New Specification

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General Certificate of Education

Centre Number

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Mathematics

StemUp Model Solution

Assessment Unit A2 1
assessing
Pure Mathematics

AMT11

[AMT11]
Assessment

TIME

2 hours 30 minutes.

Assessment Level of Control:
Tick the relevant box (✓)

Controlled Conditions	
Other	

INSTRUCTIONS TO CANDIDATES

Write your Centre Number and Candidate Number in the spaces provided at the top of this page.
You must answer **all twelve** questions in the spaces provided.
Do not write outside the boxed area on each page or on blank pages or tracing paper.
Complete in black ink only. **Do not write with a gel pen**.
Questions which require drawing or sketching should be completed using an HB pencil.
Show clearly the full development of your answers. **Answers without working may not gain full credit**.
Answers should be given to three significant figures unless otherwise stated.
You are permitted to use a graphic or scientific calculator in this paper.

INFORMATION FOR CANDIDATES

The total mark for this paper is 150
Figures in brackets printed down the right-hand side of pages indicate the marks awarded to each question or part question.
A copy of the **Mathematical Formulae and Tables booklet** is provided.
Throughout the paper the logarithmic notation used is $\ln z$ where it is noted that $\ln z \equiv \log_e z$

12355

1 **Fig. 1** below shows the rotor blades for the propeller of a boat.

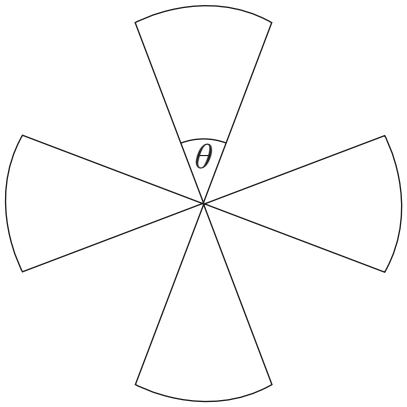


Fig. 1

They are made up of four equal sectors of a circle of radius 30 cm.

(i) If the total area of the blades is 300π cm^2, find the exact value of θ in radians. [3]

Area of one sector = $\frac{1}{2}r^2\theta$

Total area = $4 \times \frac{1}{2}(30)^2\theta = 1800\theta$

$1800\theta = 300\pi$

$\theta = \frac{300\pi}{1800}$ $\therefore$ $\theta = \frac{\pi}{6}$ radians

(ii) Hence find the total perimeter of the blades. [3]

Arc length of one sector = $r\theta = 30 \times \frac{\pi}{6} = 5\pi$ cm

Straight edges: 4 sectors × 2 radii = 8 × 30 = 240 cm

Total perimeter = $4(5\pi) + 8(30)$

$= 20\pi + 240$

$= (20\pi + 240)$ cm $(\approx 303$ cm$)$

[Turn over

2 **(a)** State whether the following sequences diverge, converge or oscillate:

(i) $\dfrac{n}{n^2+1}$ [1]

$u_n = \dfrac{n}{n^2+1} = \dfrac{1}{n+1/n} \to 0$ as $n \to \infty$

Converges (to 0)

(ii) $\sin\left(\dfrac{n\pi}{6}\right)$ [1]

Terms: ½, √3/2, 1, √3/2, ½, 0, −½, −√3/2, −1, −√3/2, −½, 0, ½, ...

The terms repeat every 12 terms between −1 and 1

Oscillates

(b) (i) Use the Binomial Theorem to expand

$$\sqrt{1-3x}$$

in ascending powers of x up to and including the term in x^3 [5]

$\sqrt{(1-3x)} = (1-3x)^{1/2}$

$= 1 + (\frac{1}{2})(-3x) + \frac{(\frac{1}{2})(-\frac{1}{2})}{2!}(-3x)^2 + \frac{(\frac{1}{2})(-\frac{1}{2})(-\frac{3}{2})}{3!}(-3x)^3 + \ldots$

$= 1 - \frac{3}{2}x + \left(\frac{-1}{8}\right)(9x^2) + \left(\frac{1}{16}\right)(-27x^3) + \ldots$

$= 1 - \frac{3}{2}x - \frac{9}{8}x^2 - \frac{27}{16}x^3 + \ldots$

[Turn over

(ii) State the range of values of x for which the expansion is valid. [1]

Valid for $|-3x| < 1$, i.e. $|x| < \frac{1}{3}$

$-\frac{1}{3} < x < \frac{1}{3}$

3 **(a)** Use the Trapezium Rule with 3 ordinates to find an approximate value for

$$\int_2^4 x\cos x\,\mathrm{d}x \qquad [5]$$

3 ordinates ⇒ 2 strips, $h = (4 - 2) \div 2 = 1$

$x = 2$: $y = 2\cos 2 = -0.832293...$

$x = 3$: $y = 3\cos 3 = -2.969977...$

$x = 4$: $y = 4\cos 4 = -2.614574...$

$$\int_2^4 x\cos x\,dx \approx \frac{h}{2}[y_0 + 2y_1 + y_2]$$

$$\approx \frac{1}{2}[-0.832293... + 2(-2.969977...) + (-2.614574...)]$$

$$\approx \frac{1}{2}(-9.386823...)$$

$$= -4.693411... \approx -4.69 \text{ (3 s.f.)}$$

(b) Part of a logo for a ski company can be modelled by the area between the curves $y = 4x - x^2$ and $y = x^2 - 6$ as shown in **Fig. 2** below.

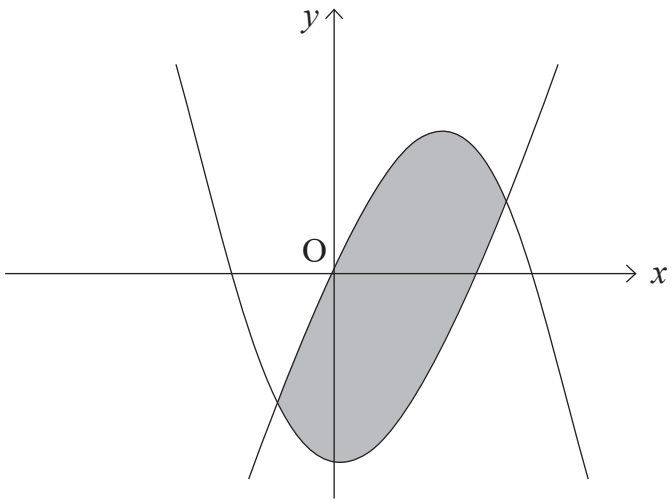


Fig. 2

Use calculus to find this area. [7]

Intersections: $x^2 - 6 = 4x - x^2$

$2x^2 - 4x - 6 = 0$

$2(x - 3)(x + 1) = 0 \Rightarrow x = 3,\ x = -1$

$\text{Area} = \int_{-1}^{3} [(4x - x^2) - (x^2 - 6)]\,dx$

$= \int_{-1}^{3} (4x - 2x^2 + 6)\,dx$

$= \left[2x^2 - \frac{2x^3}{3} + 6x\right]_{-1}^{3}$

$= (18 - 18 + 18) - \left(2 + \frac{2}{3} - 6\right)$

$= 18 + \frac{10}{3} = \frac{64}{3}$

Area = 21.3 (3 s.f.)

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[Turn over

4 **(a)** Find $\frac{dy}{dx}$ for each of the following:

(i) $y = \sec^5(2x)$ [4]

$y = \sec^5(2x) = [\sec(2x)]^5$

$\frac{dy}{dx} = 5\sec^4(2x) \times \frac{d}{dx}[\sec(2x)]$

$= 5\sec^4(2x) \times 2\sec(2x)\tan(2x)$

$\frac{dy}{dx} = 10\sec^5(2x)\tan(2x)$

(ii) $y = \dfrac{\cot x}{e^{4x}}$ [4]

$y = \dfrac{\cot x}{e^{4x}}$ $\quad u = \cot x, \quad v = e^{4x}$

$u' = -\operatorname{cosec}^2 x, \quad v' = 4e^{4x}$

$\dfrac{dy}{dx} = \dfrac{e^{4x}(-\operatorname{cosec}^2 x) - 4e^{4x}\cot x}{(e^{4x})^2}$

$= \dfrac{-e^{4x}(\operatorname{cosec}^2 x + 4\cot x)}{e^{8x}}$

$\dfrac{dy}{dx} = -\dfrac{\operatorname{cosec}^2 x + 4\cot x}{e^{4x}}$

[Turn over

(b) A vase is modelled by rotating the curve

$$y = \sqrt{x} + 2$$

between the lines $x = 0$ and $x = 4$ through 2π radians about the x-axis, as shown in **Fig. 3** below.

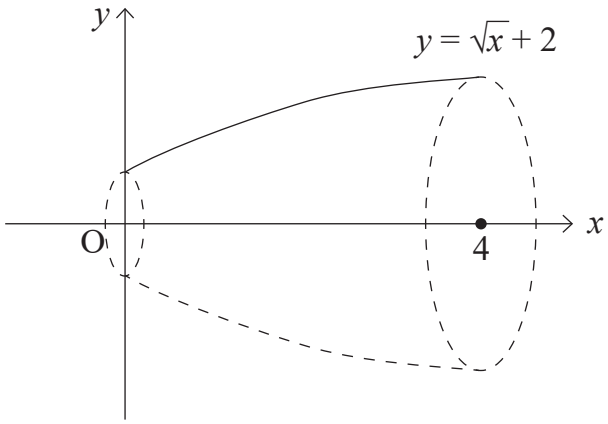


Fig. 3

Find the exact volume of the vase. [7]

$V = \pi\int_0^4 y^2\,dx = \pi\int_0^4 (\sqrt{x}+2)^2\,dx$

$= \pi\int_0^4 (x + 4\sqrt{x} + 4)\,dx = \pi\int_0^4 (x + 4x^{1/2} + 4)\,dx$

$= \pi\left[\frac{x^2}{2} + \frac{8x^{3/2}}{3} + 4x\right]_0^4$

$= \pi\left(\frac{16}{2} + \frac{8(8)}{3} + 16\right) - \pi(0)$

$= \pi\left(8 + \frac{64}{3} + 16\right)$

$V = \frac{136\pi}{3}$

[Turn over

5 **(i)** Write the following expression in partial fractions.

$$\frac{3x^2 - 10x + 5}{(x+1)(x-2)^2}$$ [6]

$\frac{3x^2 - 10x + 5}{(x+1)(x-2)^2} \equiv \frac{A}{x+1} + \frac{B}{x-2} + \frac{C}{(x-2)^2}$

$3x^2 - 10x + 5 \equiv A(x-2)^2 + B(x+1)(x-2) + C(x+1)$

$x = 2$: $12 - 20 + 5 = 3C \Rightarrow 3C = -3 \Rightarrow C = -1$

$x = -1$: $3 + 10 + 5 = 9A \Rightarrow 9A = 18 \Rightarrow A = 2$

$x = 0$: $5 = 4A - 2B + C = 8 - 2B - 1 \Rightarrow B = 1$

$\frac{3x^2 - 10x + 5}{(x+1)(x-2)^2} \equiv \frac{2}{x+1} + \frac{1}{x-2} - \frac{1}{(x-2)^2}$

[Turn over

(ii) Hence show that

$$\int_0^1 \frac{3x^2 - 10x + 5}{(x+1)(x-2)^2}\,dx = \ln 2 - \frac{1}{2}$$ [6]

$\int_0^1 \frac{3x^2 - 10x + 5}{(x+1)(x-2)^2}\,dx$

$= \int_0^1 \left(\frac{2}{x+1} + \frac{1}{x-2} - (x-2)^{-2}\right) dx$

$= \left[2\ln|x+1| + \ln|x-2| + (x-2)^{-1}\right]_0^1$

$= (2\ln 2 + \ln 1 + (-1)) - (2\ln 1 + \ln 2 + (-\tfrac{1}{2}))$

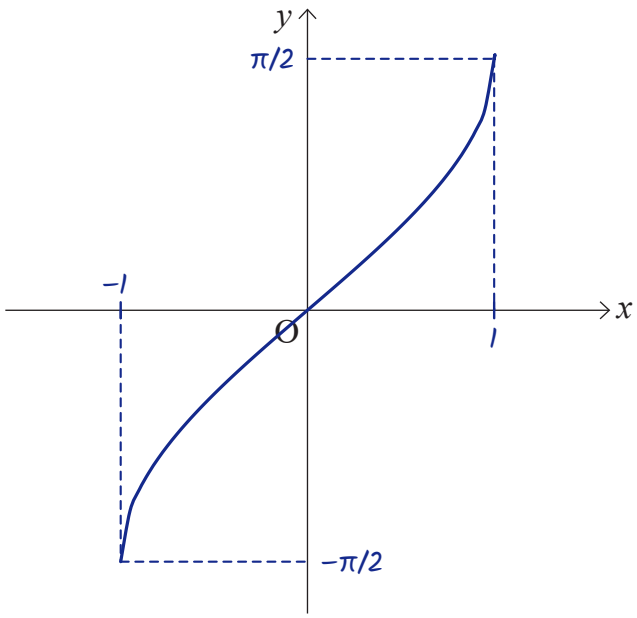
$= (2\ln 2 + 0 - 1) - (0 + \ln 2 - \tfrac{1}{2})$

$= 2\ln 2 - 1 - \ln 2 + \frac{1}{2}$

$= \ln 2 - \frac{1}{2}$ as required

[Turn over

6 **(a)** Sketch the graph of $y = \sin^{-1}(x)$ on the axes below for $-1 \leqslant x \leqslant 1$ [2]



(b) Express

$$\cos x - \sqrt{3}\sin x$$

in the form $R\cos(x + \alpha)$ where R is an integer and $0 < \alpha < 90°$ [7]

$R\cos(x + \alpha) \equiv R\cos x \cos\alpha - R\sin x \sin\alpha$

Comparing with $\cos x - \sqrt{3}\sin x$: $R\cos\alpha = 1$, $R\sin\alpha = \sqrt{3}$

$\frac{R\sin\alpha}{R\cos\alpha} = \tan\alpha = \sqrt{3} \Rightarrow \alpha = 60°$

$R^2(\cos^2\alpha + \sin^2\alpha) = 1^2 + (\sqrt{3})^2 = 4 \Rightarrow R = 2$

$\cos x - \sqrt{3}\sin x = 2\cos(x + 60°)$

[Turn over

(c) Starting with the appropriate compound angle formula, prove that

$$\sin 3\theta \equiv 3\sin\theta - 4\sin^3\theta \qquad [5]$$

$\sin 3\theta = \sin(2\theta + \theta)$

$= \sin 2\theta \cos\theta + \cos 2\theta \sin\theta$

$= (2\sin\theta\cos\theta)\cos\theta + (1 - 2\sin^2\theta)\sin\theta$

$= 2\sin\theta\cos^2\theta + \sin\theta - 2\sin^3\theta$

$= 2\sin\theta(1 - \sin^2\theta) + \sin\theta - 2\sin^3\theta$

$= 2\sin\theta - 2\sin^3\theta + \sin\theta - 2\sin^3\theta$

$= 3\sin\theta - 4\sin^3\theta$

$\therefore \sin 3\theta \equiv 3\sin\theta - 4\sin^3\theta$

7 **(a)** Solve the inequality $|2x - 1| > 5$ [4]

$|2x - 1| > 5$

$2x - 1 > 5$ or $2x - 1 < -5$

$2x > 6$ $2x < -4$

$x > 3$ or $x < -2$

(b) The graph of the function $y = f(x)$ is sketched below in **Fig. 4**
The graph cuts the x-axis at A(2, 0) and has an asymptote of $x = 1$

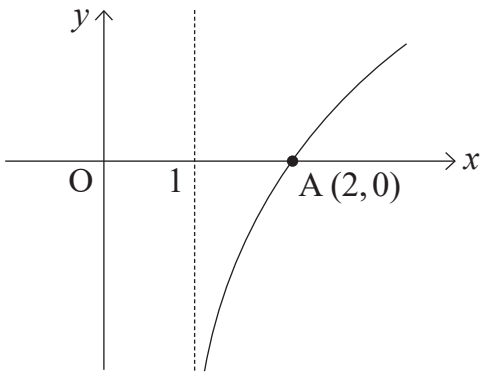
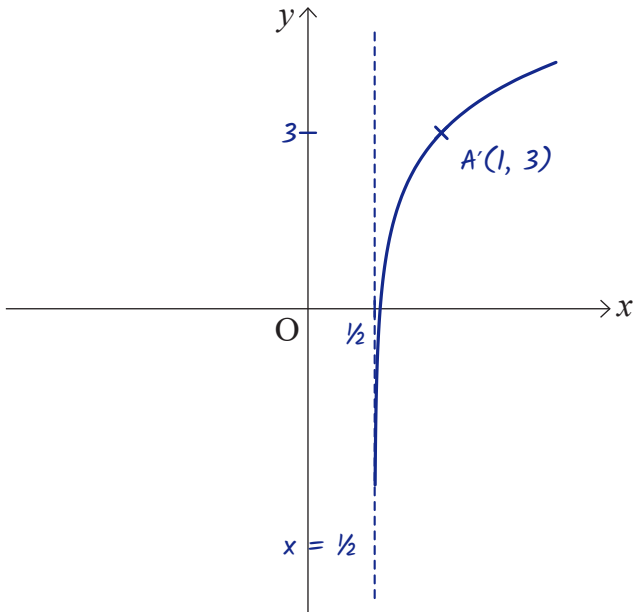


Fig. 4

(i) On the axes below, sketch the graph of

$$y = f(2x) + 3$$

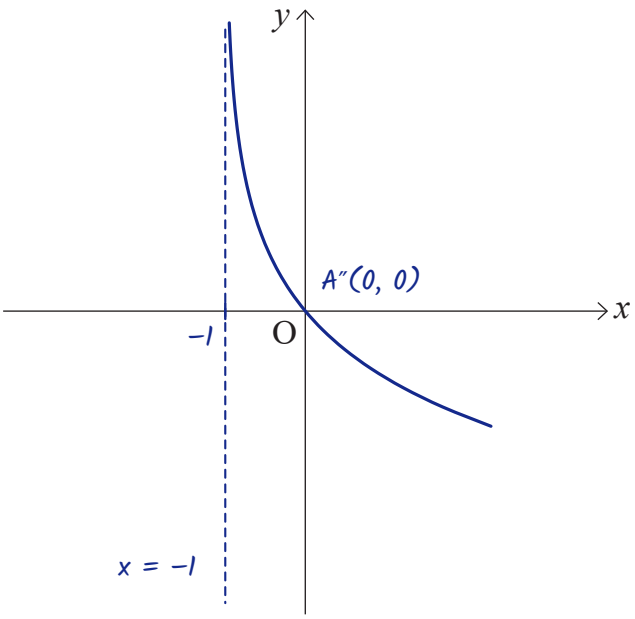
and clearly label the image of A and the asymptote. [3]



(ii) On the axes below, sketch the graph of

$$y = -f(x + 2)$$

and clearly label the image of A and the asymptote. [3]



[Turn over

(c) The functions f and g are defined as:

$$f(x) = \frac{10}{x-1} \qquad x \in \mathbb{R}, x > 1$$

$$g(x) = e^{3x} \qquad x \in \mathbb{R}, x > 0$$

(i) State the range of f(x) and the range of g(x). [2]

Range of f: $f(x) > 0$

Range of g: $g(x) > 1$

(ii) Find the inverse $f^{-1}(x)$, stating its domain. [4]

Let $y = \frac{10}{x-1}$

$y(x-1) = 10$

$yx - y = 10 \Rightarrow yx = 10 + y$

$x = \frac{10+y}{y}$

$f^{-1}(x) = \frac{10+x}{x}$ Domain: $x \in \mathbb{R}, x > 0$



(iii) Find the composite function fg(x), stating its domain. [3]

$fg(x) = f(e^{3x}) = \dfrac{10}{e^{3x} - 1}$

Domain: $x \in \mathbb{R}, x > 0$

[Turn over

8 Find the equation of the tangent to the curve

$$x^2 + 5y\ln x - xy^2 = 0$$

at the point (1, –1). [11]

$x^2 + 5y \ln x - xy^2 = 0$

Differentiating implicitly with respect to x (product rule):

$2x + 5 \ln x \frac{dy}{dx} + \frac{5y}{x} - y^2 - 2xy \frac{dy}{dx} = 0$

At (1, –1):

$2 + 5(0) \frac{dy}{dx} + \frac{5(-1)}{1} - (-1)^2 - 2(1)(-1) \frac{dy}{dx} = 0$

$2 - 5 - 1 + 2 \frac{dy}{dx} = 0 \Rightarrow 2 \frac{dy}{dx} = 4$

$\frac{dy}{dx} = 2$

Tangent: $y - (-1) = 2(x - 1)$

$y + 1 = 2x - 2$

$y = 2x - 3$

9 **(a)** Find

$$\int \ln x \, dx$$ [5]

$\int \ln x \, dx = \int 1 \times \ln x \, dx$

$u = \ln x, \quad \frac{du}{dx} = \frac{1}{x}; \qquad \frac{dv}{dx} = 1, \quad v = x$

$= x \ln x - \int x \times \frac{1}{x} \, dx$

$= x \ln x - \int 1 \, dx$

$= x \ln x - x + c$

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(Questions continue overleaf)

[Turn over

(b) Using the substitution $x = \frac{2}{3}\sin\theta$ evaluate

$$\int_0^{\frac{1}{3}} \sqrt{4 - 9x^2}\, dx$$

[12]

$x = \frac{2}{3}\sin\theta \Rightarrow \frac{dx}{d\theta} = \frac{2}{3}\cos\theta, \quad dx = \frac{2}{3}\cos\theta\, d\theta$

$x = 0 \Rightarrow \theta = 0; \quad x = \frac{1}{3} \Rightarrow \sin\theta = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{6}$

$\int_0^{1/3} \sqrt{(4 - 9x^2)}\, dx = \int_0^{\pi/6} \sqrt{(4 - 9 \cdot \frac{4}{9}\sin^2\theta)} \cdot \frac{2}{3}\cos\theta\, d\theta$

$= \int_0^{\pi/6} \sqrt{(4 - 4\sin^2\theta)} \cdot \frac{2}{3}\cos\theta\, d\theta$

$= \int_0^{\pi/6} \sqrt{(4\cos^2\theta)} \cdot \frac{2}{3}\cos\theta\, d\theta = \int_0^{\pi/6} 2\cos\theta \cdot \frac{2}{3}\cos\theta\, d\theta$

$= \int_0^{\pi/6} \frac{4}{3}\cos^2\theta\, d\theta$

$= \int_0^{\pi/6} \frac{4}{3} \cdot \frac{1}{2}(\cos 2\theta + 1)\, d\theta = \frac{2}{3}\int_0^{\pi/6} (\cos 2\theta + 1)\, d\theta$

$= \frac{2}{3}\left[\frac{1}{2}\sin 2\theta + \theta\right]_0^{\pi/6}$

$= \frac{2}{3}\left(\frac{1}{2} \cdot \frac{\sqrt{3}}{2} + \frac{\pi}{6}\right) - \frac{2}{3}(0 + 0)$

$= \frac{\sqrt{3}}{6} + \frac{\pi}{9}$

$= 0.637740\ldots = 0.638 \ (3\ \text{s.f.})$

[Turn over

10 (i) Find the four values of x for which the curve

$$y = \sqrt{2}\,e^{-x}\cos x$$

has stationary points in the range $0 < x < 4\pi$

Write your values in ascending order. [8]

$y = \sqrt{2}\,e^{-x}\cos x$

$\frac{dy}{dx} = \sqrt{2}(-e^{-x})\cos x + \sqrt{2}\,e^{-x}(-\sin x)$

$= -\sqrt{2}\,e^{-x}\sin x - \sqrt{2}\,e^{-x}\cos x$

Stationary points: $\frac{dy}{dx} = 0$

$-\sqrt{2}\,e^{-x}(\sin x + \cos x) = 0$

$e^{-x} \neq 0$, so $\sin x + \cos x = 0$

$\sin x = -\cos x \Rightarrow \tan x = -1 \quad (\cos x \neq 0)$

$x = \frac{3\pi}{4}, \frac{7\pi}{4}, \frac{11\pi}{4}, \frac{15\pi}{4}$

[Turn over

(ii) Show that the corresponding y values of the stationary points found in **(i)** form the first four terms of a geometric progression. [4]

$x = \frac{3\pi}{4}$: $y = \sqrt{2}\, e^{-3\pi/4} \cos\frac{3\pi}{4} = \sqrt{2}\, e^{-3\pi/4}\left(-\frac{1}{\sqrt{2}}\right) = -e^{-3\pi/4}$

$x = \frac{7\pi}{4}$: $y = \sqrt{2}\, e^{-7\pi/4} \cdot \frac{1}{\sqrt{2}} = e^{-7\pi/4}$

$x = \frac{11\pi}{4}$: $y = \sqrt{2}\, e^{-11\pi/4}\left(-\frac{1}{\sqrt{2}}\right) = -e^{-11\pi/4}$

$x = \frac{15\pi}{4}$: $y = \sqrt{2}\, e^{-15\pi/4} \cdot \frac{1}{\sqrt{2}} = e^{-15\pi/4}$

$\frac{u_2}{u_1} = \frac{e^{-7\pi/4}}{-e^{-3\pi/4}} = -e^{-\pi}$

$\frac{u_3}{u_2} = \frac{-e^{-11\pi/4}}{e^{-7\pi/4}} = -e^{-\pi}$

$\frac{u_4}{u_3} = \frac{e^{-15\pi/4}}{-e^{-11\pi/4}} = -e^{-\pi}$

The common ratio is the same, $r = -e^{-\pi}$, so the y values form a geometric progression with $a = -e^{-3\pi/4}$, $r = -e^{-\pi}$

(iii) For the geometric progression defined in **(ii)**, find an exact value for the sum to infinity. [2]

$S_\infty = \frac{a}{1-r} = \frac{-e^{-3\pi/4}}{1-(-e^{-\pi})} = \frac{-e^{-3\pi/4}}{1+e^{-\pi}}$

$= -\frac{e^{\pi/4}}{e^{\pi}+1}$

[Turn over

11 The path of a train on a model railway can be represented by the parametric equations

$$x = 2t^2 + 4t \quad \text{and} \quad y = \sin 2t$$

(i) Find the co-ordinates of the stationary point on this curve for $0 < t < \frac{\pi}{2}$ [6]

$\frac{dx}{dt} = 4t + 4, \qquad \frac{dy}{dt} = 2\cos 2t$

$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{2\cos 2t}{4t + 4}$

Stationary point: $\frac{dy}{dx} = 0 \Rightarrow 2\cos 2t = 0 \quad (4t + 4 \neq 0)$

$\cos 2t = 0 \Rightarrow 2t = \frac{\pi}{2} \Rightarrow t = \frac{\pi}{4} \qquad (0 < t < \frac{\pi}{2})$

$x = 2\left(\frac{\pi}{4}\right)^2 + 4\left(\frac{\pi}{4}\right) = \frac{\pi^2}{8} + \pi = 4.375293\ldots \approx 4.38$

$y = \sin\left(2 \times \frac{\pi}{4}\right) = \sin\frac{\pi}{2} = 1$

Stationary point: $\left(\frac{\pi^2}{8} + \pi,\ 1\right) \approx (4.38,\ 1)$



(ii) Using $\frac{d^2y}{dx^2}$ classify this stationary point. [6]

$\frac{d^2y}{dx^2} = \frac{d}{dt}\left(\frac{2\cos 2t}{4t+4}\right) \times \frac{dt}{dx}$

$= \frac{(4t+4)(-4\sin 2t) - (2\cos 2t)(4)}{(4t+4)^2} \times \frac{1}{4t+4}$

At $t = \frac{\pi}{4}$: $\sin 2t = 1$, $\cos 2t = 0$, $4t + 4 = \pi + 4$

$\frac{d^2y}{dx^2} = \frac{(\pi+4)(-4) - 0}{(\pi+4)^2} \times \frac{1}{\pi+4}$

$= \frac{-4}{(\pi+4)^2} = -0.0784\ldots < 0$

$\therefore \left(\frac{\pi^2}{8} + \pi,\ 1\right)$ is a maximum stationary point.

[Turn over

12 Water leaks out of a small hole in the bottom of a paddling pool.

The rate at which the water leaves the pool is proportional to the square root of the volume of water that remains in the pool.

The initial volume of water is V_0

If $\frac{5}{9}$ of the initial volume leaks out in the first hour, find how long it takes for the pool to empty. [10]

$\frac{dV}{dt} \propto -\sqrt{V} \quad \Rightarrow \quad \frac{dV}{dt} = -k\sqrt{V} \qquad (k > 0 \text{ constant})$

$\int \frac{1}{\sqrt{V}}\, dV = \int -k\, dt$

$\int V^{-1/2}\, dV = -kt + c \quad \Rightarrow \quad 2V^{1/2} = -kt + c$

$t = 0,\ V = V_0: \quad 2\sqrt{V_0} = c$

$2\sqrt{V} = -kt + 2\sqrt{V_0}$

$t = 1,\ V = V_0 - \frac{5}{9}V_0 = \frac{4}{9}V_0:$

$2\sqrt{(\frac{4}{9}V_0)} = -k + 2\sqrt{V_0} \quad \Rightarrow \quad \frac{4}{3}\sqrt{V_0} = -k + 2\sqrt{V_0}$

$k = \frac{2}{3}\sqrt{V_0}$

$2\sqrt{V} = -\frac{2}{3}\sqrt{V_0}\, t + 2\sqrt{V_0} \quad \Rightarrow \quad \sqrt{V} = \sqrt{V_0} - \frac{\sqrt{V_0}}{3}t$

Pool is empty when $V = 0$:

$0 = \sqrt{V_0} - \frac{\sqrt{V_0}}{3}t \quad \Rightarrow \quad \frac{t}{3} = 1$

$t = 3$ hours

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Question Number	**Marks**
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