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ADVANCED
General Certificate of Education
2022 Reserve Series

Centre Number

Candidate Number

Mathematics

StemUp Model Solution

Assessment Unit A2 1
assessing
Pure Mathematics

AMT11

[AMT11]
MONDAY 27 JUNE, MORNING

TIME

2 hours 30 minutes.

INSTRUCTIONS TO CANDIDATES

Write your Centre Number and Candidate Number in the spaces provided at the top of this page.
You must answer **all ten** questions in the spaces provided.
Do not write outside the boxed area on each page or on blank pages or tracing paper.
Complete in black ink only. **Do not write with a gel pen**.
Questions which require drawing or sketching should be completed using an HB pencil.
Show clearly the full development of your answers. **Answers without working may not gain full credit.**
Answers should be given to three significant figures unless otherwise stated.
You are permitted to use a graphic or scientific calculator in this paper.

INFORMATION FOR CANDIDATES

The total mark for this paper is 150
Figures in brackets printed down the right-hand side of pages indicate the marks awarded to each question or part question.
A copy of the **Mathematical Formulae and Tables booklet** is provided.
Throughout the paper the logarithmic notation used is $\ln z$ where it is noted that $\ln z \equiv \log_e z$

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1 **(a)** State whether the following sequences converge, diverge or oscillate.

(i) $\dfrac{2n^2}{n+3}$ [1]

As $n \to \infty$, $\dfrac{2n^2}{n+3} = \dfrac{2n}{1+3/n} \to \infty$

Diverges

(ii) $\cos\left(\dfrac{n\pi}{3}\right)$ [1]

$\cos(n\pi/3) = 1/2, -1/2, -1, -1/2, 1/2, 1, \ldots$ repeating

Oscillates

(iii) $\dfrac{5n}{3n-1}$ [1]

$\dfrac{5n}{3n-1} = \dfrac{5}{3-1/n} \to \dfrac{5}{3}$ as $n \to \infty$

Converges

(b) Find $\sum_{r=1}^{\infty} 3\left(\frac{2}{5}\right)^r$ [5]

$\Sigma\ 3(2/5)^r = 3(\frac{2}{5}) + 3(\frac{2}{5})^2 + 3(\frac{2}{5})^3 + \ldots$

This is a GP with $a = \frac{6}{5}$, $r = \frac{2}{5}$ ($|r| < 1$ so S_∞ exists)

$S_\infty = \frac{a}{1-r} = \frac{6/5}{1-2/5} = \frac{6/5}{3/5}$

$= 2$

[Turn over

(c) The first three terms of an arithmetic progression are

$$x,\ 3x + 1,\ 3x^2 \qquad \text{where } x > 0$$

(i) Find the value of x. [4]

$(3x + 1) - x = 3x^2 - (3x + 1)$ (common difference)

$2x + 1 = 3x^2 - 3x - 1$

$3x^2 - 5x - 2 = 0$

$(3x + 1)(x - 2) = 0$

$x = -\frac{1}{3}$ or $x = 2$

Since $x > 0$, $x = 2$

(ii) Find the sum of the first 10 terms of this arithmetic progression. [4]

Terms are 2, 7, 12, ... so $a = 2$

$d = 3x + 1 - x = 2x + 1 = 5$

$S_n = \frac{n}{2}\{2a + (n-1)d\}$

$S_{10} = \frac{10}{2}\{2(2) + 9(5)\}$

$= 5 \times 49$

$= 245$

2 The shaded region R, as shown in **Fig. 1** below, is the major segment of a circle of radius r cm with angle AOB = $\frac{\pi}{3}$ radians.

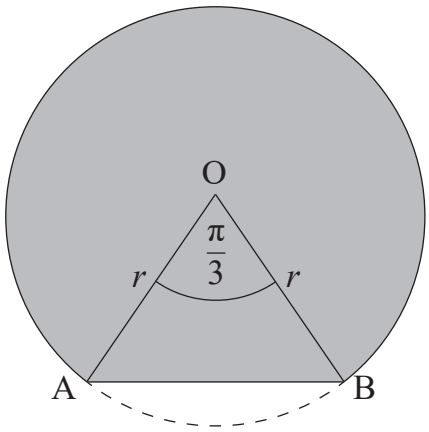


Fig. 1

(i) Find the perimeter of R in terms of r. [3]

Major arc AB = $r(2\pi - \frac{\pi}{3}) = \frac{5\pi r}{3}$

Triangle AOB is equilateral, so chord AB = r

Perimeter of R = $\frac{5\pi r}{3} + r$ cm

The perimeter of R is $(10\pi + 6)$ cm.

(ii) Find the value of r. [2]

$\frac{5\pi r}{3} + r = 10\pi + 6$

$r(\frac{5\pi}{3} + 1) = 6(\frac{5\pi}{3} + 1)$

$r = 6$

(iii) Find the area of R. [5]

Area of triangle AOB $= \frac{1}{2} \times 6^2 \times \sin\frac{\pi}{3} = 9\sqrt{3}$

Area of minor sector AOB $= \frac{1}{2} \times 6^2 \times \frac{\pi}{3} = 6\pi$

Area of minor segment $= 6\pi - 9\sqrt{3}$

Area of R $= \pi \times 6^2 - (6\pi - 9\sqrt{3})$

$= 36\pi - 6\pi + 9\sqrt{3}$

$= (30\pi + 9\sqrt{3})$ cm^2 (= 110 cm^2 to 3 s.f.)

[Turn over

3 **(a)** Simplify as far as possible

$$\frac{4x^2 - 25}{3x^2 + 14x + 8} \div \frac{6x - 15}{x + 4}$$

[5]

$\frac{4x^2 - 25}{3x^2 + 14x + 8} \div \frac{6x - 15}{x + 4} = \frac{4x^2 - 25}{3x^2 + 14x + 8} \times \frac{x + 4}{6x - 15}$

$= \frac{(2x - 5)(2x + 5)}{(3x + 2)(x + 4)} \times \frac{x + 4}{3(2x - 5)}$

$= \frac{2x + 5}{3(3x + 2)}$

(b) Find, in ascending powers of x, the expansion of

$$\frac{x+3}{\sqrt{x+2}}$$

up to and including the term in x^2 [6]

$(x+3)(x+2)^{-1/2} = 2^{-1/2}(x+3)(1+\frac{x}{2})^{-1/2}$

$(1+\frac{x}{2})^{-1/2} = 1 + (-\frac{1}{2})(\frac{x}{2}) + \frac{(-1/2)(-3/2)}{2!}(\frac{x}{2})^2 + \ldots$

$= 1 - \frac{x}{4} + \frac{3x^2}{32} + \ldots$

$\therefore \frac{x+3}{\sqrt{(x+2)}} = \frac{1}{\sqrt{2}}(x+3)(1-\frac{x}{4}+\frac{3x^2}{32}+\ldots)$

$= \frac{1}{\sqrt{2}}(3 + x - \frac{3x}{4} - \frac{x^2}{4} + \frac{9x^2}{32} + \ldots)$

$= \frac{1}{\sqrt{2}}(3 + \frac{x}{4} + \frac{x^2}{32} + \ldots)$

$= \frac{3}{\sqrt{2}} + \frac{x}{4\sqrt{2}} + \frac{x^2}{32\sqrt{2}} + \ldots$ (valid for $|x| < 2$)

[Turn over

4 **(a)** The graph of the function $y = f(x)$ is shown in **Fig. 2** below.

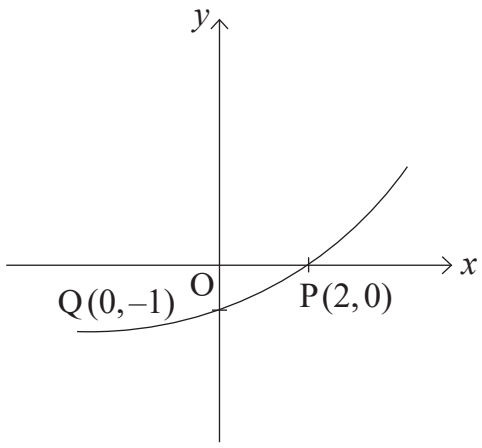
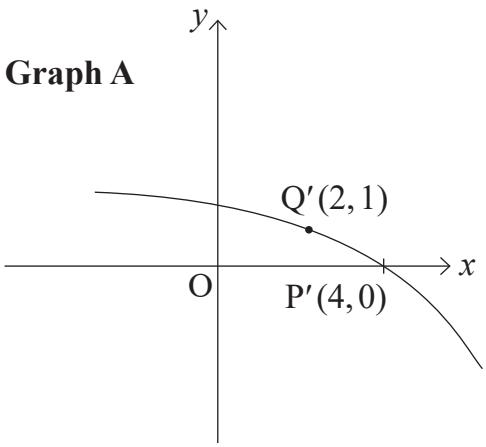


Fig. 2

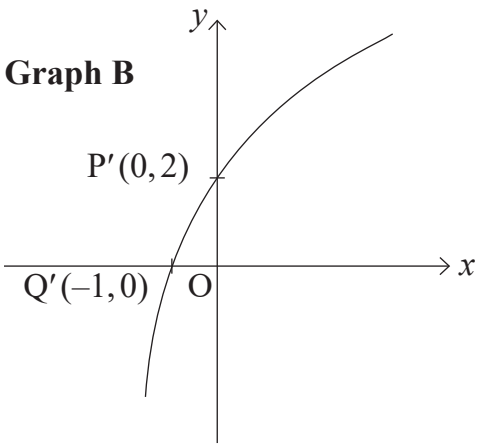
The curve cuts the axes at P(2, 0) and Q(0, –1).

Fig. 3 below shows five different transformations of $y = f(x)$.

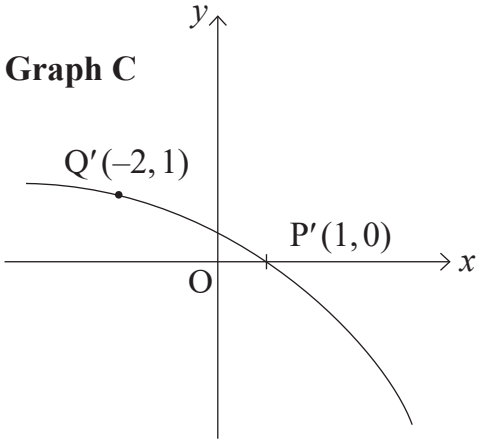
Graph A



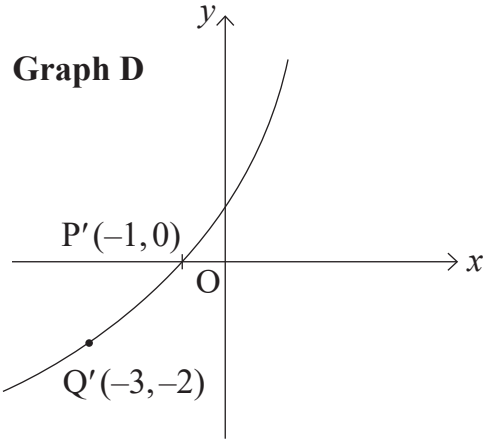
Graph B



Graph C



Graph D



Graph E

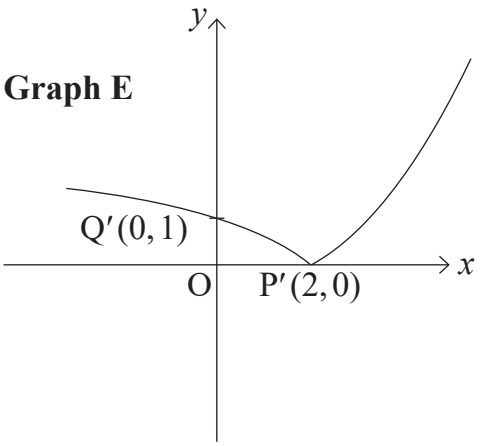


Fig. 3

Complete the following statements:

(i) $y = 2f(x + 3)$ is represented by Graph ……D…… [1]

(ii) $y = -f(x - 2)$ is represented by Graph ……A…… [1]

(iii) $y = |f(x)|$ is represented by Graph ……E…… [1]

(iv) $y = f^{-1}(x)$ is represented by Graph ……B…… [1]

[Turn over

(b) The functions g and h are defined by:

$$g(x) = 2 + \cos x \qquad x \in \mathbb{R} \quad 0 \leqslant x \leqslant \pi$$

$$h(x) = \frac{1}{1+x} \qquad x \in \mathbb{R} \quad x \neq -1$$

(i) State the range of $g(x)$. [2]

$\cos x$ decreases from 1 to -1 for $0 \leq x \leq \pi$

$1 \leq g(x) \leq 3$

(ii) Find the inverse function $h^{-1}(x)$, stating its domain. [5]

Let $y = \frac{1}{1+x}$

$y(1 + x) = 1$

$y + xy = 1 \quad \Rightarrow \quad xy = 1 - y$

$x = \frac{1-y}{y}$

$h^{-1}(x) = \frac{1-x}{x}, \quad x \in \mathbb{R}, \ x \neq 0$

(iii) Find the composite function hg(x), stating its domain. [4]

$hg(x) = h(g(x)) = h(2 + \cos x)$

$= \frac{1}{1 + 2 + \cos x}$

$hg(x) = \frac{1}{3 + \cos x}$, $x \in \mathbb{R}$, $0 \leq x \leq \pi$

[Turn over



5 The points of intersection of the curves

$$y = \text{cosec}^2 3x$$

and

$$y = x^2 + 1$$

can be found by solving the equation

$$\text{cosec}^2 3x - x^2 - 1 = 0$$

(i) Show that this equation has a root between $x = 0.3$ and $x = 0.5$ [3]

Let $f(x) = \text{cosec}^2 3x - x^2 - 1$

$f(0.3) = \text{cosec}^2(0.9) - 0.09 - 1 = 0.53972... > 0$

$f(0.5) = \text{cosec}^2(1.5) - 0.25 - 1 = -0.24497... < 0$

There is a change of sign and $f(x)$ is continuous on $[0.3, 0.5]$,

$\therefore$ there is a root between $x = 0.3$ and $x = 0.5$

It is known that there is only one root α in the interval [0.3, 0.5].

(ii) By taking $x = 0.3$ as a first approximation and using the Newton-Raphson method once, find a better approximation to α. [6]

$f(x) = \text{cosec}^2 3x - x^2 - 1$

$f'(x) = 2 \text{ cosec } 3x \times (-3 \text{ cosec } 3x \cot 3x) - 2x$

$= -6 \text{ cosec}^2 3x \cot 3x - 2x$

$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$

$x_1 = 0.3 - \frac{\text{cosec}^2(0.9) - 0.3^2 - 1}{-6 \text{ cosec}^2(0.9) \cot(0.9) - 0.6}$

$= 0.3 - \frac{0.53972...}{-8.35961...}$

$x_1 = 0.36456...$

$\alpha \approx 0.365$ (3 s.f.)

[Turn over

6 **(a)** Prove that

$$\frac{\cos 2\theta - \cos\theta + 1}{\sin 2\theta - \sin\theta} \equiv \cot\theta \qquad [6]$$

LHS $= \frac{\cos 2\theta - \cos\theta + 1}{\sin 2\theta - \sin\theta}$

$= \frac{2\cos^2\theta - 1 - \cos\theta + 1}{2\sin\theta\cos\theta - \sin\theta}$

$= \frac{2\cos^2\theta - \cos\theta}{2\sin\theta\cos\theta - \sin\theta}$

$= \frac{\cos\theta(2\cos\theta - 1)}{\sin\theta(2\cos\theta - 1)}$

$= \frac{\cos\theta}{\sin\theta}$

$= \cot\theta = RHS \qquad \therefore$ identity proved

(b) Solve the equation

$$\tan(\theta - 45°) = 6\tan\theta \qquad \text{where } 0° \leqslant \theta \leqslant 360° \qquad [7]$$

$\tan(\theta - 45°) = \dfrac{\tan\theta - \tan 45°}{1 + \tan\theta \tan 45°} = 6\tan\theta$

Let $t = \tan\theta$: $\dfrac{t-1}{1+t} = 6t$

$t - 1 = 6t + 6t^2$

$6t^2 + 5t + 1 = 0$

$(3t + 1)(2t + 1) = 0$

$t = -\dfrac{1}{2}$ or $t = -\dfrac{1}{3}$

$\tan\theta = -\dfrac{1}{2}$: $\theta = 153.43...°,\ 333.43...°$

$\tan\theta = -\dfrac{1}{3}$: $\theta = 161.56...°,\ 341.56...°$

$\theta = 153°, 162°, 333°, 342°$

[Turn over

7 **(a)** A curve is defined by the parametric equations

$$x = \sin t + \cos t$$
$$y = 4 - 3\sin 2t$$

Find the Cartesian equation of this curve. [7]

$x = \sin t + \cos t$

$x^2 = (\sin t + \cos t)^2$

$= \sin^2 t + 2 \sin t \cos t + \cos^2 t$

$= 1 + \sin 2t \quad \Rightarrow \quad \sin 2t = x^2 - 1$

$y = 4 - 3 \sin 2t \quad \Rightarrow \quad 3 \sin 2t = 4 - y$

$\therefore 3(x^2 - 1) = 4 - y$

$3x^2 - 3 = 4 - y$

$y = 7 - 3x^2$

[Turn over

(b) A curve is given by the equation

$$3x + 5xy^2 - 16 = 2(x + y)^2$$

Find the equation of the tangent to the curve at the point (1, 3).
Leave your answer in the form $ax + by + c = 0$, where a, b and c are integers. [9]

$3x + 5xy^2 - 16 = 2(x + y)^2$

Differentiating implicitly with respect to x:

$3 + 5y^2 + 10xy\frac{dy}{dx} = 4(x + y)(1 + \frac{dy}{dx})$

At (1, 3):

$3 + 45 + 30\frac{dy}{dx} = 16(1 + \frac{dy}{dx})$

$48 + 30\frac{dy}{dx} = 16 + 16\frac{dy}{dx}$

$14\frac{dy}{dx} = -32 \quad \Rightarrow \quad \frac{dy}{dx} = -\frac{16}{7}$

Tangent: $y - 3 = -\frac{16}{7}(x - 1)$

$7y - 21 = -16x + 16$

$16x + 7y - 37 = 0$

[Turn over

8 **(a)** The curved surface of a glass bowl can be modelled by rotating the curve

$$y = 3 \tan 4x$$

between $x = 0$ and $x = \frac{\pi}{16}$ through 2π radians about the x-axis.

Find, in terms of π, the maximum volume of liquid that the bowl can contain. [9]

$V = \pi \int_0^{\pi/16} y^2 \, dx = \pi \int_0^{\pi/16} 9 \tan^2 4x \, dx$

$= 9\pi \int_0^{\pi/16} (\sec^2 4x - 1) \, dx$

$= 9\pi \left[\frac{1}{4} \tan 4x - x \right]_0^{\pi/16}$

$= 9\pi \left[\left(\frac{1}{4} \tan \frac{\pi}{4} - \frac{\pi}{16} \right) - (0 - 0) \right]$

$= 9\pi \left(\frac{1}{4} - \frac{\pi}{16} \right)$

$= \frac{9\pi}{16} (4 - \pi)$ cubic units

[Turn over

(b) Find $\int xe^{2x}\,dx$ [7]

$u = x, \qquad \frac{dv}{dx} = e^{2x}$

$\frac{du}{dx} = 1, \qquad v = \frac{1}{2}e^{2x}$

$\int xe^{2x}\,dx = \frac{1}{2}xe^{2x} - \int \frac{1}{2}e^{2x}\,dx$

$= \frac{1}{2}xe^{2x} - \frac{1}{4}e^{2x} + c$

[Turn over

(c) Using the substitution

$$u = x - 3$$

find the exact value of

$$\int_4^6 \frac{x(x-5)}{(x-3)^2}\,dx \qquad [9]$$

$u = x - 3 \Rightarrow x = u + 3, \quad \frac{du}{dx} = 1$

$x = 4 \Rightarrow u = 1; \qquad x = 6 \Rightarrow u = 3$

$\int_4^6 \frac{x(x-5)}{(x-3)^2}\,dx = \int_1^3 \frac{(u+3)(u-2)}{u^2}\,du$

$= \int_1^3 \frac{u^2 + u - 6}{u^2}\,du = \int_1^3 \left(1 + \frac{1}{u} - \frac{6}{u^2}\right)du$

$= \left[u + \ln|u| + \frac{6}{u}\right]_1^3$

$= (3 + \ln 3 + 2) - (1 + \ln 1 + 6)$

$= \ln 3 - 2$

[Turn over

9 **(i)** Show that $(x + 3)$ is a factor of

$$2x^3 + 11x^2 + 12x - 9$$ [3]

Let $f(x) = 2x^3 + 11x^2 + 12x - 9$

$f(-3) = 2(-27) + 11(9) + 12(-3) - 9$

$= -54 + 99 - 36 - 9 = 0$

$\therefore$ by the factor theorem, $(x + 3)$ is a factor



(ii) Hence find

$$\int \frac{x^2 + 3x + 35}{2x^3 + 11x^2 + 12x - 9}\,dx \qquad [15]$$

$2x^3 + 11x^2 + 12x - 9 = (x + 3)(2x^2 + bx - 3)$

comparing x^2 terms: $6 + b = 11 \Rightarrow b = 5$

$= (x + 3)(2x^2 + 5x - 3)$

$= (x + 3)(2x - 1)(x + 3) = (x + 3)^2(2x - 1)$

$\frac{x^2 + 3x + 35}{(x + 3)^2(2x - 1)} \equiv \frac{A}{x + 3} + \frac{B}{(x + 3)^2} + \frac{C}{2x - 1}$

$x^2 + 3x + 35 \equiv A(x + 3)(2x - 1) + B(2x - 1) + C(x + 3)^2$

$x = \frac{1}{2}$: $\frac{1}{4} + \frac{3}{2} + 35 = C(\frac{7}{2})^2 \Rightarrow \frac{147}{4} = \frac{49}{4}C \Rightarrow C = 3$

$x = -3$: $9 - 9 + 35 = -7B \Rightarrow B = -5$

x^2 terms: $1 = 2A + C \Rightarrow A = -1$

$\int \frac{x^2 + 3x + 35}{2x^3 + 11x^2 + 12x - 9}\,dx$

$= \int \left(\frac{-1}{x + 3} - \frac{5}{(x + 3)^2} + \frac{3}{2x - 1}\right) dx$

$= -\ln|x + 3| + \frac{5}{x + 3} + \frac{3}{2}\ln|2x - 1| + c$

[Turn over

10 In a factory, a biological substance is placed in a large tank.

During the production process, the biological substance reproduces at a rate of $0.25A$ kg per hour, where A kg is the amount of substance present at time t hours.

At the same time, the biological substance is pumped out of the tank at a constant rate of 50 kg per hour.

(i) By setting up and solving a suitable differential equation, show that

$$A = 4ke^{0.25t} + 200 \qquad [7]$$

where k is a constant.

$\frac{dA}{dt} = 0.25A - 50$

$\int \frac{1}{0.25A - 50}\, dA = \int 1\, dt$

$4\ln|0.25A - 50| = t + c$

$\ln|0.25A - 50| = 0.25t + 0.25c$

$0.25A - 50 = e^{0.25c}e^{0.25t} = ke^{0.25t}$

$0.25A = ke^{0.25t} + 50$

$A = 4ke^{0.25t} + 200$

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At time $t = 0$, $A = 190$

(ii) Show that

$$A = 200 - 10e^{0.25t}$$ [2]

$t = 0, A = 190$: $190 = 4k + 200$

$4k = -10$

$\therefore A = 200 - 10e^{0.25t}$

[Turn over

The tank must be completely empty by the end of each production shift.

(iii) Find the minimum length of time for each shift.
Note that a production shift always lasts for a whole number of hours. [3]

Tank empty when $A = 0$: $200 - 10e^{0.25t} = 0$

$e^{0.25t} = 20$

$0.25t = \ln 20$

$t = 4 \ln 20 = 11.98...$ hours

Minimum length of each shift is 12 hours

The factory management decides to change to 8-hour production shifts.

They plan to manage this by simply reducing the amount of biological substance in the tank at time $t = 0$

(iv) Find the maximum amount of biological substance at time $t = 0$ which makes this change feasible, giving your answer to 1 decimal place. [5]

$t = 0, A = A_0$: $A_0 = 4k + 200 \Rightarrow 4k = A_0 - 200$

$A = (A_0 - 200)e^{0.25t} + 200$

Need $A \leq 0$ when $t = 8$:

$(A_0 - 200)e^2 + 200 \leq 0$

$A_0 \leq \frac{200e^2 - 200}{e^2} = 200 - 200e^{-2}$

$A_0 \leq 172.93...$

Maximum amount = 172.9 kg (1 d.p.)

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Question Number	**Marks**
1	
2	
3	
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8	
9	
10	
Total Marks	

Examiner Number

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