
$(z-4)^2=-1$

$z-4=\pm i$

$z=4\pm i$

So the roots of $f(z)=0$ are

$3+2i,\ 3-2i,\ 4+i$ and $4-i$



6 a $\sqrt{1^2+(-1)^2+2^2}=\sqrt{6}$

$$\hat{\mathbf{n}}=\begin{pmatrix}\frac{1}{\sqrt{6}}\\ -\frac{1}{\sqrt{6}}\\ \frac{2}{\sqrt{6}}\end{pmatrix}$$

b $\mathbf{r}=\begin{pmatrix}0\\3\\-1\end{pmatrix}+\lambda\begin{pmatrix}2\\4\\-3\end{pmatrix}$

The cartesian coordinates of any point on this line are

$x=2\lambda$

$y=3+4\lambda$

$z=-1-3\lambda$

Sub these coordiantes into the equation of Π:

$1(2\lambda)-1(3+4\lambda)+2(-1-3\lambda)=3$

$-8\lambda-8=0$

$-8\lambda=8\Rightarrow\lambda=-1$

Now use $\lambda=-1$ to find coordiantes of P:

$x=2(-1)\Rightarrow x=-2$

$y=3+4(-1)\Rightarrow y=-1$

$z=-1-3(-1)\Rightarrow z=1$

Coordinates of P: $(-2,-1,\ 2)$

Now let θ be the angle between the line l and the *normal* to the plane.

$$\cos\theta=\frac{\begin{pmatrix}1\\-1\\2\end{pmatrix}\cdot\begin{pmatrix}2\\4\\-3\end{pmatrix}}{\sqrt{1^2+1^2+2^2}\sqrt{2^2+4^2+3^2}}$$

$$=\frac{|2-4-6|}{\sqrt{6}\sqrt{29}}=0.606$$

$\theta=0.92$

Then the angle between l and Π is

$\frac{\pi}{2}-\theta=\frac{\pi}{2}-0.92=0.65$

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7 a $4x - x^2 = 0$

$x(4-x) = 0$

$x = 0$ or $x = 4$

$$V = \pi\int_0^4 \left(4x - x^2\right)^2 \,\mathrm{d}x$$

$$= \pi\int_0^4 x^4 - 8x^3 + 16x^2 \,\mathrm{d}x$$

$$= \pi\left[\frac{x^5}{5} - \frac{8x^4}{4} + \frac{16x^3}{3}\right]_0^4$$

$$= \pi\left[\frac{x^5}{5} - 2x^4 + \frac{16x^3}{3}\right]_0^4$$

$$= \pi\left[\left(\frac{(4)^5}{5} - 2(4)^4 + \frac{16(4)^3}{3}\right) - 0\right]$$

$$= \frac{512\pi}{15} \text{ mm}^3$$

Cost of silver for 500 beads:

$$£0.05 \times 500 \times \frac{512\pi}{15} = £2680.83$$

b The model does not account for a hole through the middle of the bead.

8 a $\det\mathbf{M} = \begin{vmatrix} -\frac{5}{\sqrt{2}} & -\frac{5}{\sqrt{2}} \\ \frac{5}{\sqrt{2}} & -\frac{5}{\sqrt{2}} \end{vmatrix}$

$$= -\frac{5}{\sqrt{2}} \times -\frac{5}{\sqrt{2}} - \left(-\frac{5}{\sqrt{2}}\right) \times \frac{5}{\sqrt{2}}$$

$$= \frac{25}{2} + \frac{25}{2} = 25$$

Area scale factor $= 25$

Linear scale factor of enlargement $= \sqrt{25} = 5$

b $$\begin{pmatrix} -\frac{5}{\sqrt{2}} & -\frac{5}{\sqrt{2}} \\ \frac{5}{\sqrt{2}} & -\frac{5}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} 5 & 0 \\ 0 & 5 \end{pmatrix}$$

$$= \begin{pmatrix} 5\cos\theta & -5\sin\theta \\ 5\sin\theta & 5\cos\theta \end{pmatrix}$$

$$5\sin\theta = \frac{5}{\sqrt{2}}$$

$$\sin\theta = \frac{5}{5\sqrt{2}} = \frac{1}{\sqrt{2}}$$

$\theta = 45^\circ \Rightarrow 180^\circ - 45^\circ = 135^\circ$

Check using the lower-right element:

$5\cos 135^\circ = -\frac{5}{\sqrt{2}}$ so $\theta = 135^\circ$

So **M** is a rotation anticlockwise throught 135°.

c Let the coordinates of P be (x, y). Then

$$\begin{pmatrix} -\frac{5}{\sqrt{2}} & -\frac{5}{\sqrt{2}} \\ \frac{5}{\sqrt{2}} & -\frac{5}{\sqrt{2}} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \mathbf{M}^{-1}\begin{pmatrix} a \\ b \end{pmatrix}$$

$$= \frac{1}{25}\begin{pmatrix} -\frac{5}{\sqrt{2}} & \frac{5}{\sqrt{2}} \\ -\frac{5}{\sqrt{2}} & -\frac{5}{\sqrt{2}} \end{pmatrix} \cdot \begin{pmatrix} a \\ b \end{pmatrix}$$

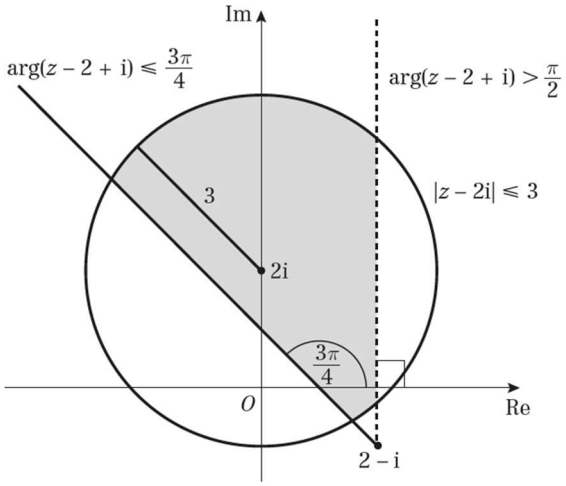
$$= \begin{pmatrix} -\frac{1}{5\sqrt{2}} & \frac{1}{5\sqrt{2}} \\ -\frac{1}{5\sqrt{2}} & -\frac{1}{5\sqrt{2}} \end{pmatrix} \cdot \begin{pmatrix} a \\ b \end{pmatrix}$$

$$= \begin{pmatrix} \frac{-a+b}{5\sqrt{2}} \\ \frac{-a-b}{5\sqrt{2}} \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$$

The coordinates of P are $\left(\frac{-a+b}{5\sqrt{2}}, \frac{-a-b}{5\sqrt{2}}\right)$

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9



$|z-2\mathrm{i}|=3$ represents a circle centred $(0,\ 2)$ with radius 3.

$|z-2\mathrm{i}|\leq 3$ is the area inside this circle

$\arg(z-2+\mathrm{i})=\dfrac{3\pi}{4}$ is the half-line from the point $2-\mathrm{i}$ which makes an angle $\dfrac{3\pi}{4}$ with the positive real axis.

$\arg(z-2+\mathrm{i})=\dfrac{\pi}{2}$ is is the half-line from the point $2-\mathrm{i}$ which makes an angle $\dfrac{\pi}{2}$ with the positive real axis.

$$\left\{z\in\mathbb{C}:|z-2\mathrm{i}|\leqslant 3\right\}\cap\left\{z\in\mathbb{C}:\frac{\pi}{2}<\arg(z-2+\mathrm{i})\leqslant\frac{3\pi}{4}\right\}$$

is the area inside the circle $|z-2\mathrm{i}|\leqslant 3$ which lies between the two half-lines $\arg(z-2+\mathrm{i})=\dfrac{\pi}{2}$ and $\arg(z-2+\mathrm{i})=\dfrac{3\pi}{4}$

The half line $\arg(z-2+\mathrm{i})=\dfrac{\pi}{2}$ is not included in the region, so is shown dotted on the Argand diagram.

10 a The stolen car is travelling along the line

$$\overrightarrow{AB}=\begin{pmatrix}5\\3\\-1\end{pmatrix}+t\begin{pmatrix}2\\-8\\9\end{pmatrix}$$

So a general point on the line $\overrightarrow{AB}$ has position vector $\begin{pmatrix}5+2t\\3-8t\\-1+9t\end{pmatrix}$.

The stinger can intercept the car if the perpendicular distance from the origin to $\overrightarrow{AB}$ is less than 6 m.

The perpendicular distance occurs when

$$\begin{pmatrix}5+2t\\3-8t\\-1+9t\end{pmatrix}\cdot\begin{pmatrix}2\\-8\\9\end{pmatrix}=0$$

$$10+4t-24+64t-9+81t=0$$
$$149t-23=0$$
$$149t=23$$
$$t=\frac{23}{149}$$

Substitute $t=\dfrac{23}{149}$ into the expression for a general point on the line $\overrightarrow{AB}$:

$$5+2\left(\frac{23}{149}\right)=\frac{791}{149}$$
$$3-8\left(\frac{23}{149}\right)=\frac{263}{149}$$
$$-1+9\left(\frac{23}{149}\right)=\frac{58}{149}$$

$$\left|\overrightarrow{AB}\right|_{\min}=\sqrt{\frac{791^2+263^2+58^2}{149^2}}$$
$$=5.6<6$$

The stolen car passes 5.6 m from the origin.
The police can stop the car.

b Limitation of the model – the car is unlikely to drive exactly in a straight line, as roads are not straight.

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