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General Certificate of Education
2023

Centre Number

Candidate Number

Mathematics

StemUp Model Solution

Assessment Unit A2 2
assessing
Applied Mathematics

AMT21

[AMT21]
TUESDAY 13 JUNE, AFTERNOON

TIME

1 hour 30 minutes.

INSTRUCTIONS TO CANDIDATES

Write your Centre Number and Candidate Number in the spaces provided at the top of this page.
You must answer the questions in the spaces provided.
Do not write outside the boxed area on each page or on blank pages.
Complete in black ink only. Questions which require drawing or sketching should be completed using an HB pencil. **Do not write with a gel pen.**
Candidates must answer **all** questions from sections A and B.
Equal time should be spent on each section. Show clearly the full development of your answers.
Answers without working may not gain full credit.
Answers should be given to three significant figures unless otherwise stated.
You are permitted to use a graphic or scientific calculator in this paper.

INFORMATION FOR CANDIDATES

The total mark for this paper is 100. The total available mark for each section of this paper is 50.
Figures in brackets printed down the right-hand side of pages indicate the marks awarded to each question or part question.
Answers should include diagrams where appropriate and marks may be awarded for them.
Take $g = 9.8\,\text{m}\,\text{s}^{-2}$, unless specified otherwise.
A copy of the **Mathematical Formulae and Tables booklet** is provided.
Throughout the paper the logarithmic notation used is $\ln z$ where it is noted that $\ln z \equiv \log_e z$

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Answer all questions.

SECTION A

Mechanics

1 A cannon has a mass of M kg.

The cannon, which is initially at rest, fires a cannonball of mass m kg with a speed of u m s^{-1}

(i) Find the speed of recoil of the cannon in terms of M, m and u. [5]

Let the velocity of the cannon be v m s^{-1}, taking the direction of the cannonball as positive.

Conservation of linear momentum:

$(M + m) \times 0 = Mv + mu$

$Mv = -mu$

$v = -\frac{m}{M}u$

$\therefore$ speed of recoil $= \frac{mu}{M}$ m s^{-1}

The duration of the force acting on the cannon due to the explosion is 0.002 s.

(ii) Find the average force that is exerted on the cannon in terms of m and u. [5]

Impulse on cannon = change in momentum of cannon

$I = Mv - M \times 0$

$I = M \times \left(\frac{-mu}{M}\right) - 0$

$I = -mu$ N s

$Ft = I$

$F \times 0.002 = -mu$

$F = \frac{-mu}{0.002} = -500mu$

Average force on cannon = 500mu N, acting in the direction opposite to the motion of the cannonball.

[Turn over

2 (In this question, **i** and **j** are perpendicular unit vectors in horizontal and vertical directions, respectively.)

A ball is projected from a point whose position vector is $10\,\mathbf{j}$ m relative to a fixed point O.

The ball's velocity of projection is $(23\,\mathbf{i} + 49\,\mathbf{j})\,\text{m s}^{-1}$

The ball moves under gravity.

(i) Show that at time t seconds after projection, the position vector of the ball relative to O is

$$[23t\,\mathbf{i} + (10 + 49t - 4.9t^2)\,\mathbf{j}]\ \text{m}$$

[5]

$s = ut + \frac{1}{2}at^2$

$u = (23i + 49j)\ m\ s^{-1}$, $a = -gj = -9.8j\ m\ s^{-2}$

Displacement from start $= (23i + 49j)t + \frac{1}{2}(-9.8j)t^2$

$= 23t\ i + (49t - 4.9t^2)\ j$

Position vector relative to O = 10j + displacement

$r = 10j + 23t\ i + (49t - 4.9t^2)\ j$

$\therefore r = [23t\ i + (10 + 49t - 4.9t^2)\ j]\ m$ as required

(ii) Hence, or otherwise, find the value of t when the ball is at its greatest height above the horizontal through O. [5]

Vertical component (height above O):

$s_y = 10 + 49t - 4.9t^2$

$\frac{ds_y}{dt} = 49 - 9.8t$

At greatest height $\frac{ds_y}{dt} = 0$: $49 - 9.8t = 0$

$9.8t = 49$

$t = 5$

(iii) What modelling assumption have you made? [1]

The ball is modelled as a particle and air resistance is negligible (the only force acting is gravity).

3 A uniform thin plank AB has mass M kg and length $6L$ metres.

C is a point on the plank where AC = $4L$ metres.

A carpenter places end A on rough horizontal ground and uses a smooth fixed cylinder to support the plank at C as shown in **Fig. 1** below.

The plank rests in equilibrium when inclined at α degrees to the horizontal.

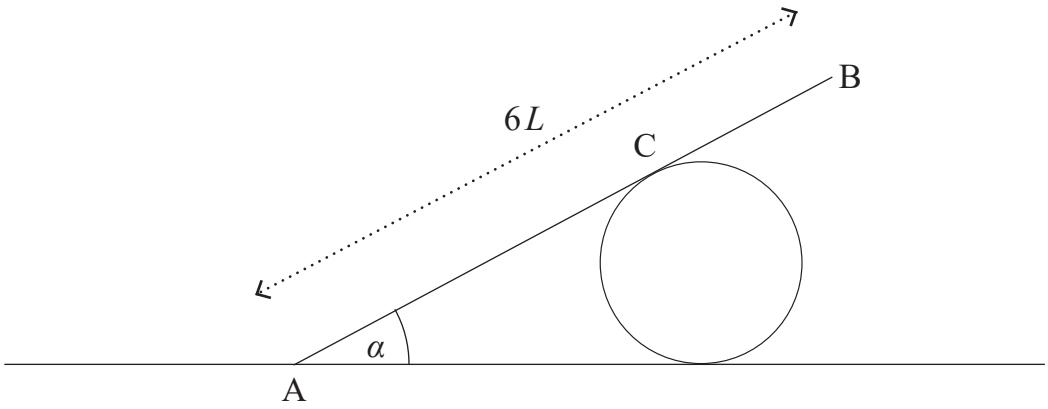
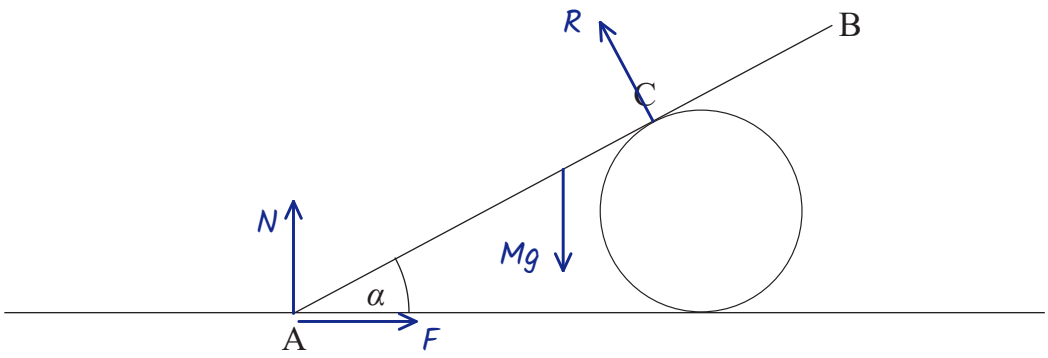


Fig. 1

(i) Complete the diagram below showing all the external forces acting on the plank.



[3]

The reaction at C between the cylinder and the plank is $\frac{3}{5}Mg$ newtons.

(ii) By taking moments about A find the value of α. [5]

Weight acts at the midpoint, 3L from A. R = ⅗Mg acts at C, 4L from A.

M(A): Mg cos α × 3L = R × 4L

Mg cos α (3L) = $\frac{3}{5}$ Mg (4L)

3 cos α = $\frac{12}{5}$

cos α = $\frac{4}{5}$

α = $\cos^{-1}(0.8)$ = 36.869...°

α = 36.9° (3 s.f.)

[Turn over

(iii) Hence, if the plank is about to slip, find the value of the coefficient of friction between the plank and the ground. [7]

$\cos\alpha = \frac{4}{5} \Rightarrow \sin\alpha = \frac{3}{5}$

Resolve horizontally ($\rightarrow$): $F = R\sin\alpha$

$F = \frac{3}{5} \times \frac{3}{5} Mg = \frac{9}{25} Mg$

Resolve vertically ($\uparrow$): $N + R\cos\alpha = Mg$

$N = Mg - \frac{3}{5} \times \frac{4}{5} Mg = \frac{13}{25} Mg$

About to slip, so friction is limiting: $F = \mu N$

$\frac{9}{25} Mg = \mu \times \frac{13}{25} Mg$

$\mu = \frac{9}{13}$ (= 0.692 to 3 s.f.)

4 A particle moves along a straight line.
Points A, B and C lie on this line.

At time t seconds, the particle's displacement x metres from A is given by

$$x = \frac{1}{3}t^3 + \frac{3}{4}t^2 + 7t \quad (0 \leqslant t \leqslant 2)$$

When $t = 2$ the particle is at B.

(i) Find the displacement of B from A. [2]

When $t = 2$:

$x = \frac{1}{3}(2)^3 + \frac{3}{4}(2)^2 + 7(2)$

$x = \frac{8}{3} + 3 + 14 = \frac{59}{3}$

Displacement of B from A $= \frac{59}{3}$ m $(= 19.7 \text{ m})$

(ii) Find its velocity at B. [4]

$v = \frac{dx}{dt} = t^2 + \frac{3}{2}t + 7$

At $t = 2$:

$v = 2^2 + \frac{3}{2}(2) + 7 = 4 + 3 + 7$

$v = 14 \text{ m s}^{-1}$ (in the direction from A to B)

For $2 \leqslant t < 4$ the velocity v m s^{-1} of the particle can now be modelled as

$$v = 14\sec^2\left(\frac{\pi t}{4} - \frac{\pi}{2}\right)$$

At $t = 3$ the particle is at C.

(iii) Find the displacement of C from A. [6]

$x = \int 14\sec^2\left(\frac{\pi t}{4} - \frac{\pi}{2}\right) dt$

$x = 14 \times \frac{4}{\pi}\tan\left(\frac{\pi t}{4} - \frac{\pi}{2}\right) + c$

$x = \frac{56}{\pi}\tan\left(\frac{\pi t}{4} - \frac{\pi}{2}\right) + c$

At $t = 2$, $x = \frac{59}{3}$: $\frac{59}{3} = \frac{56}{\pi}\tan 0 + c \Rightarrow c = \frac{59}{3}$

At $t = 3$: $x = \frac{56}{\pi}\tan\left(\frac{\pi}{4}\right) + \frac{59}{3} = \frac{56}{\pi} + \frac{59}{3}$

Displacement of C from A $= \left(\frac{56}{\pi} + \frac{59}{3}\right)$ m

$= 37.492\ldots = 37.5$ m (3 s.f.)

[Turn over

(iv) If the particle models a train on a straight track, what is the problem with using this model as t gets closer to 4? [2]

As $t \to 4$, $(\pi t/4 - \pi/2) \to \pi/2$, so $\sec^2(\pi t/4 - \pi/2) \to \infty$.
The velocity (and displacement) would tend to infinity,
which is unrealistic for a train on a track.

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SECTION B

Statistics

5 An economist studies the correlation between unemployment percentage and average income for a large city. He sets up a hypothesis test on ρ, the population correlation coefficient. His hypotheses are:

$$H_0: \rho = 0$$

$$H_1: \rho < 0$$

(i) Will the test be one-tailed or two-tailed? [1]

One-tailed (H_1: $\rho < 0$ only tests for negative correlation)

The economist uses a random sample of records over an extended period of time. He calculates the product–moment correlation coefficient for his sample and finds the p-value for the test to be 0.00845

(ii) Explain what is meant by the term p-value. [2]

The p-value is the probability that the test statistic takes a value equal to, or more extreme than, the observed value, assuming the null hypothesis H_0 is true.

(iii) Using a significance level of 1%, complete the hypothesis test. [3]

p-value = 0.00845 < 0.01 (1% significance level)

∴ reject H_0.

There is sufficient evidence at the 1% level to suggest a negative correlation between unemployment percentage and average income for the city.

(iv) If instead the economist had tested $H_1: \rho \neq 0$, what would his conclusion have been? [2]

Two-tailed test: p-value doubles to 2 × 0.00845 = 0.0169

0.0169 > 0.01, so do not reject H_0.

Insufficient evidence at the 1% level to suggest any correlation between unemployment percentage and average income for the city.

6 A brewery produces bottles of cider whose volumes are normally distributed with mean μ and variance σ^2

Given that the upper quartile of the volume of cider in a bottle is 571.71 ml and that 5% of the bottles contain less than 558.95 ml, calculate the values of μ and σ.

[10]

Let $V \sim N(\mu, \sigma^2)$ be the volume in ml.

Upper quartile: $P(V < 571.71) = 0.75$

$\Phi(z_1) = 0.75 \Rightarrow z_1 = 0.6745$

$\frac{571.71 - \mu}{\sigma} = 0.6745$

$571.71 - \mu = 0.6745\sigma \quad \ldots (1)$

$P(V < 558.95) = 0.05$

$\Phi(z_2) = 0.05 \Rightarrow z_2 = -1.6449$

$\frac{558.95 - \mu}{\sigma} = -1.6449$

$558.95 - \mu = -1.6449\sigma \quad \ldots (2)$

$(1) - (2)$: $12.76 = 2.3194\sigma$

$\sigma = 12.76 \div 2.3194 = 5.5016\ldots$

$\sigma = 5.50$ ml (3 s.f.)

$\mu = 571.71 - 0.6745 \times 5.5016 = 567.999\ldots$

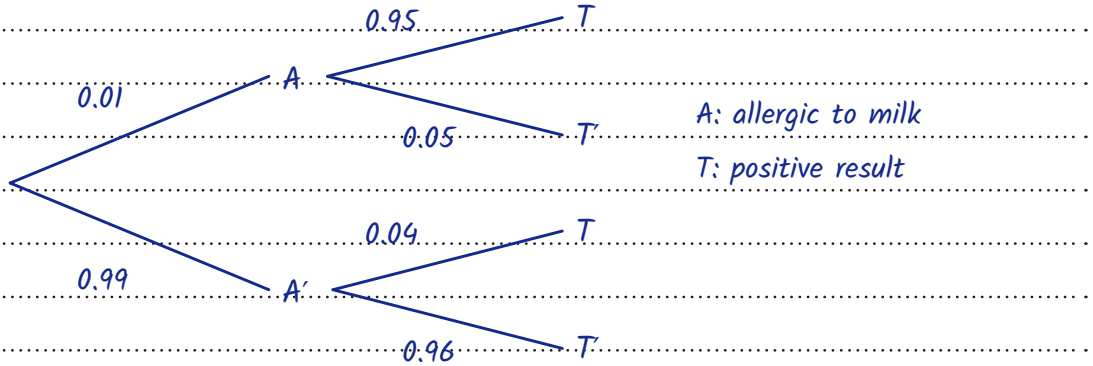
$\mu = 568$ ml (3 s.f.)

7 It is believed that 1 person in 100 is allergic to milk.

A test has been designed to try to detect the allergy.

For a person with the allergy, there is a 95% chance that the test will give a positive result.
For a person who does not have the allergy, there is a 96% chance that the test will give a negative result.

(i) Using a tree diagram, or otherwise, calculate the probability that a randomly selected person gets a positive result on the test. [3]



$P(T) = P(A \text{ and } T) + P(A' \text{ and } T)$

$= 0.01 \times 0.95 + 0.99 \times 0.04$

$= 0.0095 + 0.0396$

$= 0.0491$

A randomly selected person is tested and the result indicates that they are allergic to milk.

(ii) Find the probability that this person is actually allergic to milk. [4]

$P(A \mid T) = \dfrac{P(A \cap T)}{P(T)}$

$= \dfrac{0.01 \times 0.95}{0.01 \times 0.95 + 0.99 \times 0.04}$

$= \dfrac{0.0095}{0.0491}$

$= 0.19348\ldots = 0.193$ (3 s.f.)

(iii) Comment on your answer to **(ii)**. [2]

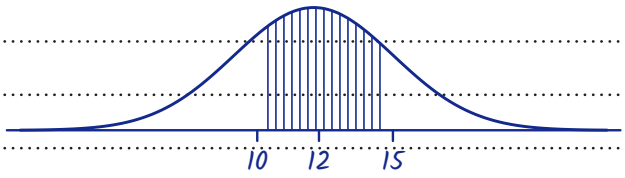
The probability is low: only about 19% (about 1 in 5) of positive results are correct, so the test is not dependable.

[Turn over

8 The time taken to answer a call arriving at a switchboard follows a normal distribution with mean 12 seconds and standard deviation 3.5 seconds.

(i) Calculate the probability that a random call arriving at the switchboard takes between 10 and 15 seconds to be answered. [4]

$X \sim N(12, 3.5^2)$



$P(10 < X < 15) = P\left(\frac{10-12}{3.5} < Z < \frac{15-12}{3.5}\right)$

$= P(-0.5714 < Z < 0.8571)$

$= \Phi(0.8571) - [1 - \Phi(0.5714)]$

$= 0.80430 - 0.28385$

$= 0.52046... = 0.520$ (3 s.f.)

The office manager introduces a bonus scheme designed to reduce the amount of time taken for calls to be answered.

He claims that the scheme has been successful.

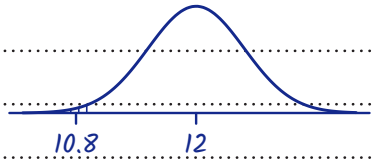
Following the introduction of the scheme, the office manager calculates the average time taken to answer calls in a random sample of 40 to be 10.8 seconds.

(ii) Test the office manager's claim at the 5% level of significance. [7]

H_0: $\mu = 12$

H_1: $\mu < 12$ (one-tailed test at the 5% level)

Under H_0: $\bar{X} \sim N\left(12, \frac{3.5^2}{40}\right)$



Reject H_0 if $z < -1.645$, or if $P(\bar{X} < 10.8) < 0.05$

$z = \frac{10.8 - 12}{3.5 \div \sqrt{40}}$

$z = -2.1684...$

$P(\bar{X} < 10.8) = P(Z < -2.168) = 0.0151$

Since $-2.168 < -1.645$ (and $0.0151 < 0.05$), z lies in the critical region, so reject H_0.

There is sufficient evidence at the 5% level of significance to suggest that the mean time to answer calls has decreased, so the office manager's claim is supported.

[Turn over

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9 Daniel took a multiple choice examination in which there were five possible responses to each of the questions.

His teacher suspects that Daniel had not revised for the examination and was just guessing his answers.

The teacher decides to test her suspicion at the 5% level of significance.

(i) Explain why a one-tailed test is appropriate in this context. [1]

If Daniel is just guessing, P(correct answer) = 1/5 = 0.2.
If he had revised, the probability of a correct answer
would be greater than 0.2, so we only test for an increase:
a one-tailed test is appropriate.

(ii) Write down the null and alternative hypotheses for this test. [2]

Let p = probability that Daniel answers a question correctly.

H_0: $p = 0.2$

H_1: $p > 0.2$

(iii) Describe the meaning of the term level of significance. [2]

The level of significance is the probability of rejecting the null hypothesis H_0 when H_0 is actually true (here a 5% chance of incorrectly rejecting H_0).

The examination contained 40 questions and Daniel got 12 of his answers correct.

(iv) Complete the hypothesis test, stating clearly whether the teacher's suspicion is justified. [7]

Let X = number of correct answers out of 40.

Under H_0: $X \sim B(40, 0.2)$

$P(X \geqslant 12) = 1 - P(X \leqslant 11)$

$= 1 - 0.91249...$

$= 0.0875$ (3 s.f.)

$0.0875 > 0.05$, so do not reject H_0.

There is insufficient evidence at the 5% level of significance to suggest that Daniel was doing better than guessing his answers.

Hence the teacher's suspicion that Daniel was just guessing is justified.

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Examiner Number

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