Rewarding Learning

ADVANCED
General Certificate of Education
2025

Centre Number

Candidate Number

Mathematics

StemUp Model Solution

Assessment Unit A2 1
assessing
Pure Mathematics

[AMT11]

AMT11

THURSDAY 29 MAY, MORNING

TIME

2 hours 30 minutes.

INSTRUCTIONS TO CANDIDATES

Write your Centre Number and Candidate Number in the spaces provided at the top of this page.
You must answer **all twelve** questions in the spaces provided.
Do not write outside the boxed area on each page or on blank pages or tracing paper.
Complete in black ink only. **Do not write with a gel pen**.
Questions which require drawing or sketching should be completed using an HB pencil.
Show clearly the full development of your answers. **Answers without working may not gain full credit.**
Answers should be given to three significant figures unless otherwise stated.
You are permitted to use a graphic or scientific calculator in this paper.

INFORMATION FOR CANDIDATES

The total mark for this paper is 150
Figures in brackets printed down the right-hand side of pages indicate the marks awarded to each question or part question.
A copy of the **Mathematical Formulae and Tables booklet** is provided.
Throughout the paper the logarithmic notation used is $\ln z$ where it is noted that $\ln z \equiv \log_e z$

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1 List the terms of the following sequences for $n = 1, 2, 3, 4$

State whether each sequence diverges, converges or oscillates.

(i) $\dfrac{2n^2}{3+n}$ [2]

$u_n = \dfrac{2n^2}{3+n}$

$n = 1$: $\dfrac{2}{4} = \dfrac{1}{2}$ $n = 2$: $\dfrac{8}{5}$ $n = 3$: $\dfrac{18}{6} = 3$ $n = 4$: $\dfrac{32}{7}$

Terms: $\dfrac{1}{2}, \dfrac{8}{5}, 3, \dfrac{32}{7}$

As $n \to \infty$, $u_n \to \infty$ (terms increase without limit)

The sequence diverges

(ii) $\dfrac{1-2n}{3n}$ [2]

$n = 1$: $\dfrac{-1}{3}$ $n = 2$: $\dfrac{-3}{6} = -\dfrac{1}{2}$ $n = 3$: $\dfrac{-5}{9}$ $n = 4$: $\dfrac{-7}{12}$

Terms: $-\dfrac{1}{3}, -\dfrac{1}{2}, -\dfrac{5}{9}, -\dfrac{7}{12}$

$u_n = \dfrac{1}{3n} - \dfrac{2}{3} \to -\dfrac{2}{3}$ as $n \to \infty$

The sequence converges (to $-\dfrac{2}{3}$)

[Turn over

2 The 4th and 9th terms in an arithmetic progression are 89 and 69 respectively.

(i) Find the first term and the common difference for this arithmetic progression. [5]

$u_n = a + (n - 1)d$

$u_4 = a + 3d = 89$ (1)

$u_9 = a + 8d = 69$ (2)

(2) − (1): $5d = -20$

$d = -4$

$a + 3(-4) = 89$

$a = 101$

First term $a = 101$, common difference $d = -4$

(ii) In this arithmetic progression, the last term is the only negative term.

Find the number of terms in the progression. [4]

The last term u_n is negative:

$u_n = a + (n-1)d < 0$

$101 - 4(n-1) < 0$

$105 - 4n < 0$

$n > \frac{105}{4} = 26.25$

$u_{26} = 101 - 4(25) = 1 > 0, \quad u_{27} = 101 - 4(26) = -3 < 0$

Only the last term is negative, so the last term is u_{27}

Number of terms = 27

[Turn over

3 **Fig. 1** below shows the graph of $y = f(x)$

The points P(0, 4) and Q(2, 2) are labelled as shown.

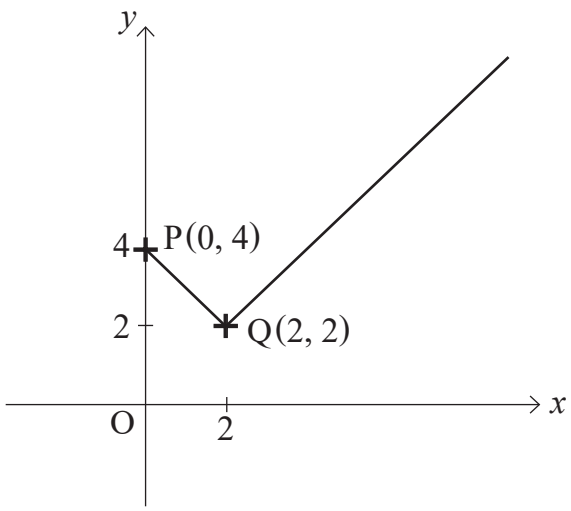
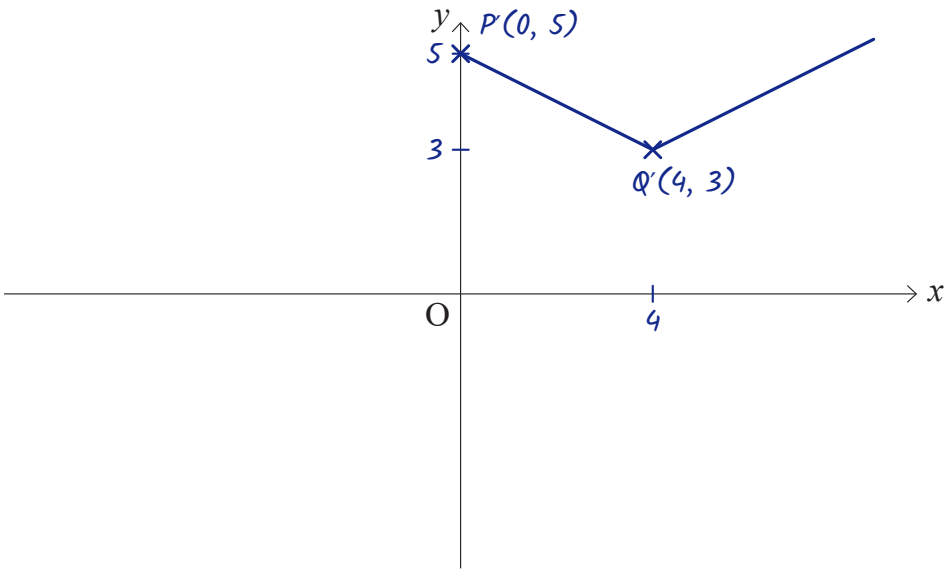


Fig. 1

(i) On the axes below, sketch the graph of

$$y = 1 + f\left(\frac{x}{2}\right)$$

Clearly label the images of the points P and Q.

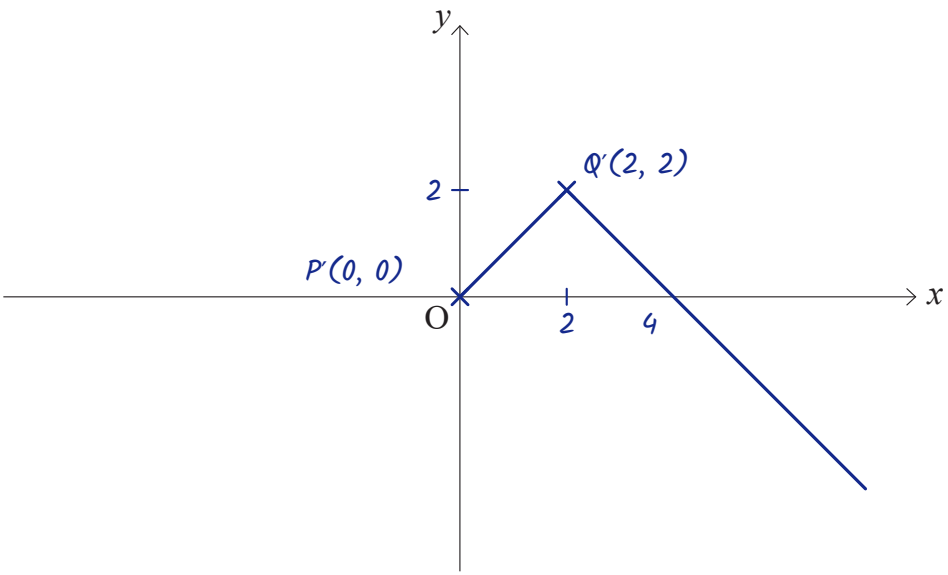


[3]

(ii) On the axes below, sketch the graph of

$$y = 4 - f(x)$$

Clearly label the images of the points P and Q.



[3]

[Turn over

4 Ella wants to make party hats for her son's birthday party.

She creates a template by cutting a major sector AOB from circular card.

The card has centre O and radius 10 cm.

The major arc AB subtends an angle θ radians at O, as shown in **Fig. 2** below.

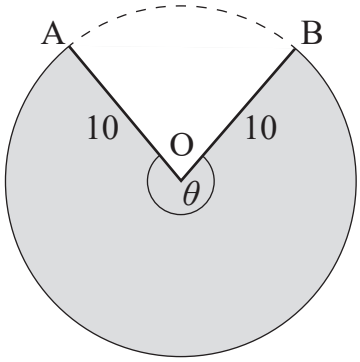


Fig. 2

She will then fold the cards into cones before decorating them.

The template must have a major arc length of 14π cm.

(i) Find the exact value of θ that Ella should use when creating her sector template.

[2]

Arc length = $r\theta$

$14\pi = 10 \times \theta$

$\theta = \frac{14\pi}{10} = \frac{7\pi}{5}$ radians

(ii) Find the area of card wasted in creating this template. [4]

Major sector area $= \frac{1}{2} r^2\theta = \frac{1}{2} \times 10^2 \times \frac{7\pi}{5}$

$= 70\pi \text{ cm}^2$

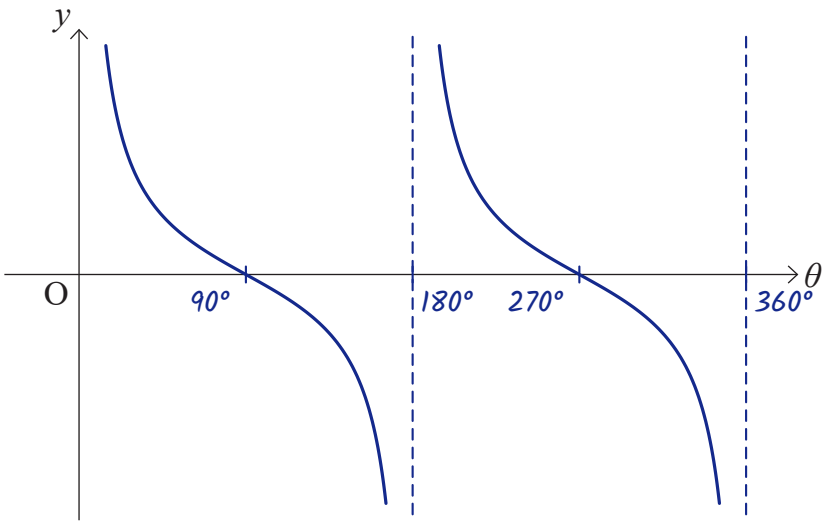
Area of circle $= \pi r^2 = \pi \times 10^2 = 100\pi \text{ cm}^2$

Wasted area $= 100\pi - 70\pi$

$= 30\pi \text{ cm}^2 \quad (= 94.2 \text{ cm}^2)$

[Turn over

5 **(a)** On the axes below, sketch the graph of $y = \cot\theta$ for $0° \leqslant \theta \leqslant 360°$



[3]

(b) **(i)** Prove the identity

$$2\cot 2\theta \equiv \cot\theta - \tan\theta$$ [4]

LHS = 2 cot 2θ

$= \frac{2}{\tan 2\theta}$

$= 2 \div \frac{2\tan\theta}{1 - \tan^2\theta}$ (double angle formula)

$= \frac{2(1 - \tan^2\theta)}{2\tan\theta}$

$= \frac{1 - \tan^2\theta}{\tan\theta}$

$= \frac{1}{\tan\theta} - \frac{\tan^2\theta}{\tan\theta}$

$= \cot\theta - \tan\theta = RHS$

$\therefore$ 2 cot 2θ ≡ cot θ − tan θ

(ii) Hence, or otherwise, find the solutions of

$$\cot\theta - \tan\theta = -2 \quad \text{for} \quad 0° \leqslant \theta \leqslant 360° \qquad [5]$$

cot θ − tan θ = −2

⇒ 2 cot 2θ = −2

cot 2θ = −1 ⇒ tan 2θ = −1

0° ≤ θ ≤ 360° ⇒ 0° ≤ 2θ ≤ 720°

2θ = 135°, 315°, 495°, 675°

θ = 67.5°, 157.5°, 247.5°, 337.5°

[Turn over

6 **(a)** Differentiate the function f(x) with respect to x, where f(x) is defined as

$$f(x) = e^{2x+1} \operatorname{cosec} 3x \qquad [4]$$

$f(x) = e^{2x+1} \operatorname{cosec} 3x$

Product rule: $u = e^{2x+1}$, $v = \operatorname{cosec} 3x$

$u' = 2e^{2x+1}$, $v' = -3 \operatorname{cosec} 3x \cot 3x$

$f'(x) = u'v + uv'$

$f'(x) = 2e^{2x+1} \operatorname{cosec} 3x + e^{2x+1}(-3 \operatorname{cosec} 3x \cot 3x)$

$f'(x) = 2e^{2x+1} \operatorname{cosec} 3x - 3e^{2x+1} \operatorname{cosec} 3x \cot 3x$

(b) A curve is given by the equation

$$2x + \frac{5x^2}{y^3} = 3$$

Find the equation of the tangent to the curve at the point (–1, 1).

Leave your answer in the form $ax + by + c = 0$, where a, b and c are integers. [9]

$2x + \frac{5x^2}{y^3} = 3 \Rightarrow 2xy^3 + 5x^2 = 3y^3$

Differentiating implicitly with respect to x:

$2y^3 + 6xy^2\frac{dy}{dx} + 10x = 9y^2\frac{dy}{dx}$

At (–1, 1):

$2 - 6\frac{dy}{dx} - 10 = 9\frac{dy}{dx}$

$-8 = 15\frac{dy}{dx}$

$\Rightarrow \quad \frac{dy}{dx} = -\frac{8}{15}$

Tangent: $y - y_1 = m(x - x_1)$

$y - 1 = -\frac{8}{15}(x + 1)$

$15(y - 1) = -8(x + 1)$

$15y - 15 = -8x - 8$

$8x + 15y - 7 = 0$

7 **(i)** Show that

$$\int \sin^2 x \, dx = \frac{1}{2}x - \frac{1}{2}\sin x \cos x + c \qquad [5]$$

$\cos 2x = 1 - 2\sin^2 x$

$\Rightarrow \quad \sin^2 x = \frac{1}{2}(1 - \cos 2x)$

$\int \sin^2 x \, dx = \int \frac{1}{2}(1 - \cos 2x) \, dx$

$= \frac{1}{2}x - \frac{1}{4}\sin 2x + c$

$= \frac{1}{2}x - \frac{1}{4}(2 \sin x \cos x) + c$

$= \frac{1}{2}x - \frac{1}{2}\sin x \cos x + c$ as required

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(Questions continue overleaf)

(ii) A vase is modelled by rotating the curve

$$y = 2 - \sin x$$

between the lines $x = 0$ and $x = 2\pi$ through 2π radians about the x-axis, as shown in **Fig. 3** below.

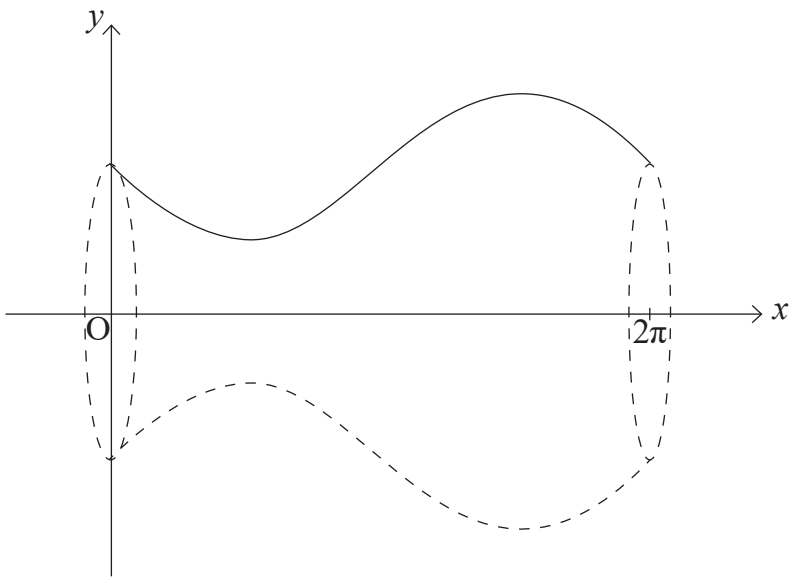


Fig. 3

Using the answer shown in **(i)**, or otherwise, find the exact volume of the vase. [7]

$V = \pi\int_0^{2\pi} y^2\,dx = \pi\int_0^{2\pi} (2 - \sin x)^2\,dx$

$= \pi\int_0^{2\pi} (4 - 4\sin x + \sin^2 x)\,dx$

$= \pi\left[4x + 4\cos x + \frac{1}{2}x - \frac{1}{2}\sin x\cos x\right]_0^{2\pi}$

$= \pi[(8\pi + 4 + \pi - 0) - (0 + 4 + 0 - 0)]$

$= \pi \times 9\pi$

$V = 9\pi^2$ cubic units

[Turn over

8 The graph of the function $f(x) = 3(1 + \ln x)^2$ is shown in **Fig. 4** below.

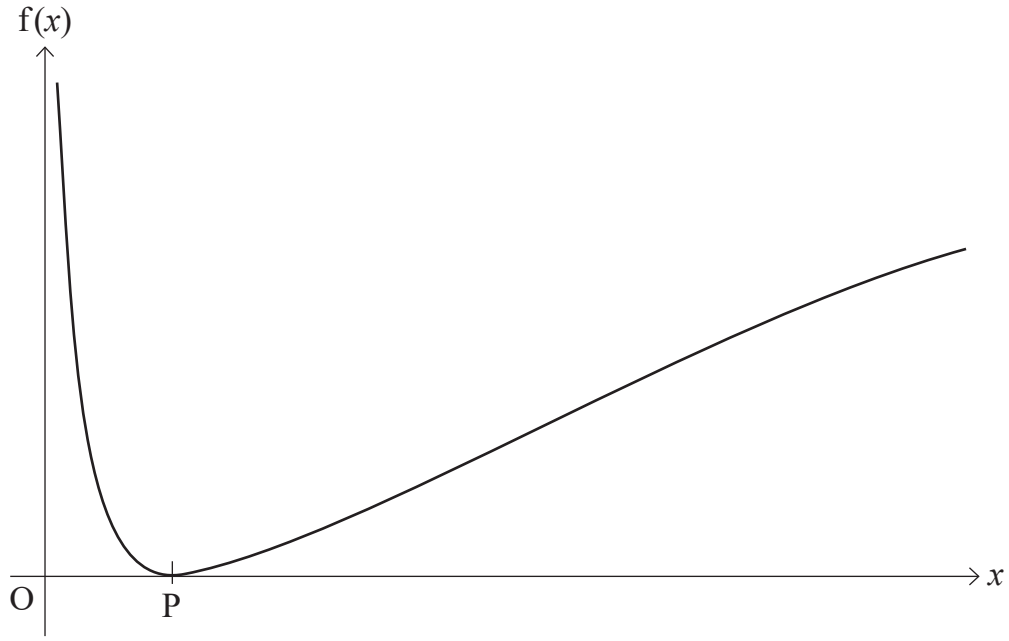


Fig. 4

The stationary point P lies on the x-axis.

(i) Using differentiation, find the exact value of the x coordinate of P.

Confirm that P is a minimum turning point. [9]

$f(x) = 3(1 + \ln x)^2$

$f'(x) = 6(1 + \ln x) \times \frac{1}{x} = \frac{6(1 + \ln x)}{x}$

Stationary point: $f'(x) = 0 \Rightarrow \frac{6}{x}(1 + \ln x) = 0$

$\frac{6}{x} \neq 0 \quad \therefore \quad 1 + \ln x = 0$

$\ln x = -1$

$x = e^{-1} = \frac{1}{e}$

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Product rule on $f'(x) = 6(1 + \ln x) \times x^{-1}$:

$f''(x) = -\frac{6}{x^2}(1 + \ln x) + \frac{6}{x} \times \frac{1}{x}$

$= -\frac{6}{x^2}(1 + \ln x) + \frac{6}{x^2}$

$= -\frac{6 \ln x}{x^2}$

When $x = \frac{1}{e}$: $f''(x) = -6e^2 \ln(e^{-1}) = 6e^2$

$6e^2 > 0 \quad \therefore \quad f''(x) > 0$

$\therefore$ P is a minimum turning point

x coordinate of $P = \frac{1}{e}$

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(ii) State the equation of the vertical asymptote of the function f(x). [1]

As $x \to 0^+$, $\ln x \to -\infty$ so $f(x) \to \infty$

Vertical asymptote: $x = 0$

(iii) State the domain and range of the function f(x). [2]

Domain: $x \in \mathbb{R}$, $x > 0$

Range: $f(x) \in \mathbb{R}$, $f(x) \geq 0$

f(x) can be considered as a composite function such that f(x) = hg(x), where

$$g(x) = \ln x - 1 \quad x \in \mathbb{R}, \quad x > 0$$

and h(x) has domain $x \in \mathbb{R}$

(iv) Write down the function h(x). [1]

$1 + \ln x = (\ln x - 1) + 2 = g(x) + 2$

$h(x) = 3(x + 2)^2, \quad x \in \mathbb{R}$

(v) Find the inverse function $g^{-1}(x)$, stating its domain. [4]

Let $y = \ln x - 1$

$\ln x = y + 1$

$x = e^{y+1}$

$g^{-1}(x) = e^{x+1}$

Domain of g^{-1} = range of g: $x \in \mathbb{R}$

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9 **(i)** Use the Trapezium Rule with 4 strips to find an approximate value for

$$\int_0^1 \frac{2}{5+\sqrt{x}}\,dx \qquad [5]$$

$y = \frac{2}{5+\sqrt{x}}$, $h = \frac{1-0}{4} = 0.25$

$x_0 = 0$ $\quad y_0 = \frac{2}{5} = 0.4$

$x_1 = 0.25$ $\quad y_1 = \frac{4}{11} = 0.363636...$

$x_2 = 0.5$ $\quad y_2 = 0.350440...$

$x_3 = 0.75$ $\quad y_3 = 0.340946...$

$x_4 = 1$ $\quad y_4 = \frac{1}{3} = 0.333333...$

$\int_0^1 \frac{2}{5+\sqrt{x}}\,dx \approx \frac{h}{2}[y_0 + y_4 + 2(y_1 + y_2 + y_3)]$

$\approx \frac{0.25}{2}[0.4 + 0.333333... + 2(0.363636... + 0.350440...$

$+ 0.340946...)]$

$\approx 0.125 \times 2.843379...$

$\approx 0.355422...$

≈ 0.355 (3 s.f.)

[Turn over

(ii) Using the substitution $u = 5 + \sqrt{x}$, find the value of

$$\int_0^1 \frac{2}{5+\sqrt{x}}\,dx \qquad [8]$$

$u = 5 + \sqrt{x} \Rightarrow \sqrt{x} = u - 5$

$\frac{du}{dx} = \frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}}$

$dx = 2\sqrt{x}\,du = 2(u-5)\,du$

When $x = 0$, $u = 5$; when $x = 1$, $u = 6$

$\int_0^1 \frac{2}{5+\sqrt{x}}\,dx = \int_5^6 \frac{2}{u} \times 2(u-5)\,du$

$= \int_5^6 \frac{4(u-5)}{u}\,du$

$= \int_5^6 \left(4 - \frac{20}{u}\right)du$

$= [4u - 20\ln u]_5^6$

$= (24 - 20\ln 6) - (20 - 20\ln 5)$

$= 4 + 20\ln\left(\frac{5}{6}\right)$

$= 0.353568\ldots = 0.354$ (3 s.f.)

(iii) Explain how the use of the Trapezium Rule in **(i)** could be modified to obtain a better approximation to the integral

$$\int_0^1 \frac{2}{5+\sqrt{x}}\,dx \qquad [1]$$

Use more strips (more ordinates, i.e. a smaller strip width h) to improve the approximation.

[Turn over

10 (a) The expression $2\sqrt{2}\cos x + \sin x$ can be written in the form $R\cos(x - \alpha)$, where R is an integer and $0° < \alpha < 90°$

(i) Find the values of R and α. [6]

$2\sqrt{2}\cos x + \sin x \equiv R\cos(x - \alpha)$

$\equiv R\cos x\cos\alpha + R\sin x\sin\alpha$

Comparing: $R\cos\alpha = 2\sqrt{2}$, $R\sin\alpha = 1$

$\tan\alpha = \frac{1}{2\sqrt{2}} \Rightarrow \alpha = \tan^{-1}\left(\frac{1}{2\sqrt{2}}\right) = 19.471...°$

$\alpha = 19.5°$

$R^2 = (2\sqrt{2})^2 + 1^2 = 8 + 1 = 9$

$R = 3$

$\therefore 2\sqrt{2}\cos x + \sin x \equiv 3\cos(x - 19.5°)$

(ii) Hence determine the minimum value of

$$3 + 4\sqrt{2}\cos x + 2\sin x$$ [3]

$3 + 4\sqrt{2}\cos x + 2\sin x = 3 + 2(2\sqrt{2}\cos x + \sin x)$

$= 3 + 2(3\cos(x - 19.5°))$

$= 3 + 6\cos(x - 19.5°)$

Minimum when $\cos(x - 19.5°) = -1$

Minimum value $= 3 + 6(-1) = -3$

(b) A toy car is moving around a circular track such that its coordinates relative to a fixed origin O are given parametrically by

$$x = 3\cos t + 4\sin t + 2 \qquad \text{and} \qquad y = 4\cos t - 3\sin t - 1$$

where t is the time in seconds.

(i) By using parametric differentiation, show that

$$\frac{dy}{dx} = \frac{2 - x}{y + 1} \qquad [5]$$

$\frac{dx}{dt} = -3\sin t + 4\cos t$

$\frac{dy}{dt} = -4\sin t - 3\cos t$

$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx} = \frac{-4\sin t - 3\cos t}{4\cos t - 3\sin t}$

$2 - x = -3\cos t - 4\sin t \quad \text{and} \quad y + 1 = 4\cos t - 3\sin t$

$\frac{dy}{dx} = \frac{-3\cos t - 4\sin t}{4\cos t - 3\sin t} = \frac{2 - x}{y + 1}$ as required

Initially, the car was at the point with coordinates (5, 3).

(ii) By solving the differential equation shown in **(i)**, find the equation of the track. [7]

$\frac{dy}{dx} = \frac{2 - x}{y + 1}$

Separating variables: $\int (y + 1)\, dy = \int (2 - x)\, dx$

$\frac{1}{2}y^2 + y = 2x - \frac{1}{2}x^2 + c$

When $x = 5$, $y = 3$:

$\frac{9}{2} + 3 = 10 - \frac{25}{2} + c$

$7.5 = -2.5 + c \quad \Rightarrow \quad c = 10$

$\frac{1}{2}y^2 + y = 2x - \frac{1}{2}x^2 + 10$

$y^2 + 2y = 4x - x^2 + 20$

$x^2 + y^2 - 4x + 2y - 20 = 0$

[Turn over

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11 (i) Write the following expression in partial fractions.

$$\frac{4x^2 + 11x + 7}{(1 + 2x)(3 + x)}$$

[9]

Numerator and denominator have the same degree, so divide:

$(1 + 2x)(3 + x) = 2x^2 + 7x + 3$

$(4x^2 + 11x + 7) \div (2x^2 + 7x + 3)$: quotient 2

$2(2x^2 + 7x + 3) = 4x^2 + 14x + 6$

Remainder $= (4x^2 + 11x + 7) - (4x^2 + 14x + 6) = -3x + 1$

$$\frac{4x^2 + 11x + 7}{(1 + 2x)(3 + x)} \equiv 2 + \frac{-3x + 1}{(1 + 2x)(3 + x)}$$

$$\frac{-3x + 1}{(1 + 2x)(3 + x)} \equiv \frac{A}{1 + 2x} + \frac{B}{3 + x}$$

$-3x + 1 \equiv A(3 + x) + B(1 + 2x)$

$x = -3$: $10 = 0 - 5B \Rightarrow B = -2$

$x = -\frac{1}{2}$: $\frac{5}{2} = \frac{5}{2}A + 0 \Rightarrow A = 1$

$$\frac{4x^2 + 11x + 7}{(1 + 2x)(3 + x)} \equiv 2 + \frac{1}{1 + 2x} - \frac{2}{3 + x}$$

[Turn over

(ii) Hence, using the binomial theorem, expand

$$\frac{4x^2 + 11x + 7}{(1 + 2x)(3 + x)}$$

in ascending powers of x, up to and including the term in x^2 [8]

$\frac{4x^2 + 11x + 7}{(1 + 2x)(3 + x)} \equiv 2 + \frac{1}{1 + 2x} - \frac{2}{3 + x}$

$\frac{1}{1 + 2x} = (1 + 2x)^{-1} = 1 + (-1)(2x) + \frac{(-1)(-2)}{2!}(2x)^2 + \ldots$

$= 1 - 2x + 4x^2 + \ldots$

$\frac{2}{3 + x} = 2(3 + x)^{-1} = 2 \times 3^{-1}(1 + \frac{x}{3})^{-1} = \frac{2}{3}(1 + \frac{x}{3})^{-1}$

$= \frac{2}{3}[1 + (-1)(\frac{x}{3}) + \frac{(-1)(-2)}{2!}(\frac{x}{3})^2 + \ldots]$

$= \frac{2}{3}[1 - \frac{x}{3} + \frac{x^2}{9} + \ldots] = \frac{2}{3} - \frac{2x}{9} + \frac{2x^2}{27} + \ldots$

Hence the expansion is

$2 + (1 - 2x + 4x^2 + \ldots) - (\frac{2}{3} - \frac{2x}{9} + \frac{2x^2}{27} + \ldots)$

$= \frac{7}{3} - \frac{16}{9}x + \frac{106}{27}x^2 + \ldots$

[Turn over

12 The graphs of

$$y = \ln x, \quad y = x - 2 \quad \text{and} \quad x = 1$$

are shown in **Fig. 5** below.

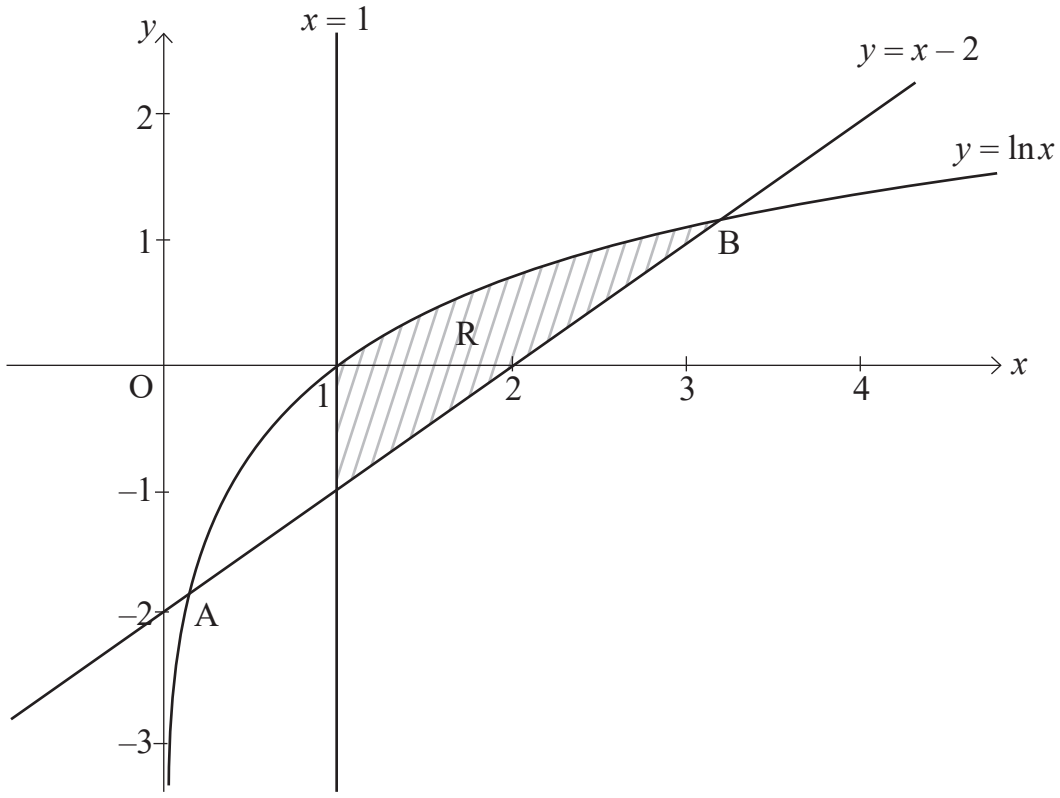


Fig. 5

The graphs of $y = \ln x$ and $y = x - 2$ intersect at the points A and B.

Point B has an x coordinate close to 3

(i) Show that the x coordinate of B lies between $x = 3$ and $x = 4$ [4]

At B: $\ln x = x - 2$

Let $f(x) = \ln x - x + 2 = 0$

$f(3) = \ln 3 - 3 + 2 = 0.09861... > 0$

$f(4) = \ln 4 - 4 + 2 = -0.61370... < 0$

There is a change of sign in $f(x)$ between $x = 3$ and $x = 4$

and $f(x)$ is continuous in this range,

$\therefore$ the x coordinate of B lies between $x = 3$ and $x = 4$

[Turn over

(ii) Starting with $x_0 = 3$, use the Newton-Raphson method once to find a better approximation for the x coordinate of B.

Give your answer correct to three decimal places. [5]

$f(x) = \ln x - x + 2$

$f'(x) = \frac{1}{x} - 1$

$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$

$x_1 = 3 - \frac{\ln 3 - 3 + 2}{(1 \div 3) - 1}$

$= 3 - \frac{0.098612...}{-0.666666...} = 3.147918...$

$x_1 = 3.148$ (3 d.p.)

The shaded region R, as shown in **Fig. 5**, is the area enclosed between the two lines and the curve.

(iii) Using the approximation you have found in **(ii)**, find the area of R. [10]

Area of $R = \int_1^{3.148} (\ln x - (x - 2))\, dx = \int_1^{3.148} (\ln x - x + 2)\, dx$

Integration by parts for $\int \ln x\, dx$:

$u = \ln x$, $\frac{dv}{dx} = 1 \Rightarrow \frac{du}{dx} = \frac{1}{x}$, $v = x$

$\int \ln x\, dx = x \ln x - \int x \times \frac{1}{x}\, dx = x \ln x - x$

Area $= \left[x \ln x - x - \frac{x^2}{2} + 2x\right]_1^{3.148}$

$= \left[x \ln x - \frac{x^2}{2} + x\right]_1^{3.148}$

$= \left(3.148 \ln 3.148 - \frac{3.148^2}{2} + 3.148\right) - \left(0 - \frac{1}{2} + 1\right)$

$= 1.803071\ldots - 0.5$

$= 1.303071\ldots$

Area of $R = 1.30$ units^2 (3 s.f.)

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Total Marks	

Examiner Number

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