Please check the examination details below before entering your candidate information

Candidate surname StemUp Model Solution | Other names

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# Pearson Edexcel Level 3 GCE

## Monday 26 June 2023

Afternoon (Time: 1 hour 30 minutes) | Paper reference **9FM0/4A**

# Further Mathematics

## Advanced

## PAPER 4A: Further Pure Mathematics 2

**You must have:**
Mathematical Formulae and Statistics Tables (Green), calculator

Total Marks

**Candidates may use any calculator permitted by Pearson regulations. Calculators must not have the facility for symbolic algebraic manipulation, differentiation and integration, or have retrievable mathematical formulae stored in them.**

### Instructions

- Use **black** ink or ball-point pen.
- If pencil is used for diagrams/sketches/graphs it must be dark (HB or B).
- **Fill in the boxes** at the top of this page with your name, centre number and candidate number.
- Answer **all** questions and ensure that your answers to parts of questions are clearly labelled.
- Answer the questions in the spaces provided
  – *there may be more space than you need.*
- You should show sufficient working to make your methods clear. Answers without working may not gain full credit.
- Inexact answers should be given to three significant figures unless otherwise stated.

### Information

- A booklet 'Mathematical Formulae and Statistical Tables' is provided.
- There are 10 questions in this question paper. The total mark for this paper is 75.
- The marks for **each** question are shown in brackets
  – *use this as a guide as to how much time to spend on each question.*

### Advice

- Read each question carefully before you start to answer it.
- Try to answer every question.
- Check your answers if you have time at the end.

*Turn over* ▶

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**1.**

$$\mathbf{A} = \begin{pmatrix} -1 & a \\ 3 & 8 \end{pmatrix}$$

where $a$ is a constant.

(a) Determine, in expanded form in terms of $a$, the characteristic equation for $\mathbf{A}$.

**(2)**

(b) Hence use the Cayley-Hamilton theorem to determine values of $a$ and $b$ such that

$$\mathbf{A}^3 = \mathbf{A} + b\mathbf{I}$$

where $\mathbf{I}$ is the $2 \times 2$ identity matrix.

**(4)**

(a) $\det(A - \lambda I) = \begin{vmatrix} -1-\lambda & a \\ 3 & 8-\lambda \end{vmatrix} = 0$

$(-1 - \lambda)(8 - \lambda) - 3a = 0$

$-8 + \lambda - 8\lambda + \lambda^2 - 3a = 0$

$\lambda^2 - 7\lambda - 3a - 8 = 0$

(b) By Cayley-Hamilton: $A^2 - 7A - (3a + 8)I = 0$

$\Rightarrow A^2 = 7A + (3a + 8)I$

$\Rightarrow A^3 = 7A^2 + (3a + 8)A$

$= 7[7A + (3a + 8)I] + (3a + 8)A$

$= (3a + 57)A + 7(3a + 8)I$

Compare with $A^3 = A + bI$:

A: $3a + 57 = 1 \Rightarrow 3a = -56 \Rightarrow a = -\frac{56}{3}$

I: $b = 7(3a + 8) = 7(-56 + 8) = 7 \times (-48) = -336$

$a = -\frac{56}{3}$, $b = -336$

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**Question 1 continued**

**(Total for Question 1 is 6 marks)**

**2.** A complex number $z$ is represented by the point $P$ in the complex plane.

Given that $z$ satisfies

$$|z - 6| = 2|z + 3i|$$

(a) show that the locus of $P$ passes through the origin and the points $-4$ and $-8i$

**(2)**

(b) Sketch on an Argand diagram the locus of $P$ as $z$ varies.

**(2)**

(c) On your sketch, shade the region which satisfies both

$$|z - 6| \geqslant 2|z + 3i| \quad \text{and} \quad |z| \leqslant 4$$

**(2)**

(a) $z = -4$: $|-4 - 6| = |-10| = 10$

$2|-4 + 3i| = 2\sqrt{(4^2 + 3^2)} = 2 \times 5 = 10$, equal so $-4$ is on the locus

$z = -8i$: $|-6 - 8i| = \sqrt{(36 + 64)} = 10$

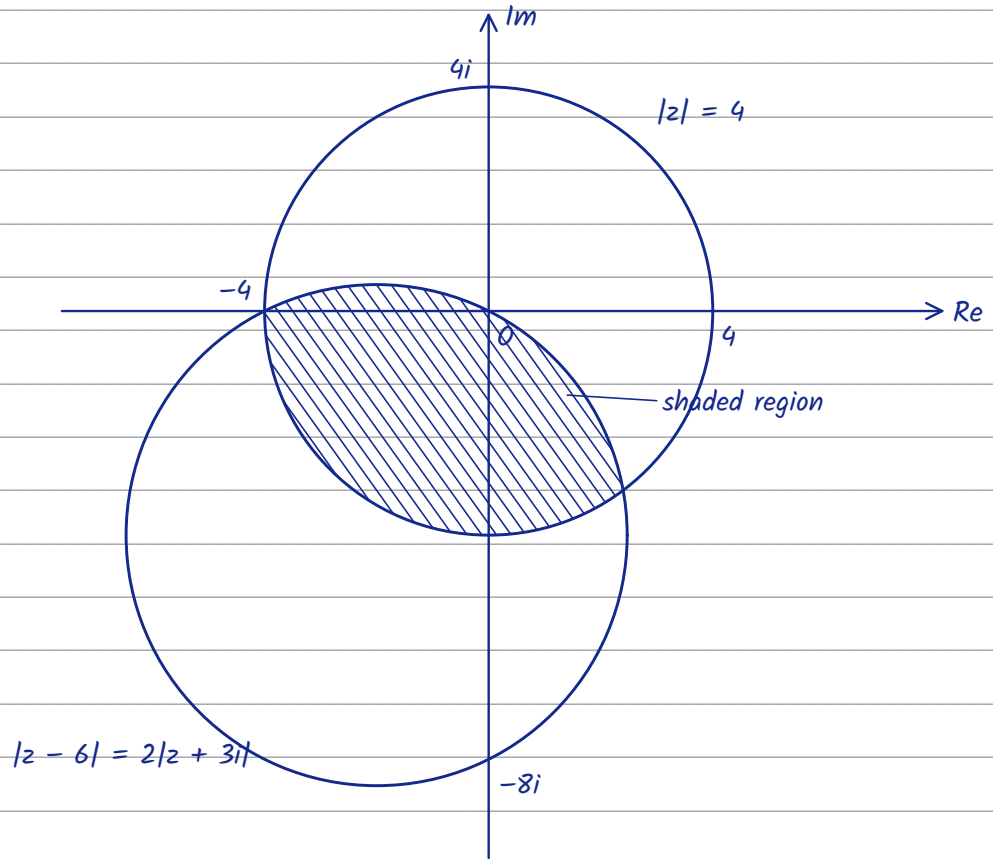
$2|-8i + 3i| = 2|-5i| = 2 \times 5 = 10$, equal so $-8i$ is on the locus

$z = 0$: $|-6| = 6$ and $2|3i| = 2 \times 3 = 6$, so the origin is on the locus

(Cartesian: $(x - 6)^2 + y^2 = 4[x^2 + (y + 3)^2] \Rightarrow x^2 + y^2 + 4x + 8y = 0$

$\Rightarrow (x + 2)^2 + (y + 4)^2 = 20$: circle centre $(-2, -4)$, radius $2\sqrt{5}$)

(b), (c)




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**Question 2 continued**

**Question 2 continued**

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**Question 2 continued**

**(Total for Question 2 is 6 marks)**

**3.** In a model for the number of subscribers to a new social media channel it is assumed that

- each week 20% of the subscribers at the start of the week cancel their subscriptions
- between the start and end of week $n$ the channel gains $20n$ new subscribers

Given that at the end of week 1 there were 25 subscribers,

(a) explain why the number of subscribers at the end of week $n$, $U_n$, is modelled by the recurrence relation

$$U_1 = 25 \qquad U_{n+1} = 0.8U_n + 20(n+1) \qquad n = 1,2,3,...$$

**(2)**

(b) Prove by induction that for $n \geqslant 1$

$$U_n = 325\left(\frac{4}{5}\right)^{n-1} + 100n - 400$$

**(5)**

Given that 6 months after starting the channel there were approximately 1800 subscribers,

(c) evaluate the model in the light of this information.

**(2)**

(a) $U_1 = 25$ because there are 25 subscribers at the end of week 1.

At the start of week $n + 1$ there are $U_n$ subscribers; 20% of them cancel, so 80% remain, i.e. $0.8U_n$ subscribers.

During week $n + 1$ the channel gains $20(n + 1)$ new subscribers.

Hence $U_{n+1} = 0.8U_n + 20(n + 1)$, $U_1 = 25$

(b) $n = 1$: $325(\frac{4}{5})^0 + 100(1) - 400 = 325 + 100 - 400 = 25 = U_1$

so the result is true for $n = 1$

Assume true for $n = k$: $U_k = 325(\frac{4}{5})^{k-1} + 100k - 400$

Then $U_{k+1} = 0.8U_k + 20(k + 1)$

$= \frac{4}{5}[\,325(\frac{4}{5})^{k-1} + 100k - 400\,] + 20k + 20$

$= 325 \times \frac{4}{5} \times (\frac{4}{5})^{k-1} + 80k - 320 + 20k + 20$

$= 325(\frac{4}{5})^k + 100k - 300$

$= 325(\frac{4}{5})^{(k+1)-1} + 100(k + 1) - 400$

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**Question 3 continued**

which is the result with $n = k + 1$.

If the result is true for $n = k$ then it is true for $n = k + 1$, and as it is true for $n = 1$, by mathematical induction it is true for all $n \geq 1$.

(c) 6 months $\approx$ 26 weeks:

$U_{26} = 325\left(\frac{4}{5}\right)^{25} + 100(26) - 400 = 2201.227...$

So the model predicts about 2201 subscribers after 6 months.

This overestimates the actual figure of 1800 by about 400, so the model is not very accurate after 6 months and needs refining (e.g. the number of new subscribers each week should not keep increasing by 20).

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**Question 3 continued**

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**Question 3 continued**

**(Total for Question 3 is 9 marks)**

**4.** (a) Use the Euclidean algorithm to show that the highest common factor of 168 and 66 is 6

**(2)**

(b) Use back substitution to determine integers $a$ and $b$ such that

$$168a + 66b = 6$$

**(3)**

(c) Explain why there are no integer solutions to the equation

$$168x + 66y = 10$$

**(1)**

(d) Solve the congruence equation

$$11v \equiv 8 \pmod{28}$$

**(3)**

(a) $168 = 2 \times 66 + 36$

$66 = 1 \times 36 + 30$

$36 = 1 \times 30 + 6$

$30 = 5 \times 6 + 0$

The last non-zero remainder is 6, so the HCF of 168 and 66 is 6

(b) $6 = 36 - 1 \times 30$

$= 36 - (66 - 36) = 2 \times 36 - 66$

$= 2(168 - 2 \times 66) - 66$

$= 2 \times 168 - 5 \times 66$

$a = 2, \quad b = -5$

(c) $168x + 66y = 6(28x + 11y)$, which is a multiple of 6 for all integers x, y.

10 is not a multiple of 6 ($6 \nmid 10$), so there are no integer solutions.

(d) From (b), dividing by 6: $1 = 2 \times 28 - 5 \times 11$

$\Rightarrow -5 \times 11 \equiv 1 \pmod{28}$, so $-5$ is the inverse of 11 (mod 28)

$11v \equiv 8 \pmod{28} \Rightarrow -5 \times 11v \equiv -5 \times 8 \pmod{28}$

$v \equiv -40 \pmod{28}$

$v \equiv 16 \pmod{28}$

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**Question 4 continued**

**Question 4 continued**

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**Question 4 continued**

**(Total for Question 4 is 9 marks)**

**5.** (i) A security code is made up of 4 numerical digits followed by 3 **distinct** uppercase letters.

Given that the digits must be from the set {1, 2, 3, 4, 5} and the letters from the set {A, B, C, D}

(a) determine the total number of possible codes using this system.

To enable more codes to be generated, the system is adapted so that the 3 letters can appear anywhere in the code **but** no letter can be next to another letter.

(b) Determine the increase in the number of codes using this adapted system.

**(4)**

(ii) A combination lock code consists of four **distinct** digits that can be read as a positive integer, $N = abcd$, satisfying

- all the digits are odd
- $N$ is divisible by 9
- the digits appear in either ascending or descending order
- $N \equiv e \pmod{ab}$ where $ab$ is read as a two-digit number and $e$ is the odd digit that is not used in the code

(a) Use the first two properties to determine the four digits used in the code.

(b) Hence determine the code on the lock.

**(4)**

(i)(a) Digits: $5^4$; letters (distinct): $4 \times 3 \times 2$

Number of codes = $5^4 \times 4 \times 3 \times 2 = 625 \times 24 = 15\,000$

(i)(b) No two letters adjacent: letters go in 3 of the 5 gaps around the 4 digits: _ D _ D _ D _ D _ so ${}^5C_3 = 10$ arrangements

New number of codes = $10 \times 15\,000 = 150\,000$

Increase = $150\,000 - 15\,000 = 135\,000$

(ii)(a) Odd digits 1, 3, 5, 7, 9 have sum 25; four are used, e is left out.

$9 \mid N \Rightarrow 9 \mid (a + b + c + d)$ so $a + b + c + d = 9k$

$a + b + c + d = 25 - e$, and $16 \leq 25 - e \leq 24 \Rightarrow 25 - e = 18$

$\Rightarrow e = 7$ (check: $1 + 3 + 5 + 9 = 18 = 2 \times 9$)

The digits are 1, 3, 5 and 9

(ii)(b) In order: N = 1359 or N = 9531, with e = 7

$1359 = 104 \times 13 + 7 \Rightarrow 1359 \equiv 7 \pmod{13}$ ✓

$9531 = 100 \times 95 + 31 \Rightarrow 9531 \equiv 31 \pmod{95} \neq 7$

The code is 1359

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**Question 5 continued**

**Question 5 continued**

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**Question 5 continued**

**(Total for Question 5 is 8 marks)**

**6.** Determine a closed form for the recurrence relation

$$u_0 = 1 \qquad u_1 = 4$$

$$u_{n+2} = 2u_{n+1} - \frac{4}{3}u_n + n \qquad n \geqslant 0 \qquad \textbf{(7)}$$

Auxiliary equation: $\lambda^2 = 2\lambda - \frac{4}{3} \Rightarrow 3\lambda^2 - 6\lambda + 4 = 0$

$\lambda = \frac{6 \pm \sqrt{(36 - 48)}}{6} = \frac{6 \pm 2\sqrt{3}\,i}{6} = \frac{3 \pm i\sqrt{3}}{3} = 1 \pm \frac{\sqrt{3}}{3}i$

$|\lambda| = \sqrt{(1 + \frac{1}{3})} = \frac{2}{\sqrt{3}} = \frac{2\sqrt{3}}{3}, \quad \arg\lambda = \pm \arctan(\frac{1}{\sqrt{3}}) = \pm \frac{\pi}{6}$

$\lambda = \frac{2\sqrt{3}}{3} e^{\pm i\pi/6}$

CF: $u_n = (\frac{2\sqrt{3}}{3})^n (P\cos(\frac{n\pi}{6}) + Q\sin(\frac{n\pi}{6}))$

PS: try $u_n = kn + l$

$k(n + 2) + l = 2[k(n + 1) + l] - \frac{4}{3}(kn + l) + n$

$n$: $\quad k = 2k - \frac{4}{3}k + 1 \Rightarrow \frac{1}{3}k = 1 \Rightarrow k = 3$

constant: $2k + l = 2k + 2l - \frac{4}{3}l \Rightarrow \frac{1}{3}l = 0 \Rightarrow l = 0$

General solution: $u_n = (\frac{2\sqrt{3}}{3})^n (P\cos(\frac{n\pi}{6}) + Q\sin(\frac{n\pi}{6})) + 3n$

$u_0 = 1$: $P\cos 0 + Q\sin 0 + 0 = 1 \Rightarrow P = 1$

$u_1 = 4$: $\frac{2\sqrt{3}}{3}(P\cos(\frac{\pi}{6}) + Q\sin(\frac{\pi}{6})) + 3 = 4$

$\frac{2}{\sqrt{3}}(\frac{\sqrt{3}}{2} + \frac{Q}{2}) = 1 \Rightarrow 1 + \frac{Q}{\sqrt{3}} = 1 \Rightarrow Q = 0$

$u_n = (\frac{2\sqrt{3}}{3})^n \cos(\frac{n\pi}{6}) + 3n$

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**Question 6 continued**

**Question 6 continued**

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**Question 6 continued**

**(Total for Question 6 is 7 marks)**

**7.** The set $G = \mathbb{R} - \left\{-\frac{3}{2}\right\}$ with the operation of $x \bullet y = 3(x + y + 1) + 2xy$ forms a group.

(a) Determine the identity element of this group.

**(2)**

(b) Determine the inverse of a general element $x$ in this group.

**(3)**

(c) Explain why the value $-\frac{3}{2}$ must be excluded from $G$ in order for this to be a group.

**(1)**

(a) $x \cdot e = x \Rightarrow 3(x + e + 1) + 2xe = x$

$3e + 2xe = x - 3x - 3 = -2x - 3$

$e(2x + 3) = -(2x + 3) \Rightarrow e = -1$

Check: $(-1) \cdot x = 3(-1 + x + 1) + 2(-1)x = 3x - 2x = x$

Identity element is $-1$

(b) Let $y$ be the inverse of $x$: $x \cdot y = -1$

$3(x + y + 1) + 2xy = -1$

$3y + 2xy = -1 - 3x - 3 = -3x - 4$

$y(2x + 3) = -(3x + 4)$

$x^{-1} = -\frac{3x + 4}{2x + 3}$

(c) When $x = -\frac{3}{2}$, $2x + 3 = 0$ so $x^{-1} = -\frac{3x + 4}{2x + 3}$ would need division by 0.

Indeed $\left(-\frac{3}{2}\right) \cdot y = 3\left(-\frac{3}{2} + y + 1\right) - 3y = -\frac{3}{2}$ for every $y$,

so $\left(-\frac{3}{2}\right) \cdot y$ can never equal the identity $-1$: $-\frac{3}{2}$ would have no inverse,

so it must be excluded from $G$ for $G$ to be a group.

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**Question 7 continued**

**(Total for Question 7 is 6 marks)**

**8.**

$$I_n = \int_0^2 (x-2)^n \mathrm{e}^{4x}\,\mathrm{d}x \qquad n \geqslant 0$$

(a) Prove that for $n \geqslant 1$

$$I_n = -a^{n-2} - \frac{n}{4} I_{n-1}$$

where $a$ is a constant to be determined.

**(4)**

(b) Hence determine the exact value of

$$\int_0^2 (x-2)^2 \mathrm{e}^{4x}\,\mathrm{d}x$$

**(3)**

(a) $I_n = \int_0^2 (x-2)^n e^{4x}\,dx$

$u = (x-2)^n \Rightarrow du/dx = n(x-2)^{n-1}$; $dv/dx = e^{4x} \Rightarrow v = \frac{1}{4}e^{4x}$

$I_n = \left[\frac{1}{4}(x-2)^n e^{4x}\right]_0^2 - \int_0^2 \frac{n}{4}(x-2)^{n-1}e^{4x}\,dx$

$= \left(0 - \frac{1}{4}(-2)^n e^0\right) - \frac{n}{4} I_{n-1}$ $\quad (n \geq 1$ so $0^n = 0)$

$= -\frac{(-2)^n}{(-2)^2} - \frac{n}{4} I_{n-1}$ $\quad$ since $4 = (-2)^2$

$I_n = -(-2)^{n-2} - \frac{n}{4} I_{n-1}$ $\quad$ so $a = -2$

(b) $I_2 = -(-2)^0 - \frac{2}{4} I_1 = -1 - \frac{1}{2} I_1$

$I_1 = -(-2)^{-1} - \frac{1}{4} I_0 = \frac{1}{2} - \frac{1}{4} I_0$

$I_0 = \int_0^2 e^{4x}\,dx = \left[\frac{1}{4}e^{4x}\right]_0^2 = \frac{1}{4}e^8 - \frac{1}{4}$

$I_1 = \frac{1}{2} - \frac{1}{16}e^8 + \frac{1}{16} = \frac{9}{16} - \frac{1}{16}e^8$

$I_2 = -1 - \frac{1}{2}\left(\frac{9}{16} - \frac{1}{16}e^8\right) = -1 - \frac{9}{32} + \frac{1}{32}e^8$

$\int_0^2 (x-2)^2 e^{4x}\,dx = \frac{1}{32}e^8 - \frac{41}{32}$

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**Question 8 continued**

**Question 8 continued**

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**Question 8 continued**

**(Total for Question 8 is 7 marks)**

**9.**

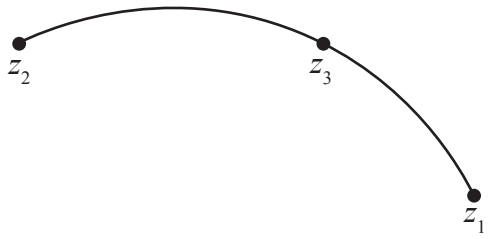



**Figure 1**

Figure 1 shows a locus in the complex plane.

The locus is an arc of a circle from the point represented by $z_1 = 3 + 2\text{i}$ to the point represented by $z_2 = a + 4\text{i}$, where $a$ is a constant, $a \neq 1$

Given that

- the point $z_3 = 1 + 4\text{i}$ also lies on the locus
- the centre of the circle has real part equal to $-1$

(a) determine the value of $a$.

**(2)**

(b) Hence determine a complex equation for the locus, giving any angles in the equation as positive values.

**(3)**

(a) $z_2 = a + 4i$ and $z_3 = 1 + 4i$ have the same imaginary part, so the perpendicular bisector of $z_2z_3$ is vertical and passes through the centre:

$\text{Re(centre)} = \frac{a+1}{2} = -1 \Rightarrow a + 1 = -2 \Rightarrow a = -3$

(b) The arc has equation of the form $\arg(z - z_1) - \arg(z - z_2) = \theta$

$\theta = \arg\left(\frac{z_3 - z_1}{z_3 - z_2}\right) = \arg\left(\frac{1 + 4i - 3 - 2i}{1 + 4i + 3 - 4i}\right) = \arg\left(\frac{-2 + 2i}{4}\right)$

$= \arg\left(-\frac{1}{2} + \frac{1}{2}i\right) = \frac{3\pi}{4}$ (second quadrant)

$\arg\left(\frac{z - 3 - 2i}{z + 3 - 4i}\right) = \frac{3\pi}{4}$

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**Question 9 continued**

**(Total for Question 9 is 5 marks)**

**10.**

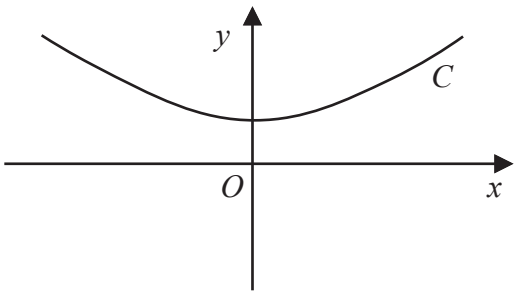



**Figure 2**

A solid playing piece for a board game is modelled by rotating the curve $C$, shown in Figure 2, through $2\pi$ radians about the $x$-axis.

The curve $C$ has equation

$$y = \sqrt{1 + \frac{x^2}{9}} \qquad -4 \leqslant x \leqslant 4$$

with units as centimetres.

(a) Show that the total surface area, $S$ cm$^2$, of the playing piece is given by

$$S = p\pi \int_{-4}^{4} \sqrt{81 + 10x^2}\, \mathrm{d}x + q\pi$$

where $p$ and $q$ are constants to be determined.

**(6)**

Using the substitution $x = \frac{9}{\sqrt{10}} \sinh u$, or another algebraic integration method, and showing all your working,

(b) determine the total surface area of the playing piece, giving your answer to the nearest cm$^2$

**(6)**

(a) $y = (1 + \frac{x^2}{9})^{1/2} \Rightarrow \frac{dy}{dx} = \frac{1}{2}(1 + \frac{x^2}{9})^{-1/2} \times \frac{2x}{9} = \frac{x}{9}(1 + \frac{x^2}{9})^{-1/2}$

Curved surface $= 2\pi \int_{-4}^{4} y \sqrt{(1 + (\frac{dy}{dx})^2)}\, dx = 2\pi \int_{-4}^{4} \sqrt{(y^2 + y^2(\frac{dy}{dx})^2)}\, dx$

$y^2(\frac{dy}{dx})^2 = (1 + \frac{x^2}{9}) \times \frac{x^2}{81}(1 + \frac{x^2}{9})^{-1} = \frac{x^2}{81}$

$y^2 + y^2(\frac{dy}{dx})^2 = 1 + \frac{x^2}{9} + \frac{x^2}{81} = \frac{81 + 9x^2 + x^2}{81} = \frac{81 + 10x^2}{81}$

Curved surface $= 2\pi \int_{-4}^{4} \frac{1}{9}\sqrt{(81 + 10x^2)}\, dx = \frac{2\pi}{9} \int_{-4}^{4} \sqrt{(81 + 10x^2)}\, dx$

Flat ends at $x = \pm 4$: radius $y = \sqrt{(1 + \frac{16}{9})} = \frac{5}{3}$, area $= \pi(\frac{5}{3})^2 = \frac{25\pi}{9}$ each

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**Question 10 continued**

$S = \frac{2}{9}\pi\int_{-4}^{4}\sqrt{(81 + 10x^2)}\,dx + 2 \times \frac{25\pi}{9} = \frac{2}{9}\pi\int_{-4}^{4}\sqrt{(81 + 10x^2)}\,dx + \frac{50}{9}\pi$

$p = \frac{2}{9}, \quad q = \frac{50}{9}$

(b) $x = \frac{9}{\sqrt{10}}\sinh u \quad \Rightarrow \quad \frac{dx}{du} = \frac{9}{\sqrt{10}}\cosh u$

$81 + 10x^2 = 81 + 81\sinh^2 u = 81\cosh^2 u \quad \Rightarrow \quad \sqrt{(81 + 10x^2)} = 9\cosh u$

$x = \pm 4 \Rightarrow \sinh u = \pm\frac{4\sqrt{10}}{9} \Rightarrow u = \pm\operatorname{arsinh}\left(\frac{4\sqrt{10}}{9}\right) = \pm 1.14115\ldots$

$\int\sqrt{(81 + 10x^2)}\,dx = \int 9\cosh u \times \frac{9}{\sqrt{10}}\cosh u\,du = \frac{81}{\sqrt{10}}\int\cosh^2 u\,du$

$= \frac{81}{\sqrt{10}}\int\frac{1}{2}(1 + \cosh 2u)\,du = \frac{81}{\sqrt{10}}\left[\frac{u}{2} + \frac{1}{4}\sinh 2u\right]$

Let $\alpha = \operatorname{arsinh}\left(\frac{4\sqrt{10}}{9}\right)$: $\left[\frac{u}{2} + \frac{1}{4}\sinh 2u\right]_{-\alpha}^{\alpha} = 1.78271\ldots - (-1.78271\ldots)$

$= 3.56543\ldots$

Curved surface $= \frac{2\pi}{9} \times \frac{81}{\sqrt{10}} \times 3.56543\ldots = 63.758\ldots$

$S = 63.758\ldots + \frac{50\pi}{9} = 63.758\ldots + 17.453\ldots = 81.211\ldots$

Total surface area $\approx 81\ \text{cm}^2$

**Question 10 continued**

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**Question 10 continued**

**Question 10 continued**

**(Total for Question 10 is 12 marks)**

**TOTAL FOR PAPER IS 75 MARKS**

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