Please check the examination details below before entering your candidate information

Candidate surname StemUp Model Solution | Other names

Centre Number | Candidate Number

Pearson Edexcel Level 3 GCE

Monday 24 June 2024

Afternoon (Time: 1 hour 30 minutes) | Paper reference **9FM0/4A**

Further Mathematics

Advanced

PAPER 4A: Further Pure Mathematics 2

You must have:
Mathematical Formulae and Statistical Tables (Green), calculator

Total Marks

Candidates may use any calculator allowed by Pearson regulations. Calculators must not have the facility for symbolic algebra manipulation, differentiation and integration, or have retrievable mathematical formulae stored in them.

Instructions

- Use **black** ink or ball-point pen.
- If pencil is used for diagrams/sketches/graphs it must be dark (HB or B).
- **Fill in the boxes** at the top of this page with your name, centre number and candidate number.
- Answer **all** questions and ensure that your answers to parts of questions are clearly labelled.
- Answer the questions in the spaces provided
  – there may be more space than you need.
- You should show sufficient working to make your methods clear.
  Answers without working may not gain full credit.
- Inexact answers should be given to three significant figures unless otherwise stated.

Information

- A booklet 'Mathematical Formulae and Statistical Tables' is provided.
- There are 8 questions in this question paper. The total mark for this paper is 75.
- The marks for **each** question are shown in brackets
  – use this as a guide as to how much time to spend on each question.

Advice

- Read each question carefully before you start to answer it.
- Try to answer every question.
- Check your answers if you have time at the end.

Turn over ▶

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Pearson

1. **In this question you must show detailed reasoning.**

Use Fermat's Little Theorem to determine the least positive residue of

$$21^{80} \pmod{23}$$

(4)

23 is prime and 23 does not divide 21, so by Fermat's Little Theorem

$21^{22} \equiv 1 \pmod{23}$

$80 = 3 \times 22 + 14$

$\Rightarrow 21^{80} = (21^{22})^3 \times 21^{14} \equiv 1^3 \times 21^{14} \equiv 21^{14} \pmod{23}$

$21 \equiv -2 \pmod{23} \Rightarrow 21^{14} \equiv (-2)^{14} = 2^{14} = (2^4)^3 \times 2^2 = 16^3 \times 4$

$16 \equiv -7 \pmod{23} \Rightarrow 16^2 \equiv 49 \equiv 3 \pmod{23}$

$\Rightarrow 16^3 \equiv 3 \times 16 = 48 \equiv 2 \pmod{23}$

$\Rightarrow 21^{14} \equiv 2 \times 4 = 8 \pmod{23}$

(check: $2^{14} = 16384 = 712 \times 23 + 8$)

$\therefore$ least positive residue: $21^{80} \equiv 8 \pmod{23}$

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Question 1 continued

(Total for Question 1 is 4 marks)

2. Determine a closed form for the recurrence system

$$u_1 = 4 \qquad u_2 = 6$$
$$u_{n+2} = 6u_{n+1} - 9u_n \qquad (n = 1, 2, 3, \ldots)$$

(5)

Auxiliary equation: $r^2 - 6r + 9 = 0$

$(r - 3)^2 = 0 \Rightarrow r = 3$ (repeated root)

General form: $u_n = (A + Bn)3^n$

$n = 1$: $4 = 3(A + B) \Rightarrow 3A + 3B = 4$ (1)

$n = 2$: $6 = 9(A + 2B) \Rightarrow 3A + 6B = 2$ (2)

(2) − (1): $3B = -2 \Rightarrow B = -\frac{2}{3}$

$A = \frac{4}{3} - B = \frac{4}{3} + \frac{2}{3} = 2$

$u_n = \left(2 - \frac{2}{3}n\right)3^n$ or $u_n = (6 - 2n)3^{n-1}$

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Question 2 continued

Question 2 continued

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Question 2 continued

(Total for Question 2 is 5 marks)

3. **In this question you must show all stages of your working.**
Solutions relying on calculator technology are not acceptable.

(a) Use the Euclidean Algorithm to determine the highest common factor h of 234 and 96

(3)

(b) Hence determine integers a and b such that

$$234a + 96b = h$$

(3)

(c) Solve the congruence equation

$$96x \equiv 36 \pmod{234}$$

(5)

(a) $234 = 2 \times 96 + 42$

$96 = 2 \times 42 + 12$

$42 = 3 \times 12 + 6$

$12 = 2 \times 6 + 0$

Last non-zero remainder, so $h = 6$

(b) $6 = 42 - 3 \times 12$

$= 42 - 3(96 - 2 \times 42) = 7 \times 42 - 3 \times 96$

$= 7(234 - 2 \times 96) - 3 \times 96$

$= 7 \times 234 - 17 \times 96$

$a = 7, \quad b = -17$

(c) hcf(96, 234) = 6 and 6 divides 36, so dividing through by 6:

$96x \equiv 36 \pmod{234} \Leftrightarrow 16x \equiv 6 \pmod{39}$

From (b), dividing by 6: $1 = 7 \times 39 - 17 \times 16$

$\Rightarrow -17 \times 16 \equiv 1 \pmod{39}$, so $16^{-1} \equiv -17 \equiv 22 \pmod{39}$

$\Rightarrow x \equiv 22 \times 6 = 132 \equiv 132 - 3 \times 39 = 15 \pmod{39}$

(check: $16 \times 15 = 240 = 6 \times 39 + 6$)

$x \equiv 15 \pmod{39}$

i.e. $x \equiv 15, 54, 93, 132, 171, 210 \pmod{234}$

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Question 3 continued

Question 3 continued

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Question 3 continued

(Total for Question 3 is 11 marks)

4.

$$\mathbf{A} = \begin{pmatrix} 4 & 2 & 0 \\ 2 & p & -2 \\ 0 & -2 & 2 \end{pmatrix} \quad \text{where } p \text{ is a constant}$$

Given that $\begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix}$ is an eigenvector of $\mathbf{A}$,

(a) determine the eigenvalue corresponding to this eigenvector.

(2)

(b) Hence show that $p = 3$

(1)

(c) Determine

(i) the remaining eigenvalues of $\mathbf{A}$,

(ii) corresponding eigenvectors for these eigenvalues.

(6)

(d) Hence determine a matrix $\mathbf{P}$ and a diagonal matrix $\mathbf{D}$ such that $\mathbf{A} = \mathbf{PDP}^{\mathrm{T}}$

(3)

(a) $\begin{pmatrix} 4 & 2 & 0 \\ 2 & p & -2 \\ 0 & -2 & 2 \end{pmatrix}\begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix} = \begin{pmatrix} 8-2+0 \\ 4-p-4 \\ 0+2+4 \end{pmatrix} = \begin{pmatrix} 6 \\ -p \\ 6 \end{pmatrix} = \lambda\begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix}$

Row 1: $6 = 2\lambda \Rightarrow \lambda = 3$

(b) Row 2: $4 - p - 4 = 3 \times (-1) \Rightarrow -p = -3 \Rightarrow p = 3$ as required

(c)(i) $\det(A - \lambda I) = \begin{vmatrix} 4-\lambda & 2 & 0 \\ 2 & 3-\lambda & -2 \\ 0 & -2 & 2-\lambda \end{vmatrix} = 0$

$(4 - \lambda)[(3 - \lambda)(2 - \lambda) - 4] - 2[2(2 - \lambda) - 0] + 0 = 0$

$(4 - \lambda)(\lambda^2 - 5\lambda + 2) - 8 + 4\lambda = 0$

$4\lambda^2 - 20\lambda + 8 - \lambda^3 + 5\lambda^2 - 2\lambda - 8 + 4\lambda = 0$

$-\lambda^3 + 9\lambda^2 - 18\lambda = 0 \Rightarrow \lambda^3 - 9\lambda^2 + 18\lambda = 0$

$\lambda(\lambda - 3)(\lambda - 6) = 0$

Remaining eigenvalues: $\lambda = 0$ and $\lambda = 6$

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Question 4 continued

(c)(ii) $\lambda = 0$: $4x + 2y = 0$, $2x + 3y - 2z = 0$, $-2y + 2z = 0$

$\Rightarrow y = -2x$ and $z = y = -2x$

$x = -1$: eigenvector $\begin{pmatrix} -1 \\ 2 \\ 2 \end{pmatrix}$

$\lambda = 6$: $-2x + 2y = 0$, $2x - 3y - 2z = 0$, $-2y - 4z = 0$

$\Rightarrow y = x$ and $z = -\frac{y}{2} = -\frac{x}{2}$

$x = 2$: eigenvector $\begin{pmatrix} 2 \\ 2 \\ -1 \end{pmatrix}$

(d) A is symmetric, so the eigenvectors are orthogonal; normalise them:

$|(2, -1, 2)| = |(-1, 2, 2)| = |(2, 2, -1)| = \sqrt{(4 + 1 + 4)} = \sqrt{9} = 3$

$P = \frac{1}{3}\begin{pmatrix} 2 & -1 & 2 \\ -1 & 2 & 2 \\ 2 & 2 & -1 \end{pmatrix}$ $\quad D = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 6 \end{pmatrix}$

(P is orthogonal, so $P^{-1} = P^T$ and $A = PDP^T$)

Question 4 continued

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Question 4 continued

(Total for Question 4 is 12 marks)

5. (i) A circle C in the complex plane is defined by the locus of points satisfying

$$|z - 3i| = 2|z|$$

(a) Determine a Cartesian equation for C, giving your answer in simplest form.

(3)

(b) On an Argand diagram, shade the region defined by

$$\{z \in \mathbb{C} : |z - 3i| > 2|z|\}$$

(2)

(ii) The transformation T from the z-plane to the w-plane is given by

$$w = z^3$$

(a) Describe the geometric effect of T.

(2)

The region R in the z-plane is given by

$$\left\{z \in \mathbb{C} : 0 < \arg z < \frac{\pi}{4}\right\}$$

(b) On a **different** Argand diagram, sketch the image of R under T.

(2)

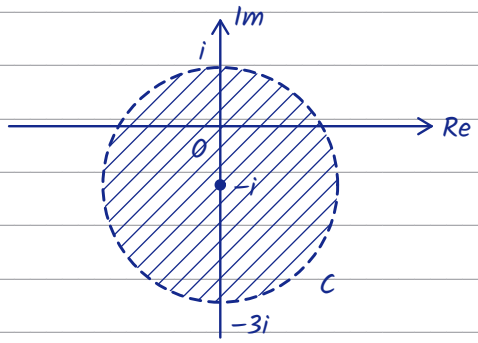
(i)(a) $z = x + iy$: $|x + (y - 3)i| = 2|x + iy|$

$x^2 + (y - 3)^2 = 4(x^2 + y^2)$

$x^2 + y^2 - 6y + 9 = 4x^2 + 4y^2 \Rightarrow 3x^2 + 3y^2 + 6y - 9 = 0$

$x^2 + y^2 + 2y - 3 = 0$ (i.e. $x^2 + (y + 1)^2 = 4$)

(i)(b) Circle centre $-i$, radius 2. $z = 0$: $|-3i| = 3 > 0$, so shade the inside



$|z - 3i| > 2|z|$ (inside, boundary not included)

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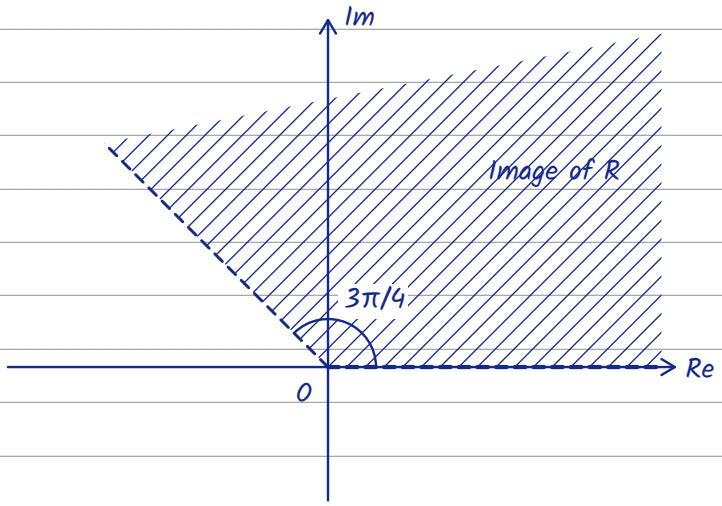
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Question 5 continued

(ii)(a) If $z = r e^{i\theta}$ then $w = z^3 = r^3 e^{3i\theta}$:

the argument is tripled, i.e. each point is rotated anticlockwise about O through 2 arg z, and the modulus is cubed (distance from O goes from $|z|$ to $|z|^3$)

(ii)(b) $0 < \arg z < \pi/4 \Rightarrow 0 < \arg w < 3\pi/4$ (all moduli possible)



Question 5 continued

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Question 5 continued

(Total for Question 5 is 9 marks)

6. **In this question you must show all stages of your working.**
Solutions relying entirely on calculator technology are not acceptable.

$$I_n = \int \frac{\cos(nx)}{\sin x}\,\mathrm{d}x \qquad n \geqslant 1$$

(a) Show that, for $n \geqslant 1$

$$I_{n+2} = \frac{2\cos(n+1)x}{n+1} + I_n$$

(6)

(b) Hence determine the exact value of

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{\cos(5x)}{\sin x}\,\mathrm{d}x$$

giving the answer in the form $a + b\ln c$ where a, b and c are rational numbers to be found.

(5)

(a) $I_{n+2} - I_n = \int \frac{\cos(n+2)x - \cos nx}{\sin x}\,dx$

$\cos P - \cos Q = -2\sin\left(\frac{P+Q}{2}\right)\sin\left(\frac{P-Q}{2}\right)$

$\Rightarrow \cos(n+2)x - \cos nx = -2\sin(n+1)x \sin x$

$I_{n+2} - I_n = \int \frac{-2\sin(n+1)x \sin x}{\sin x}\,dx = -2\int \sin(n+1)x\,dx$

$= -2 \times \left(-\frac{\cos(n+1)x}{n+1}\right) = \frac{2\cos(n+1)x}{n+1}$

$\therefore I_{n+2} = \frac{2\cos(n+1)x}{n+1} + I_n$ as required

(b) Let $J_n = \int_{\pi/4}^{\pi/3} \frac{\cos(nx)}{\sin x}\,dx$

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Question 6 continued

$$I_1 = \int \frac{\cos x}{\sin x}\,dx = \int \cot x\,dx = \ln(\sin x)$$

$$J_1 = [\ln(\sin x)]_{\pi/4}^{\pi/3} = \ln\frac{\sqrt{3}}{2} - \ln\frac{\sqrt{2}}{2} = \ln\sqrt{\left(\frac{3}{2}\right)} = \frac{1}{2}\ln\frac{3}{2}$$

$$J_3 = \left[\frac{2\cos 2x}{2}\right]_{\pi/4}^{\pi/3} + J_1 = \cos\frac{2\pi}{3} - \cos\frac{\pi}{2} + J_1 = -\frac{1}{2} + J_1$$

$$J_5 = \left[\frac{2\cos 4x}{4}\right]_{\pi/4}^{\pi/3} + J_3 = \frac{1}{2}\left(\cos\frac{4\pi}{3} - \cos\pi\right) + J_3$$

$$= \frac{1}{2}\left(-\frac{1}{2} + 1\right) + J_3 = \frac{1}{4} + J_3$$

$$\int_{\pi/4}^{\pi/3} \frac{\cos(5x)}{\sin x}\,dx = J_5 = \frac{1}{4} - \frac{1}{2} + \frac{1}{2}\ln\frac{3}{2}$$

$$= -\frac{1}{4} + \frac{1}{2}\ln\frac{3}{2} \qquad \left(a = -\frac{1}{4},\ b = \frac{1}{2},\ c = \frac{3}{2}\right)$$

Question 6 continued

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Question 6 continued

(Total for Question 6 is 11 marks)

7. The set of matrices $G = \{\mathbf{I}, \mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}, \mathbf{E}\}$ where

$$\mathbf{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \mathbf{A} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \mathbf{B} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \quad \mathbf{C} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \mathbf{D} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \quad \mathbf{E} = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$$

with the operation $\otimes_2$ of matrix multiplication with entries evaluated modulo 2, forms a group.

(a) Show that **B** is an element of order 3 in G.

(2)

(b) Determine the orders of the other elements of G.

(3)

(c) Give a reason why G is **not** isomorphic to

(i) a cyclic group of order 6

(ii) the group of symmetries of a regular hexagon.

(2)

The group H of permutations of the numbers 1, 2 and 3 contains the following elements, denoted in two-line notation,

$$e = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix} \quad a = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} \quad b = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$$

$$c = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} \quad d = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} \quad f = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}$$

(d) Determine an isomorphism between the groups G and H.

(3)

(a) $B^2 = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \equiv \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} = E \pmod 2$

$B^3 = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I \pmod 2$

$B \neq I$ and $B^2 = E \neq I$ but $B^3 = I$, so B has order 3

(b) I is the identity so has order 1; $E = B^2 = B^{-1}$, so E has order 3

$A^2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$

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Question 7 continued

$$C^2 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$$

$$D^2 = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$$

So I has order 1, B and E have order 3, A, C and D each have order 2

(c)(i) G has no element of order 6 (the largest order is 3), but a cyclic group of order 6 has a generator of order 6, so G is not cyclic

(c)(ii) The symmetries of a regular hexagon form a group of order 12 (6 rotations and 6 reflections), but G has only 6 elements (also, the hexagon group contains rotations of order 6; G does not)

(d) Orders in H: e order 1; a, b order 3 (3-cycles); c, d, f order 2

(composition: right-hand permutation applied first)

$a \circ a = b$, matching $B^2 = E$; $AB = D \pmod 2$ and $c \circ a = f$

An isomorphism $\varphi : G \to H$ is

x	I	A	B	C	D	E
$\varphi(x)$	e	c	a	d	f	b

Question 7 continued

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Question 7 continued

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8.



Figure 1

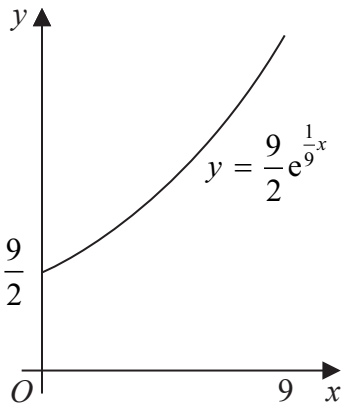


Figure 2

Figure 1 shows a French horn with a detachable bell section.

The shape of the bell section can be modelled by rotating an exponential curve through 360° about the x-axis, where units are centimetres.

The model uses the curve shown in Figure 2, with equation

$$y = \frac{9}{2}e^{\frac{1}{9}x} \qquad 0 \leqslant x \leqslant 9$$

(a) Show that, according to this model, the external surface area of the bell section is given by

$$K\int_0^9 e^{\frac{1}{9}x}\sqrt{4+e^{\frac{2}{9}x}}\,dx$$

where K is a real constant to be determined.

(3)

(b) Use the substitution $u = e^{\frac{1}{9}x}$ to show that

$$\int_0^9 e^{\frac{1}{9}x}\sqrt{4+e^{\frac{2}{9}x}}\,dx = 9\int_a^b \frac{2u+u^3}{\sqrt{4u^2+u^4}}du + 18\int_a^b \frac{1}{\sqrt{4+u^2}}du$$

where a and b are constants to be determined.

(5)

Hence, using algebraic integration,

(c) determine, according to the model, the external surface area of the bell section of the horn, giving your answer to 3 significant figures.

(5)

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Question 8 continued

(a) $S = 2\pi \int_0^9 y \sqrt{\left(1 + \left(\frac{dy}{dx}\right)^2\right)}\, dx$

$y = \frac{9}{2} e^{x/9} \Rightarrow \frac{dy}{dx} = \frac{9}{2} \times \frac{1}{9} e^{x/9} = \frac{1}{2} e^{x/9}$

$S = 2\pi \int_0^9 \frac{9}{2} e^{x/9} \sqrt{\left(1 + \frac{1}{4} e^{2x/9}\right)}\, dx$

$= 9\pi \int_0^9 e^{x/9} \times \frac{1}{2} \sqrt{(4 + e^{2x/9})}\, dx$

$= \frac{9\pi}{2} \int_0^9 e^{x/9} \sqrt{(4 + e^{2x/9})}\, dx, \quad$ so $\quad K = \frac{9\pi}{2}$

(b) $u = e^{x/9} \Rightarrow \frac{du}{dx} = \frac{1}{9} e^{x/9} = \frac{u}{9} \Rightarrow dx = \frac{9}{u}\, du$

$x = 0 \Rightarrow u = e^0 = 1; \quad x = 9 \Rightarrow u = e^1 = e$

$\int_0^9 e^{x/9} \sqrt{(4 + e^{2x/9})}\, dx = \int_1^e u \sqrt{(4 + u^2)} \times \frac{9}{u}\, du = 9 \int_1^e \sqrt{(4 + u^2)}\, du$

$= 9 \int_1^e \frac{4 + u^2}{\sqrt{(4 + u^2)}}\, du = 9 \int_1^e \frac{2 + u^2}{\sqrt{(4 + u^2)}}\, du + 9 \int_1^e \frac{2}{\sqrt{(4 + u^2)}}\, du$

$\frac{2 + u^2}{\sqrt{(4 + u^2)}} = \frac{u(2 + u^2)}{u\sqrt{(4 + u^2)}} = \frac{2u + u^3}{\sqrt{(u^2(4 + u^2))}} = \frac{2u + u^3}{\sqrt{(4u^2 + u^4)}}$

$\therefore \int_0^9 e^{x/9} \sqrt{(4 + e^{2x/9})}\, dx = 9 \int_1^e \frac{2u + u^3}{\sqrt{(4u^2 + u^4)}}\, du + 18 \int_1^e \frac{1}{\sqrt{(4 + u^2)}}\, du$

where $a = 1$ and $b = e$



Question 8 continued

(c) $9\int \frac{2u + u^3}{\sqrt{(4u^2 + u^4)}}\,du = \frac{9}{4}\int \frac{8u + 4u^3}{\sqrt{(4u^2 + u^4)}}\,du = \frac{9}{4} \times 2\sqrt{(4u^2 + u^4)}$

$= \frac{9}{2}\sqrt{(4u^2 + u^4)}$

$18\int \frac{1}{\sqrt{(4 + u^2)}}\,du = 18\,\text{arsinh}\left(\frac{u}{2}\right)$

$\int_0^9 e^{x/9}\sqrt{(4 + e^{2x/9})}\,dx = \left[\frac{9}{2}\sqrt{(4u^2 + u^4)} + 18\,\text{arsinh}\left(\frac{u}{2}\right)\right]_1^e$

$= \frac{9}{2}\sqrt{(4e^2 + e^4)} + 18\,\text{arsinh}\left(\frac{e}{2}\right) - \frac{9}{2}\sqrt{5} - 18\,\text{arsinh}\left(\frac{1}{2}\right)$

$= 42.6089...$

$S = \frac{9\pi}{2} \times 42.6089... = 602.370...$

Surface area $\approx 602\ \text{cm}^2$ (3 s.f.)

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Question 8 continued

Question 8 continued

(Total for Question 8 is 13 marks)

TOTAL FOR PAPER IS 75 MARKS

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