Please check the examination details below before entering your candidate information

Candidate surname StemUp Model Solution | Other names

Pearson Edexcel Level 3 GCE

Centre Number | Candidate Number

Tuesday 23 June 2020

Afternoon (Time: 1 hour 30 minutes) | Paper Reference **9FM0/4A**

Further Mathematics

Advanced

Paper 4A: Further Pure Mathematics 2

You must have:
Mathematical Formulae and Statistical Tables (Green), calculator

Total Marks

Candidates may use any calculator permitted by Pearson regulations. Calculators must not have the facility for symbolic algebra manipulation, differentiation and integration, or have retrievable mathematical formulae stored in them.

Instructions

- Use **black** ink or ball-point pen.
- If pencil is used for diagrams/sketches/graphs it must be dark (HB or B)
- **Fill in the boxes** at the top of this page with your name, centre number and candidate number.
- Answer **all** questions and ensure that your answers to parts of questions are clearly labelled.
- Answer the questions in the spaces provided
  – *there may be more space than you need.*
- You should show sufficient working to make your methods clear. Answers without working may not gain full credit.
- Inexact answers should be given to three significant figures unless otherwise stated.

Information

- A booklet 'Mathematical Formulae and Statistical Tables' is provided.
- There are 8 questions in this question paper. The total mark for this paper is 75.
- The marks for **each** question are shown in brackets
  – *use this as a guide as to how much time to spend on each question.*

Advice

- Read each question carefully before you start to answer it.
- Try to answer every question.
- Check your answers if you have time at the end.

Turn over ▶

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1. A small sports club has 12 adult members and 14 junior members.

The club needs to enter a team of 8 players for a particular competition.

Determine the number of ways in which the team can be selected if

(i) there are no restrictions on the team,

(1)

(ii) the team must contain 4 adults and 4 juniors,

(2)

(iii) more than half the team must be adults.

(3)

(i) ${}^{26}C_8 = 1562275$

(ii) ${}^{12}C_4 \times {}^{14}C_4 = 495 \times 1001 = 495495$

(iii) More than half adults $\Rightarrow$ 5, 6, 7 or 8 adults

5A, 3J: ${}^{12}C_5 \times {}^{14}C_3 = 792 \times 364 = 288288$

6A, 2J: ${}^{12}C_6 \times {}^{14}C_2 = 924 \times 91 = 84084$

7A, 1J: ${}^{12}C_7 \times {}^{14}C_1 = 792 \times 14 = 11088$

8A, 0J: ${}^{12}C_8 \times {}^{14}C_0 = 495 \times 1 = 495$

Total = 288288 + 84084 + 11088 + 495 = 383955

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Question 1 continued

(Total for Question 1 is 6 marks)

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2. Solve the recurrence system

$$u_1 = 1 \qquad u_2 = 4$$
$$9u_{n+2} - 12u_{n+1} + 4u_n = 3n$$

(9)

Auxiliary equation: $9r^2 - 12r + 4 = 0$

$(3r - 2)^2 = 0 \Rightarrow r = \frac{2}{3}$ (repeated root)

Complementary function: $u_n = (A + Bn)(\frac{2}{3})^n$

Particular solution: try $u_n = an + b$

$9(a(n + 2) + b) - 12(a(n + 1) + b) + 4(an + b) = 3n$

$9an + 18a + 9b - 12an - 12a - 12b + 4an + 4b = 3n$

$an + 6a + b = 3n$

$\Rightarrow a = 3, \quad 6a + b = 0 \Rightarrow b = -18$

General solution: $u_n = (A + Bn)(\frac{2}{3})^n + 3n - 18$

$u_1 = 1$: $1 = \frac{2}{3}(A + B) + 3 - 18 \Rightarrow A + B = 24$

$u_2 = 4$: $4 = \frac{4}{9}(A + 2B) + 6 - 18 \Rightarrow A + 2B = 36$

$\Rightarrow B = 12, \quad A = 12$

$u_n = 12(n + 1)(\frac{2}{3})^n + 3n - 18$

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Question 2 continued

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Question 2 continued

(Total for Question 2 is 9 marks)

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3.

$$\mathbf{M} = \begin{pmatrix} 1 & k & -2 \\ 2 & -4 & 1 \\ 1 & 2 & 3 \end{pmatrix}$$

where k is a constant.

(a) Show that, in terms of k, a characteristic equation for $\mathbf{M}$ is given by

$$\lambda^3 - (2k + 13)\lambda + 5(k + 6) = 0$$

(3)

Given that $\det \mathbf{M} = 5$

(b) (i) find the value of k

(ii) use the Cayley-Hamilton theorem to find the inverse of $\mathbf{M}$.

(7)

(a) Characteristic equation is $\det(M - \lambda I) = 0$

$$\begin{vmatrix} 1-\lambda & k & -2 \\ 2 & -4-\lambda & 1 \\ 1 & 2 & 3-\lambda \end{vmatrix} = 0$$

$(1 - \lambda)[(-4 - \lambda)(3 - \lambda) - 2] - k[2(3 - \lambda) - 1] - 2[4 - (-4 - \lambda)] = 0$

$(1 - \lambda)(\lambda^2 + \lambda - 14) - k(5 - 2\lambda) - 2(8 + \lambda) = 0$

$-\lambda^3 + 15\lambda - 14 - 5k + 2k\lambda - 16 - 2\lambda = 0$

$-\lambda^3 + (2k + 13)\lambda - 5k - 30 = 0$

$\therefore \lambda^3 - (2k + 13)\lambda + 5(k + 6) = 0$

(b)(i) $\det M = 1(-12 - 2) - k(6 - 1) - 2(4 + 4) = -30 - 5k$

$-30 - 5k = 5 \Rightarrow k = -7$

(ii) $k = -7$: $\lambda^3 - (-14 + 13)\lambda + 5(-1) = 0 \Rightarrow \lambda^3 + \lambda - 5 = 0$

By the Cayley-Hamilton theorem: $M^3 + M - 5I = 0$

Multiply by M^{-1}: $M^2 + I - 5M^{-1} = 0 \Rightarrow M^{-1} = \frac{1}{5}(M^2 + I)$

$$M^2 = \begin{pmatrix} 1 & -7 & -2 \\ 2 & -4 & 1 \\ 1 & 2 & 3 \end{pmatrix}\begin{pmatrix} 1 & -7 & -2 \\ 2 & -4 & 1 \\ 1 & 2 & 3 \end{pmatrix} = \begin{pmatrix} -15 & 17 & -15 \\ -5 & 4 & -5 \\ 8 & -9 & 9 \end{pmatrix}$$

$$M^{-1} = \frac{1}{5}\left[\begin{pmatrix} -15 & 17 & -15 \\ -5 & 4 & -5 \\ 8 & -9 & 9 \end{pmatrix} + \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right]$$

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Question 3 continued

$$M^{-1} = \frac{1}{5}\begin{pmatrix} -14 & 17 & -15 \\ -5 & 5 & -5 \\ 8 & -9 & 10 \end{pmatrix}$$

Question 3 continued

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Question 3 continued

(Total for Question 3 is 10 marks)

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4.

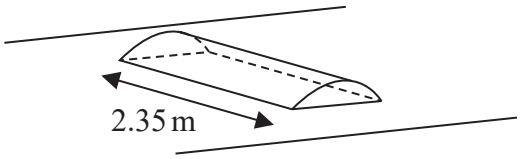


Figure 1

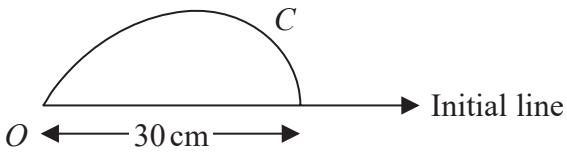


Figure 2

Figure 1 shows a sketch of a design for a road speed bump of width 2.35 metres. The speed bump has a uniform cross-section with vertical ends and its length is 30 cm. A side profile of the speed bump is shown in Figure 2.

The curve C shown in Figure 2 is modelled by the polar equation

$$r = 30(1 - \theta^2) \qquad 0 \leqslant \theta \leqslant 1$$

The units for r are centimetres and the initial line lies along the road surface, which is assumed to be horizontal.

Once the speed bump has been fixed to the road, the visible surfaces of the speed bump are to be painted.

Determine, in cm^2, the area that is to be painted, according to the model.

(10)

Painted area = 2 × (area of each vertical end) + (curved top surface)

Area of each end = $\int \frac{1}{2} r^2 \, d\theta = \frac{1}{2} \int_0^1 900(1 - \theta^2)^2 \, d\theta = 450 \int_0^1 (1 - \theta^2)^2 \, d\theta$

$= 450 \int_0^1 (1 - 2\theta^2 + \theta^4) \, d\theta = 450[\theta - \frac{2}{3}\theta^3 + \frac{1}{5}\theta^5]_0^1$

$= 450(1 - \frac{2}{3} + \frac{1}{5}) = 450 \times \frac{8}{15} = 240 \text{ cm}^2$

$r = 30(1 - \theta^2) \Rightarrow \frac{dr}{d\theta} = -60\theta$

$r^2 + (\frac{dr}{d\theta})^2 = 900(1 - 2\theta^2 + \theta^4) + (30 \times -2\theta)^2$

$= 900(1 - 2\theta^2 + \theta^4 + 4\theta^2) = 900(1 + 2\theta^2 + \theta^4) = 900(1 + \theta^2)^2$

Arc length of $C = \int_0^1 \sqrt{r^2 + (dr/d\theta)^2} \, d\theta = \int_0^1 30(1 + \theta^2) \, d\theta$

$= 30[\theta + \frac{1}{3}\theta^3]_0^1 = 30(1 + \frac{1}{3} - 0) = 40 \text{ cm}$

Curved surface = arc length × width = 40 × 235 = 9400 cm^2 (2.35 m = 235 cm)

Total area painted = 2 × 240 + 9400 = 480 + 9400

= 9880 cm^2

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Question 4 continued

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Question 4 continued

(Total for Question 4 is 10 marks)

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5. A transformation T from the z-plane to the w-plane is given by

$$w = \frac{1 - 3z}{z + 2\mathrm{i}} \qquad z \neq -2\mathrm{i}$$

The circle with equation $|z + \mathrm{i}| = 3$ is mapped by T onto the circle C.

(a) Show that the equation for C can be written as

$$3|w + 3| = |1 + (3 - w)\mathrm{i}|$$

(4)

(b) Hence find

(i) a Cartesian equation for C,

(ii) the centre and radius of C.

(6)

(a) $w(z + 2i) = 1 - 3z \Rightarrow wz + 3z = 1 - 2iw \Rightarrow z = \frac{1 - 2iw}{w + 3}$

$z + i = \frac{1 - 2iw}{w + 3} + i = \frac{1 - 2iw + iw + 3i}{w + 3} = \frac{1 + (3 - w)i}{w + 3}$

$|z + i| = 3 \Rightarrow \frac{|1 + (3 - w)i|}{|w + 3|} = 3$

$\Rightarrow |1 + (3 - w)i| = 3|w + 3|$

$\therefore 3|w + 3| = |1 + (3 - w)i|$

(b)(i) Let $w = u + iv$: $1 + (3 - u - iv)i = (1 + v) + (3 - u)i$

$3|(u + 3) + iv| = |(1 + v) + (3 - u)i|$

$9[(u + 3)^2 + v^2] = (1 + v)^2 + (3 - u)^2$

$9u^2 + 54u + 81 + 9v^2 = 1 + 2v + v^2 + 9 - 6u + u^2$

$8u^2 + 8v^2 + 60u - 2v + 71 = 0$

(ii) $u^2 + \frac{15}{2}u + v^2 - \frac{1}{4}v + \frac{71}{8} = 0$

$(u + \frac{15}{4})^2 - \frac{225}{16} + (v - \frac{1}{8})^2 - \frac{1}{64} + \frac{71}{8} = 0$

$(u + \frac{15}{4})^2 + (v - \frac{1}{8})^2 = \frac{900 + 1 - 568}{64} = \frac{333}{64}$

Centre $(-\frac{15}{4}, \frac{1}{8})$, radius $= \frac{\sqrt{333}}{8} = \frac{3\sqrt{37}}{8}$

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Question 5 continued

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Question 5 continued

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6.

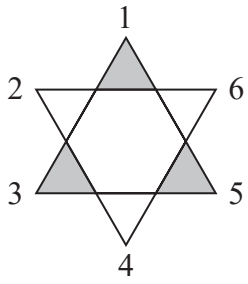


Figure 3

Figure 3 shows a plane shape made up of a regular hexagon with an equilateral triangle joined to each edge and with alternate equilateral triangles shaded.

The symmetries of this shape are the rotations and reflections of the plane that preserve the shape and its shading.

The symmetries of the shape can be represented by permutations of the six vertices labelled 1 to 6 in Figure 3. The set of these permutations with the operation of composition form a group, G.

(a) Describe geometrically the symmetry of the shape represented by the permutation

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 4 & 5 & 6 & 1 & 2 \end{pmatrix}$$

(2)

(b) Write down, in similar two-line notation, the remaining elements of the group G.

(4)

(c) Explain why each of the following statements is false, making your reasoning clear.

(i) G has a subgroup of order 4

(ii) G is cyclic.

(2)

Diagram 1, on page 23, shows an unshaded shape with the same outline as the shape in Figure 3.

(d) Shade the shape in Diagram 1 in such a way that the group of symmetries of the resulting shaded shape is isomorphic to the cyclic group of order 6

(2)

(a) Rotation about the centre of the shape through 120° anticlockwise (vertex 1 → 3 → 5, 2 → 4 → 6).

(b) Rotation through 240° anticlockwise: $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 5 & 6 & 1 & 2 & 3 & 4 \end{pmatrix}$ Identity: $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 2 & 3 & 4 & 5 & 6 \end{pmatrix}$

Reflections in the lines through vertices 1 & 4, 3 & 6 and 5 & 2:

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Question 6 continued

$\begin{pmatrix}1&2&3&4&5&6\\1&6&5&4&3&2\end{pmatrix}$ $\begin{pmatrix}1&2&3&4&5&6\\5&4&3&2&1&6\end{pmatrix}$ $\begin{pmatrix}1&2&3&4&5&6\\3&2&1&6&5&4\end{pmatrix}$

(c)(i) G has order 6. By Lagrange's theorem the order of a subgroup must divide the order of the group, and 4 does not divide 6, so G cannot have a subgroup of order 4.

(ii) For G to be cyclic it would need an element of order 6 to generate it. The rotations have order 3 (or 1) and the reflections have order 2, so no element has order 6 and G is not cyclic.

(d) Shading shown on Diagram 1: half of each point shaded the same way round, so all rotations through multiples of 60° are kept but no reflections.

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Question 6 continued

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Question 6 continued

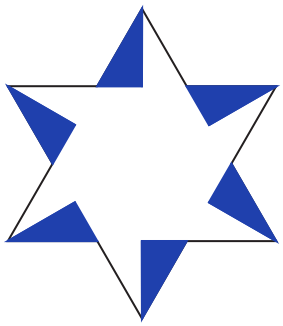
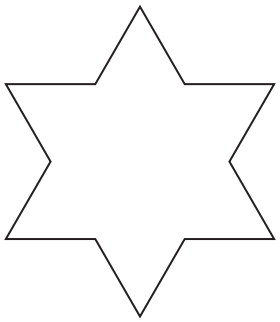


Diagram 1

Spare copy of Diagram 1



Only use this diagram if you need to redraw your answer to part (d).

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7. $$I_n = \int (4 - x^2)^{-n}\,\mathrm{d}x \qquad n > 0$$

(a) Show that, for $n > 0$

$$I_{n+1} = \frac{x}{8n(4 - x^2)^n} + \frac{2n - 1}{8n} I_n$$

(5)

(b) Find I_2

(3)

(a) $I_n = \int 1 \times (4 - x^2)^{-n}\,dx$ (by parts: $u = (4 - x^2)^{-n}$, $dv/dx = 1$)

$I_n = x(4 - x^2)^{-n} - \int x \times (-n)(4 - x^2)^{-n-1} \times (-2x)\,dx$

$= x(4 - x^2)^{-n} - 2n \int x^2(4 - x^2)^{-n-1}\,dx$

Write $x^2 = 4 - (4 - x^2)$:

$= x(4 - x^2)^{-n} - 2n \int [4(4 - x^2)^{-n-1} - (4 - x^2)(4 - x^2)^{-n-1}]\,dx$

$= x(4 - x^2)^{-n} - 8n \int (4 - x^2)^{-(n+1)}\,dx + 2n \int (4 - x^2)^{-n}\,dx$

$I_n = x(4 - x^2)^{-n} - 8nI_{n+1} + 2nI_n$

$8nI_{n+1} = x(4 - x^2)^{-n} + (2n - 1)I_n$

$\therefore \quad I_{n+1} = \frac{x}{8n(4 - x^2)^n} + \frac{2n - 1}{8n} I_n$

(b) $I_1 = \int \frac{1}{4 - x^2}\,dx = \frac{1}{2}\operatorname{artanh}\left(\frac{x}{2}\right) \quad \left(= \frac{1}{4}\ln\left|\frac{2 + x}{2 - x}\right|\right)$

$n = 1$: $I_2 = \frac{x}{8(4 - x^2)} + \frac{1}{8} I_1$

$I_2 = \frac{x}{8(4 - x^2)} + \frac{1}{16}\operatorname{artanh}\left(\frac{x}{2}\right) + c$

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Question 7 continued

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Question 7 continued

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Question 7 continued

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8. The four digit number $n = abcd$ satisfies the following properties:

(1) $n \equiv 3 \pmod 7$

(2) n is divisible by 9

(3) the first two digits have the same sum as the last two digits

(4) the digit b is smaller than any other digit

(5) the digit c is even

(a) Use property (1) to explain why $6a + 2b + 3c + d \equiv 3 \pmod 7$

(2)

(b) Use properties (2), (3) and (4) to show that $a + b = 9$

(4)

(c) Deduce that $c \equiv 5(a - 1) \pmod 7$

(2)

(d) Hence determine the number n, verifying that it is unique. You must make your reasoning clear.

(4)

(a) $n = 1000a + 100b + 10c + d$

$1000 = 142 \times 7 + 6$, $100 = 14 \times 7 + 2$, $10 = 7 + 3$, so modulo 7

$n \equiv 6a + 2b + 3c + d \pmod 7$, and $n \equiv 3 \pmod 7 \Rightarrow 6a + 2b + 3c + d \equiv 3 \pmod 7$

(b) n divisible by $9 \Rightarrow a + b + c + d = 9k$ for some integer k

$a + b = c + d \Rightarrow 2(a + b) = 9k$, so k is even

b is smaller than every other digit, so $a \geq 1$, $c \geq 1$, $d \geq 1$ and $b \geq 0$

$\Rightarrow a + b + c + d \geq 3$, and $a + b + c + d \leq 9 + 8 + 9 + 9 = 35$

So $9k$ lies between 3 and 35 with k even $\Rightarrow k = 2$ ($k = 0$ or 4 impossible)

$\Rightarrow a + b + c + d = 18 \Rightarrow 2(a + b) = 18 \Rightarrow a + b = 9$

(c) $b = 9 - a$ and $d = 9 - c$ (since $c + d = 9$). Substitute into (a):

$6a + 2(9 - a) + 3c + (9 - c) \equiv 3 \Rightarrow 4a + 2c + 27 \equiv 3 \pmod 7$

$2c \equiv -4a - 24 \equiv -4(a + 6) \equiv 3(a + 6) \equiv 3a + 18 \equiv 3(a - 1) \pmod 7$

Multiply by 4: $8c \equiv 12(a - 1) \Rightarrow c \equiv 5(a - 1) \pmod 7$

(d) $b < a$ and $a + b = 9 \Rightarrow a \geq 5$

$a = 9$: $b = 0$, $c \equiv 40 \equiv 5 \pmod 7 \Rightarrow c = 5$, but c must be even ✗

$a = 8$: $b = 1$, $c \equiv 35 \equiv 0 \Rightarrow c = 0$ or 7; $0 < b$ and 7 is odd ✗

$a = 7$: $b = 2$, $c \equiv 30 \equiv 2 \Rightarrow c = 2$ or 9; $c = 2 = b$ (b not smaller), 9 odd ✗

$a = 6$: $b = 3$, $c \equiv 25 \equiv 4 \Rightarrow c = 4$, $d = 9 - 4 = 5$; $b = 3$ is smallest ✓

$a = 5$: $b = 4$, $c \equiv 20 \equiv 6 \Rightarrow c = 6$, $d = 3 < b$ ✗

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Question 8 continued

Check: $6345 = 7 \times 906 + 3$ ✓, $6 + 3 + 4 + 5 = 18$ ✓, $6 + 3 = 4 + 5$ ✓

Only one case works, so n is unique: $n = 6345$

Question 8 continued

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Question 8 continued

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Question 8 continued

(Total for Question 8 is 12 marks)

TOTAL FOR PAPER IS 75 MARKS

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