Please check the examination details below before entering your candidate information

Candidate surname	Other names
StemUp Model Solution	

Pearson Edexcel Level 3 GCE

Centre Number	Candidate Number

Thursday 6 June 2019

Afternoon (Time: 1 hour 30 minutes) | Paper Reference **9FM0/02**

Further Mathematics

Advanced

Paper 2: Core Pure Mathematics 2

You must have:
Mathematical Formulae and Statistical Tables (Green), calculator

Total Marks

Candidates may use any calculator permitted by Pearson regulations. Calculators must not have the facility for algebraic manipulation, differentiation and integration, or have retrievable mathematical formulae stored in them.

Instructions

- Use **black** ink or ball-point pen.
- If pencil is used for diagrams/sketches/graphs it must be dark (HB or B).
- **Fill in the boxes** at the top of this page with your name, centre number and candidate number.
- Answer **all** questions and ensure that your answers to parts of questions are clearly labelled.
- Answer the questions in the spaces provided
  – *there may be more space than you need.*
- You should show sufficient working to make your methods clear. Answers without working may not gain full credit.
- Answers should be given to three significant figures unless otherwise stated.

Information

- A booklet 'Mathematical Formulae and Statistical Tables' is provided.
- There are 8 questions in this question paper. The total mark for this paper is 75.
- The marks for **each** question are shown in brackets
  – *use this as a guide as to how much time to spend on each question.*

Advice

- Read each question carefully before you start to answer it.
- Try to answer every question.
- Check your answers if you have time at the end.

Turn over ▶

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Answer ALL questions. Write your answers in the spaces provided.

1. (a) Prove that

$$\tanh^{-1}(x) = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right) \qquad -k < x < k$$

stating the value of the constant k.

(5)

(b) Hence, or otherwise, solve the equation

$$2x = \tanh\left(\ln\sqrt{2-3x}\right)$$

(5)

(a) Let $y = \tanh^{-1}x \Rightarrow x = \tanh y$

$x = \frac{\sinh y}{\cosh y} = \frac{e^y - e^{-y}}{e^y + e^{-y}} = \frac{e^{2y} - 1}{e^{2y} + 1}$

$x(e^{2y} + 1) = e^{2y} - 1$

$xe^{2y} + x = e^{2y} - 1$

$1 + x = e^{2y} - xe^{2y} = e^{2y}(1 - x)$

$e^{2y} = \frac{1+x}{1-x} \Rightarrow 2y = \ln\frac{1+x}{1-x}$

$\therefore \tanh^{-1}(x) = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$

Since $-1 < \tanh y < 1$, we need $-1 < x < 1$, so $k = 1$

(b) $2x = \tanh(\ln\sqrt{(2-3x)})$

$\Rightarrow \tanh^{-1}(2x) = \ln\sqrt{(2-3x)} = \frac{1}{2}\ln(2-3x)$

Using (a): $\frac{1}{2}\ln\left(\frac{1+2x}{1-2x}\right) = \frac{1}{2}\ln(2-3x)$

$\frac{1+2x}{1-2x} = 2 - 3x$

$1 + 2x = (2-3x)(1-2x) = 2 - 7x + 6x^2$

$6x^2 - 9x + 1 = 0$

$x = \frac{9 \pm \sqrt{(81-24)}}{12} = \frac{9 \pm \sqrt{57}}{12}$

$x = \frac{9+\sqrt{57}}{12} = 1.379\ldots$ reject, since need $-\frac{1}{2} < x < \frac{1}{2}$ $(|2x| < 1)$

$\therefore x = \frac{9-\sqrt{57}}{12}$

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Question 1 continued

(Total for Question 1 is 10 marks)

2. The roots of the equation

$$x^3 - 2x^2 + 4x - 5 = 0$$

are p, q and r.

Without solving the equation, find the value of

(i) $\frac{2}{p} + \frac{2}{q} + \frac{2}{r}$

(ii) $(p - 4)(q - 4)(r - 4)$

(iii) $p^3 + q^3 + r^3$

(8)

$p + q + r = 2, \quad pq + pr + qr = 4, \quad pqr = 5$

(i) $\frac{2}{p} + \frac{2}{q} + \frac{2}{r} = \frac{2(qr + pr + pq)}{pqr} = \frac{2 \times 4}{5} = \frac{8}{5}$

(ii) $(p - 4)(q - 4)(r - 4) = (pq - 4p - 4q + 16)(r - 4)$

$= pqr - 4pq - 4pr - 4qr + 16p + 16q + 16r - 64$

$= pqr - 4(pq + pr + qr) + 16(p + q + r) - 64$

$= 5 - 4(4) + 16(2) - 64$

$= -43$

(iii) $p^3 + q^3 + r^3 = (p + q + r)^3 - 3(p + q + r)(pq + pr + qr) + 3pqr$

$= 2^3 - 3(2)(4) + 3(5)$

$= 8 - 24 + 15$

$= -1$

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Question 2 continued

Question 2 continued

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Question 2 continued

(Total for Question 2 is 8 marks)

3.

$$f(x) = \frac{1}{\sqrt{4x^2 + 9}}$$

(a) Using a substitution, that should be stated clearly, show that

$$\int f(x)dx = A\sinh^{-1}(Bx) + c$$

where c is an arbitrary constant and A and B are constants to be found.

(4)

(b) Hence find, in exact form in terms of natural logarithms, the mean value of $f(x)$ over the interval $[0, 3]$.

(2)

(a) Let $x = \frac{3}{2}\sinh u \Rightarrow dx = \frac{3}{2}\cosh u\, du$

$\int \frac{1}{\sqrt{(4x^2 + 9)}}\, dx = \int \frac{1}{\sqrt{(9\sinh^2 u + 9)}} \times \frac{3}{2}\cosh u\, du$

$= \int \frac{3\cosh u}{2 \times 3\cosh u}\, du$ (since $9\sinh^2 u + 9 = 9\cosh^2 u$)

$= \int \frac{1}{2}\, du = \frac{1}{2}u + c$

$= \frac{1}{2}\sinh^{-1}\left(\frac{2x}{3}\right) + c$

so $A = ½$, $B = ⅔$

(b) Mean value $= \frac{1}{3 - 0}\int_0^3 \frac{1}{\sqrt{(4x^2 + 9)}}\, dx$

$= \frac{1}{3}\left[\frac{1}{2}\sinh^{-1}\left(\frac{2x}{3}\right)\right]_0^3 = \frac{1}{3}\left(\frac{1}{2}\sinh^{-1}2 - \frac{1}{2}\sinh^{-1}0\right)$

$= \frac{1}{6}\sinh^{-1}2 = \frac{1}{6}\ln(2 + \sqrt{(2^2 + 1)})$

Mean value $= \frac{1}{6}\ln(2 + \sqrt{5})$

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Question 3 continued

(Total for Question 3 is 6 marks)

4. The infinite series C and S are defined by

$$C = \cos\theta + \frac{1}{2}\cos 5\theta + \frac{1}{4}\cos 9\theta + \frac{1}{8}\cos 13\theta + \ldots$$

$$S = \sin\theta + \frac{1}{2}\sin 5\theta + \frac{1}{4}\sin 9\theta + \frac{1}{8}\sin 13\theta + \ldots$$

Given that the series C and S are both convergent,

(a) show that

$$C + iS = \frac{2e^{i\theta}}{2 - e^{4i\theta}}$$

(4)

(b) Hence show that

$$S = \frac{4\sin\theta + 2\sin 3\theta}{5 - 4\cos 4\theta}$$

(4)

(a) $C + iS = (\cos\theta + i\sin\theta) + \frac{1}{2}(\cos 5\theta + i\sin 5\theta) + \frac{1}{4}(\cos 9\theta + i\sin 9\theta) + \ldots$

$= e^{i\theta} + \frac{1}{2}e^{5i\theta} + \frac{1}{4}e^{9i\theta} + \frac{1}{8}e^{13i\theta} + \ldots$

This is a geometric series with $a = e^{i\theta}$, $r = \frac{1}{2}e^{4i\theta}$ $(|r| = \frac{1}{2} < 1)$

$C + iS = \frac{a}{1 - r} = \frac{e^{i\theta}}{1 - \frac{1}{2}e^{4i\theta}} = \frac{2e^{i\theta}}{2 - e^{4i\theta}}$ as required

(b) $C + iS = \frac{2e^{i\theta}}{2 - e^{4i\theta}} \times \frac{2 - e^{-4i\theta}}{2 - e^{-4i\theta}} = \frac{4e^{i\theta} - 2e^{-3i\theta}}{4 - 2e^{4i\theta} - 2e^{-4i\theta} + 1}$

$= \frac{4\cos\theta + 4i\sin\theta - 2\cos 3\theta + 2i\sin 3\theta}{5 - 2(e^{4i\theta} + e^{-4i\theta})}$

$= \frac{4\cos\theta - 2\cos 3\theta + i(4\sin\theta + 2\sin 3\theta)}{5 - 4\cos 4\theta}$

$S = \mathrm{Im}(C + iS)$

$S = \frac{4\sin\theta + 2\sin 3\theta}{5 - 4\cos 4\theta}$ as required

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Question 4 continued

(Total for Question 4 is 8 marks)

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5. An engineer is investigating the motion of a sprung diving board at a swimming pool.
Let E be the position of the end of the diving board when it is at rest in its equilibrium position and when there is no diver standing on the diving board.
A diver jumps from the diving board.
The vertical displacement, h cm, of the end of the diving board above E is modelled by the differential equation

$$4\frac{\mathrm{d}^2h}{\mathrm{d}t^2} + 4\frac{\mathrm{d}h}{\mathrm{d}t} + 37h = 0$$

where t seconds is the time after the diver jumps.

(a) Find a general solution of the differential equation.

(2)

When $t = 0$, the end of the diving board is 20 cm below E and is moving upwards with a speed of 55 cm s^{-1}.

(b) Find, according to the model, the maximum vertical displacement of the end of the diving board above E.

(8)

(c) Comment on the suitability of the model for large values of t.

(2)

(a) $4m^2 + 4m + 37 = 0$

$m = \frac{-4 \pm \sqrt{(16 - 592)}}{8} = \frac{-4 \pm 24i}{8} = -\frac{1}{2} \pm 3i$

$h = e^{-0.5t}(A\cos 3t + B\sin 3t)$

(b) $t = 0, h = -20$: $A = -20$

$\frac{dh}{dt} = -0.5e^{-0.5t}(A\cos 3t + B\sin 3t) + e^{-0.5t}(-3A\sin 3t + 3B\cos 3t)$

$t = 0, \frac{dh}{dt} = 55$: $-0.5A + 3B = 55 \Rightarrow 10 + 3B = 55 \Rightarrow B = 15$

$h = e^{-0.5t}(15\sin 3t - 20\cos 3t)$

$\frac{dh}{dt} = -0.5e^{-0.5t}(15\sin 3t - 20\cos 3t) + e^{-0.5t}(45\cos 3t + 60\sin 3t)$

$= e^{-0.5t}(52.5\sin 3t + 55\cos 3t)$

Maximum when $\frac{dh}{dt} = 0$: $52.5\sin 3t + 55\cos 3t = 0$

$\tan 3t = -\frac{55}{52.5} = -\frac{22}{21}$

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Question 5 continued

First stationary point after $t = 0$ (board starts below E moving up, so it is a maximum):

$3t = \pi - \tan^{-1}\left(\frac{22}{21}\right) = 2.33294...$

$t = 0.77765... = 0.778$ s

$h = e^{-0.5 \times 0.77765}(15 \sin(3 \times 0.77765) - 20 \cos(3 \times 0.77765))$

$= 0.67786... \times (10.8504... + 13.8094...)$

$= 16.7157...$

Maximum vertical displacement above E = 16.7 cm

(c) As $t \to \infty$, $e^{-0.5t} \to 0$ so $h \to 0$: the oscillations get smaller and smaller and the end of the board settles back to its equilibrium position E.

This is realistic, so the model is suitable for large values of t.

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Question 5 continued

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Question 5 continued

(Total for Question 5 is 12 marks)

6. In an Argand diagram, the points A, B and C are the vertices of an equilateral triangle with its centre at the origin. The point A represents the complex number $6 + 2i$.

(a) Find the complex numbers represented by the points B and C, giving your answers in the form $x + iy$, where x and y are real and exact.

(6)

The points D, E and F are the midpoints of the sides of triangle ABC.

(b) Find the exact area of triangle DEF.

(3)

(a) B and C are A rotated about O by $\frac{2\pi}{3}$ and $\frac{4\pi}{3}$ (120° and 240°)

$B = (6 + 2i)(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}) = (6 + 2i)(-\frac{1}{2} + \frac{\sqrt{3}}{2}i)$

$= -3 + 3\sqrt{3}\,i - i - \sqrt{3}$

B: $(-3 - \sqrt{3}) + (3\sqrt{3} - 1)i$

$C = (6 + 2i)(\cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3}) = (6 + 2i)(-\frac{1}{2} - \frac{\sqrt{3}}{2}i)$

$= -3 - 3\sqrt{3}\,i - i + \sqrt{3}$

C: $(-3 + \sqrt{3}) + (-3\sqrt{3} - 1)i$

(b) $OA = OB = \sqrt{(6^2 + 2^2)} = \sqrt{40}$, angle $AOB = 120°$

Area $AOB = \frac{1}{2} \times \sqrt{40} \times \sqrt{40} \times \sin 120° = \frac{1}{2} \times 40 \times \frac{\sqrt{3}}{2} = 10\sqrt{3}$

Area $ABC = 3 \times 10\sqrt{3} = 30\sqrt{3}$

DEF is similar to ABC with scale factor ½, so area DEF = ¼ × area ABC

Area $DEF = \frac{1}{4} \times 30\sqrt{3} = \frac{15\sqrt{3}}{2}$

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Question 6 continued

Question 6 continued

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Question 6 continued

(Total for Question 6 is 9 marks)



7.

$$\mathbf{M} = \begin{pmatrix} 2 & -1 & 1 \\ 3 & k & 4 \\ 3 & 2 & -1 \end{pmatrix} \quad \text{where } k \text{ is a constant}$$

(a) Find the values of k for which the matrix $\mathbf{M}$ has an inverse.

(2)

(b) Find, in terms of p, the coordinates of the point where the following planes intersect

$$2x - y + z = p$$
$$3x - 6y + 4z = 1$$
$$3x + 2y - z = 0$$

(5)

(c) (i) Find the value of q for which the set of simultaneous equations

$$2x - y + z = 1$$
$$3x - 5y + 4z = q$$
$$3x + 2y - z = 0$$

can be solved.

(ii) For this value of q, interpret the solution of the set of simultaneous equations geometrically.

(4)

(a) $\det M = 2(-k - 8) - (-1)(-3 - 12) + 1(6 - 3k)$

$= -2k - 16 - 15 + 6 - 3k = -5k - 25$

M has an inverse when $\det M \neq 0$: $-5k - 25 \neq 0 \Rightarrow k \neq -5$

(b) Here $k = -6$, so $\begin{pmatrix} 2 & -1 & 1 \\ 3 & -6 & 4 \\ 3 & 2 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} p \\ 1 \\ 0 \end{pmatrix}$

$\det M = -5(-6) - 25 = 5$

$M^{-1} = \frac{1}{5}\begin{pmatrix} -2 & 1 & 2 \\ 15 & -5 & -5 \\ 24 & -7 & -9 \end{pmatrix}$

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Question 7 continued

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \frac{1}{5}\begin{pmatrix} -2 & 1 & 2 \\ 15 & -5 & -5 \\ 24 & -7 & -9 \end{pmatrix}\begin{pmatrix} p \\ 1 \\ 0 \end{pmatrix} = \frac{1}{5}\begin{pmatrix} -2p+1 \\ 15p-5 \\ 24p-7 \end{pmatrix}$$

$x = \frac{1-2p}{5}$, $y = 3p - 1$, $z = \frac{24p-7}{5}$

Point of intersection: $\left(\frac{1-2p}{5}, 3p-1, \frac{24p-7}{5}\right)$

(c)(i) $2x - y + z = 1$ (1), $3x - 5y + 4z = q$ (2), $3x + 2y - z = 0$ (3)

(1) + (3): $5x + y = 1$

(2) + 4 × (3): $15x + 3y = q$

For consistency $15x + 3y = 3(5x + y) = 3 \times 1$, so $q = 3$

$q = 3$

(ii) The three planes meet in a single line (they form a sheaf),
so there are infinitely many solutions.



Question 7 continued

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Question 7 continued

(Total for Question 7 is 11 marks)

8.

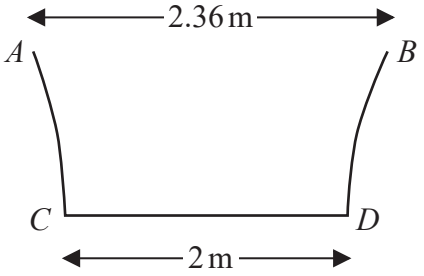


Figure 1

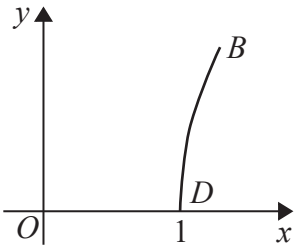


Figure 2

Figure 1 shows the central vertical cross section $ABCD$ of a paddling pool that has a circular horizontal cross section. Measurements of the diameters of the top and bottom of the paddling pool have been taken in order to estimate the volume of water that the paddling pool can contain.

Using these measurements, the curve BD is modelled by the equation

$$y = \ln(3.6x - k) \qquad 1 \leqslant x \leqslant 1.18$$

as shown in Figure 2.

(a) Find the value of k.

(1)

(b) Find the depth of the paddling pool according to this model.

(2)

The pool is being filled with water from a tap.

(c) Find, in terms of h, the volume of water in the pool when the pool is filled to a depth of h m.

(5)

Given that the pool is being filled at a constant rate of 15 litres every minute,

(d) find, in cm h^{-1}, the rate at which the water level is rising in the pool when the depth of the water is 0.2 m.

(3)

(a) At D, $x = 1$ and $y = 0$: $\ln(3.6 - k) = 0 \Rightarrow 3.6 - k = 1 \Rightarrow k = 2.6$

(b) At B, $x = 1.18$: $y = \ln(3.6 \times 1.18 - 2.6) = \ln 1.648 = 0.49956...$

Depth of pool = 0.4996 m ($\approx$ 0.500 m)

(c) $y = \ln(3.6x - 2.6) \Rightarrow e^y = 3.6x - 2.6 \Rightarrow x = \dfrac{e^y + 2.6}{3.6}$

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Question 8 continued

$V = \pi \int_0^h x^2 \, dy = \pi \int_0^h \left(\frac{e^y + 2.6}{3.6} \right)^2 dy$

$= \frac{\pi}{3.6^2} \int_0^h (e^{2y} + 5.2e^y + 6.76) \, dy$

$= \frac{\pi}{12.96} \left[\frac{1}{2} e^{2y} + 5.2e^y + 6.76y \right]_0^h$

$= \frac{\pi}{12.96} \left\{ \left(\frac{1}{2} e^{2h} + 5.2e^h + 6.76h \right) - \left(\frac{1}{2} e^0 + 5.2e^0 + 6.76(0) \right) \right\}$

$V = \frac{\pi}{12.96} \left(\frac{1}{2} e^{2h} + 5.2e^h + 6.76h - 5.7 \right) \ m^3$

(d) $\frac{dV}{dh} = \frac{\pi}{12.96} (e^{2h} + 5.2e^h + 6.76)$

$h = 0.2$: $\frac{dV}{dh} = \frac{\pi}{12.96} (e^{0.4} + 5.2e^{0.2} + 6.76) = 3.5398... \ m^2$

$\frac{dV}{dt} = 15 \text{ litres min}^{-1} = 0.015 \ m^3 \ min^{-1} = 0.9 \ m^3 \ h^{-1}$

$\frac{dh}{dt} = \frac{dV}{dt} \div \frac{dV}{dh} = \frac{0.9}{3.5398...} = 0.25424... \ m \ h^{-1}$

Water level rising at 25.4 cm h^{-1}

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Question 8 continued

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Question 8 continued

Question 8 continued

(Total for Question 8 is 11 marks)

TOTAL FOR PAPER IS 75 MARKS

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